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Confidence Intervals for the Ratio of Variances of Delta-Gamma Distributions with Applications

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Abstract: Since rainfall data often contain zero observations, the ratio of the variances of delta-gamma distributions can be used to compare the rainfall dispersion between two rainfall datasets. To this end, we constructed the confidence interval for the ratio of the variances of two delta-gamma distributions by using the fiducial quantity method, Bayesian credible intervals based on the Jeffreys, uniform, or normal-gamma-beta priors, and highest posterior density (HPD) intervals based on the Jeffreys, uniform, or normal-gamma-beta priors. The performances of the proposed confidence interval methods were evaluated in terms of their coverage probabilities and average lengths via Monte Carlo simulation. Our findings show that the HPD intervals based on Jeffreys prior and the normal-gamma-beta prior are both suitable for datasets with a small and large probability of containing zeros, respectively. Rainfall data from Phrae province, Thailand, are used to illustrate the practicability of the proposed methods with real data.

Keywords: fiducial quantities; highest posterior density; Jeffreys prior; uniform prior; normal-gamma-beta prior

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1. Introduction

For statistical inference, variance is the second central moment that gives a measure of the spread or variability of a distribution and is often used for probability and statistical inference. Many researchers have studied and constructed the confidence interval for the variance of various distributions by using several methods. For example, Harvey and van der Merwe [1] proposed Bayesian confidence interval methods for the means and variances of lognormal and bivariate lognormal distributions. Niwitpong [2] suggested the generalized confidence interval approach for a function of the variance of a lognormal distribution. Puggard et al. [3] constructed the confidence intervals for the variance and difference between the variances of several Birnbaum-Saunders distributions. Puggard et al. [4] proposed the confidence interval for comparing the variances of two independent Birnbaum–Saunders distributions.

Populations containing positive observations, such as environmental data, can be reasonably assessed by using a gamma distribution [5]. Gibbons and Coleman [6] pointed out that the use of a gamma distribution is more appropriate than a normal distribution when variability and concentration are related, as is the case with many environmental datasets. However, rainfall data often contain zero observations, which violates the necessity for positive data for gamma modeling, and so this must be taken into account when studying this phenomenon. Aitchison [7] provided guidelines for modeling populations containing zero observations whereby the probability of obtaining zeros (δ) is constrained by $0 < \delta < 1$ while the positive observations comprise the remaining probability $(1 - \delta)$. Later, Aitchison and Brown [8] coped with this issue by introducing the delta-lognormal distribution in which the number of zero observations can be viewed as a random variable with a binomial distribution while the positive observations are assumed to be from a random variable with a lognormal distribution.

Many researchers have developed various methods for constructing confidence intervals for various parameters of a delta-lognormal distribution. For example, Yosboonruang et al. [9] constructed the confidence interval for the coefficient of variation of a single delta-lognormal distribution. Maneerat et al. [10,11] suggested Bayesian confidence interval methods for the variance of a delta-lognormal distribution and the difference between the variances of delta-lognormal distributions and applied them to analyze rainfall dispersion. Maneerat et al. [12] used the Bayesian approach to construct the confidence interval for comparing the ratio of the variances of delta-lognormal-distributed rainfall dispersion datasets in Thailand. Zhang et al. [13] proposed simultaneous confidence intervals for the ratio of the means of zero-inflated lognormal populations. In a slightly different approach, Ren et al. [14] proposed simultaneous confidence intervals for the difference between the means of multiple zero-inflated gamma distributions by using one exact and two approximate fiducial methods and applied them for analyzing precipitation datasets and found that the exact method provided more accurate results. Muralidharan and Kale [15] proposed a modified gamma distribution with a singularity at zero and thereby obtained the confidence interval for the mean of the mixed distribution. Lecomte et al. [16] provided compound Poisson-gamma and delta-gamma distributions to handle zero-inflated continuous data under a variable sampling regime. Kaewprasert et al. [17] used Bayesian estimation for the mean of delta-gamma distributions with application to rainfall data in Thailand. Khooriphan et al. [18] proposed a Bayesian estimation of rainfall dispersion in Thailand using gamma distribution with excess zeros. Wang et al. [19] proposed confidence interval methods for the parameters of a zero-inflated gamma distribution.

The ratio of the variances of two populations of rainfall data containing zero observations, which can thus be modeled by using the delta-gamma distribution, is a suitable approach for comparing rainfall dispersion in two areas. Thus, we constructed the confidence interval for the ratio of the variances of delta-gamma distributions by using six Bayesian approaches: Bayesian credible intervals based on the Jeffreys (BAY-J), uniform (BAY-U), or normal-gamma-beta (BAY-NGB) priors and highest posterior density (HPD) intervals based on the Jeffreys (HPD-J), uniform (HPD-U), or normal-gamma-beta (HPD-NGB) priors and compared them with the fiducial quantity (FQ) approach.

This article is organized as follows. The theoretical background for the proposed methods for constructing the confidence interval for the ratio of variances of delta-gamma distributions is covered in Section 2. Simulation study parameters and results are presented in Section 3. The application of the methods to real datasets is reported in Section 4. Finally, conclusions based on the study are covered in Section 5.

2. The Confidence Interval for the Ratio of the Variances of Two Delta-Gamma Distributions

Let $\mathbf{X}_i = (X_{i1}, X_{i2}, ..., X_{in_i}); i = 1, 2$ be a random sample from a delta-gamma distribution, denoted as $X_i \sim \Delta(\delta_i, \alpha_i, \beta_i)$. The distribution function of a delta-gamma can be derived as

$$G(x_i; \delta_i, \alpha_i, \beta_i) = \begin{cases} \delta_i & ; \quad x_i = 0, \\ \delta_i + (1-\delta_i)F(x_i; \alpha_i, \beta_i) & ; \quad x_i > 0 \end{cases} \tag{1}$$

where $F(x_i; \alpha_i, \beta_i)$ stands for the gamma cumulative distribution function. The mean and variance of a gamma(α_i, β_i) distribution with shape parameter α_i and scale parameter β_i can be defined as $\alpha_i\beta_i$ and $\alpha_i\beta_i^2$, respectively. The zero and non-zero observed values are denoted as $n_{i,(0)}$ and $n_{i,(1)}$, respectively, where $n_i = n_{i,(0)} + n_{i,(1)}$. The zero observations follow binomial distribution $n_{i,(0)} \sim Bin(n_i, \delta_i)$ while the non-zero observations follow a gamma distribution.

The maximum likelihood estimators of parameters α_i, β_i, and δ_i can be defined as

$$\widehat{\alpha}_i = \frac{1}{2(\log \overline{X}_i - \sum_{i=1}^{n_i} \log X_i / n_i)}, \widehat{\beta}_i = \overline{X}_i / \widehat{\alpha}_i, \text{ and } \widehat{\delta}_i = n_{i,(0)}/n_i; i = 1, 2.$$

where $\overline{X}_i$ is the sample mean of X_i [20].

According to Aitchison [7], the population mean and variance of X_i can be written as

$$E(X_i) = (1 - \delta_i) \cdot (\alpha_i \beta_i) \tag{2}$$

$$Var(X_i) = \tau_i = (1 - \delta_i) \cdot (\alpha_i \beta_i^2) + \delta_i (1 - \delta_i) \cdot (\alpha_i \beta_i)^2 \tag{3}$$

Subsequently, the ratio of the two variances becomes

$$\theta = \frac{\tau_1}{\tau_2} \tag{4}$$

The methods to construct the confidence interval for θ are proposed in the following sub-sections.

2.1. The Fiducial Quantity Method

Krishnamoorthy et al. [21] developed FQs based on cube-root-transforming a sample. Let $X_i = (X_{i1}, X_{i2}, ..., X_{in_i}); i = 1, 2$ be a random sample from a delta-gamma distribution with shape parameter a_i and scale parameter b_i. For $Y_i = X_i^{\frac{1}{3}}; i = 1, 2$, then Y_i is approximately normally distributed with respective means and variances μ_i and σ_i^2 given by

$$\mu_i = (b_i a_i^{\frac{1}{3}}) \left(1 - \frac{1}{9a_i}\right) \quad \text{and} \quad \sigma_i^2 = \frac{b_i^{\frac{2}{3}}}{9a_i^{\frac{1}{3}}} \tag{5}$$

Consider the stochastic representations

$$\overline{X}_i \overset{d}{\sim} \mu_i + Z_i \frac{\sigma_i}{\sqrt{n_i}} \quad \text{and} \quad S_i^2 \overset{d}{\sim} \sigma_i^2 \frac{\chi^2_{n_i-1}}{n_i - 1} \tag{6}$$

where notation "$\overset{d}{\sim}$" means "distributed as". Let $\overline{x}_i$ and s_i are the observed values of $\overline{X}_i$ and S_i, respectively; Z_i is an independent random variable from a standard normal distribution; and $\chi^2_{n_i-1}$ is an independent random variable from Chi-squared distribution (n_i is the sample size). By solving the above equations for μ_i and σ_i^2, we arrive at

$$\begin{aligned}
\text{from} \quad & S_i^2 \overset{d}{\sim} \sigma_i^2 \tfrac{\chi^2_{n_i-1}}{n_i-1} \\
\text{then} \quad & \sigma_i^2 \overset{d}{\sim} \tfrac{(n_i-1)S_i^2}{\chi^2_{n_i-1}} \\
\text{so} \quad & \sigma_i \overset{d}{\sim} \tfrac{\sqrt{n_i-1}S_i}{\sqrt{\chi^2_{n_i-1}}} \\
\text{and} \quad & \overline{X}_i \overset{d}{\sim} \mu_i + Z_i \tfrac{\sigma_i}{\sqrt{n_i}} \\
\text{then} \quad & \overline{X}_i \overset{d}{\sim} \mu_i + \tfrac{Z_i\sqrt{n_i-1}}{\sqrt{\chi^2_{n_i-1}}} \cdot \tfrac{S_i}{\sqrt{n_i}}
\end{aligned}$$

by replacing $\overline{X}_i$ and S_i with $\overline{x}_i$ and s_i respectively. Thus the FQs of μ_i and σ_i^2 are

$$Q_{\mu_i} = \overline{x}_i + \frac{Z_i \sqrt{n_i - 1}}{\sqrt{\chi^2_{n_i-1}}} \cdot \frac{s_i}{\sqrt{n_i}} \quad \text{and} \quad Q_{\sigma_i^2} = \frac{(n_i - 1)s_i^2}{\chi^2_{n_i-1}} \tag{7}$$

The above FQs, $\overline{x}_i$ and s_i are fixed, and Z_i and $\chi^2_{n_i-1}$ are random variables whose distributions do not depend on any parameters.

Meanwhile, the respective FQs for δ_i are as follows [22]

$$Q_{\delta_i} \sim \frac{1}{2}\text{Beta}(n_{i,(1)}, n_{i,(0)} + 1) + \frac{1}{2}\text{Beta}(n_{i,(1)} + 1, n_{i,(0)}) \tag{8}$$

The FQs for the means are [5]

$$Q_{M_i} = \left\{ \frac{Q_{\mu_i}}{2} + \sqrt{\left(\frac{Q_{\mu_i}}{2}\right)^2 + Q_{\sigma_i^2}} \right\}^3 \tag{9}$$

We can express the FQs for the variances as follows:
Let $V_i = a_i b_i^2$. Thus, we can rewrite Equation (5). as

$$\mu_i = V_i^{\frac{1}{3}} b_i^{\frac{-1}{3}} \left(1 - \frac{b_i^2}{9V_i}\right) \quad \text{and} \quad \sigma_i^2 = \frac{b_i^{\frac{4}{3}}}{9V_i^{\frac{1}{3}}}$$

We can find $V_i = \left\{ \frac{\mu_i + \sqrt{\mu_i^2 + 4\sigma_i^2}}{2(9^{-1/4})(\sigma_i^2)^{-1/4}} \right\}^4$ by solving the above equations for V_i. Thus, the FQs for the variances are obtained as:

$$Q_{V_i} = \left\{ \frac{Q_{\mu_i} + \sqrt{Q_{\mu_i}^2 + 4Q_{\sigma_i^2}}}{2(9^{-1/4})(Q_{\sigma_i^2})^{-1/4}} \right\}^4 \tag{10}$$

where Q_{μ_i} and $Q_{\sigma_i^2}$ are defined as in Equation (7).

Hence, the FQs for τ_1 and τ_2 become

$$\begin{aligned} Q_{\tau_1} &= (1 - Q_{\delta_1}) \cdot Q_{V_1} + Q_{\delta_1}(1 - Q_{\delta_1}) \cdot Q_{M_1}^2 \\ Q_{\tau_2} &= (1 - Q_{\delta_2}) \cdot Q_{V_2} + Q_{\delta_2}(1 - Q_{\delta_2}) \cdot Q_{M_2}^2 \end{aligned} \tag{11}$$

Thus, the FQs for the ratio of the variances of two delta-gamma distributions can be derived as

$$Q_\theta = \frac{Q_{\tau_1}}{Q_{\tau_2}} \tag{12}$$

Therefore, the equal-tails $100(1-\alpha)\%$ FQ interval for the ratio of variances can be defined by

$$CI_{FQ} = [Q_\theta(\alpha/2), Q_\theta(1-\alpha/2)] \tag{13}$$

where $Q_\theta(\alpha/2)$ and $Q_\theta(1-\alpha/2)$ are the $100(\alpha/2)-$th and $100(1-\alpha/2)-$th percentiles of the distribution of Q_θ, respectively.

The confidence interval for the ratio of variances θ can be obtained by executing Algorithm 1.

Algorithm 1 FQ

1: For a given sample from $X_i \sim \Delta(\delta_i, \alpha_i, \beta_i)$, compute $\bar{x}_i$ and s_i^2 of the cube-root-transformed sample.
2: Generate standard normal variate Z_i and Chi-squared variate $\chi^2_{n_i-1}$.
3: Generate Beta$(n_{i,(1)}, n_{i,(0)}+1)$ and Beta$(n_{i,(1)}+1, n_{i,(0)})$.
4: Compute Q_{μ_i}, $Q_{\sigma_i^2}$, and Q_{δ_i} from Equations (7). and (8).
5: Compute the FQs for the mean (Q_{M_i}) and variance (Q_{V_i}) of a gamma distribution from Equations (9). and (10), respectively.
6: Compute Q_{τ_i} and Q_θ from Equations (11) and (12), respectively.
7: Repeat Steps 2–6 5000 times and obtains an array of Q_θ.
8: Compute the 95% confidence interval for θ from Equation (13).
9: Repeat Steps 1–8 10,000 times to compute the coverage probability (CP) and the average length (AL).

2.2. The Bayesian Methods

The Bayesian credible interval involves estimating the parameter of interest from the posterior distribution [23], while HPD intervals are based on the Bayesian approach where the posterior density for every parameter value within the confidence region is higher than those outside of the region [24]. HPD is regarded as the narrowest possible interval for the parameter of interest for probability $100(1-\alpha)\%$ [25]. Box and Tiao [26] described the HPD definition as follows:

Definition 1. *Let $p(\theta|y)$ be a posterior density function. A region R in the parameter space of θ is called a HPD region of content $(1-\alpha)$ if*
(i) $Pr(\theta \in R|y) = 1-\alpha$,
(ii) For $\theta_1 \in R$ and $\theta_2 \notin R$, $p(\theta_1 R|y) \geq p(\theta_2 R|y)$.

We can explain that the HPD interval has two main properties for a given probability level $1-\alpha$, the interval has the narrowest length and every point within the interval has a higher probability density than the points outside of it.

In this section, the Bayesian credible interval approaches based on the Jeffreys, uniform, and normal-gamma-beta priors are presented.

2.2.1. The Bayesian Methods Using the Jeffreys Prior

This is derived from the square root of the Fisher information matrix [27]; i.e., $p(\theta) \propto \sqrt{|I(\theta)|}$. Since $X_i; i = 1,2$ are random samples from a delta-gamma distribution. For, $Y_i = X_i^{\frac{1}{3}}; i = 1,2$, then Y_i is approximately normally distributed with means μ_i and variance and σ_i^2. The delta-gamma distribution for three unknown parameters can be denoted as $\omega_i = (\delta_i, \mu_i, \sigma_i^2)$ with likelihood function

$$p(x_i|\omega_i) \propto \delta^{n_{i,(0)}}(1-\delta^{n_{i,(1)}}) \prod_{i=1}^{n_i} (\sigma_i^2)^{-1/2} \left(-\tfrac{1}{2\sigma_i^2}(x_i - \mu_i) \right)$$

Therefore, the Fisher information for ω_i becomes

$$I(\omega_i) = diag \left[\tfrac{n_i}{\delta_i(1-\delta_i)} \; \tfrac{n_{i,(1)}}{\sigma_i^2} \; \tfrac{n_{i,(1)}}{2\sigma_i^2} \right]$$

Bolstad and Curran [25] defined the Jeffreys prior for δ_i in a binomial distribution as $p(\delta_i) \propto (\delta_i)^{-\frac{1}{2}}(1-\delta_i)^{\frac{1}{2}}$. This allows us to obtain the marginal posterior distribution of δ_i as

$$\delta_{i(jef)}|x_i \sim Beta\left(n_{i,(0)} + \frac{1}{2}, n_{i,(1)} + \frac{3}{2}\right) \tag{14}$$

Subsequently, the Jeffreys prior for σ_i^2 in a lognormal distribution is $p(\sigma_i^2) \propto \sigma_i^{-2}$. Therefore, the marginal posterior distribution of σ_i^2 is

$$\sigma_{i(jef)}^2 | x_i \sim IG\left(\frac{n_{i,(1)}}{2}, \frac{\sum_{i=1}^{n_i}(x_i - \mu_i)^2}{2}\right) \tag{15}$$

and the marginal posterior distribution of μ_i is

$$\mu_{i(jef)} | \sigma_{i(jef)}^2, x_i \sim N(\overline{x}_i, \sigma_{i(jef)}^2 / n_{i,(1)}) \tag{16}$$

We compute the mean and variance of the gamma distribution by using $\mu_{i(jef)} | \sigma_{i(jef)}^2, x_i$ and $\sigma_{i(jef)}^2 | x_i$, respectively, as follows:

$$M_{i(BAY-J)} = \left\{ \frac{\mu_{i(jef)}}{2} + \sqrt{\left(\frac{\mu_{i(jef)}}{2}\right)^2 + \sigma_{i(jef)}^2} \right\}^3 \tag{17}$$

$$V_{i(BAY-J)} = \left\{ \frac{\mu_{i(jef)} + \sqrt{\mu_{i(jef)} + 4\sigma_{i(jef)}^2}}{2(9^{-1/4})(\sigma_{i(jef)}^2)^{-1/4}} \right\}^4 \tag{18}$$

Subsequently,

$$\widehat{\tau}_{i(BAY-J)} = (1 - \delta_{i(jef)}) \cdot V_{i(BAY-J)} + \delta_{i(jef)}(1 - \delta_{i(jef)}) \cdot M_{i(BAY-J)}^2 \tag{19}$$

such that

$$\widehat{\theta}_{BAY-J} = \frac{\widehat{\tau}_{1(BAY-J)}}{\widehat{\tau}_{2(BAY-J)}} \tag{20}$$

The Bayesian credible interval and HPD interval for the ratio of variances of delta-gamma distributions are respectively obtained as

$$CI_{BAY-J} = [\widehat{\theta}_{BAY-J}(\alpha/2), \widehat{\theta}_{BAY-J}(1 - \alpha/2)] \tag{21}$$

2.2.2. The Bayesian Methods Using the Uniform Prior

For the uniform prior that gives equally likely a priori to all possible values [28], the prior probability is a constant function [29]. The uniform prior for δ_i in binomial distribution is $p(\delta_i) \propto 1$ [25], which leads to obtaining the marginal posterior distribution of δ_i as

$$\delta_{i(unif)} | x_i \sim Beta(n_{i,(0)} + 1, n_{i,(1)} + 1) \tag{22}$$

Kalkur and Rao [30] defined the uniform prior of σ_i^2 as $\sigma_i^2 \propto 1$, with the marginal posterior distribution of σ_i^2 being

$$\sigma_{i(unif)}^2 | x_i \sim IG\left(\frac{n_{i,(1)} - 2}{2}, \frac{\sum_{i=1}^{n_i}(x_i - \mu_i)^2}{2}\right) \tag{23}$$

Similarly, the marginal posterior distribution of μ_i is

$$\mu_{i(unif)} | \sigma_{i(unif)}^2, x_i \sim N(\overline{x}_i, \sigma_{i(unif)}^2 / n_{i,(1)}) \tag{24}$$

We can compute the mean and variance of gamma by using $\mu_{i(unif)} | \sigma_{i(unif)}^2, x_i$ and $\sigma_{i(unif)}^2 | x_i$, respectively, as follows:

$$M_{i(BAY-U)} = \left\{ \frac{\mu_{i(unif)}}{2} + \sqrt{\left(\frac{\mu_{i(unif)}}{2}\right)^2 + \sigma_{i(unif)}^2} \right\}^3 \tag{25}$$

$$V_{i(BAY-U)} = \left\{ \frac{\mu_{i(unif)} + \sqrt{\mu^2_{i(unif)} + 4\sigma^2_{i(unif)}}}{2(9^{-1/4})(\sigma^2_{i(unif)})^{-1/4}} \right\}^4 \tag{26}$$

Hence,

$$\widehat{\tau}_{i(BAY-U)} = (1 - \delta_{i(unif)}) \cdot V_{i(BAY-U)} + \delta_{i(unif)}(1 - \delta_{i(unif)}) \cdot M^2_{i(BAY-U)} \tag{27}$$

such that

$$\widehat{\theta}_{BAY-U} = \frac{\widehat{\tau}_{1(BAY-U)}}{\widehat{\tau}_{2(BAY-U)}} \tag{28}$$

Thus, the Bayesian credible interval and HPD interval for the ratio of the variances of two delta-gamma distributions are respectively obtained as

$$CI_{BAY-U} = [\widehat{\theta}_{BAY-U}(\alpha/2), \widehat{\theta}_{BAY-U}(1-\alpha/2)] \tag{29}$$

2.2.3. The Bayesian Methods Using the Normal-Gamma-Beta Prior

Maneerat and Niwitpong [31] proposed an HPD-NGB for the common mean of several delta-lognormal distributions, which performed well for small-to-large sample sizes and better than HPD-J derived by Harvey and van der Merwe [1]. Let $Y = \log W$ be a random variable from a normal distribution with mean $\mu = (\mu_1, \mu_2)$ and precision $\lambda = (\lambda_1, \lambda_2)$ where $W \sim LN(\mu, \lambda)$ and $\lambda_i = \sigma_i^{-2}$. The HPD based on the normal-gamma-beta prior of $\theta = (\mu_i, \lambda_i, \delta_i)$ is defined as $p(\theta) \propto \prod_{i=1}^{k} \lambda_i^{-1}[\delta_i(1-\delta_i)]^{-1/2}$, where μ_i, λ_i follows a normal-gamma distribution, and δ_i follows a beta distribution. Thus, the respective marginal posterior distributions of δ_i, σ_i^2 and μ_i are as follows:

$$\delta_{i(NGB)}|x_i \sim Beta\left(n_{i,(0)} + \frac{1}{2}, n_{i,(1)} + \frac{1}{2}\right) \tag{30}$$

$$\sigma^2_{i(NGB)}|x_i \sim IG\left(\frac{n_{i,(1)} - 1}{2}, \frac{\sum_{i=1}^{n_{i,(1)}}(x_i - \mu_i)^2}{2}\right) \tag{31}$$

$$\mu_{i(NGB)}|x_i \sim t_{2(n_{i,(1)}-1)}\left(\overline{x}_i, \frac{\sum_{i=1}^{n_i}(x_i - \overline{x}_i)^2}{n_{i,(1)}(n_{i,(1)} - 1)}\right) \tag{32}$$

where $t_{2(n_{i,(1)}-1)}$ denotes a Student's t distribution with $2(n_{i,(1)} - 1)$ degrees of freedom.

We compute the mean and variance of gamma using $\mu_{i(NGB)}|x_i$ and $\sigma^2_{i(NGB)}|x_i$, respectively, as follows

$$M_{i(BAY-NGB)} = \left\{ \frac{\mu_{i(NGB)}}{2} + \sqrt{\left(\frac{\mu_{i(NGB)}}{2}\right)^2 + \sigma^2_{i(NGB)}} \right\}^3 \tag{33}$$

$$V_{i(BAY-NGB)} = \left\{ \frac{\mu_{i(NGB)} + \sqrt{\mu^2_{i(NGB)} + 4\sigma^2_{i(NGB)}}}{2(9^{-1/4})(\sigma^2_{i(NGB)})^{-1/4}} \right\}^4 \tag{34}$$

Hence,

$$\widehat{\tau}_{i(BAY-NGB)} = (1 - \delta_{i(NGB)}) \cdot V_{i(BAY-NGB)} + \delta_{i(NGB)}(1 - \delta_{i(NGB)}) \cdot M^2_{i(BAY-NGB)} \tag{35}$$

such that

$$\widehat{\theta}_{BAY-NGB} = \frac{\widehat{\tau}_{1(BAY-NGB)}}{\widehat{\tau}_{2(BAY-NGB)}} \tag{36}$$

The credible interval and HPD interval based on the BAY-NGB and HPD-NGB methods for the ratio of variances of delta-gamma distributions are respectively obtained as

$$CI_{BAY-NGB} = [\widehat{\theta}_{BAY-NGB}(\alpha/2), \widehat{\theta}_{BAY-NGB}(1-\alpha/2)] \quad (37)$$

The confidence interval for the ratio of variances θ can be obtained by executing Algorithm 2.

Algorithm 2 **Bayesian interval**

1: Generate $X_i \sim \Delta(\delta_i, \alpha_i, \beta_i)$ and compute $\bar{x}_i$ and s_i^2 of the cube-root-transformed sample.
2: Generate $\delta_i|x_i$ from Equations (14), (22), and (30).
3: Generate $\sigma_i^2|x_i$ from Equations (15), (23), and (31).
4: Generate $\mu_i|\sigma_i^2, x_i$ from Equations (16), (24), and (32).
5: Compute the mean and variance of a gamma distribution.
6: Compute $\widehat{\tau}_i$ and $\widehat{\theta}$.
7: Compute the 95% credible intervals and HPD intervals for $\widehat{\theta}$ by using Equations (21), (29), and (37).
8: Repeat Steps 1–7 10,000 times to compute the CP and the AL.

3. The Simulation Study and Results

A simulation study to generate the confidence interval for the ratio of the variance of two independent delta-gamma distributions by using the proposed methods was conducted with 10,000 replications (M), 5000 repetitions (m) for FQ, and the nominal confidence level set as 0.95 using R statistical software version 4.1.0. For equal sample sizes ($n_1 = n_2$), we used (30,30), (50,50), (100,100), or (200,200), and for unequal sample sizes ($n_1 \neq n_2$), we used (30,50), (50,100), or (100,200). For the two probabilities of data containing zeros $(\delta_1, \delta_2) = (0.2, 0.2)$, we set shape parameters (α_1, α_2) as (7.00,7.00), (7.00,7.50), (7.50,7.00), or (7.50,7.50); for $(\delta_1, \delta_2) = (0.5, 0.5)$, we set (α_1, α_2) as (2.00,2.00), (2.00,2.50), (2.50,2.00), or (2.50,2.50); and for $(\delta_1, \delta_2) = (0.8, 0.8)$, we set (α_1, α_2) as (1.25,1.25), (1.25,1.50), (1.50,1.25), or (1.50,1.50); we set rate parameters (β_1, β_2) as (1,1) for all cases. The performances of FQ, BAY-J, HPD-J, BAY-U, HPD-U, BAY-NGB, and HPD-NGB were assessed by comparing their CPs and ALs, with the best-performing one for a particular scenario having a CP close to or greater than 0.95 and the shortest AL.

The efficacies of the various methods for the nominal 95% two-sided confidence interval for the ratio of the variances of delta-gamma distributions with equal and unequal sample sizes in terms of their CPs and ALs are reported in Tables 1 and 2 and Figures 1–4: Tables 1 and 2 report the simulation results, while Figures 1–4 summarize the CPs and ALs from Tables 1 and 2.

The findings show that FQ, HPD-J, HPD-U, BAY-NGB, and HPD-NGB attained CPs greater than or close to the nominal confidence level of 0.95. For small-to-moderate sample sizes, FQ, BAY-NGB, and HPD-NGB performed well for both small and large δ whereas the HPD-J and HPD-U performed well for small δ. For large δ, the ALs of HPD-NGB were the shortest. For large sample sizes, FQ and HPD-J performed well for small δ whereas HPD-U, BAY-NGB, and HPD-NGB performed well for large δ. For small δ, the ALs of FQ and HPD-J were shorter than the other methods whereas for large δ, the ALs of HPD-NGB were the shortest. The results in Figures 1 and 3 reveal that FQ, BAY-NGB, and HPD-NGB performed well in almost all cases. Figures 2 and 4 show that BAY-J and HPD-J provide the shortest ALs.

Maneerat et al. [12] proposed the confidence interval for the ratio of the variances of two delta-lognormal distributions using an HPD based on the normal-gamma prior (HPD-NG), as well as the method of variance estimates recovery (MOVER). These proposed methods were compared with existing HPD-J, HPD based on the Jeffreys' rule prior, the generalized confidence interval (GCI), and the fiducial GCI. They found that HPD-NG performed very well in various situations while MOVER could be recommended for scenarios with small equal sample sizes. From the simulation results of the present study, it can be seen that HPD-NGB performed well for moderate-to-large sample sizes, while HPD-J and HPD-NGB both performed well for small-to-large sample sizes. Hence, both methods can be recommended for constructing the confidence interval for the ratio of the variances of two delta-gamma distributions.

Table 1. The CPs and (ALs) of nominal 95% two-sided confidence interval for the ratio of variances of delta-gamma distributions ($n_1 = n_2$).

n_1, n_2	δ_1, δ_2	α_1, α_2	CP (AL)						
			FQ	**BAY-J**	**HPD-J**	**BAY-U**	**HPD-U**	**BAY-NGB**	**HPD-NGB**
30, 30	0.2, 0.2	7.00, 7.00	**0.9697**	0.9456	0.9487	0.9401	0.9443	**0.9812**	**0.9798**
			(*1.3406*)	(1.1720)	(1.1228)	(1.1637)	(1.1147)	(1.4517)	(1.3708)
		7.00, 7.50	**0.9684**	0.9449	**0.9500**	0.9399	0.9471	**0.9800**	**0.9773**
			(*1.1818*)	(1.0453)	(0.9999)	(1.0355)	(0.9907)	(1.2858)	(1.2139)
		7.50, 7.00	**0.9690**	0.9447	0.9488	0.9393	0.9447	**0.9805**	**0.9795**
			(*1.4761*)	(1.3029)	(1.2500)	(1.2892)	(1.2372)	(1.6069)	(1.5201)
		7.50, 7.50	**0.9758**	**0.9542**	**0.9568**	**0.9506**	**0.9520**	**0.9846**	**0.9830**
			(1.3018)	(1.1622)	(1.1136)	(1.1470)	(*1.0995*)	(1.4230)	(1.3457)
	0.5, 0.5	2.00, 2.00	**0.9543**	0.8024	0.8214	0.8558	0.8755	**0.9609**	**0.9590**
			(3.5398)	(1.9996)	(1.8291)	(2.4537)	(2.1768)	(3.5356)	(*3.0144*)
		2.00, 2.50	**0.9579**	0.8046	0.8101	0.8570	0.8679	**0.9633**	**0.9540**
			(2.2508)	(1.2767)	(1.1654)	(1.5877)	(1.4052)	(2.2075)	(*1.9006*)
		2.50, 2.00	**0.9528**	0.8064	0.8274	0.8589	0.8746	**0.9593**	**0.9637**
			(4.6023)	(2.6154)	(2.4322)	(3.1508)	(2.8538)	(4.6543)	(*4.0229*)
		2.50, 2.50	**0.9531**	0.8046	0.8194	0.8619	0.8647	**0.9594**	**0.9607**
			(2.9341)	(1.6747)	(1.5549)	(2.0430)	(1.8489)	(2.9196)	(*2.5507*)
	0.8, 0.8	1.25, 1.25	**0.9604**	0.8441	0.8689	0.9474	0.9456	**0.9749**	**0.9692**
			(54.7822)	(15.1368)	(9.0762)	(187.644)	(47.1799)	(60.6257)	(*27.3089*)
		1.25, 1.50	**0.9662**	0.8464	0.8663	**0.9510**	0.9470	**0.9786**	**0.9686**
			(33.1790)	(9.4095)	(5.7188)	(111.917)	(29.4096)	(33.5318)	(*15.7383*)
		1.50, 1.25	**0.9636**	0.8481	0.8797	0.9495	0.9468	**0.9771**	**0.9727**
			(57.4290)	(16.7939)	(10.7268)	(166.582)	(45.5388)	(62.8024)	(*30.5099*)
		1.50, 1.50	**0.9676**	0.8568	0.8731	**0.9554**	0.9491	**0.9803**	**0.9743**
			(34.5386)	(10.4576)	(6.7205)	(102.669)	(28.7895)	(36.1227)	(*17.8842*)
50, 50	0.2, 0.2	7.00, 7.00	**0.9733**	0.9485	0.9498	0.9454	0.9464	**0.9821**	**0.9818**
			(*0.9710*)	(0.8531)	(0.8311)	(0.8440)	(0.8224)	(1.0532)	(1.0167)
		7.00, 7.50	**0.9694**	0.9482	0.9471	0.9448	0.9430	**0.9822**	**0.9795**
			(*0.8638*)	(0.7674)	(0.7468)	(0.7583)	(0.7382)	(0.9417)	(0.9087)
		7.50, 7.00	**0.9708**	**0.9516**	**0.9505**	0.9481	0.9474	**0.9814**	**0.9835**
			(1.0778)	(0.9573)	(*0.9334*)	(0.9453)	(0.9221)	(1.1762)	(1.1364)
		7.50, 7.50	**0.9723**	**0.9540**	**0.9543**	0.9499	0.9499	**0.9852**	**0.9810**
			(0.9496)	(0.8521)	(*0.8301*)	(0.8401)	(0.8188)	(1.0408)	(1.0053)

Table 1. *Cont.*

n_1, n_2	δ_1, δ_2	α_1, α_2	CP (AL)						
			FQ	**BAY-J**	**HPD-J**	**BAY-U**	**HPD-U**	**BAY-NGB**	**HPD-NGB**
	0.5, 0.5	2.00, 2.00	**0.9555**	0.8074	0.8170	0.8393	0.8481	**0.9595**	**0.9596**
			(2.1853)	(1.3055)	(1.2436)	(1.4396)	(1.3608)	(2.1998)	(*1.9997*)
		2.00, 2.50	**0.9524**	0.8000	0.8074	0.8360	0.8367	**0.9579**	0.9489
			(1.4219)	(0.8503)	(0.8090)	(0.9437)	(0.8905)	(1.4160)	(*1.2965*)
		2.50, 2.00	**0.9533**	0.7989	0.8138	0.8307	0.8463	**0.9592**	**0.9625**
			(2.9496)	(1.7637)	(1.6953)	(1.9309)	(1.8438)	(2.9917)	(*2.7412*)
		2.50, 2.50	**0.9566**	0.7966	0.8056	0.8297	0.8389	**0.9596**	**0.9593**
			(1.8968)	(1.1338)	(1.0888)	(1.2506)	(1.1928)	(1.9017)	(*1.7557*)
	0.8, 0.8	1.25, 1.25	**0.9621**	0.8403	0.8591	0.9079	0.9075	**0.9742**	**0.9648**
			(9.7449)	(4.7376)	(3.8265)	(7.7233)	(5.3985)	(10.3437)	(*7.3304*)
		1.25, 1.50	**0.9583**	0.8422	0.8540	0.9045	0.9088	**0.9712**	**0.9634**
			(6.7461)	(3.3189)	(2.6700)	(5.6071)	(3.8850)	(6.9155)	(*4.9549*)
		1.50, 1.25	**0.9605**	0.8470	0.8746	0.9055	0.9201	**0.9727**	**0.9703**
			(12.0438)	(6.0132)	(4.9732)	(9.2776)	(6.7417)	(13.0570)	(*9.4220*)
		1.50, 1.50	**0.9621**	0.8454	0.8643	0.9086	0.9138	**0.9736**	**0.9681**
			(8.1384)	(4.1292)	(3.4129)	(6.4968)	(4.7229)	(8.5396)	(*6.2547*)
100, 100	0.2, 0.2	7.00, 7.00	**0.9669**	0.9412	0.9408	0.9391	0.9403	**0.9810**	**0.9786**
			(*0.6607*)	(0.5822)	(0.5738)	(0.5777)	(0.5694)	(0.7181)	(0.7042)
		7.00, 7.50	**0.9720**	0.9456	0.9468	0.9433	0.9452	**0.9841**	**0.9826**
			(*0.5827*)	(0.5186)	(0.5109)	(0.5145)	(0.5068)	(0.6364)	(0.6240)
		7.50, 7.00	**0.9731**	**0.9519**	**0.9528**	0.9499	**0.9502**	**0.9828**	**0.9808**
			(0.7294)	(0.6491)	(0.6399)	(0.6433)	(*0.6344*)	(0.7966)	(0.7816)
		7.50, 7.50	**0.9742**	**0.9555**	**0.9534**	**0.9534**	**0.9518**	**0.9846**	**0.9835**
			(0.6452)	(0.5804)	(0.5718)	(0.5751)	(*0.5669*)	(0.7083)	(0.6949)
	0.5, 0.5	2.00, 2.00	**0.9537**	0.8008	0.8045	0.8160	0.8188	**0.9581**	**0.9538**
			(1.3306)	(0.8227)	(0.8030)	(0.8586)	(0.8366)	(1.3450)	(*1.2799*)
		2.00, 2.50	0.9496	0.7950	0.7932	0.8103	0.8101	**0.9539**	0.9465
			(0.8744)	(0.5394)	(0.5260)	(0.5645)	(0.5497)	(0.8795)	(0.8403)
		2.50, 2.00	**0.9497**	0.7949	0.8051	0.8095	0.8213	**0.9533**	**0.9600**
			(1.8266)	(1.1256)	(1.1039)	(1.1716)	(1.1474)	(1.8542)	(*1.7716*)
		2.50, 2.50	**0.9532**	0.7870	0.7912	0.8026	0.8081	**0.9565**	**0.9532**
			(1.1874)	(0.7289)	(0.7142)	(0.7614)	(0.7449)	(1.1976)	(*1.1488*)
	0.8, 0.8	1.25, 1.25	**0.9553**	0.8310	0.8440	0.8621	0.8755	**0.9668**	**0.9637**
			(3.7848)	(2.2414)	(2.0428)	(2.5366)	(2.2693)	(4.0075)	(*3.3653*)
		1.25, 1.50	**0.9592**	0.8386	0.8420	0.8708	0.8723	**0.9702**	**0.9588**
			(2.5811)	(1.5459)	(1.4084)	(1.7612)	(1.5750)	(2.6887)	(*2.2802*)
		1.50, 1.25	**0.9565**	0.8366	0.8580	0.8687	0.8841	**0.9673**	**0.9727**
			(4.8404)	(2.9052)	(2.6763)	(3.2390)	(2.9372)	(5.1819)	(*4.3894*)
		1.50, 1.50	**0.9639**	0.8433	0.8571	0.8750	0.8848	**0.9740**	**0.9690**
			(3.2934)	(2.0003)	(1.8424)	(2.2460)	(2.0369)	(3.4707)	(*2.9718*)
200, 200	0.2, 0.2	7.00, 7.00	**0.9709**	0.9448	0.9428	0.9431	0.9436	**0.9813**	**0.9803**
			(*0.4566*)	(0.4027)	(0.3990)	(0.4010)	(0.3973)	(0.4966)	(0.4908)
		7.00, 7.50	**0.9721**	**0.9511**	0.9486	0.9479	0.9470	**0.9841**	**0.9828**
			(0.4044)	(*0.3601*)	(0.3567)	(0.3584)	(0.3551)	(0.4419)	(0.4367)
		7.50, 7.00	**0.9711**	**0.9510**	**0.9505**	0.9486	0.9492	**0.9819**	**0.9821**
			(0.5060)	(*0.4506*)	(0.4465)	(0.4484)	(0.4444)	(0.5529)	(0.5465)
		7.50, 7.50	**0.9744**	**0.9529**	**0.9523**	**0.9506**	**0.9521**	**0.9853**	**0.9847**
			(0.4477)	(0.4030)	(0.3992)	(0.4009)	(*0.3972*)	(0.4919)	(0.4862)

Table 1. *Cont.*

n_1, n_2	δ_1, δ_2	α_1, α_2	CP (AL)						
			FQ	**BAY-J**	**HPD-J**	**BAY-U**	**HPD-U**	**BAY-NGB**	**HPD-NGB**
	0.5, 0.5	2.00, 2.00	**0.9531**	0.7896	0.7923	0.7974	0.7988	**0.9562**	**0.9532**
			(0.8838)	(0.5547)	(0.5473)	(0.5659)	(0.5581)	(0.8953)	(*0.8716*)
		2.00, 2.50	**0.9502**	0.7895	0.7845	0.7979	0.7916	**0.9533**	0.9470
			(0.5793)	(0.3621)	(0.3570)	(0.3700)	(0.3646)	(0.5851)	(*0.5708*)
		2.50, 2.00	0.9478	0.7830	0.7887	0.7900	0.7968	**0.9506**	**0.9566**
			(1.2195)	(0.7615)	(0.7529)	(0.7765)	(0.7675)	(1.2360)	(*1.2058*)
		2.50, 2.50	**0.9536**	0.7918	0.7927	0.8012	0.8012	**0.9575**	**0.9574**
			(0.7949)	(0.4934)	(0.4876)	(0.5037)	(0.4976)	(0.8034)	(*0.7852*)
	0.8, 0.8	1.25, 1.25	**0.9534**	0.8263	0.8346	0.8425	0.8487	**0.9645**	**0.9607**
			(2.0672)	(1.3182)	(1.2591)	(1.3814)	(1.3145)	(2.1786)	(*1.9832*)
		1.25, 1.50	**0.9556**	0.8360	0.8333	0.8490	0.8501	**0.9668**	**0.9573**
			(1.4357)	(0.9237)	(0.8822)	(0.9709)	(0.9235)	(1.5058)	(*1.3793*)
		1.50, 1.25	**0.9579**	0.8316	0.8474	0.8470	0.8614	**0.9667**	**0.9718**
			(2.6965)	(1.7363)	(1.6663)	(1.8096)	(1.7314)	(2.8603)	(*2.6168*)
		1.50,1.50	**0.9605**	0.8403	0.8446	0.8532	0.8592	**0.9707**	**0.9642**
			(1.8546)	(1.2043)	(1.1559)	(1.2597)	(1.2054)	(1.9569)	(*1.8021*)

Table 2. The CPs and (ALs) of nominal 95% two-sided confidence interval for the ratio of variances of delta-gamma distributions ($n_1 \neq n_2$).

n_1, n_2	δ_1, δ_2	α_1, α_2	CP (AL)						
			FQ	**BAY-J**	**HPD-J**	**BAY-U**	**HPD-U**	**BAY-NGB**	**HPD-NGB**
30, 50	0.2, 0.2	7.00, 7.00	**0.9697**	0.9432	0.9445	0.9387	0.9475	**0.9808**	**0.9786**
			(*1.1937*)	(1.0045)	(0.9722)	(1.0338)	(0.9966)	(1.2593)	(1.2036)
		7.00, 7.50	**0.9738**	0.9491	0.9469	0.9461	**0.9501**	**0.9844**	**0.9804**
			(*1.0568*)	(0.8970)	(0.8678)	(0.9217)	(0.8883)	(1.1195)	(1.0700)
		7.50, 7.00	**0.9727**	**0.9512**	**0.9511**	0.9468	**0.9532**	**0.9816**	**0.9838**
			(1.3123)	(1.1185)	(1.0848)	(1.1449)	(*1.1065*)	(1.3939)	(1.3349)
		7.50, 7.50	**0.9711**	**0.9515**	0.9495	0.9463	**0.9533**	**0.9819**	**0.9822**
			(1.1640)	(1.0001)	(0.9692)	(1.0229)	(*0.9878*)	(1.2405)	(1.1878)
	0.5, 0.5	2.00, 2.00	**0.9537**	0.7999	0.8181	0.8427	0.8689	**0.9586**	**0.9581**
			(3.0302)	(1.7232)	(1.5584)	(2.1715)	(1.8961)	(2.8930)	(*2.5193*)
		2.00, 2.50	**0.9520**	0.8020	0.8088	0.8451	0.8628	**0.9558**	**0.9516**
			(2.0151)	(1.1480)	(1.0336)	(1.4625)	(1.2705)	(1.9013)	(*1.6621*)
		2.50, 2.00	**0.9544**	0.8059	0.8238	0.8446	0.8726	**0.9571**	**0.9659**
			(3.9549)	(2.2632)	(2.0781)	(2.7946)	(2.4900)	(3.8104)	(*3.3632*)
		2.50, 2.50	**0.9521**	0.7984	0.8141	0.8402	0.8645	**0.9564**	**0.9578**
			(2.5824)	(1.4775)	(1.3537)	(1.8419)	(1.6371)	(2.4630)	(*2.1867*)
	0.8, 0.8	1.25, 1.25	**0.9619**	0.8431	0.8749	0.9197	**0.9569**	**0.9753**	**0.9718**
			(40.0928)	(11.3179)	(6.8256)	(156.047)	(40.3903)	(36.1470)	(*16.9783*)
		1.25, 1.50	**0.9611**	0.8384	0.8654	0.9198	**0.9548**	**0.9745**	**0.9703**
			(27.1107)	(7.7188)	(4.6975)	(107.454)	(27.9729)	(23.8861)	(*11.3980*)
		1.50, 1.25	**0.9592**	0.8421	0.8754	0.9165	**0.9576**	**0.9731**	**0.9757**
			(41.5169)	(12.6202)	(8.0997)	(144.186)	(39.7663)	(37.9083)	(*19.2347*)
		1.50, 1.50	**0.9633**	0.8504	0.8767	0.9244	**0.9591**	**0.9765**	**0.9752**
			(27.9288)	(8.5264)	(5.4643)	(98.4976)	(27.3681)	(24.8532)	(*12.6009*)

Table 2. *Cont.*

n_1, n_2	δ_1, δ_2	α_1, α_2	CP (AL)						
			FQ	**BAY-J**	**HPD-J**	**BAY-U**	**HPD-U**	**BAY-NGB**	**HPD-NGB**
50, 100	0.2, 0.2	7.00, 7.00	**0.9723**	0.9448	0.9447	0.9431	0.9465	**0.9822**	**0.9823**
			(*0.8419*)	(0.7190)	(0.7066)	(0.7296)	(0.7159)	(0.8958)	(0.8735)
		7.00, 7.50	**0.9700**	0.9441	0.9404	0.9434	0.9429	**0.9809**	**0.9807**
			(*0.7452*)	(0.6412)	(0.6300)	(0.6503)	(0.6379)	(0.7959)	(0.7760)
		7.50, 7.00	**0.9750**	**0.9528**	**0.9505**	0.9499	**0.9531**	**0.9857**	**0.9832**
			(0.9235)	(0.7996)	(*0.7869*)	(0.8093)	(0.7954)	(0.9911)	(0.9675)
		7.50, 7.50	**0.9743**	**0.9532**	**0.9524**	**0.9503**	**0.9549**	**0.9843**	**0.9857**
			(0.8218)	(0.7160)	(*0.7042*)	(0.7240)	(0.7113)	(0.8835)	(0.8623)
	0.5, 0.5	2.00, 2.00	**0.9511**	0.7965	0.8092	0.8201	0.8364	**0.9548**	**0.9550**
			(1.8500)	(1.1130)	(1.0490)	(1.2452)	(1.1620)	(1.8027)	(*1.6689*)
		2.00, 2.50	**0.9529**	0.8045	0.7980	0.8286	0.8354	**0.9597**	**0.9509**
			(1.2170)	(0.7311)	(0.6876)	(0.8217)	(0.7651)	(1.1809)	(*1.0969*)
		2.50, 2.00	0.9494	0.7868	0.7985	0.8093	0.8317	**0.9515**	**0.9589**
			(2.4361)	(1.4641)	(1.3922)	(1.6248)	(1.5318)	(2.3858)	(*2.2273*)
		2.50, 2.50	**0.9564**	0.7990	0.8009	0.8259	0.8338	**0.9596**	**0.9584**
			(1.6124)	(0.9668)	(0.9177)	(1.0792)	(1.0155)	(1.5698)	(*1.4699*)
	0.8, 0.8	1.25, 1.25	**0.9585**	0.8335	0.8642	0.8827	0.9191	**0.9701**	**0.9687**
			(7.7112)	(3.8873)	(3.1102)	(6.9755)	(4.7917)	(7.2172)	(*5.3521*)
		1.25, 1.50	**0.9586**	0.8444	0.8598	0.8885	0.9210	**0.9707**	**0.9642**
			(5.4096)	(2.7443)	(2.1945)	(4.9852)	(3.4183)	(5.0004)	(*3.7310*)
		1.50, 1.25	**0.9583**	0.8389	0.8723	0.8782	0.9251	**0.9683**	**0.9734**
			(9.2636)	(4.7883)	(3.9429)	(8.0576)	(5.7833)	(8.8128)	(*6.7038*)
		1.50, 1.50	**0.9584**	0.8428	0.8637	0.8847	0.9225	**0.9708**	**0.9698**
			(6.4244)	(3.3469)	(2.7510)	(5.6927)	(4.0797)	(6.0353)	(*4.6216*)
100, 200	0.2, 0.2	7.00, 7.00	**0.9726**	0.9471	0.9469	0.9448	0.9474	**0.9837**	**0.9828**
			(*0.5701*)	(0.4950)	(0.4901)	(0.4972)	(0.4922)	(0.6133)	(0.6047)
		7.00, 7.50	**0.9712**	0.9477	0.9476	0.9453	0.9475	**0.9832**	**0.9813**
			(*0.5074*)	(0.4435)	(0.4391)	(0.4453)	(0.4408)	(0.5475)	(0.5398)
		7.50, 7.00	**0.9741**	**0.9547**	**0.9551**	**0.9544**	**0.9544**	**0.9845**	**0.9850**
			(0.6277)	(0.5523)	(*0.5472*)	(0.5541)	(0.5489)	(0.6803)	(0.6711)
		7.50, 7.50	**0.9719**	**0.9535**	**0.9521**	**0.9511**	**0.9530**	**0.9849**	**0.9839**
			(0.5578)	(0.4943)	(*0.4897*)	(0.4958)	(0.4911)	(0.6065)	(0.5983)
	0.5, 0.5	2.00, 2.00	**0.9528**	0.7967	0.7992	0.8076	0.8126	**0.9565**	**0.9553**
			(1.1466)	(0.7116)	(0.6903)	(0.7464)	(0.7228)	(1.1408)	(*1.0965*)
		2.00, 2.50	**0.9510**	0.7922	0.7890	0.8073	0.8071	**0.9553**	0.9469
			(0.7627)	(0.4719)	(0.4571)	(0.4964)	(0.4800)	(*0.7566*)	(0.7288)
		2.50, 2.00	**0.9508**	0.7853	0.7953	0.7988	0.8084	**0.9541**	**0.9584**
			(1.5382)	(0.9498)	(0.9258)	(0.9938)	(0.9668)	(1.5336)	(*1.4799*)
		2.50, 2.50	**0.9528**	0.7896	0.7911	0.8018	0.8060	**0.9574**	**0.9557**
			(1.0160)	(0.6249)	(0.6084)	(0.6556)	(0.6371)	(1.0104)	(*0.9772*)
	0.8, 0.8	1.25, 1.25	**0.9558**	0.8257	0.8417	0.8491	0.8706	**0.9661**	**0.9645**
			(3.0752)	(1.8653)	(1.6982)	(2.1607)	(1.9264)	(3.0750)	(*2.6785*)
		1.25, 1.50	**0.9582**	0.8405	0.8464	0.8625	0.8798	**0.9678**	**0.9601**
			(2.1497)	(1.3110)	(1.1934)	(1.5267)	(1.3603)	(2.1382)	(*1.8719*)
		1.50, 1.25	**0.9587**	0.8350	0.8555	0.8563	0.8855	**0.9684**	**0.9725**
			(3.8343)	(2.3578)	(2.1723)	(2.6898)	(2.4330)	(3.8684)	(*3.4072*)
		1.50, 1.50	**0.9614**	0.8415	0.8575	0.8601	0.8875	**0.9711**	**0.9711**
			(2.6930)	(1.6670)	(1.5355)	(1.9107)	(1.7274)	(2.7070)	(*2.3962*)

The CPs greater than the nominal confidence level of 0.95 be in bold and the shortest AL be in italic.

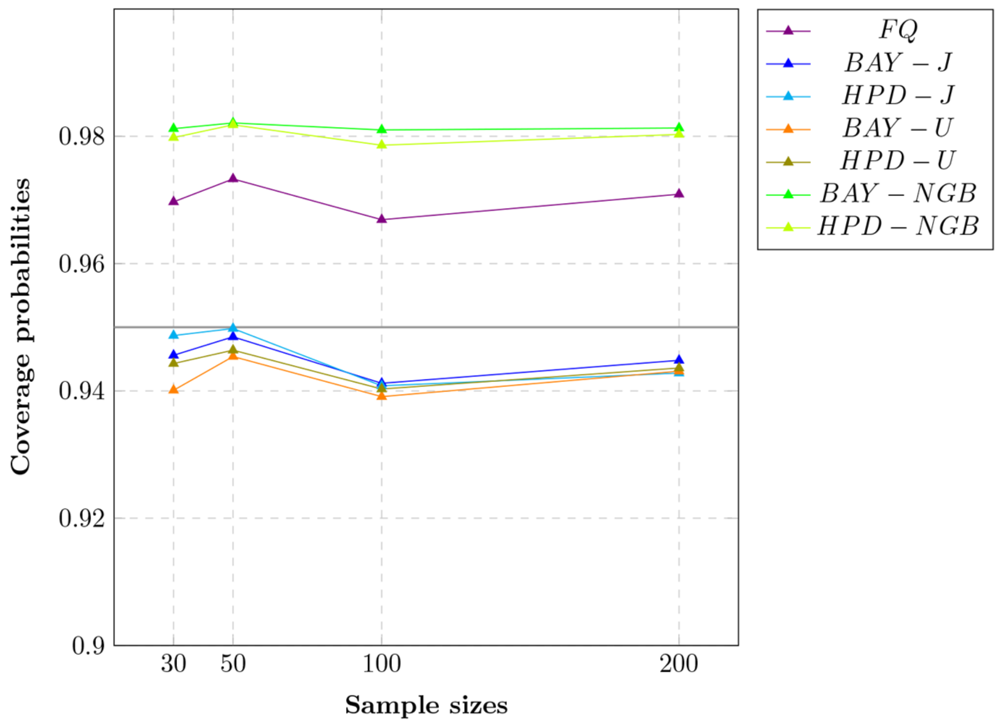

Figure 1. Line graphs of the CPs of the methods in the simulated scenario with different sample sizes.

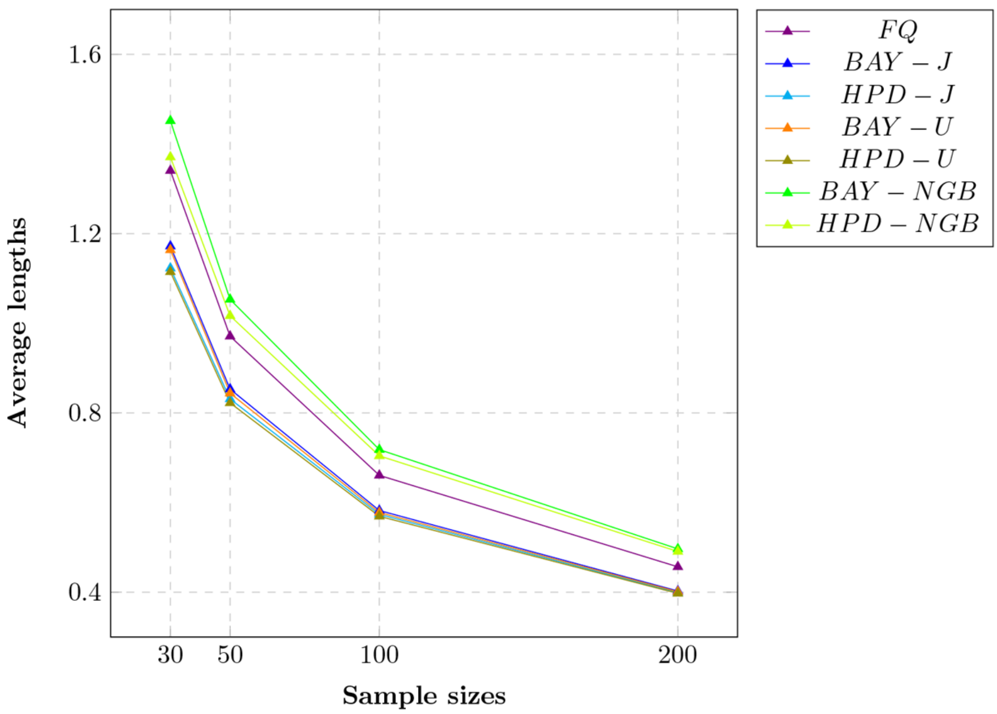

Figure 2. Line graphs of the ALs of the methods in the simulated scenario with different sample sizes.

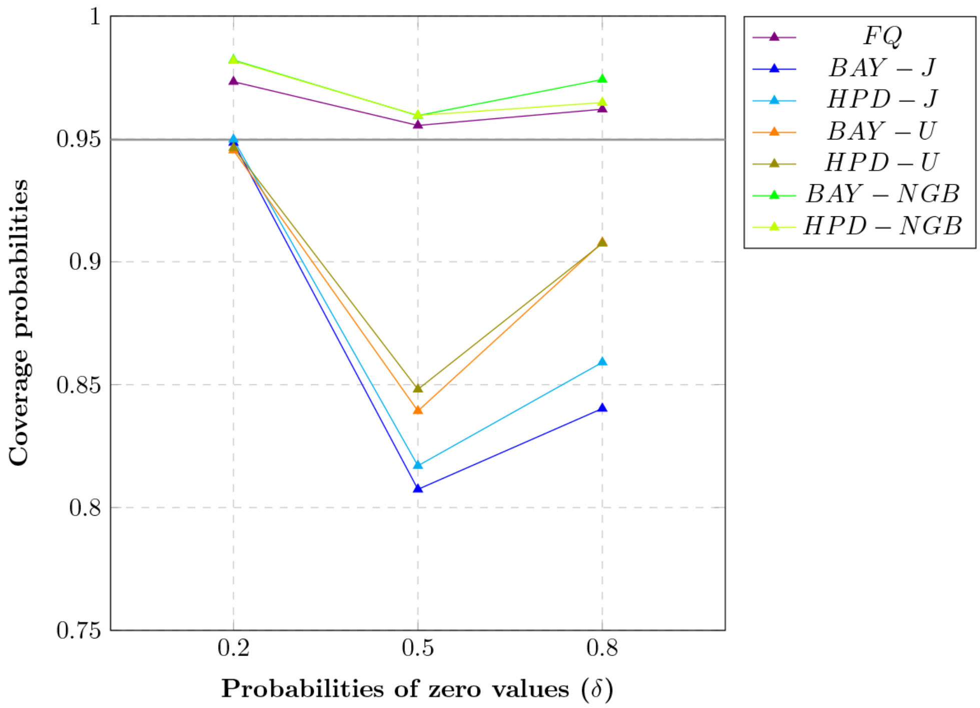

Figure 3. Line graphs of the CPs of the methods in the simulated scenario with different probabilities of zero values.

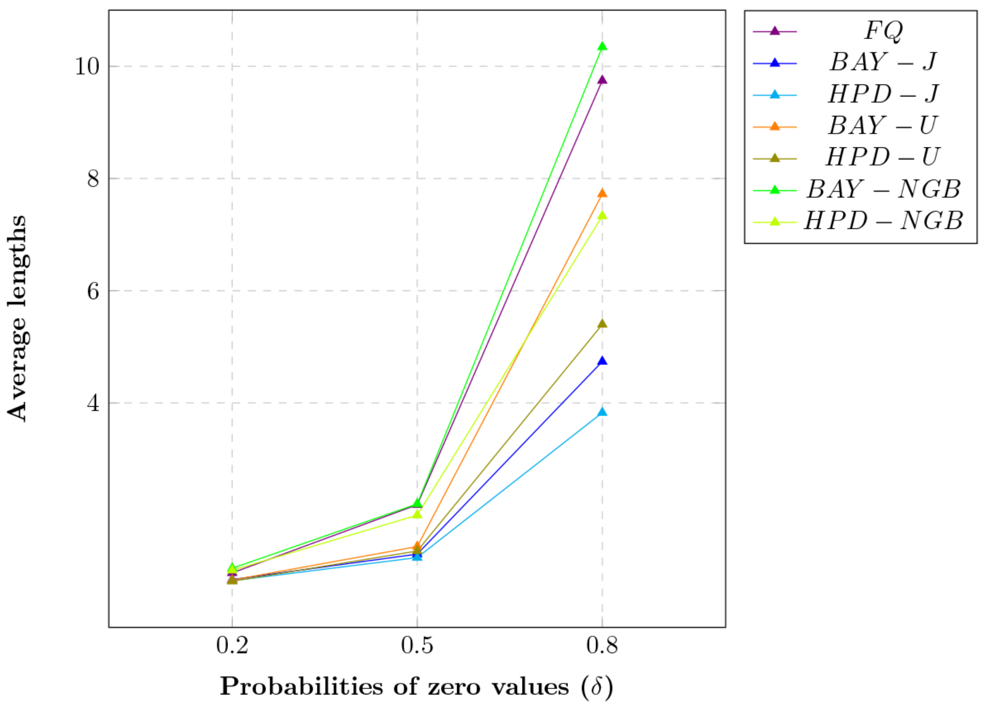

Figure 4. Line graphs of the ALs of the methods in the simulated scenario with different probabilities of zero values.

4. Application of the Methods with Real Data

The performances of confidence interval methods were compared by analyzing rainfall data reported by the Upper Northern Region Irrigation Hydrology Center, Phrae province, Thailand.

4.1. Application of the Ratio of Variances of Two Delta-Gamma Distributions with Equal Sample Sizes

For $n_1 = n_2$, we used monthly rainfall data from January and February 1980 to 2021 in Song district, Phrae province, Thailand. The densities of the rainfall data are shown in Figure 5.

First, we attempted to fit the positive rainfall data using four models normal, lognormal, Cauchy, and gamma by using the Akaike information criterion (AIC), the results of which are reported in Table 3; the lowest AIC value was obtained by fitting with the gamma distribution, which is thus the most suitable distribution. Q-Q plots of positive rainfall data are shown in Figure 6.

Table 3. AIC results of positive rainfall data.

Rainfall Station	Normal	Lognormal	Cauchy	Gamma
Song (January)	198.35	178.43	198.48	175.61
Song (February)	205.29	183.56	198.98	178.75
Rong Kwang (January)	158.92	145.36	164.03	143.15
Rong Kwang (February)	216.35	196.28	211.51	195.57

The summary statistics for the rainfall in the February dataset from the Song station, $\bar{x}_1 = 16.9478, n_1 = 42, n_{1,(1)} = 23, n_{1,(0)} = 19$, while the maximum likelihood estimators for $\delta_1, \alpha_1, \beta_1$, and τ_1 are $\hat{\delta}_1 = 0.45, \hat{\alpha}_1 = 0.6242, \hat{\beta}_1 = 27.1507$, and $\hat{\tau}_1 = 323.1406$, respectively. Similarly, the summary statistics for the rainfall in the January dataset from the Song station are $\bar{x}_2 = 23.1142, n_2 = 42, n_{2,(1)} = 21, n_{2,(0)} = 21$, while the maximum likelihood estimators for $\delta_2, \alpha_2, \beta_2$, and τ_2 are $\hat{\delta}_2 = 0.5, \hat{\alpha}_2 = 0.5666, \hat{\beta}_2 = 40.7940$, and $\hat{\tau}_2 = 605.0296$ are respectively. The 95% two-sided confidence intervals results for θ reported in Table 4 indicate that the AL provided by HPD-U was the shortest, and thus it is the best approach for constructing the confidence interval for the ratio of the variances of two rainfall datasets with equal sample sizes from the Rong Kwang district, Phrae province, Thailand.

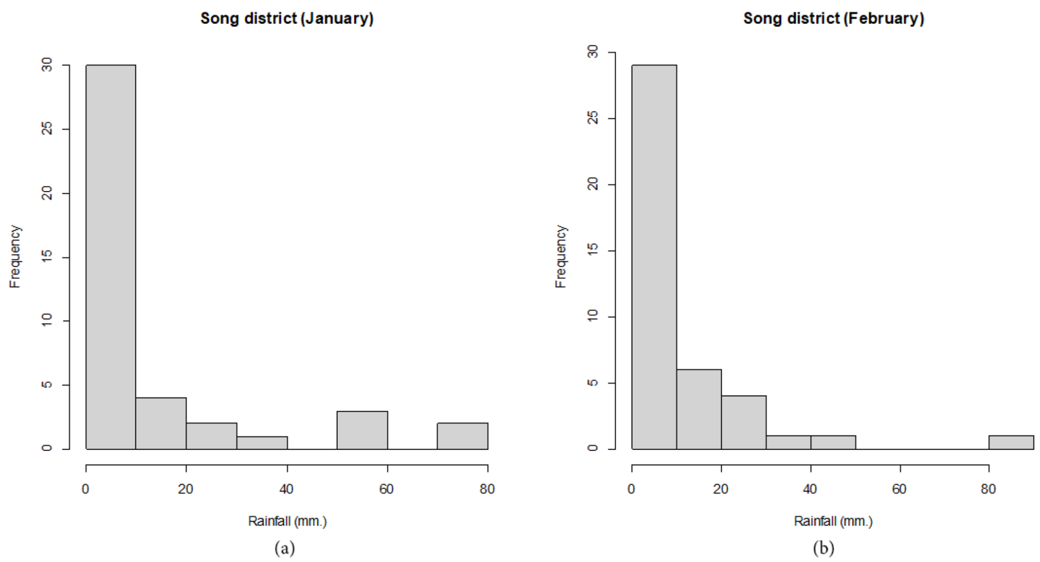

Figure 5. The densities of the rainfall data from Song district station, Phrae province, Thailand, for (**a**) January and (**b**) February from 1980–2021.

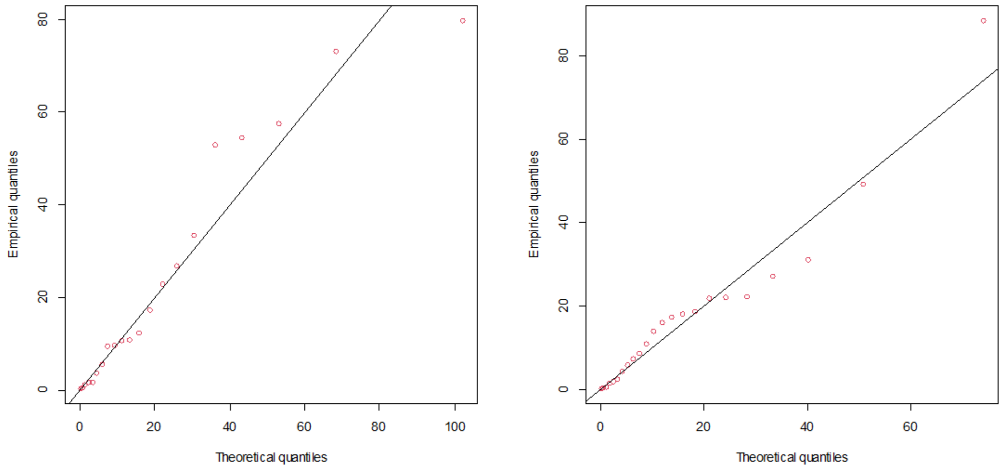

Figure 6. Q-Q plots for distribution fitting of the positive rainfall data from the Song district station, Phrae province, Thailand, for (**a**) January and (**b**) February from 1980–2021.

Table 4. The 95% two-sided confidence intervals for the ratio of variances of rainfall datasets from Song district, Phrae province, Thailand.

Methods	Confidence Intervals for θ		Length of Intervals
	Lower	**Upper**	
FQ	0.0011	5.5189	5.5178
BAY-J	0.1374	0.1790	0.0416
HPD-J	0.1366	0.1776	0.0410
BAY-U	0.1399	0.1800	0.0401
HPD-U	0.1395	0.1790	0.0395
BAY-NGB	0.1386	0.1791	0.0405
HPD-NGB	0.1384	0.1788	0.0404

4.2. Application of Variances of Two Delta-Gamma Distributions with Unequal Sample Sizes

For $n_1 \neq n_2$, we used monthly rainfall data in January from 1969 to 2021 and February 1953 to 2021 in Rong Kwang district, Phrae province, Thailand. The densities of the rainfall data are shown in Figure 7.

Fitting of the positive rainfall data was attempted with four models: normal, lognormal, Cauchy, and gamma, the AIC values for which are reported in Table 3. The results show that the gamma distribution is the best fit. Q-Q plots of positive rainfall data are shown in Figure 8.

The summary statistics for the rainfall in the February dataset from the Rong Kwang station, $\bar{x}_1 = 19.9666, n_1 = 65, n_{1,(1)} = 24, n_{1,(0)} = 41$, while the maximum likelihood estimators for $\delta_1, \alpha_1, \beta_1$, and τ_1 are $\hat{\delta}_1 = 0.63, \hat{\alpha}_1 = 0.9649, \hat{\beta}_1 = 20.6922$, and $\hat{\tau}_1 = 245.3991$, respectively. Similarly, the summary statistics for the rainfall in the January dataset from the Rong Kwang station as $\bar{x}_2 = 22.9294, n_2 = 50, n_{2,(1)} = 17, n_{2,(0)} = 33$, while the maximum likelihood estimators for $\delta_2, \alpha_2, \beta_2$, and τ_2 are $\hat{\delta}_2 = 0.66, \hat{\alpha}_2 = 0.6001, \hat{\beta}_2 = 38.2077$, and $\hat{\tau}_2 = 415.8480$ are respectively. The 95% two-sided confidence intervals results for θ reported in Table 5 indicate that the AL provided by HPD-J was the shortest, and thus it is the best approach for constructing the confidence interval for the ratio of variances of two rainfall datasets with unequal sample sizes from the Rong Kwang district, Phrae province, Thailand.

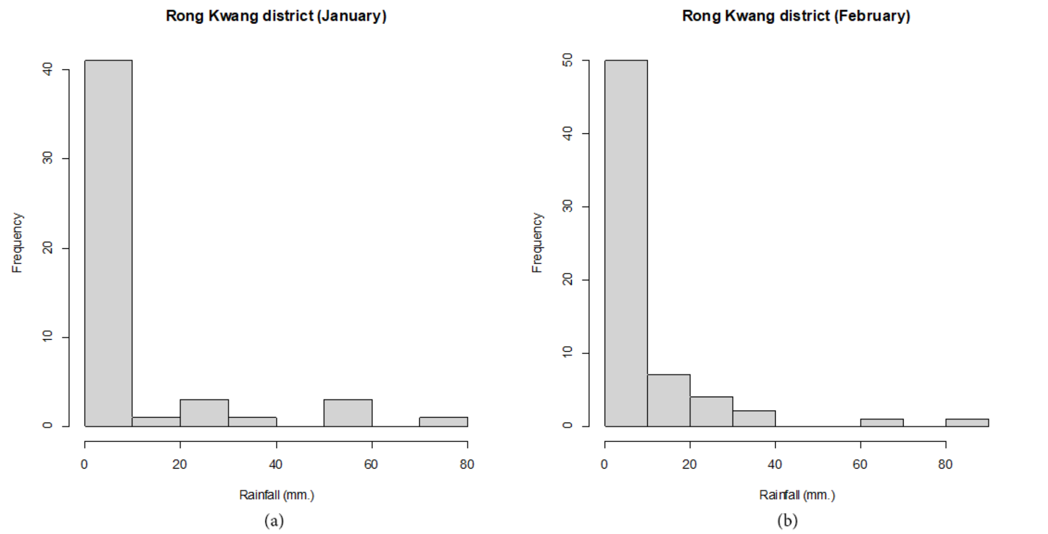

Figure 7. The densities of the rainfall data from Rong Kwang district station, Phrae province, Thailand, for (**a**) January from 1969–2021 and (**b**) February from 1953–2021.

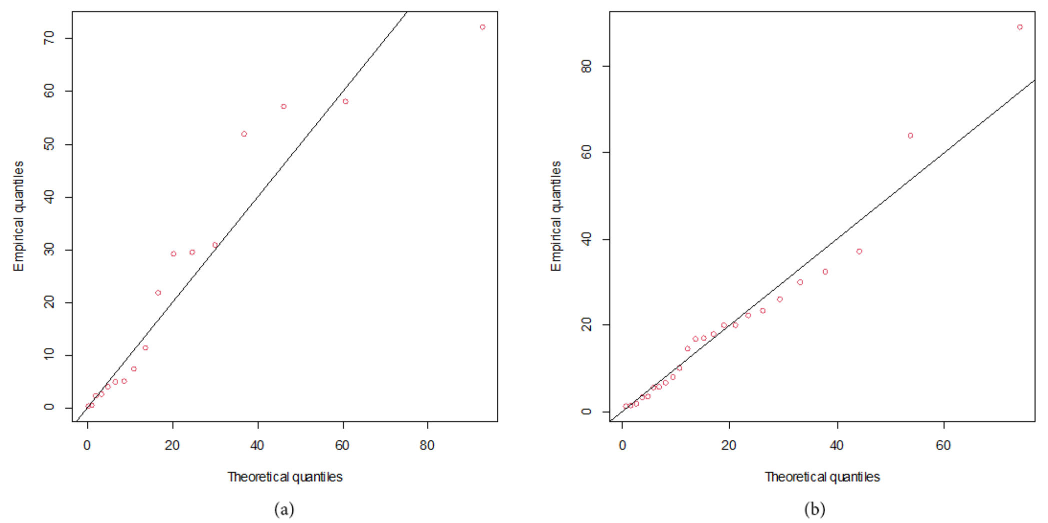

Figure 8. Q-Q plots for distribution fitting of the positive rainfall data from the Rong Kwang district station, Phrae province, Thailand, for (**a**) January from 1969–2021 and (**b**) February from 1953–2021.

Table 5. The 95% two-sided confidence intervals for the ratio of variances of rainfall datasets from Rong Kwang district, Phrae province, Thailand.

Methods	Confidence Intervals for θ		Length of Intervals
	Lower	Upper	
FQ	0.0023	18.8012	18.7989
BAY-J	0.3574	0.5822	0.2248
HPD-J	0.3483	0.5685	0.2202
BAY-U	0.3534	0.5904	0.2370
HPD-U	0.3420	0.5715	0.2295
BAY-NGB	0.3597	0.5924	0.2327
HPD-NGB	0.3511	0.5757	0.2246

5. Conclusions

We constructed the confidence interval for the ratio of the variances of two delta-gamma distributions by using the FQ, BAY-J, HPD-J, BAY-U, HPD-U, BAY-NGB, and HPD-NGB approaches. The CPs and ALs as performance measures for the methods were assessed via Monte Carlo simulation. Our findings show that for small and large δ, HPD-J and HPD-NGB can be recommended for constructing the confidence interval for this scenario. Maybe other priors are more effective. Therefore, choosing priors is very important in the Bayesian method.

Author Contributions: W.K.: performed the experiments, analyzed the data, authored or reviewed drafts of the paper; S.-A.N.: concived and designed the experiments, approved the final draft; S.N.: contributed analysis tools, prepared tables. All authors have read and agreed to the published version of the manuscript.

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