axioms

Article
Schwarzschild Spacetimes: Topology

Demeter Krupka [1,2,†,‡] and Ján Brajerčík [3,*,‡]

1 Lepage Research Institute, 081 16 Presov, Slovakia
2 Faculty of Mathematics and Computer Science, Transilvania University in Brasov, 500091 Brasov, Romania
3 Faculty of Humanities and Natural Sciences, University of Presov in Presov, 081 16 Presov, Slovakia
* Correspondence: jan.brajercik@unipo.sk
† Current address: Faculty of Humanities and Natural Sciences, University of Presov in Presov, 17. Novembra 1, 081 16 Presov, Slovakia.
‡ These authors contributed equally to this work.

Abstract: This paper is devoted to the geometric theory of a Schwarzschild spacetime, a basic objective in applications of the classical general relativity theory. In a broader sense, a *Schwarzschild spacetime* is a smooth manifold, endowed with an action of the special orthogonal group SO(3) and a *Schwarzschild metric*, an SO(3)-invariant metric field, satisfying the Einstein equations. We prove the existence of and find *all* Schwarzschild metrics on two topologically non-equivalent manifolds, $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ and $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$. The method includes a classification of SO(3)-invariant, time-translation invariant and time-reflection invariant metrics on $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ and a winding mapping of the real line $\mathbf{R}$ onto the circle S^1. The resulting family of Schwarzschild metrics is parametrized by an arbitrary function and two real parameters, the integration constants. For any Schwarzschild metric, one of the parameters determines a submanifold, where the metric is not defined, the *Schwarzschild sphere*. In particular, the family admits a global metric whose Schwarzschild sphere is empty. These results transfer to $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ by the winding mapping. All our assertions are derived independently of the signature of the Schwarzschild metric; the signature can be chosen as an independent axiom.

Keywords: manifold topology; Einstein equations; spherical symmetry; Schwarzschild spacetime; special orthogonal group; SO(3)-action; invariant metric

MSC: 83C05; 58E30; 58E40; 53C25

check for updates

Citation: Krupka, D.; Brajerčík, J. Schwarzschild Spacetimes: Topology. *Axioms* **2022**, *11*, 693. https://doi.org/10.3390/axioms11120693

Academic Editor: Sidney A. Morris

Received: 14 November 2022
Accepted: 2 December 2022
Published: 4 December 2022

1. Introduction

In this paper, a *Schwarzschild spacetime*, or a *spherically symmetric spacetime*, is a smooth 4-dimensional manifold X endowed with a left action of the special orthogonal group SO(3) and a non-singular, symmetric $(0,2)$-tensor field g, satisfying the following two conditions:

(1) g is SO(3)-invariant.
(2) g solves the Einstein vacuum equations.

where g is a *Schwarzschild metric* on X.

Standard topological properties are required: X is *Hausdorff*, *second countable*, and *connected*. As g can be understood as an extremal of an integral variational functional, the *Hilbert variational functional*, no a priori restrictions of the signature of g are imposed.

In this paper, we revisit and extend several constructions of classical general relativity theory, especially the theory of spherically symmetric spacetimes (Einstein 1915 [1], Hilbert 1915 [2], Schwarzschild 1916 [3]). Since Schwarzschild, spherically symmetric models became a principal application of the theory, stimulating extensive research on the basis of classical differential geometry on Riemannian spaces (see Hawking, Ellis 1973 [4] and, for a more comprehensive contemporary discussion De Felice, Clarke 1990 [5], and Kriele 1999 [6]). Less is known, however, on the effort focused on a deeper understanding of what is going on from the topological point of view. For first steps in this direction, we refer to Clarke 1987 [7], and Siegl 1990 [8], 1992 [9]; different approaches can be found in the book Sachs, Wu 1977 [10], and the papers Szenthe 2000 [11], 2004 [12], and Tupper, Keane, Carot 2012 [13].

We do not consider in this paper physical aspects and physical motivation of the theory.

Our main objective is the existence and uniqueness of the Schwarzschild metrics on two topologically non-equivalent product manifolds, $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ and $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$. We wish to give an independent and more complete exposition of basic theorems and their proofs.

To study globally defined $(0,2)$-tensor fields, we need smooth structures on $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ and $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ explicitly, in terms of some smooth atlases. In Section 2, a spherical atlas on $\mathbf{R}^3 \setminus \{(0,0,0)\}$ consisting of two charts is introduced; we follow the choice of Krupka [14] and Tanaka, Krupka [15] with minor modifications only. Section 3 includes basic information on the special orthogonal group SO(3). Clearly, SO(3) acts as a transformation group on the second factor of $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$; its elements are the *rotations*. The translations and the reflection of the real line $\mathbf{R}$ induce the *time translations* and the *time reflection* of $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$. In Section 4, we derive chart formulas for $(0,2)$-tensor fields on $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$, *invariant* with respect to rotations, time translations and the time reflection. Section 5 briefly summarizes the basic notation, terminology and conventions related to the Einstein equations.

In Section 6, Einstein equations for a $(0,2)$-tensor field g on $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ are considered. We search for solutions, *invariant* with respect to rotations, time translations, and the time reflection. On the contrary to familiar approaches, no assumption on the signature of g, and no arguments outside mathematics, are applied (cf. De Felice, Clarke [5], Oas [16]). Our basic results are summarized in two theorems:

(a) First, a family of solutions, the *Schwarzschild metrics*, is obtained in terms of specific charts, close to the spherical charts. The family is parametrized by a strictly monotonic function $q = q(r)$, where r is the radial spherical coordinate, and by two real parameters, C and C', appearing as integration constants. A notable fact is that the family labelled by q, C and C', represents *all* solutions of the Einstein equations on the underlying chart neighborhood.

(b) Second, we show that the solutions defined in chart neighborhoods can be globalized; in other words, for any fixed q, the integration constants C and C' can be chosen in such a way that the solutions on the chart neighborhoods coincide on their intersection. Thus, as in the charts, we have a family of (global) solutions, parametrized by q, C, and C'.

For any Schwarzschild metric, one of the parameters, C, determines a submanifold of $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$, where the metric is not defined, the *Schwarzschild sphere*. It should be pointed out, however, that the family of solutions admits a metric whose Schwarzschild sphere is empty.

Finally, in Section 7, we search for spherically symmetric solutions of the Einstein equations on $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$. A specific method is implied. A winding mapping κ_0 of the real line $\mathbf{R}$ onto the circle S^1 is introduced, inducing a surjection κ of $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ onto $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$, and for any $(0,2)$-tensor fields on $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$, the pull-back $\kappa^* h$ on $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$. The chart expression of h can be determined by means of a standard atlas on the circle S^1 and the spherical atlas on $\mathbf{R}^3 \setminus \{(0,0,0)\}$. Given a Schwarzshild metric g on $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$, these construction allows us to consider condition $\kappa^* h = g$ as an equation for h. Our basic results can now be expressed parallelly to Section 6:

(a) For any Schwarzschild metric g on $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ there exists exactly one Schwarzschild metric h on a chart neighborhood in $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ such that the pull-back $\kappa^* h$ coincides with g, that is, $\kappa^* h = g$. Explicit expression of h in charts is given.

(b) For any Schwarzschild metric g on $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$, there exists exactly one Schwarzschild metric h on $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ such that $\kappa^* h = g$.

2. Spherical Atlas

In this Section, we define an atlas on the open subset $X = \mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ in the Euclidean space $\mathbf{R}^4$. This atlas consists of two charts employing spherical charts on $\mathbf{R}^3 \setminus \{(0,0,0)\}$. First, we describe spherical charts on $\mathbf{R}^3 \setminus \{(0,0,0)\}$. For this purpose, we use real-valued function arccos, which is defined as the inverse of the function cos with domain of definition $\langle 0, \pi \rangle$.

Let us denote by $U, \overline{U}, V$ open subsets of $\mathbf{R}^3$ determined as

$$
\begin{aligned}
U &= \mathbf{R}^3 \setminus \{(x,y,z) \in \mathbf{R}^3 |\, x \geq 0, y = 0\}, \\
\overline{U} &= \mathbf{R}^3 \setminus \{(x,y,z) \in \mathbf{R}^3 |\, x \leq 0, z = 0\}, \\
V &= (0,\infty) \times (0, 2\pi) \times (0, \pi),
\end{aligned}
$$

and by $\Lambda : V \ni (r, \varphi, \vartheta) \to (x,y,z) \in U$ the mapping, defined by equations

$$
x = r \cos\varphi \sin\vartheta, \quad y = r \sin\varphi \sin\vartheta, \quad z = r\cos\vartheta.
$$

Since the determinant of the Jacobi matrix of Λ is $-r^2 \sin\vartheta$, the map Λ is a local diffeomorphism. The inverse diffeomorphism $\Psi = \Lambda^{-1}$, $\Psi : U \ni (x,y,z) \to (r, \varphi, \vartheta) \in V$, is given by equations

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2}, \\
\varphi &= \begin{cases} \arccos \dfrac{x}{\sqrt{x^2+y^2}}, & y \geq 0, \\ 2\pi - \arccos \dfrac{x}{\sqrt{x^2+y^2}}, & y < 0, \end{cases} \\
\vartheta &= \arccos \frac{z}{\sqrt{x^2+y^2+z^2}}.
\end{aligned}
$$

Analogously, denoting by $\overline{\Lambda} : V \ni (\bar{r}, \bar{\varphi}, \bar{\vartheta}) \to (x,y,z) \in \overline{U}$ the mapping, defined by equations

$$
x = -\bar{r} \cos\bar{\varphi} \sin\bar{\vartheta}, \quad y = -\bar{r}\cos\bar{\vartheta}, \quad z = -\bar{r}\sin\bar{\varphi}\sin\bar{\vartheta},
$$

its inverse $\overline{\Psi} = \overline{\Lambda}^{-1}$, $\overline{\Psi} : \overline{U} \ni (x,y,z) \to (\bar{r}, \bar{\varphi}, \bar{\vartheta}) \in V$, is given by

$$
\begin{aligned}
\bar{r} &= \sqrt{x^2 + y^2 + z^2} \\
\bar{\varphi} &= \begin{cases} \arccos \dfrac{-x}{\sqrt{x^2+z^2}}, & z \leq 0, \\ 2\pi - \arccos \dfrac{-x}{\sqrt{x^2+z^2}}, & z > 0, \end{cases} \\
\bar{\vartheta} &= \arccos \frac{-y}{\sqrt{x^2+y^2+z^2}}.
\end{aligned}
$$

Lemma 1. *The set* $\{(U, \Psi), (\overline{U}, \overline{\Psi})\}$ *represents a smooth atlas on* $\boldsymbol{R}^3 \setminus \{(0,0,0)\}$.

Proof. The union $U \cup \overline{U}$ covers $\mathbf{R}^3 \setminus \{(0,0,0)\}$. The coordinate transformation $\overline{\Psi} \circ \Psi^{-1} : \Psi(U \cap \overline{U}) \to \overline{\Psi}(U \cap \overline{U})$, where $\Psi(U \cap \overline{U}) = \overline{\Psi}(U \cap \overline{U}) = V \setminus ((0,\infty) \times (\pi/2,\ 3\pi/2) \times \{\pi/2\})$, is a diffeomorphism as a composition of two diffeomorphisms, given by equations

$$\bar{r} = r, \quad \cos\bar{\varphi} = \frac{-\cos\varphi \sin\vartheta}{\sqrt{1-\sin^2\varphi \sin^2\vartheta}}, \quad \cos\bar{\vartheta} = -\sin\varphi\sin\vartheta. \tag{1}$$

□

This atlas is called the *spherical atlas* on $\mathbf{R}^3 \setminus \{(0,0,0)\}$; the charts (U, Ψ), $(\overline{U}, \overline{\Psi})$ are called the *first* and the *second spherical charts* on $\mathbf{R}^3 \setminus \{(0,0,0)\}$.

Remark 1. *The charts (U, Ψ), $(\overline{U}, \overline{\Psi})$ on $\mathbf{R}^3 \setminus \{(0,0,0)\}$ are related through the rotation ν of $\mathbf{R}^3$, in canonical coordinates expressed by the equations*

$$x \circ \nu = -x \quad y \circ \nu = -z \quad z \circ \nu = -y.$$

More exactly, $\overline{U} = \nu(U)$, and $\overline{\Psi} = \Psi \circ \nu$. Because ν is an involution, $U = \nu(\overline{U})$, and $\Psi = \overline{\Psi} \circ \nu$ also hold.

It is well-known that the manifold $\mathbf{R}^3 \setminus \{(0,0,0)\}$ is diffeomorphic with the manifold $(0,\infty) \times S^2$. The two-dimensional submanifold S^2 of $\mathbf{R}^3$ is defined by setting $r = 1$. If we denote

$$W = U \cap S^2, \quad \psi = (\phi, \theta), \quad \phi = \varphi|_{S^2}, \quad \theta = \vartheta|_{S^2},$$
$$\overline{W} = \overline{U} \cap S^2, \quad \overline{\psi} = (\bar{\phi}, \bar{\theta}), \quad \bar{\phi} = \bar{\varphi}|_{S^2}, \quad \bar{\theta} = \bar{\vartheta}|_{S^2},$$

the pairs (W, ψ) and $(\overline{W}, \overline{\psi})$ are charts on S^2 defining an atlas on S^2; we will call them the *first* and the *second charts on* S^2. Coordinate transformation $\overline{\psi} \circ \psi^{-1} : \psi(W \cap \overline{W}) \to \overline{\psi}(W \cap \overline{W})$ between the charts can be obtained from (1), and reads

$$\cos\bar{\phi} = \frac{-\cos\phi\sin\theta}{\sqrt{1-\sin^2\phi\sin^2\theta}}, \quad \cos\bar{\theta} = -\sin\phi\sin\theta.$$

Let us denote by s the canonical coordinate on $(0,\infty)$, and consider the product $(0,\infty) \times S^2$ with the product smooth manifold structure. The coordinate expressions $(r, \varphi, \vartheta) \to (s, \phi, \theta)$ of the mapping $U \to (0,\infty) \times W$, and $(\bar{r}, \bar{\varphi}, \bar{\vartheta}) \to (s, \bar{\phi}, \bar{\theta})$ of the mapping $\overline{U} \to (0,\infty) \times \overline{W}$, are identities on the domain $V = (0,\infty) \times (0, 2\pi) \times (0,\pi)$, so $\Theta : \mathbf{R}^3 \setminus \{(0,0,0)\} \to (0,\infty) \times S^2$ is a diffeomorphism.

Now consider $X = \mathbf{R} \times \mathbf{R}^3 \setminus \{(0,0,0)\}$ with the atlas formed by two charts $(\mathbf{R} \times U, \Phi)$, $(\mathbf{R} \times \overline{U}, \overline{\Phi})$, where $\Phi = (t, \Psi) = (t, r, \varphi, \vartheta)$, $\overline{\Phi} = (t, \overline{\Psi}) = (t, \bar{r}, \bar{\varphi}, \bar{\vartheta})$, and t is the canonical coordinate on $\mathbf{R}$.

In this paper, we call this atlas the *spherical atlas* on X; the charts $(\mathbf{R} \times U, \Phi)$, $(\mathbf{R} \times \overline{U}, \overline{\Phi})$ are called the *first* and *second spherical charts* on X.

3. The Special Orthogonal Group

The special orthogonal group SO(3) of $\mathbf{R}^3$ consists of orthogonal matrices with determinant +1 representing rotations of $\mathbf{R}^3$ around a point $(0,0,0)$. Such rotations are generated by the set of rotations around the axis x, y, z of the canonical frame in $\mathbf{R}^3$. In a positive-oriented frame, the equations of rotations about the x-axis, the y-axis and the z-axis are

$$\bar{x} = x, \quad \bar{y} = y\cos\beta_1 - z\sin\beta_1, \quad \bar{z} = y\sin\beta_1 + z\cos\beta_1,$$
$$\bar{x} = x\cos\beta_2 + z\sin\beta_2, \quad \bar{y} = y, \quad \bar{z} = -x\sin\beta_2 + z\cos\beta_2,$$
$$\bar{x} = x\cos\beta_3 - y\sin\beta_3, \quad \bar{y} = x\sin\beta_3 + y\cos\beta_3, \quad \bar{z} = z,$$

respectively, where β_1, β_2 and β_3 are the corresponding rotation parameters—angles (measured counter-clockwise from the point of view of positive orientation of the corresponding axis). The matrices of these rotations are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta_1 & -\sin\beta_1 \\ 0 & \sin\beta_1 & \cos\beta_1 \end{pmatrix}, \begin{pmatrix} \cos\beta_2 & 0 & \sin\beta_2 \\ 0 & 1 & 0 \\ -\sin\beta_2 & 0 & \cos\beta_2 \end{pmatrix}, \begin{pmatrix} \cos\beta_3 & -\sin\beta_3 & 0 \\ \sin\beta_3 & \cos\beta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The generators of rotations around the coordinate axes z, x, and y are expressed in canonical coordinates by

$$\xi = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \zeta = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad \lambda = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

For these vector fields, $[\xi,\zeta] = -\lambda$, $[\zeta,\lambda] = -\xi$, $[\lambda,\xi] = -\zeta$. In the first spherical coordinates,

$$\xi = \frac{\partial}{\partial\varphi}, \quad \zeta = -\cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi} - \sin\varphi\frac{\partial}{\partial\vartheta}, \quad \lambda = -\sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial}{\partial\vartheta}, \tag{2}$$

and in the second spherical coordinates,

$$\xi = \sin\bar{\varphi}\cot\bar{\vartheta}\frac{\partial}{\partial\bar{\varphi}} - \cos\bar{\varphi}\frac{\partial}{\partial\bar{\vartheta}}, \quad \zeta = \cos\bar{\varphi}\cot\bar{\vartheta}\frac{\partial}{\partial\bar{\varphi}} + \sin\bar{\varphi}\frac{\partial}{\partial\bar{\vartheta}}, \quad \lambda = -\frac{\partial}{\partial\bar{\varphi}}. \tag{3}$$

4. Invariance: SO(3), Time Translations, Time Reflection

Consider a $(0,2)$-tensor field g on an n-dimensional manifold X. If such tensor field is everywhere non-degenerate and symmetric, it is called a *metric tensor* on X, or a *metric* of X.

A $(0,2)$-tensor field g on a manifold X is said to be *invariant* with respect to a diffeomorphism $\alpha : X \to X$, if its pullback $\alpha^* g$ satisfies

$$\alpha^* g = g.$$

In such a case, we also say that α is an *invariance transformation* of g.

This definition can be naturally transformed to vector fields by means of the local one-parameter groups of diffeomorphisms. It is also applicable to an action of a Lie group on a manifold X, where vector fields on X become the generators of the corresponding group action on X.

Let ξ be a vector field on X. We say that ξ is the *generator of invariance transformations* of a metric field g if one-parameter group of ξ consists of the invariance transformations of g. This condition for ξ is equivalent to the *Killing equation*

$$\partial_\xi g = 0, \tag{4}$$

where ∂_ξ denotes the Lie derivative by a vector field ξ. If g and ξ are expressed in a chart (U,φ), $\varphi = (x^i)$, on X, by

$$g = g_{ij} dx^i \otimes dx^j, \quad \xi = \xi^i \frac{\partial}{\partial x^i},$$

then

$$\partial_\xi g = \left(\frac{\partial g_{kl}}{\partial x^p}\xi^p + g_{il}\frac{\partial \xi^i}{\partial x^k} + g_{kj}\frac{\partial \xi^j}{\partial x^l} \right) dx^k \otimes dx^l.$$

If a tensor field g on X is required to be invariant with respect to the one-parameter group of transformations, generated by given ξ, Equation (4) can be understood as a condition for g.

Now we apply (4) to find a tensor field g invariant with respect to the standard action of the special orthogonal group SO(3) on $\mathbf{R}^3 \setminus \{(0,0,0)\}$,

$$\mathrm{SO}(3) \times (\mathbf{R}^3 \setminus \{(0,0,0)\}) \ni (A, x) \mapsto A \cdot x \in \mathbf{R}^3 \setminus \{(0,0,0)\}. \tag{5}$$

Consider a $(0,2)$-tensor field g on the manifold $\mathbf{R}^3 \setminus \{(0,0,0)\}$. In the first spherical chart,

$$\begin{aligned} g = {} & g_{rr} dr \otimes dr + g_{r\varphi} dr \otimes d\varphi + g_{r\vartheta} dr \otimes d\vartheta \\ & + g_{\varphi r} d\varphi \otimes dr + g_{\varphi\varphi} d\varphi \otimes d\varphi + g_{\varphi\vartheta} d\varphi \otimes d\vartheta \\ & + g_{\vartheta r} d\vartheta \otimes dr + g_{\vartheta\varphi} d\vartheta \otimes d\varphi + g_{\vartheta\vartheta} d\vartheta \otimes d\vartheta. \end{aligned}$$

We wish to find the solution g_{rr}, $g_{r\varphi}$, $g_{r\vartheta}$, $g_{\varphi\varphi}$, $g_{\varphi\vartheta}$, $g_{\vartheta\vartheta}$ of the Killing equations

$$\partial_\xi g = 0, \quad \partial_\zeta g = 0, \quad \partial_\lambda g = 0, \tag{6}$$

where

$$\xi = \frac{\partial}{\partial \varphi}, \quad \zeta = -\cos\varphi \cot\vartheta \frac{\partial}{\partial \varphi} - \sin\varphi \frac{\partial}{\partial \vartheta}, \quad \lambda = -\sin\varphi \cot\vartheta \frac{\partial}{\partial \varphi} + \cos\varphi \frac{\partial}{\partial \vartheta}.$$

Theorem 1. *If $(0,2)$-tensor field g on $\mathbf{R}^3 \setminus \{(0,0,0)\}$ is invariant with respect to the action (5) of* SO(3)*, then in the first spherical coordinates, it is of the form*

$$g = P(r) dr \otimes dr + Q(r)(\sin^2\vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta), \tag{7}$$

where P and Q are functions, depending on r only.

Proof. The result follows from the solution of the Killing Equation (6); see also [14]. □

An analogous result can be obtained in the second spherical chart on $\mathbf{R}^3 \setminus \{(0,0,0)\}$. We now formally describe the globalization of our local result to the whole manifold $\mathbf{R}^3 \setminus \{(0,0,0)\}$, considering with atlas $\{(U, \Psi), (\overline{U}, \overline{\Psi})\}$.

Theorem 2. *Let*

$$g_U = P(r) dr \otimes dr + Q(r)(\sin^2\vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta)$$

be an SO(3)*-invariant $(0,2)$-tensor field on U, and let*

$$g_{\overline{U}} = \overline{P}(\bar{r}) d\bar{r} \otimes d\bar{r} + \overline{Q}(\bar{r})(\sin^2\bar{\vartheta} d\bar{\varphi} \otimes d\bar{\varphi} + d\bar{\vartheta} \otimes d\bar{\vartheta})$$

be an SO(3)*-invariant $(0,2)$-tensor field on $\overline{U}$. Then $g_U = g_{\overline{U}}$ on $U \cap \overline{U}$ if and only if*

$$\overline{P}(\bar{r}(x)) = P(r(x)), \quad \overline{Q}(\bar{r}(x)) = Q(r(x)) \tag{8}$$

for all $x \in U \cap \overline{U}$.

Proof. Since on the intersection $U \cap \overline{U}$,

$$dr \otimes dr = d\bar{r} \otimes d\bar{r}, \quad \sin^2\vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta = \sin^2\bar{\vartheta} d\bar{\varphi} \otimes d\bar{\varphi} + d\bar{\vartheta} \otimes d\bar{\vartheta},$$

the assertion is obvious. □

Condition (8) means that the function P can be naturally extended to the set $U \cup \overline{U}$; when no misunderstanding may possibly arise, we denote the extended function by the same symbol, P. A similar convention is applied to Q. This construction leads to globally defined functions P, Q on $\mathbf{R}^3 \setminus \{(0,0,0)\}$. Thus Theorem 2 constitutes a one-to-one correspondence between SO(3)-invariant $(0,2)$-tensor fields on $\mathbf{R}^3 \setminus \{(0,0,0)\}$ and the pairs of functions (P,Q), defined on $\mathbf{R}^3 \setminus \{(0,0,0)\}$.

Conversely, any two functions $P : \mathbf{R}^3 \setminus \{(0,0,0)\} \to \mathbf{R}$ and $Q : \mathbf{R}^3 \setminus \{(0,0,0)\} \to \mathbf{R}$ define an SO(3)-invariant $(0,2)$-tensor field on $\mathbf{R}^3 \setminus \{(0,0,0)\}$ by Theorem 2.

Analogously, if condition (8) is satisfied, then the formula

$$g(x) = \begin{cases} g_U(x), & x \in U \\ g_{\overline{U}}(x), & x \in \overline{U} \end{cases}$$

defines a SO(3)-invariant $(0,2)$-tensor field on $\mathbf{R}^3 \setminus \{(0,0,0)\}$.

Now our aim is to determine all $(0,2)$-tensor fields g on $X = \mathbf{R} \times \mathbf{R}^3 \setminus \{(0,0,0)\}$ invariant with respect to the left action of the group SO(3) on X defined by

$$\mathrm{SO}(3) \times X \ni (A,(t,x)) \mapsto (t, A \cdot x) \in X, \tag{9}$$

induced by canonical left action (5) of SO(3) on $\mathbf{R}^3 \setminus \{(0,0,0)\}$. We consider the atlas on X formed by the first and the second spherical charts, $(\mathbf{R} \times U, \Phi)$, and $(\mathbf{R} \times \overline{U}, \overline{\Phi})$, respectively (Section 2).

Theorem 3. *If $(0,2)$-tensor field g on X is invariant with respect to the action (9) of* SO(3)*, then in the first spherical coordinates, it is of the form*

$$\begin{aligned} g = {} & J(t,r)dt \otimes dt + K(t,r)(dt \otimes dr + dr \otimes dt) \\ & + P(t,r)dr \otimes dr + Q(t,r)(\sin^2\vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta), \end{aligned}$$

where J, K, P and Q are arbitrary functions of t and r on $\mathbf{R} \times U$.

Proof. Consider a $(0,2)$-tensor field g on the manifold X. In the first spherical chart on X,

$$\begin{aligned} g = {} & g_{tt}dt \otimes dt + g_{tr}dt \otimes dr + g_{t\varphi}dt \otimes d\varphi + g_{t\vartheta}dt \otimes d\vartheta \\ & + g_{rt}dr \otimes dt + g_{rr}dr \otimes dr + g_{r\varphi}dr \otimes d\varphi + g_{r\vartheta}dr \otimes d\vartheta \\ & + g_{\varphi t}d\varphi \otimes dt + g_{\varphi r}d\varphi \otimes dr + g_{\varphi\varphi}d\varphi \otimes d\varphi + g_{\varphi\vartheta}d\varphi \otimes d\vartheta \\ & + g_{\vartheta t}d\vartheta \otimes dt + g_{\vartheta r}d\vartheta \otimes dr + g_{\vartheta\varphi}d\vartheta \otimes d\varphi + g_{\vartheta\vartheta}d\vartheta \otimes d\vartheta. \end{aligned}$$

The solution

$$g_{tt},\ g_{tr},\ g_{t\varphi},\ g_{t\vartheta},\ g_{rr},\ g_{r\varphi},\ g_{r\vartheta},\ g_{\varphi\varphi},\ g_{\varphi\vartheta},\ g_{\vartheta\vartheta}$$

of the Killing equations $\partial_\xi g = 0, \partial_\zeta g = 0, \partial_\lambda g = 0$ for vector fields ξ, ζ, λ given by (2), is

$$\begin{gathered} g_{tt} = J(t,r),\ g_{tr} = K(t,r),\ g_{t\varphi} = 0,\ g_{t\vartheta} = 0, \\ g_{rr} = P(t,r),\ g_{r\varphi} = 0,\ g_{r\vartheta} = 0, \\ g_{\varphi\varphi} = Q(t,r)\sin^2\vartheta,\ g_{\varphi\vartheta} = 0,\ g_{\vartheta\vartheta} = Q(t,r). \end{gathered}$$

□

The similar result we analogously obtain can analogously be obtained in the second spherical chart on X for vector fields ξ, ζ, λ given by (3).

The following is an analogue of Theorem 2.

Theorem 4. *Let*

$$\begin{aligned} g_{\mathbf{R}\times U} = {} & J(t,r)dt \otimes dt + K(t,r)(dt \otimes dr + dr \otimes dt) \\ & + P(t,r)dr \otimes dr + Q(t,r)(\sin^2\vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta), \end{aligned}$$

be an SO(3)*-invariant* $(0,2)$*-tensor field on* U*, and let*

$$
\begin{aligned}
g_{\mathbf{R}\times\overline{U}} &= \overline{J}(t,\bar{r})dt\otimes dt + \overline{K}(t,\bar{r})(dt\otimes d\bar{r} + d\bar{r}\otimes dt) \\
&\quad + \overline{P}(t,\bar{r})d\bar{r}\otimes d\bar{r} + \overline{Q}(t,\bar{r})(\sin^2\bar{\vartheta}d\bar{\varphi}\otimes d\bar{\varphi} + d\bar{\vartheta}\otimes d\bar{\vartheta})
\end{aligned}
$$

be an SO(3)*-invariant* $(0,2)$*-tensor field on* $\mathbf{R}\times\overline{U}$*. Then* $g_{\mathbf{R}\times U} = g_{\mathbf{R}\times\overline{U}}$ *on* $(\mathbf{R}\times U)\cap(\mathbf{R}\times\overline{U})$ *if and only if*

$$
\begin{aligned}
\overline{J}(t,\bar{r}) &= J(t,r), \quad \overline{K}(t,\bar{r}) = K(t,r), \\
\overline{P}(t,\bar{r}) &= P(t,r), \quad \overline{Q}(t,\bar{r}) = Q(t,r),
\end{aligned}
$$

on $(\mathbf{R}\times U)\cap(\mathbf{R}\times\overline{U})$*.*

Proof. The assertion follows from the transformation equations between the first and the second spherical charts, and is analogous to the proof of Theorem 2. □

By the *time translation* in $X = \mathbf{R}\times(\mathbf{R}^3\setminus\{(0,0,0)\})$ we mean any transformation of the form

$$
\mathbf{R}\times(\mathbf{R}\times(\mathbf{R}^3\setminus\{(0,0,0)\})) \ni (\varepsilon,(t,x)) \to \tau_\varepsilon(t,x) = (t+\varepsilon,x) \in \mathbf{R}\times(\mathbf{R}^3\setminus\{(0,0,0)\}). \tag{10}
$$

Clearly, time translations define a left action of the additive group of real numbers $\mathbf{R}$ on X. The generator of the translations is the vector field

$$
\tau = \frac{\partial}{\partial t}.
$$

The *time reflection* in X is a transformation σ of X,

$$
\mathbf{R}\times(\mathbf{R}^3\setminus\{(0,0,0)\}) \ni (t,x) \to \sigma(t,x) = (-t,x) \in \mathbf{R}\times(\mathbf{R}^3\setminus\{(0,0,0)\}). \tag{11}
$$

We wish to determine all $(0,2)$-tensor fields g on X invariant with respect to the action (9), the time translations (10), and the time reflection (11).

Theorem 5. *Each* $(0,2)$*-tensor field* g *on* X *invariant with respect to the action* (9) *of* SO(3)*, with respect to the translations* (10)*, and to the transformation* (11)*, is in the first spherical chart expressed by*

$$
g = J(r)dt\otimes dt + P(r)dr\otimes dr + Q(r)(\sin^2\vartheta d\varphi\otimes d\varphi + d\vartheta\otimes d\vartheta),
$$

where J*,* P*, and* Q *are arbitrary functions on* $\mathbf{R}\times U$*, of the variable* r*.*

Proof. In the first spherical chart, a $(0,2)$-tensor field g on X invariant with respect to the action (9) is given by (7),

$$
\begin{aligned}
g &= J(t,r)dt\otimes dt + K(t,r)(dt\otimes dr + dr\otimes dt) \\
&\quad + P(t,r)dr\otimes dr + Q(t,r)(\sin^2\vartheta d\varphi\otimes d\varphi + d\vartheta\otimes d\vartheta),
\end{aligned}
$$

where J,K,P,Q are arbitrary functions on $\mathbf{R}\times U$, depending on t and r only. Equation $\partial_\tau g = 0$ implies that J,K,P,Q do not depend on t. Finally, invariance of g with respect to the transformation (11) yields $K=0$. □

The same consideration can be made in the second spherical chart, and we obtain the following result.

Theorem 6. *Let*

$$g_{\mathbf{R}\times U} = J(r)dt \otimes dt + P(r)dr \otimes dr + Q(r)(\sin^2 \vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta)$$

be an $(0,2)$*-tensor field on* $\mathbf{R} \times U$*, invariant with respect to the action* (9)*, the time translations* (10)*, and the time reflection* (11)*, and let*

$$g_{\mathbf{R}\times \overline{U}} = \overline{J}(\bar{r})dt \otimes dt + \overline{P}(\bar{r})d\bar{r} \otimes d\bar{r} + \overline{Q}(\bar{r})(\sin^2 \bar{\vartheta} d\bar{\varphi} \otimes d\bar{\varphi} + d\bar{\vartheta} \otimes d\bar{\vartheta})$$

be an $(0,2)$*-tensor field on* $\mathbf{R} \times \overline{U}$*, invariant with respect to the action* (9)*, the time translations* (10)*, and the time reflection* (11)*. Then* $g_{\mathbf{R}\times U} = g_{\mathbf{R}\times \overline{U}}$ *on* $(\mathbf{R} \times U) \cap (\mathbf{R} \times \overline{U})$ *if and only if*

$$\overline{J}(\bar{r}) = J(r), \quad \overline{P}(\bar{r}) = P(r), \quad \overline{Q}(\bar{r}) = Q(r),$$

on $(\mathbf{R} \times U) \cap (\mathbf{R} \times \overline{U})$*.*

Remark 2. *Theorem 6 does not imply that the tensor field g is regular, or of a certain signature. Such assumptions should be applied independently.*

5. Einstein Equations

We shall briefly recall basic definitions and conventions. Let X be a smooth manifold of dimension n. By a *metric* on X, we mean a symmetric, regular $(0,2)$-tensor field g on X. Note that in this definition, the signature of g is not specified. Let us have a metric g on an n-dimensional manifold X, expressed in a chart (U,φ), $\varphi = (x^i)$, on X by

$$g = g_{ij} dx^i \otimes dx^j.$$

The symmetry requirement is in this chart expression represented by the condition $g_{ij} = g_{ji}$ for all i, j; regularity means that $\det(g_{ij}) \neq 0$ everywhere. The functions

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

where g^{kl} are functions defined by $g_{jk} g^{kl} = \delta^l_j$, are the *Christoffel symbols*, the components of the *Levi–Civita connection associated with the metric g*, in a chart (U,φ). The *curvature tensor* of the Levi–Civita connection is a $(1,3)$-tensor field on X, expressed by

$$R^l_{kij} \frac{\partial}{\partial x^l} \otimes dx^k \otimes dx^i \otimes dx^j,$$

where

$$R^l_{kij} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik}.$$

The *Ricci tensor* is a $(0,2)$-tensor field on X, expressed by

$$R_{ij} dx^i \otimes dx^j,$$

where the components R_{ij} are defined by a $(1,3)$-contraction of the curvature tensor,

$$R_{ij} = R^k_{ijk}.$$

Contracting the $(1,1)$-tensor field $R^i_j = g^{im} R_{mj}$, we obtain a function R on X, the *scalar curvature* of g, or the *Ricci scalar*. In coordinates,

$$R = g^{ij} R_{ij}.$$

Extremals of the Hilbert variational functional, in which the scalar curvature stands for the Lagrangian, are determined by the *Einstein equations*:

$$R_{ij} - \frac{1}{2} R\, g_{ij} = 0.$$

The Einstein equations represent a system of second-order partial differential equations for the components g_{ij} of a metric g; the problem is to find solutions of the Einstein equations defined on X.

6. The Schwarzschild Solution: $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$

Let us consider invariant metric g, in the chart $(\mathbf{R} \times U, \Phi)$, on $X = \mathbf{R} \times \mathbf{R}^3 \setminus \{(0,0,0)\}$,

$$g = J(r) dt \otimes dt + P(r) dr \otimes dr + Q(r)(\sin^2 \vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta), \tag{12}$$

as introduced by Theorem 5. From (12), we can determine the left sides of the Einstein equations explicitly. Non-trivial equations yield

$$R_{tt} - \frac{1}{2} R\, g_{tt} = 0, \quad R_{rr} - \frac{1}{2} R\, g_{rr} = 0,$$
$$R_{\varphi\varphi} - \frac{1}{2} R\, g_{\varphi\varphi} = 0, \quad R_{\vartheta\vartheta} - \frac{1}{2} R\, g_{\vartheta\vartheta} = 0.$$

Since

$$0 = R_{\varphi\varphi} - \frac{1}{2} R\, g_{\varphi\varphi} = \sin^2 \vartheta \left(R_{\vartheta\vartheta} - \frac{1}{2} R\, g_{\vartheta\vartheta} \right),$$

only three equations of the system are independent. Writing these equations for the class of metrics (12), we obtain the following system:

$$\begin{aligned} \frac{J}{PQ}\left(\frac{1}{2}\frac{P'Q'}{P} + \frac{1}{4}\frac{(Q')^2}{Q} - Q'' + P \right) &= 0, \\ \frac{1}{Q}\left(\frac{1}{2}\frac{J'Q'}{J} + \frac{1}{4}\frac{(Q')^2}{Q} - P \right) &= 0, \\ \frac{1}{4JP}\left(-J'Q' - 2JQ'' - 2J''Q + \frac{JP'Q'}{P} + \frac{J'P'Q}{P} + \frac{(J')^2 Q}{J} + \frac{(Q')^2 J}{Q} \right) &= 0, \end{aligned} \tag{13}$$

where $'$ denotes the derivative with respect to r. (13) represents the system of three ordinary differential equations for unknown functions J, P, Q of the variable r.

Since, from the regularity condition, the functions J, P, Q are non-zero at every point of their domain, the system (13) is equivalent to the system

$$\begin{aligned} \frac{P'}{P}\frac{Q'}{Q} + \frac{1}{2}\left(\frac{Q'}{Q}\right)^2 - 2\frac{Q''}{Q} + 2\frac{P}{Q} &= 0, \\ \frac{J'}{J}\frac{Q'}{Q} + \frac{1}{2}\left(\frac{Q'}{Q}\right)^2 - 2\frac{P}{Q} &= 0, \\ -\frac{J'}{J}\frac{Q'}{Q} - 2\frac{Q''}{Q} - 2\frac{J''}{J} + \frac{P'}{P}\frac{Q'}{Q} + \frac{J'}{J}\frac{P'}{P} + \left(\frac{J'}{J}\right)^2 + \left(\frac{Q'}{Q}\right)^2 &= 0. \end{aligned} \tag{14}$$

Remark 3. *The system (14) is equivalent to the Einstein equations on the considered coordinate neighborhood. It should be pointed out, however, that the system (14) was derived without any assumption on the signature of an unknown metric. A standard approach following Schwarzschild [3] is based on a priori fixing of the signature—the Lorentz type signature (see [5,16]).*

From the first equation of (14), for the function $Q(r)$ of the variable r, we have that $Q'(r) \neq 0$ for every r from the domain; otherwise, we obtain $P = 0$, which is in contradiction to the assumption $P \neq 0$. According to the inverse function theorem, for any r, there exist connected neighborhoods U_0 of r, and V_0 of $Q(r)$ such that there exists a smooth map $Q^{-1} : V_0 \to U_0$, i.e., Q is invertible on the corresponding domain.

Due to the assumption $Q(r) \neq 0$, for every r and smoothness of Q, we have that $Q(r) > 0$, or $Q(r) < 0$ for every r. First, let us suppose $Q(r) > 0$ for every r. It enables us to denote $q(r) = \sqrt{Q(r)}$, and to replace the coordinates $(t, r, \varphi, \vartheta)$, on $\mathbf{R} \times U$, by $(t, q, \varphi, \vartheta)$. Setting

$$j(q) = J(r), \quad p(q) = P(r)\left(\frac{dr}{dq}\right)^2,$$

a metric g (12) can be rewritten in the form

$$g = j(q)dt \otimes dt + p(q)dq \otimes dq + q^2(\sin^2\vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta). \tag{15}$$

If $Q(r) < 0$ for all r, then we denote $q = \sqrt{-Q}$, and proceed as above.

Now, we give an assertion on the solution of the Einstein equations on the open set $\mathbf{R} \times U \subset \mathbf{R} \times (\mathbf{R}^3 \setminus (0,0,0))$ for the metrics determined by (15). The unknown g is expressed in the form (15).

Theorem 7. ***(Schwarzschild solution)*** *For any constants C, C', where $C' \neq 0$, formulas*

$$j(q) = C'\left(1 - \frac{C}{q}\right), \quad p(q) = \left(1 - \frac{C}{q}\right)^{-1}, \tag{16}$$

define a solution of the Einstein equations. The domain of definition of this solution is an open set of $\mathbf{R} \times U$ defined by $q \neq C$.

Proof. Consider the metric g on $\mathbf{R} \times U$ expressed by (15). Then, non-zero metric components of g on $\mathbf{R} \times U$ are

$$g_{tt} = j(q), \quad g_{rr} = p(q), \quad g_{\varphi\varphi} = q^2\sin^2\vartheta, \quad g_{\vartheta\vartheta} = q^2,$$

which implies

$$g^{tt} = \frac{1}{j(q)}, \quad g^{rr} = \frac{1}{p(q)}, \quad g^{\varphi\varphi} = \frac{1}{q^2\sin^2\vartheta}, \quad g^{\vartheta\vartheta} = \frac{1}{q^2},$$

and $g^{ik} = 0$ for each pair of mutually different indices i, k. Let us denote j', j'' and p', p'' the first and the second derivatives by q of the functions j, p, respectively.

The system (14) for unknown functions $j(q)$, $p(q)$ of one variable q, representing the Einstein equations, is then rewritten in the form

$$\begin{aligned} \frac{j}{qp}\left(\frac{p'}{p} - \frac{1}{q}(1-p)\right) &= 0, \\ -\frac{1}{q}\left(\frac{j'}{j} + \frac{1}{q}(1-p)\right) &= 0, \\ -\frac{1}{4}\frac{q^2}{p}\left(\frac{2}{q}\left(\frac{j'}{j} - \frac{p'}{p}\right) + 2\frac{j''}{j} - \frac{j'p'}{jp} - \frac{(j')^2}{j^2}\right) &= 0. \end{aligned} \tag{17}$$

A direct integration of the first equation of (17)

$$\frac{p'}{p} = \frac{1}{q}(1-p),$$

gives

$$p(q) = \left(1 - \frac{C}{q}\right)^{-1} \tag{18}$$

for any real constant C. Note that p is not defined on S, where S a subset of $\mathbf{R} \times U$ defined by $q = C$. Then $(\mathbf{R} \times U) \setminus S$ is a submanifold of $\mathbf{R} \times U$, consisting of two connected components determined by $0 < q < C$, and $q > C$, respectively.

Substituting (18) to the second equation of (17), we obtain

$$q\,\frac{j'}{j} = \frac{C}{q - C}.$$

Its solution is

$$j(q) = C'\left(1 - \frac{C}{q}\right),$$

where C' is a non-zero constant.

The solution $(j(q), p(q))$ fulfils the third equation of (17). This ends the proof. □

Remark 4. *Due to the invertibility of $q(r)$ as mentioned above, we are able to express the solution in the first spherical chart.*

The same assertion can be proved for the chart $(\mathbf{R} \times \overline{U}, \overline{\Phi})$. We obtain the solution

$$\bar{j}(\bar{q}) = \overline{C}'\left(1 - \frac{\overline{C}}{\bar{q}}\right), \quad \bar{p}(\bar{q}) = \left(1 - \frac{\overline{C}}{\bar{q}}\right)^{-1},$$

on $(\mathbf{R} \times \overline{U}) \setminus \overline{S}$ for constants $\overline{C}, \overline{C}'$, where $\overline{C}' \neq 0$, and $\overline{S} = \{x \in \mathbf{R} \times \overline{U} \mid \bar{q}(x) = 0\}$. Now we are in a position to globalize our results to the whole manifold $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0\})$.

Theorem 8. *Let*

$$g_{(\mathbf{R}\times U,\Phi)} = C'\left(1 - \frac{C}{q}\right)dt \otimes dt + \left(1 - \frac{C}{q}\right)^{-1} dq \otimes dq + q^2(\sin^2\vartheta d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta) \tag{19}$$

be the solution of the Einstein equations in the chart $(\mathbf{R} \times U, \Phi)$, and let

$$g_{(\mathbf{R}\times\overline{U},\overline{\Phi})} = \overline{C}'\left(1 - \frac{\overline{C}}{\bar{q}}\right)dt \otimes dt + \left(1 - \frac{\overline{C}}{\bar{q}}\right)^{-1} d\bar{q} \otimes d\bar{q} + \bar{q}^2(\sin^2\bar{\vartheta} d\bar{\varphi} \otimes d\bar{\varphi} + d\bar{\vartheta} \otimes d\bar{\vartheta}) \tag{20}$$

be the solution of the Einstein equations in the chart $(\mathbf{R} \times \overline{U}, \overline{\Phi})$. If

$$C = \overline{C}, \quad C' = \overline{C}', \tag{21}$$

then,
(a) *$S \cup \overline{S}$ is a submanifold of $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0\})$, given by equations $q = C$, $\bar{q} = \overline{C}$*
(b) *Formula*

$$g(x) = \begin{cases} g_{(\mathbf{R}\times U,\Phi)}(x), & x \in \mathbf{R} \times U \\ g_{(\mathbf{R}\times\overline{U},\overline{\Phi})}(x), & x \in \mathbf{R} \times \overline{U}, \end{cases} \tag{22}$$

defines a metric on the complement of $S \cup \overline{S}$ in $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0\})$.

Proof. Conditions (21) imply that on the intersection $(\mathbf{R} \times U) \cap (\mathbf{R} \times \overline{U})$ the set $S \cap \overline{S}$ is defined by equation $q = \bar{q}$. Also, expressions (19) and (20) satisfy assumptions of Theorem 6. □

We call the submanifold $S \cup \overline{S}$ the *Schwarzschild sphere* of the *Schwarzschild radius* $q = C = \overline{C} = \bar{q}$. For simplicity, we denote the Schwarzschild sphere just by S, and the Schwarzschild radius just by C.

Remark 5. *We can take in Theorem 8 for q the radial coordinate r. Note that in this case, Theorem 8 admits the value $C \leq 0$. However, condition $r = C$ has no sense, which means that the Schwarzschild sphere S is empty. In other words, the corresponding solution g is defined globally on* $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0\})$.

For any fixed q, Theorem 8 defines a metric g on $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0\})$. We obtain a family parametrized by the constants C and C'. Any element of this family is called a *Schwarzschild metric*. The manifold $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0\})$ endowed with a Schwarzschild metric g is a *Schwarzschild spacetime*.

Remark 6. *Considering $q = r$, $C' = -1$, and $C \neq 0$, we obtain the* classical Schwarzschild metric*, as known from the literature (e.g.,* [5]*).*

7. Extension: Spherical Symmetry on $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$

In this section, we consider the canonical product manifold structure on the topological space $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$. On the second factor $\mathbf{R}^3 \setminus \{(0,0,0)\}$, we use the atlas introduced in Section 2. It will be convenient to consider S^1 with the atlas defined by parallel projections along coordinate axes. Next, we introduce a *winding mapping* κ_0 from $\mathbf{R}$ to S^1, assigning to a point $t \in \mathbf{R}$ the point $(\cos t, \sin t)$ belonging to $S^1 \subset \mathbf{R}^2$. Indeed, κ_0 can be canonically extended to the projection mapping κ from $\mathbf{R} \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$ to $S^1 \times (\mathbf{R}^3 \setminus \{(0,0,0)\})$. Our objective will be to consider the pull-back of metric fields h by κ; we shall search for h such that $g = \kappa^* h$ is the Schwarzschild metric.

Consider the circle $S^1 \subset \mathbf{R}^2$ defined by $S^1 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$, and its subsets

$$U_1^+ = \{(x,y) \in S^1 \mid x > 0\}, \quad U_1^- = \{(x,y) \in S^1 \mid x < 0\},$$
$$U_2^+ = \{(x,y) \in S^1 \mid y > 0\}, \quad U_2^- = \{(x,y) \in S^1 \mid y < 0\}.$$

Define mappings $\varphi_1^+ : U_1^+ \to (-1,1)$, $\varphi_1^- : U_1^- \to (-1,1)$, $\varphi_2^+ : U_2^+ \to (-1,1)$, $\varphi_2^- : U_2^- \to (-1,1)$, by

$$\varphi_1^+(x,y) = y, \quad \varphi_1^-(x,y) = y, \quad \varphi_2^+(x,y) = x, \quad \varphi_2^-(x,y) = x.$$

Then the set $\mathcal{A} = \{(U_1^+, \varphi_1^+), (U_1^-, \varphi_1^-), (U_2^+, \varphi_2^+), (U_2^-, \varphi_2^-)\}$ is a smooth atlas on S^1. Indeed, the union $U_1^+ \cup U_1^- \cup U_2^+ \cup U_2^-$ covers S^1,

$$\varphi_1^+(U_1^+ \cap U_2^+) = \varphi_2^+(U_1^+ \cap U_2^+) = \varphi_1^-(U_1^- \cap U_2^+) = \varphi_2^-(U_1^+ \cap U_2^-) = (0,1),$$
$$\varphi_1^-(U_1^- \cap U_2^-) = \varphi_2^-(U_1^- \cap U_2^-) = \varphi_1^+(U_1^+ \cap U_2^-) = \varphi_2^+(U_1^- \cap U_2^+) = (-1,0),$$

and the coordinate transformations

$$\varphi_2^+ \circ (\varphi_1^+)^{-1} : \varphi_1^+(U_1^+ \cap U_2^+) \to \varphi_2^+(U_1^+ \cap U_2^+), \quad t \mapsto \sqrt{1-t^2},$$
$$\varphi_1^+ \circ (\varphi_2^+)^{-1} : \varphi_2^+(U_1^+ \cap U_2^+) \to \varphi_1^+(U_1^+ \cap U_2^+), \quad t \mapsto \sqrt{1-t^2},$$
$$\varphi_2^+ \circ (\varphi_1^-)^{-1} : \varphi_1^-(U_1^- \cap U_2^+) \to \varphi_2^+(U_1^- \cap U_2^+), \quad t \mapsto -\sqrt{1-t^2},$$
$$\varphi_1^- \circ (\varphi_2^+)^{-1} : \varphi_2^+(U_1^- \cap U_2^+) \to \varphi_1^-(U_1^- \cap U_2^+), \quad t \mapsto \sqrt{1-t^2},$$
$$\varphi_2^- \circ (\varphi_1^-)^{-1} : \varphi_1^-(U_1^- \cap U_2^-) \to \varphi_2^-(U_1^- \cap U_2^-), \quad t \mapsto -\sqrt{1-t^2},$$
$$\varphi_1^- \circ (\varphi_2^-)^{-1} : \varphi_2^-(U_1^- \cap U_2^-) \to \varphi_1^-(U_1^- \cap U_2^-), \quad t \mapsto -\sqrt{1-t^2},$$
$$\varphi_2^- \circ (\varphi_1^+)^{-1} : \varphi_1^+(U_1^+ \cap U_2^-) \to \varphi_2^-(U_1^+ \cap U_2^-), \quad t \mapsto \sqrt{1-t^2},$$
$$\varphi_1^+ \circ (\varphi_2^-)^{-1} : \varphi_2^-(U_1^+ \cap U_2^-) \to \varphi_1^+(U_1^+ \cap U_2^-), \quad t \mapsto -\sqrt{1-t^2},$$

are obviously smooth mappings on the corresponding domains. The circle S^1 will be always considered with the smooth structure defined by the atlas $\mathcal{A}$.

Setting

$$V_k = ((k - \tfrac{1}{2})\pi, (k + \tfrac{1}{2})\pi), \quad W_k = (k\pi, (k+1)\pi),$$

we obtain a family of open intervals in $\mathbf{R}$, indexed by the integers $k \in Z$. The sets V_k, W_k cover $\mathbf{R}$. Obviously, $V_i \cap V_k = W_i \cap W_k = \emptyset$ for each pair of different indices i, k. The intersection $V_i \cap W_k$ is non-empty if and only if $i = k$, or $i = k + 1$. The following assertion introduces a mapping $\kappa_0 : \mathbf{R} \to S^1$ as a periodic mapping with the period 2π.

Lemma 2. *There exists a smooth mapping $\kappa_0 : \mathbf{R} \to S^1$ whose coordinate expressions satisfy*

$$\begin{aligned} &(\varphi_1^+ \circ \kappa_0)|_{V_k} = \sin|_{V_k}, \quad (\varphi_2^+ \circ \kappa_0)|_{W_k} = \cos|_{W_k}, \quad k \text{ even}, \\ &(\varphi_1^- \circ \kappa_0)|_{V_k} = \sin|_{V_k}, \quad (\varphi_2^- \circ \kappa_0)|_{W_k} = \cos|_{W_k}, \quad k \text{ odd}. \end{aligned}$$

Proof. Straightforward. □

The mapping $\kappa_0 : \mathbf{R} \to S^1$ induces the mapping

$$\kappa : \mathbf{R} \times \mathbf{R}^3 \setminus \{(0,0,0)\} \to S^1 \times \mathbf{R}^3 \setminus \{(0,0,0)\}, \qquad \kappa(t, x) = (\kappa_0(t), x).$$

In what follows, we denote for simplicity

$$\alpha = \varphi_1^+, \quad \beta = \varphi_2^+, \quad \gamma = \varphi_1^-, \quad \delta = \varphi_2^-.$$

Next, consider the spherical charts (U, Ψ), $\Psi = (r, \varphi, \vartheta)$, and $(\overline{U}, \overline{\Psi})$, $\overline{\Psi} = (\bar{r}, \bar{\varphi}, \bar{\vartheta})$, on $\mathbf{R}^3 \setminus \{(0,0,0)\}$ (Section 2). The manifold $Y = S^1 \times \mathbf{R}^3 \setminus \{(0,0,0)\}$ can be covered by eight sets

$$\begin{aligned} &W_1 = U_1^+ \times U, \quad W_2 = U_2^+ \times U, \quad W_3 = U_1^- \times U, \quad W_4 = U_2^- \times U, \\ &\overline{W}_1 = U_1^+ \times \overline{U}, \quad \overline{W}_2 = U_2^+ \times \overline{U}, \quad \overline{W}_3 = U_1^- \times \overline{U}, \quad \overline{W}_4 = U_2^- \times \overline{U}. \end{aligned} \tag{23}$$

The corresponding coordinates on the sets of (23) are

$$\begin{aligned} &(\alpha, \Psi), \quad (\beta, \Psi), \quad (\gamma, \Psi), \quad (\delta, \Psi), \\ &(\alpha, \overline{\Psi}), \quad (\beta, \overline{\Psi}), \quad (\gamma, \overline{\Psi}), \quad (\delta, \overline{\Psi}), \end{aligned}$$

where the coordinates $r, \bar{r}$ can be replaced, in the sense of Section 6, by $q, \bar{q}$, respectively.

Our aim is to find a metric h on $Y = S^1 \times \mathbf{R}^3 \setminus \{(0,0,0)\}$ corresponding through the mapping κ with the Schwarzschild metric g (22), on X. We construct h by means of charts. However, the chart expressions of h in different charts of our atlas turn out to be quite analogous. For this reason, we restrict the formulation of the following theorem to a fixed chart on Y.

Theorem 9. *Let g be a Schwarzschild metric (22) on $\mathbf{R} \times \mathbf{R}^3 \setminus \{(0,0,0)\}$. There exists a unique metric h on W^1 such that $g = \kappa^* h$. In the coordinates $(\alpha, q, \varphi, \vartheta))$, h is expressed by*

$$h = C'\left(1 - \frac{C}{q}\right)\frac{1}{1-\alpha^2}\, d\alpha \otimes d\alpha + \left(1 - \frac{C}{q}\right)^{-1} dq \otimes dq + q^2 \sin^2\vartheta\, d\varphi \otimes d\varphi + q^2\, d\vartheta \otimes d\vartheta.$$

This expression is defined on an open subset of W_1, determined by $q \neq C$, and satisfies the Einstein equations.

Proof. The coordinate expression of h is obtained by comparing the pull-back by κ of a $(0,2)$-tensor field h, in the coordinate chart $(W_1, (\alpha, q, \varphi, \vartheta))$, on Y, expressed by

$$\begin{aligned} h = {} & h_{\alpha\alpha} d\alpha \otimes d\alpha + h_{\alpha q} d\alpha \otimes dq + h_{\alpha\varphi} dt \otimes d\varphi + h_{\alpha\vartheta} dt \otimes d\vartheta \\ & + h_{q\alpha} dq \otimes d\alpha + h_{qq} dq \otimes dq + h_{q\varphi} dq \otimes d\varphi + h_{q\vartheta} dq \otimes d\vartheta \\ & + h_{\varphi\alpha} d\varphi \otimes d\alpha + h_{\varphi q} d\varphi \otimes dq + h_{\varphi\varphi} d\varphi \otimes d\varphi + h_{\varphi\vartheta} d\varphi \otimes d\vartheta \\ & + h_{\vartheta\alpha} d\vartheta \otimes d\alpha + h_{\vartheta q} d\vartheta \otimes dq + h_{\vartheta\varphi} d\vartheta \otimes d\varphi + h_{\vartheta\vartheta} d\vartheta \otimes d\vartheta, \end{aligned}$$

and the expression of the Schwarzshild metric g (19) in the coordinate chart $(\mathbf{R} \times U, \Phi)$. Then, computing the Christoffel symbols from the components of h, we obtain that the components of corresponding Ricci tensor vanish. □

For globalization, we need coordinate expressions of h in all charts of our atlas on Y (23). According to Theorem 9, we obtain

$$\begin{aligned} & C'\left(1 - \frac{C}{q}\right) \frac{1}{1-\alpha^2} d\alpha \otimes d\alpha + \left(1 - \frac{C}{q}\right)^{-1} dq \otimes dq + q^2 \sin^2 \vartheta \, d\varphi \otimes d\varphi + q^2 \, d\vartheta \otimes d\vartheta, \text{ on } W_1 \\ & C'\left(1 - \frac{C}{q}\right) \frac{1}{1-\beta^2} d\beta \otimes d\beta + \left(1 - \frac{C}{q}\right)^{-1} dq \otimes dq + q^2 \sin^2 \vartheta \, d\varphi \otimes d\varphi + q^2 \, d\vartheta \otimes d\vartheta, \text{ on } W_2 \\ & C'\left(1 - \frac{C}{q}\right) \frac{1}{1-\gamma^2} d\gamma \otimes d\gamma + \left(1 - \frac{C}{q}\right)^{-1} dq \otimes dq + q^2 \sin^2 \vartheta \, d\varphi \otimes d\varphi + q^2 \, d\vartheta \otimes d\vartheta, \text{ on } W_3 \\ & C'\left(1 - \frac{C}{q}\right) \frac{1}{1-\delta^2} d\delta \otimes d\delta + \left(1 - \frac{C}{q}\right)^{-1} dq \otimes dq + q^2 \sin^2 \vartheta \, d\varphi \otimes d\varphi + q^2 \, d\vartheta \otimes d\vartheta, \text{ on } W_4 \\ & C'\left(1 - \frac{C}{\bar q}\right) \frac{1}{1-\alpha^2} d\alpha \otimes d\alpha + \left(1 - \frac{C}{\bar q}\right)^{-1} d\bar q \otimes d\bar q + \bar q^2 \sin^2 \bar\vartheta \, d\bar\varphi \otimes d\bar\varphi + \bar q^2 \, d\bar\vartheta \otimes d\bar\vartheta, \text{ on } \overline{W}_1 \\ & C'\left(1 - \frac{C}{\bar q}\right) \frac{1}{1-\beta^2} d\beta \otimes d\beta + \left(1 - \frac{C}{\bar q}\right)^{-1} d\bar q \otimes d\bar q + \bar q^2 \sin^2 \bar\vartheta \, d\bar\varphi \otimes d\bar\varphi + \bar q^2 \, d\bar\vartheta \otimes d\bar\vartheta, \text{ on } \overline{W}_2 \\ & C'\left(1 - \frac{C}{\bar q}\right) \frac{1}{1-\gamma^2} d\gamma \otimes d\gamma + \left(1 - \frac{C}{\bar q}\right)^{-1} d\bar q \otimes d\bar q + \bar q^2 \sin^2 \bar\vartheta \, d\bar\varphi \otimes d\bar\varphi + \bar q^2 \, d\bar\vartheta \otimes d\bar\vartheta, \text{ on } \overline{W}_3 \\ & C'\left(1 - \frac{C}{\bar q}\right) \frac{1}{1-\delta^2} d\delta \otimes d\delta + \left(1 - \frac{C}{\bar q}\right)^{-1} d\bar q \otimes d\bar q + \bar q^2 \sin^2 \bar\vartheta \, d\bar\varphi \otimes d\bar\varphi + \bar q^2 \, d\bar\vartheta \otimes d\bar\vartheta, \text{ on } \overline{W}_4. \end{aligned} \tag{24}$$

Theorem 10. *Let g be a Schwarzschild metric (22) determined by q, C, C', defined on the open subset of $X = \mathbf{R} \times \mathbf{R}^3 \setminus \{(0,0,0)\}$, where $q \neq C$. Then, h (24) is the metric on an open subset of $Y = S^1 \times \mathbf{R}^3 \setminus \{(0,0,0)\}$, determined by $q \neq C$. h satisfies the Einstein equations.*

Proof. Since the constants $C, C' \neq 0$ are the same in all charts the corresponding components of h transform as the components of a metric according to

$$\bar h_{ij} = \frac{\partial x^k}{\partial \bar x^i} \frac{\partial x^l}{\partial \bar x^j} h_{kl}.$$

On each of the charts on Y, computing the Christoffel symbols from the components of the metric h, we obtain that the components of corresponding Ricci tensor vanish, which means that the metric h fulfils the Einstein equations on Y. □

Remark 7. *The product manifold $Y = S^1 \times \mathbf{R}^3 \setminus \{(0,0,0)\}$ has the structure of a fibered manifold over S^1; its base S^1 is compact. An analogous assertion is not true for the fibered manifold $X = \mathbf{R} \times \mathbf{R}^3 \setminus \{(0,0,0)\}$ over $\mathbf{R}$.*

Author Contributions: The authors equally contributed to conceptualization, methodology, original draft preparation, and preparation of comments on reviews of the paper.

Funding: This research received no external funding.

Acknowledgments: This work was supported by the Transilvania Fellowship Program for Visiting Professors. The first author (D.K.) highly appreciates excellent research conditions extended to him by the Department of Mathematics and Computer Science of the Transilvania University in Brasov, Romania.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Einstein, A. Die Feldgleichungen der Gravitation. *Sitzungsberichte Der KöNiglich Preuss. Akad. Der Wiss. Berl.* **1915**, *25*, 844–847.
2. Hilbert, D. Die Grundlangen der Physik. Nachrichten von der Kon. Gessellschaft der Wiss. Gottingen. *Math.-phys. K l* **1915**, 395–407.
3. Schwarzschild, K. Uber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte Der KöNiglich PreußIschen Akad. Der Wiss. Berl.* **1916**, 189–196.
4. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure of Space-Time*; Cambridge University Press: Cambridge, UK, 1973.
5. De Felice, F.; Clarke, C.J.S. Relativity on Curved Manifolds. In *Cambridge Monographs on Mathematical Physics*; Cambridge University Press: Cambridge, UK, 1990.
6. Kriele, M. *Spacetime—Foundations of General Relativity and Differential Geometry*; Springer: Berlin/Heidelberg, Germany, 1999.
7. Clarke, C.J.S. Spherical symmetry does not imply a direct product. *Class. Quantum Grav.* **1987**, *4*, L37–L40. [CrossRef]
8. Siegl, R. Spherically symmetric vacuum spacetimes: Global approach. In Proceedings of the Differential Geometry and Its Applications, Brno, Czechoslovakia, 27 August–2 September 1989; World Scientific: Singapore, 1990; pp. 432–440.
9. Siegl, R. Some underlying manifolds of the Schwarzschild solution. *Class. Quantum Grav.* **1992**, *9*, 239–240. [CrossRef]
10. Sachs, R.K.; Wu, H. *General Relativity for Mathematicians*; Graduate Texts in Mathematics, 48; Springer: New York, NY, USA, 1977.
11. Szenthe, J. On the global geometry of spherically symmetric space-times. *Math. Proc. Camb. Philos. Soc.* **2004**, *137*, 741–754. [CrossRef]
12. Szenthe, J. On topology of spherically symmetric space-times. *Cent. Eur. J. Math.* **2004**, *2*, 725–731. [CrossRef]
13. Tupper, B.O.J.; Keane, A.J.; Carot, J. A classification of spherically symmetric spacetimes. *Class. Quantum Grav.* **2012**, *29*, 145016. [CrossRef]
14. Krupka, D. *The Kruskal-Szekeres Globalization. Lecture Notes*; Lepage Research Institute Library: Prešov, Slovakia, 2022; Volume 7, 39p.
15. Tanaka, E.; Krupka, D. On metrizability of invariant affine connections. *Int. J. Geom. Methods Mod. Phys.* **2012**, *9*, 1250014. [CrossRef]
16. Oas, G. Full Derivation of the Schwarzschild Solution. EPGY Summer Institute. Available online: https://www.scribd.com/document/410275080/SchwarzschildSolution-pdf (accessed on 14 November 2022).