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On a Surface Associated to the Catalan Triangle

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Abstract: We define a surface that interpolates the ballot numbers in the Catalan triangle corresponding to every pair of nonnegative integers (except for the origin). We study the geometric properties of this surface and prove that it contains exactly five half-lines. The mean curvature and the Gauss curvature of the surface are also calculated.

Keywords: Catalan numbers; Catalan triangle; immersions; invariants; Gauss curvature; mean curvature; gamma function; digamma function

MSC: 53A05; 41A05; 11B75

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1. Introduction: Catalan Numbers and Catalan Triangle

The Catalan numbers are one of the most well-known sequences of positive integers, being comparable with Fibonacci or Lucas numbers. Richard Stanley [1] collected 214 combinatorial interpretations of Catalan numbers, illustrating their ubiquity. The book also contains a history of the multiple (re)discoveries of Catalan numbers (Appendix B of [1], written by Igor Pak).

The most important combinatorial interpretations of Catalan numbers are synthesised by the following theorem ([1] Theorem 1.5.1):

Theorem 1. *The Catalan number C_n counts the following :*

(i) *Triangulations of a convex polygon with $n+2$ vertices.*
(ii) *Binary trees with n vertices.*
(iii) *Plane trees with $n+1$ vertices.*
(iv) *Bracketings of a string of $n+1$ x's subject to a nonassociative binary operation.*
(v) *Ballot sequences of length $2n$.*
(vi) *Dyck paths of length $2n$.*

The mathematical expression of these magnificent numbers (deeply connected to the binomial coefficients) is (see [2]):

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{1}$$

or, equivalently,

$$C_n = \frac{(2n)!}{n!(n+1)!} = \binom{2n}{n} - \binom{2n}{n-1}. \tag{2}$$

The Catalan numbers are closely related to the *ballot numbers*[3]. These numbers occur in the solution of the *ballot problem*, which can be formulated as follows: two candidates P and Q receive in an election p and q votes, respectively; supposing that P wins $(p > q)$, what is the probability that P stays (strictly) ahead of Q during the counting of votes? The solution was given by J. Bertrand in 1887 [4] and was also found by D. André using the *reflection principle* [5] and reformulating the problem in terms of lattice paths: a counting of votes such that P stays (strictly) ahead of Q corresponds to a lattice path from $(0,0)$ to (p,q) with steps $(1,0)$ and $(0,1)$, staying under the line $y = x$ (and never touching it, except for $(0,0)$). It can be proved (see [6]) that among the $\binom{p+q}{p}$ possible lattice paths (ways of counting the votes), there are exactly $\frac{p-q}{p+q}\binom{p+q}{p}$ that satisfy this condition, which means that the required probability is $\frac{p-q}{p+q}$. The numbers

$$B(p,q) = \frac{p-q}{p+q}\binom{p+q}{p} \tag{3}$$

are called *ballot numbers*.

If the equality of votes is admitted (and in this case we have $p \geq q$), then the number of possible ways of counting is

$$B(p+1,q) = \frac{p-q+1}{p+1}\binom{p+q}{p}.$$

A *ballot sequence* of length $2n$ is a sequence of n 1's and n -1's, such that every partial sum is nonnegative. From the relation above, we obtain that the number of ballot sequences of length $2n$ is the Catalan number C_n:

$$B(n+1,n) = \frac{1}{n+1}\binom{2n}{n} = C_n.$$

If we write the numbers $B(p,q)$, for every $p = 1,2,\ldots$ and $q = 0,1,\ldots,p$, we obtain a triangle where the sequence of Catalan numbers appears twice (see (4)).

$$\begin{array}{ccccccccc}
\mathbf{1} \\
\mathbf{1} & \mathbf{1} \\
1 & \mathbf{2} & \mathbf{2} \\
1 & 3 & \mathbf{5} & \mathbf{5} \\
1 & 4 & 9 & \mathbf{14} & \mathbf{14} \\
1 & 5 & 14 & 28 & \mathbf{42} & \mathbf{42} \\
1 & 6 & 20 & 48 & 90 & \mathbf{132} & \mathbf{132} \\
1 & 7 & 27 & 75 & 165 & 297 & \mathbf{429} & \mathbf{429} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{4}$$

This triangle is known as the Catalan triangle, being recorded as the sequence A009766 in the On-line Encyclopedia of Integer Sequences [7]. It is not the only triangular arrangement of integers known as "Catalan triangle". Another famous example is the one introduced by Shapiro in [8]. Although there are several triangles known as the "Catalan triangle" (see, for instance, [9]), this one consisting of the ballot numbers is "the most-standing form" [10].

In the next section, we use the gamma function to extend the Catalan triangle (4) to a continuous surface explicitly defined by a function $z = z(x,y)$. As we know, such a surface related to a Catalan triangle has not been considered until now.

2. The Surface Associated to the Catalan Triangle

The Pascal surface, which extends the Pascal triangle to real (positive) numbers is defined by the function

$$w(x,y) = \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)}. \tag{5}$$

The geometric properties of this surface are studied in [11], while [12] highlights the relation between the Pascal surface and the coefficients of the reliability polynomials of some networks.

In this paper, we study the surface (S) associated to the Catalan triangle (4), defined in the three-dimensional Euclidean space by the immersion

$$(S): \ f(x,y) = (x, y, z(x,y)),$$

where

$$z(x,y) = \frac{(x-y)\Gamma(x+y)}{\Gamma(x+1)\Gamma(y+1)} = \frac{x-y}{x+y} \cdot w(x,y) \tag{6}$$

is defined for every $(x,y) \in [0,\infty) \times [0,\infty) - \{(0,0)\}$. If x and y are nonnegative integers, then $z(x,y) = \frac{x-y}{x+y}\binom{x+y}{x}$, which are exactly the numbers in the Catalan triangle (4), completed (for $x \leq y$) to an (infinite) antisymmetric matrix:

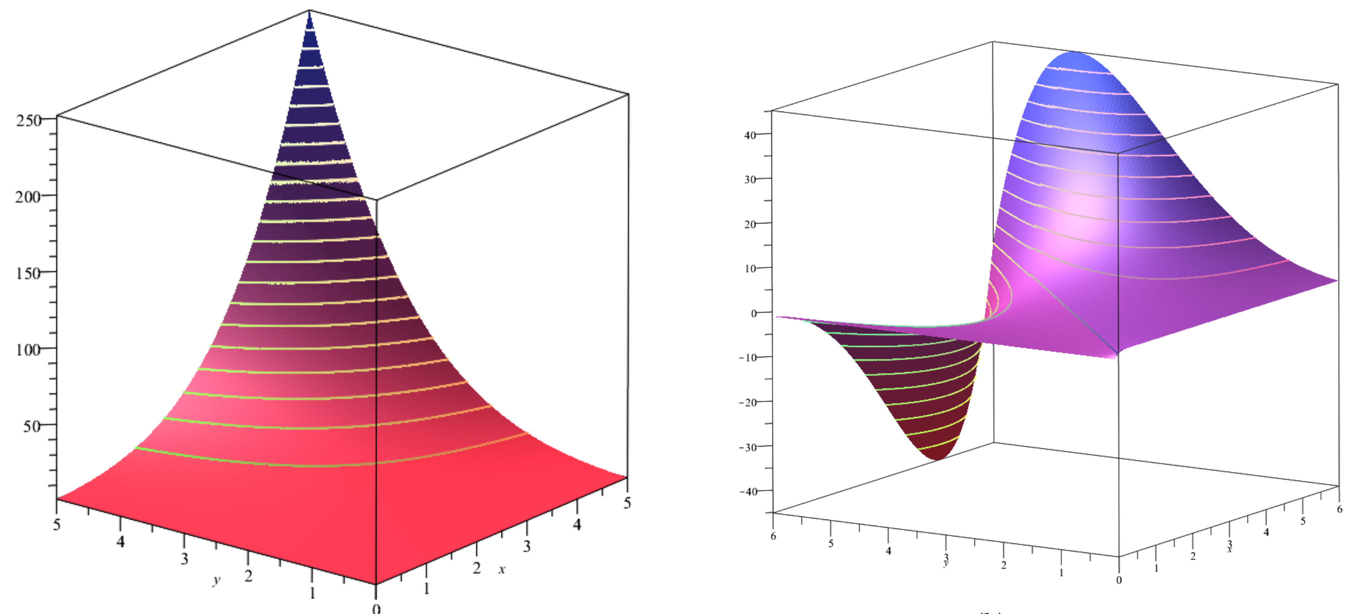
$$\begin{array}{cccccccccc}
* & \mathbf{-1} & \mathbf{-1} & -1 & -1 & -1 & -1 & -1 & -1 & \ldots \\
\mathbf{1} & 0 & \mathbf{-1} & \mathbf{-2} & -3 & -4 & -5 & -6 & -7 & \ldots \\
\mathbf{1} & \mathbf{1} & 0 & \mathbf{-2} & \mathbf{-5} & -9 & -14 & -20 & -27 & \ldots \\
1 & \mathbf{2} & \mathbf{2} & 0 & \mathbf{-5} & \mathbf{-14} & -28 & -48 & -75 & \ldots \\
1 & 3 & \mathbf{5} & \mathbf{5} & 0 & \mathbf{-14} & \mathbf{-42} & -90 & -165 & \ldots \\
1 & 4 & 9 & \mathbf{14} & \mathbf{14} & 0 & \mathbf{-42} & \mathbf{-132} & -297 & \ldots \\
1 & 5 & 14 & 28 & \mathbf{42} & \mathbf{42} & 0 & \mathbf{-132} & \mathbf{-429} & \ldots \\
1 & 6 & 20 & 48 & 90 & \mathbf{132} & \mathbf{132} & 0 & \mathbf{-429} & \ldots \\
1 & 7 & 27 & 75 & 165 & 297 & \mathbf{429} & \mathbf{429} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{7}$$

We remark that

$$\lim_{x \to 0} z(x,0) = 1, \text{ and } \lim_{y \to 0} z(0,y) = -1,$$

so the limit $\lim\limits_{(x,y)\to(0,0)} z(x,y)$ does not exist.

The surface explicitly defined by $z = z(x,y)$, as well as the Pascal surface defined by $z = w(x,y)$ are represented in Figure 1.



(a) (b)

Figure 1. (**a**) Pascal surface; (**b**) the surface associated to the Catalan triangle (4).

The function $z(x,y)$ can be also written as

$$z(x,y) = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y+1)} - \frac{\Gamma(x+y)}{\Gamma(x+1)\Gamma(y)} = w(x-1,y) - w(x,y-1).$$

First of all, since

$$z(y,x) = -z(x,y),$$

we notice the symmetry of the surface with respect to the straight line

$$L_1 : \; x = y, z = 0.$$

We also remark that for $x = y+1$ and $x = y+2$, respectively, where $y = n \in \mathbb{N}$, the Catalan numbers are obtained. Thus, we have:

$$z(n+1,n) = C_n, \quad z(n+2,n) = C_{n+1}$$

and

$$z(n,n+1) = -C_n, \quad z(n,n+2) = -C_{n+1}.$$

These points are represented in Figure 2.

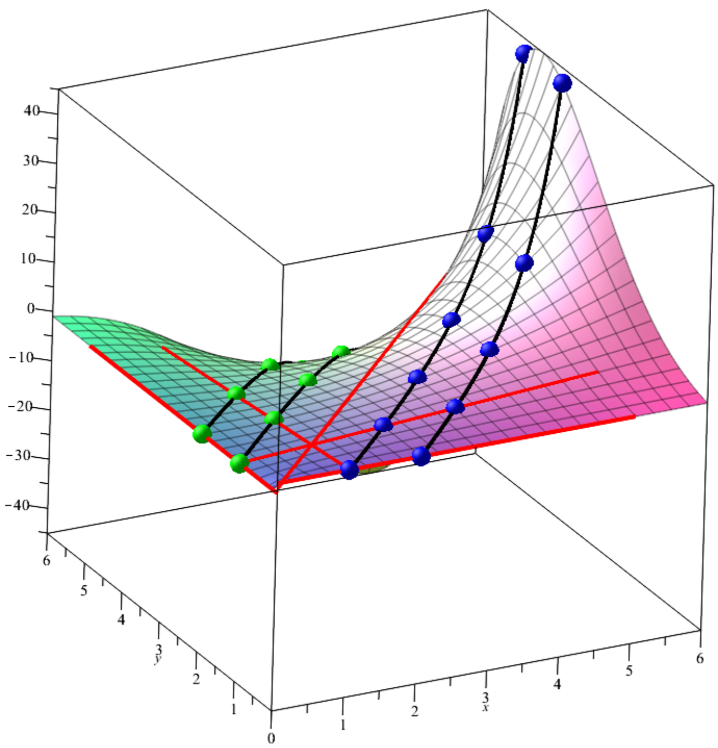


Figure 2. The points corresponding to Catalan numbers C_n (the blue ones) and negative Catalan numbers $-C_n$ (the green ones); the (red) lines $L_1, \ldots, L_5$ contained in the surface (S).

It is easy to see that, besides the axis of symmetry L_1, the surface contains four more straight lines (see Figure 2):

$$\begin{aligned} &L_2 : y = 0, \; z = 1 \\ &L_3 : y = 1, \; z = x-1 \\ &L_4 : x = 0, \; z = -1 \\ &L_5 : x = 1, \; z = 1-y \end{aligned}$$

Theorem 2. *The lines L_i, $i = 1, \ldots, 5$ are the only straight lines contained in the surface (S).*

Proof. First, we prove that the intersection of the immersed surface (S) with a plane $y = k$ (where $k \geq 0$ is a constant) is a straight line if and only if $k = 0$ or $k = 1$. Thus, for $k = 0$ and $k = 1$, the straight lines L_2 and L_3, respectively, are obtained.

Suppose that $y = k > 0, k \neq 1$. Then, for every $x \geq 0$ we have:

$$z(x,k) = \frac{(x-k)\Gamma(x+k)}{\Gamma(x+1)\Gamma(k+1)},$$

hence $z(0,k) = -1$ and $z(1,k) = 1-k$. The straight line defined by the points $(0,k,-1)$ and $(1,k,1-k)$ is

$$y = k,\ z = (2-k)x - 1.$$

Since $z(k,k) = 0$ and $k \neq 1$, it follows that the surface (S) does not contain this line.

Similarly, L_3 and L_4 are the only straight lines obtained by intersecting the surface (S) with planes of the form $x = k$ (L_3 is obtained for $k = 0$ and L_4, for $k = 1$).

Now, let us suppose that

$$y = \alpha x + \beta,$$

with $\alpha \neq 0$, defines a straight line on the surface (S). We prove that, if $\beta = 0$, then $\alpha = 1$ (and so the line L_1 is obtained). For every $x > 0$, we have:

$$z(x,\alpha x) = \frac{(1-\alpha)\Gamma(\alpha x + x)}{\Gamma(x)\Gamma(\alpha x + 1)}. \tag{8}$$

Writing (8) for $x = 1,2,3$, we obtain the following points:

$$(1,\alpha,1-\alpha),\ (2,2\alpha,(1-\alpha)(1+2\alpha)),\ \left(3,3\alpha,\frac{1}{2}(1-\alpha)(1+3\alpha)(2+3\alpha)\right).$$

The only positive value of α for which the points are collinear is $\alpha = 1$.

Now, suppose that $\alpha > 0$ and $\beta > 0$. We have:

$$z(x,\alpha x+\beta) = \frac{((1-\alpha)x-\beta)\Gamma((\alpha+1)x+\beta)}{\Gamma(x+1)\Gamma(\alpha x+\beta+1)}. \tag{9}$$

Taking $x = 0$, $x = 1$ and $x = 2$ in Equation (9), we obtain the points:

$$(0,\beta,-1),\ (1,\alpha+\beta,1-\alpha-\beta) \text{ and } \left(2,2\alpha+\beta,\frac{1}{2}(2-2\alpha-\beta)(1+2\alpha+\beta)\right).$$

The straight line determined by the first two points is defined by:

$$y = \alpha x + \beta,\ z = (2-\alpha-\beta)x - 1. \tag{10}$$

The third point is on this line if and only if the following condition is fulfilled:

$$(2-2\alpha-\beta)(1+2\alpha+\beta) = 2(3-2\alpha-2\beta),$$

or, equivalently,

$$\beta^2 + (4\alpha-5)\beta + 4\alpha^2 - 6\alpha + 4 = 0. \tag{11}$$

This equation has real solutions if and only $\alpha \leq \frac{9}{16}$.

We know that $z(x,x) = 0$ on the surface (S). Since on line (10)

$$x = y \ \Leftrightarrow\ x = \frac{\beta}{1-\alpha},$$

it follows that a necessary condition for line (10) to be contained in (S) is

$$(2-\alpha-\beta)\frac{\beta}{1-\alpha}-1=0,$$

or, equivalently,

$$(\beta-1)(\alpha+\beta+1)=0.$$

If $\alpha+\beta=1$ then (11) becomes $\alpha^2+\alpha=0$, an equation with no positive solutions.

In the other case, if $\beta=1$, then we obtain from (11) that $\alpha=\frac{1}{2}$, so the equations of line (10) are written:

$$y=\tfrac{1}{2}x+1,\ z=\tfrac{1}{2}x-1. \tag{12}$$

It can be easily verified that line (12) is not contained in the surface (S), although they have three common points.

If, in the equation $y=\alpha x+\beta$, we have $\alpha>0$ and $\beta<0$, then we can write

$$x=\frac{1}{\alpha}y-\frac{\beta}{\alpha}$$

and, from the reasoning above, we obtain that no such line is contained into (S).

Finally, we consider the case when $\alpha<0$ and $\beta>0$. Since $y=\alpha x+\beta\geq 0$, we obtain that $x\in\left[0,-\frac{\beta}{\alpha}\right]$. The straight line determined by the points

$$(0,\beta,-1) \text{ and } \left(-\frac{\beta}{\alpha},0,1\right)$$

has the equations:

$$y=\alpha x+\beta, \quad z=-\frac{2\alpha}{\beta}x-1, \tag{13}$$

and, from the condition $z(x,x)=0$, we obtain that $\alpha=-1$, so Equations (13) are written

$$y=-x+\beta, \quad z=\frac{2}{\beta}x-1.$$

We should have

$$z(x,-x+\beta)=(2x-\beta)\frac{\Gamma(\beta)}{\Gamma(x+1)\Gamma(-x+\beta+1)}=\frac{2x-\beta}{\beta},$$

for any $x\in[0,\beta]$. Thus, we obtain that

$$\Gamma(x+1)\Gamma(-x+\beta+1)=\Gamma(\beta+1)$$

for every $x\in[0,\beta]$, $x\neq\beta/2$, which is not possible.

Hence, the only straight lines on the surface (S) are L_i, $i=1,\ldots,5$. □

At the end of this section, we prove the following result regarding the cross section of the surface associated to the Catalan triangle with planes of the form $x=n\in\mathbb{N}$ and $y=n\in\mathbb{N}$, respectively.

Proposition 1. *The curves of the intersection of surface (6) with planes of the form $x=n\in\mathbb{N}$ or $y=n\in\mathbb{N}$ are the polynomials of degree n.*

Proof. Suppose that $y=n\in\mathbb{N}$. Then,

$$z(x,n)=(x-n)\cdot\frac{\Gamma(x+n)}{\Gamma(x+1)\Gamma(n+1)}=\frac{1}{n!}(x-n)(x+n-1)(x+n-2)\ldots(x+1),$$

which is a polynomial of degree n. The case $x=n\in\mathbb{N}$ can be treated similarly. □

3. Geometric Properties of the Surface Associated to the Catalan Triangle

We use the digamma function ψ, which is defined as the logarithmic derivative of the gamma function (see [13]):

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For formulas on the geometry of surfaces, we refer the reader to [14,15].

We have:

$$\frac{\partial z}{\partial x} = z(x,y)\left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y}\right),$$
$$\frac{\partial z}{\partial y} = z(x,y)\left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x}\right).$$

The coefficients of the first fundamental form are given by

$$g_{11} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle,\ g_{12} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle,\ g_{22} = \left\langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y} \right\rangle,$$

hence

$$g_{11} = 1 + z^2(x,y)\left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y}\right)^2,$$
$$g_{12} = z^2(x,y)\left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y}\right)\left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x}\right),$$
$$g_{22} = 1 + z^2(x,y)\left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x}\right)^2.$$

Using these formulas, we determine

$$\begin{aligned}\det g = g_{11}g_{22} - g_{12}^2 &= 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \\ &= 1 + z^2(x,y)\left[\left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y}\right)^2 + \left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x}\right)^2\right].\end{aligned}$$

The unit normal vector $\mathbf{n}$ to the surface is given by

$$\mathbf{n} = \frac{1}{\sqrt{\det g}} \frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y} = \frac{1}{\sqrt{\det g}}\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right).$$

By straightforward differentiation we obtain:

$$\frac{\partial^2 z}{\partial x^2} = z(x,y)\left[\left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y}\right)^2 + \psi'(x+y) - \psi'(x+1) - \frac{1}{(x-y)^2}\right],$$
$$\frac{\partial^2 z}{\partial y^2} = z(x,y)\left[\left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x}\right)^2 + \psi'(x+y) - \psi'(y+1) - \frac{1}{(y-x)^2}\right],$$
$$\frac{\partial^2 z}{\partial x \partial y} = z(x,y)\left[\left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y}\right)\left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x}\right) + \psi'(x+y) + \frac{1}{(x-y)^2}\right]$$

It follows that the coefficients of the second fundamental form,

$$h_{11} = \left\langle \frac{\partial^2 f}{\partial x^2}, \mathbf{n} \right\rangle, \quad h_{22} = \left\langle \frac{\partial^2 f}{\partial y^2}, \mathbf{n} \right\rangle, \quad h_{12} = \left\langle \frac{\partial^2 f}{\partial x \partial y}, \mathbf{n} \right\rangle$$

are given by the formulas:

$$h_{11} = \frac{z(x,y)}{\sqrt{\det g}} \left[\left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y} \right)^2 + \psi'(x+y) - \psi'(x+1) - \frac{1}{(x-y)^2} \right],$$
$$h_{22} = \frac{z(x,y)}{\sqrt{\det g}} \left[\left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x} \right)^2 + \psi'(x+y) - \psi'(y+1) - \frac{1}{(x-y)^2} \right],$$
$$h_{12} = \frac{z(x,y)}{\sqrt{\det g}} \left[\left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y} \right) \left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x} \right) + \psi'(x+y) + \frac{1}{(x-y)^2} \right].$$

Now, we can compute the mean curvature H, the main extrinsic invariant of the surface (S), and the Gauss curvature G, the main intrinsic invariant, respectively:

$$\begin{aligned} H(x,y) &= \frac{g_{22}h_{11} - 2g_{12}h_{12} + g_{11}h_{22}}{2\det g} \\ &= \frac{z(x,y)}{2(\det g)^{3/2}} \Bigg\{ \left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y} \right)^2 + \left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x} \right)^2 \\ &+ 2\psi'(x+y) - \psi'(x+1) - \psi'(y+1) - \frac{2}{(x-y)^2} \\ &+ z^2(x,y) \Bigg[\psi'(x+y) \left(\psi(x+1) - \psi(y+1) - \frac{2}{x-y} \right)^2 - \frac{1}{(x-y)^2} \Big(2\psi(x+y) - \psi(x+1) - \psi(y+1) \Big)^2 \\ &- \psi'(x+1) \left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x} \right)^2 - \psi'(y+1) \left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y} \right)^2 \Bigg] \Bigg\}. \end{aligned}$$

$$\begin{aligned} G(x,y) &= \frac{h_{11}h_{22} - h_{12}^2}{\det g} \\ &= \frac{z^2(x,y)}{(\det g)^2} \Bigg\{ \psi'(x+y) \left[\left(\psi(x+1) - \psi(y+1) - \frac{2}{x-y} \right)^2 - \frac{4}{(x-y)^2} \right] \\ &- \frac{1}{(x-y)^2} \Big(2\psi(x+y) - \psi(x+1) - \psi(y+1) \Big)^2 - \left(\psi'(x+y) - \frac{1}{(x-y)^2} \right) \Big(\psi'(x+1) + \psi'(y+1) \Big) \\ &- \psi'(x+1) \left(\psi(x+y) - \psi(y+1) + \frac{1}{y-x} \right)^2 - \psi'(y+1) \left(\psi(x+y) - \psi(x+1) + \frac{1}{x-y} \right)^2 \Bigg\}. \end{aligned}$$

Remark 1. *If $z(x,y)$ is an antisymmetric function, that is,*

$$z(y,x) = -z(x,y),$$

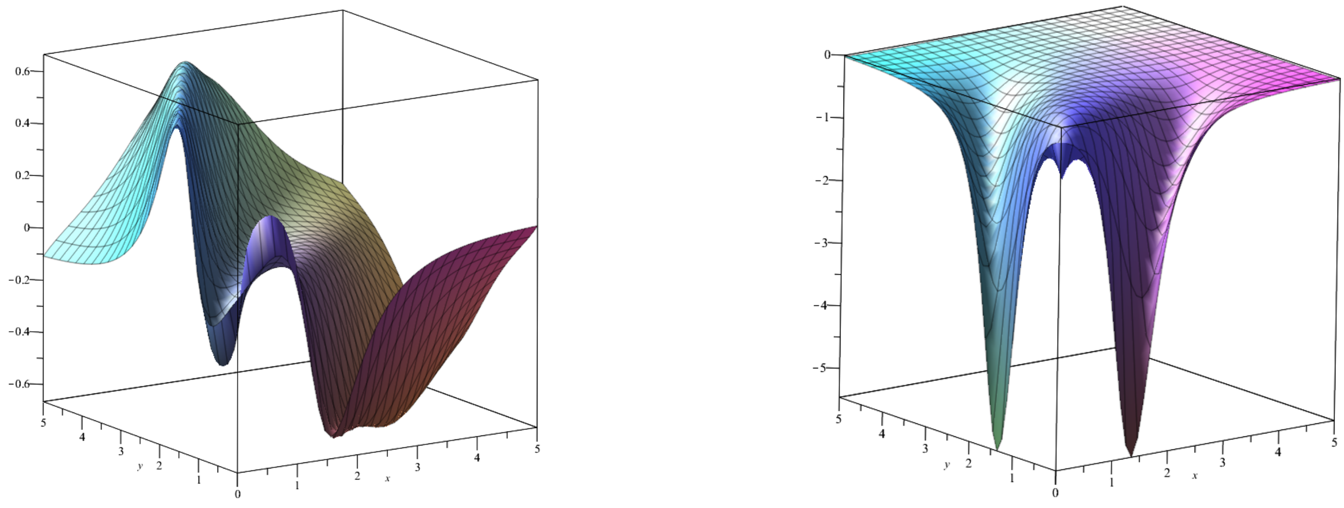
then the surface explicitly defined by $z = z(x,y)$ has the mean curvature $H(x,y)$ with the same property,

$$H(y,x) = -H(x,y),$$

while the Gauss curvature $G(x,y)$ is a symmetric function:

$$G(y,x) = G(x,y).$$

These properties can be easily observed in Figure 3, which presents the mean curvature and the Gauss curvature of the surface (S).



(**a**) (**b**)

Figure 3. (**a**) The mean curvature $H(x, y)$; (**b**) the Gauss curvature $G(x, y)$

4. Conclusions

In our paper, we studied mainly the basic geometrical properties of the surface associated to the Catalan triangle (for example, we determined the main extrinsic and intrinsic invariants of that surface, namely, the mean curvature and the Gauss curvature, and plotted their visualizations). We intend to continue our work in the future, from a theoretical point of view, as well as looking for applications, for instance in generalized Riemann spaces. To our knowledge, such a surface related to a Catalan triangle, immersed into the three-dimensional Euclidean space, has not been considered until now.

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