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Some Properties of the Solution to a System of Quaternion Matrix Equations

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Abstract: This paper investigates the properties of the ϕ-skew-Hermitian solution to the system of quaternion matrix equations involving ϕ-skew-Hermicity with four unknowns $A_iX_i(A_i)_\phi + B_iX_{i+1}(B_i)_\phi = C_i,\ (i = 1,2,3),\ A_4X_4(A_4)_\phi = C_4$. We present the general ϕ-skew-Hermitian solution to this system. Moreover, we derive the $\beta(\phi)$-signature bounds of the ϕ-skew-Hermitian solution X_1 in terms of the coefficient matrices. We also give some necessary and sufficient conditions for the system to have $\beta(\phi)$-positive semidefinite, $\beta(\phi)$-positive definite, $\beta(\phi)$-negative semidefinite and $\beta(\phi)$-negative definite solutions.

Keywords: quaternion algebra; matrix decompositions; matrix equations; ϕ-skew-Hermicity; signature bounds

MSC: 15A06; 15A23; 15A24; 15B57

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1. Introduction

The quaternion matrix can be used in quantum mechanics [1], color image processing (e.g., [2–4]), and signal processing [5], etc. Some researchers have studied the solvability conditions and solutions to some quaternion matrix equations (e.g., [6–9]).

Hermitian solutions to quaternion matrix equations have been discussed in many papers. Rodman investigated the definitions of ϕ-Hermitian, ϕ-skew-Hermitian quaternion matrices (Definition 3.6.1 in [10]) and presented a decomposition of ϕ-skew-Hermitian quaternion matrix (see Lemma 1). Since then, some researchers have considered the applications of ϕ(-skew)-Hermitian quaternion matrices in various aspects. Aghamollaei and Rahjoo [11] established the numerical ranges with respect to nonstandard involutions on quaternionic. Rahjoo et al. [12] studied the numerical ranges with respect to nonstandard involutions. He et al. [13] considered two systems of quaternion matrix equations

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Z - YB_2 = C_2, \end{cases} \quad Z = Z_\phi,$$

and

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Y - ZB_2 = C_2, \end{cases} \quad Z = Z_\phi.$$

Wang and Jiang [14] derived the ranks of the skew-Hermitian solution to a classical quaternion matrix equation with two unknowns. The η-Hermitian quaternion matrix decompositions have applications in signal processing and linear modeling (e.g., [15–18]). Moreover, He [19] has been investigated the structure, properties and applications of a simultaneous decomposition for quaternion matrices involving ϕ-skew-Hermitian. He et al. [20]

presented some solvability conditions to a system of quaternion matrix equations involving ϕ-skew-Hermicity

$$\begin{cases} A_1X_1(A_1)_\phi + B_1X_2(B_1)_\phi = C_1, \\ A_2X_2(A_2)_\phi + B_2X_3(B_2)_\phi = C_2, \\ A_3X_3(A_3)_\phi + B_3X_4(B_3)_\phi = C_3, \\ A_4X_4(A_4)_\phi = C_4, \end{cases} \quad X_i = -(X_i)_\phi, \tag{1}$$

where $A_i \in \mathbb{H}^{p_i \times t_i}$, $B_i \in \mathbb{H}^{p_i \times t_{i+1}}$, $C_i \in \mathbb{H}^{p_i \times p_i}$, and C_i are ϕ-skew-Hermitian matrices. As we know, the solution of the system (1) has not been studied. On the other hand, a special case of the system (1)

$$A_1X_1(A_1)_\phi = C_1 \tag{2}$$

can be used in statistics and vibration theory (e.g., [21,22]). The matrix Equation (2) can be used to consider an inverse problem arising in structural modification of the dynamic behaviour (e.g., [23,24]). We conjecture that the main system (1) will also play an important role in the statistics, vibration theory and dynamic behaviour. Inspired by the Hermitian solutions to quaternion matrix equations have widely applications in system and control theory, we consider the expression and properties of the solution to the system (1) in this paper.

The remainder of this paper is organized as follows. In Section 2, we review some definitions and introduce some notations. In Section 3, we provide the general solution to the system (1). In Section 4, we give the $\beta(\phi)$-signature bounds of the solution X_1 to the system (1) and give some necessary and sufficient conditions for the system (1) to have $\beta(\phi)$-positive semidefinite, $\beta(\phi)$-positive definite, $\beta(\phi)$-negative semidefinite and $\beta(\phi)$-negative definite solutions.

2. Preliminaries

In this section, we review some definitions.

Let $\mathbb{R}$ denote the fields of the real numbers. Let $\mathbb{H}$ be a four dimensional vector space over $\mathbb{R}$ with an ordered basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ [25]. Note that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfies

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \ \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \ \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

A real quaternion simply called quaternion is a vector $\boldsymbol{x} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}$ with real coefficients a_0, a_1, a_2, a_3.

The definition of nonstandard involution is giving as follows.

Definition 1 (Non-standard Involution [10])**.** *Let ϕ be an anti-endomorphism of $\mathbb{H}$. Assume that ϕ does not map $\mathbb{H}$ into zero. Then, ϕ is one-to-one and onto $\mathbb{H}$. Thus, ϕ is an anti-automorphism. Moreover, ϕ is real linear and can be represented as a 4×4 real matrix with respect to the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Then, ϕ is a non-standard involution if and only if*

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},$$

where T is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$.

Rodman [10] considered some properties of nonstandard involution. Next, we review the definition of ϕ-skew-Hermitian.

Definition 2 (ϕ-skew-Hermitian [10])**.** *$A \in \mathbb{H}^{n \times n}$ is said to be ϕ-skew-Hermitian if $A = -(A)_\phi$, where ϕ is a nonstandard involution.*

The canonical form of a ϕ-skew-Hermitian matrix is presented in [10]. First, we review the definition of ϕ-congruent.

Definition 3 (ϕ-congruent [10])**.** *The quaternion matrices $A, B \in \mathbb{H}^{n\times n}$ are ϕ-congruent if $A = SBS_{\phi}$, where S is an invertible quaternion matrix and ϕ is a nonstandard involution.*

Lemma 1 ([10])**.** *Let ϕ be a nonstandard involution. For every ϕ-skew-Hermitian matrix $A \in \mathbb{H}^{n\times n}$, there exists an invertible matrix $S \in \mathbb{H}^{n\times n}$ such that*

$$SAS_{\phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta I_p & 0 \\ 0 & 0 & -\beta I_q \end{pmatrix}, \quad \beta = \beta(\phi), \tag{3}$$

where the unit ϕ-skew-Hermitian quaternion β is fixed and denoted by $\beta(\phi)$. Moreover, the integers p and q are uniquely determined by A (for a fixed $\beta(\phi)$).

According to Lemma 1, the definition of $\beta(\phi)$-signature of a ϕ-skew-Hermitian quaternion matrix A is provided.

Definition 4 ($\beta(\phi)$-signature [10])**.** *We say that the ordered triple of nonnegative integers*

$$(\ln_{+}(A), \ln_{-}(A), \ln_{0}(A)) := (p, q, n-p-q)$$

is the $\beta(\phi)$-signature of a ϕ-skew-Hermitian quaternion matrix A, as in Lemma 1. The matrix A is said to be $\beta(\phi)$-positive definite, $\beta(\phi)$-positive semidefinite, if $\ln_{+}(A) = n$, $\ln_{+}(A) + \ln_{0}(A) = n$, respectively. Analogously, $\beta(\phi)$-negative definite and $\beta(\phi)$-negative semidefinite ϕ-skew-Hermitian quaternion matrices are defined.

3. The General ϕ-Skew-Hermitian Solution to the System (1)

In this section, we provide the general ϕ-skew-Hermitian solution to the system (1).

Using the results of Lemma 1 in [20], there exist nonsingular matrices $\widehat{T}_i \in \mathbb{H}^{t_i\times t_i}$, $\widehat{P}_i \in \mathbb{H}^{p_i\times p_i}$, $(i = 1, 2, 3)$, $\widehat{T}_4 \in \mathbb{H}^{t_4\times t_4}$, $\widehat{P}_4 \in \mathbb{H}^{p_4\times p_4}$ such that

$$\widehat{P}_i A_i \widehat{T}_i = S_{a_i}, \quad \widehat{P}_i B_i \widehat{T}_{i+1} = S_{b_i}, \quad \widehat{P}_4 A_4 \widehat{T}_4 = S_{a_4}.$$

Therefore, the system (1) is equivalent to the following system:

$$\begin{cases} S_{a_1}\widehat{X}_1(S_{a_1})_{\phi} + S_{b_1}\widehat{X}_2(S_{b_1})_{\phi} = D_{kj}^{(1)}, \\ S_{a_2}\widehat{X}_2(S_{a_2})_{\phi} + S_{b_2}\widehat{X}_3(S_{b_2})_{\phi} = D_{kj}^{(2)}, \\ S_{a_3}\widehat{X}_3(S_{a_3})_{\phi} + S_{b_3}\widehat{X}_4(S_{b_3})_{\phi} = D_{kj}^{(3)}, \\ S_{a_4}\widehat{X}_4(S_{a_4})_{\phi} = D_{kj}^{(4)}, \end{cases} \tag{4}$$

where $X_i = -(X_i)_{\phi}$, $D_{kj}^{(i)} = \widehat{P}_i C_i (\widehat{P}_i)_{\phi}$, S_{a_i} and S_{b_i} have the following form

$$
\begin{pmatrix} S_{a_1} & S_{b_1} & & \\ \hline & S_{a_2} & S_{b_2} & \\ \hline & & S_{a_3} & S_{b_3} \\ \hline & & & S_{a_4} \end{pmatrix}
$$

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\end{array}
\end{array}\right). \tag{5}
$$

The above idea and symbols are presented in [20].

In order to give the general ϕ-skew-Hermitian solution to the system (1), we need to obtain the general ϕ-skew-Hermitian solution to the system (4). The following theorem gives the general ϕ-skew-Hermitian solution to the system (1).

Theorem 1. *Assume that the system (1) is consistent. The general ϕ-skew-Hermitian solution to the system (1) can be expressed as*

$$
X_1 = \widehat{T}_1 \widehat{X}_1 (\widehat{T}_1)_\phi, \ X_2 = \widehat{T}_2 \widehat{X}_2 (\widehat{T}_2)_\phi, \ X_3 = \widehat{T}_3 \widehat{X}_3 (\widehat{T}_3)_\phi, \ X_4 = \widehat{T}_4 \widehat{X}_4 (\widehat{T}_4)_\phi, \tag{6}
$$

where

$$\widehat{X}_1 = \begin{pmatrix} D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} & D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{16}^{(1)} - X_{16}^{(2)} & D_{17}^{(1)} & X_{18}^{(1)} \\ -(D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)})_\phi & D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)} & D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} & D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)} & D_{25}^{(1)} - D_{25}^{(2)} & D_{26}^{(1)} - X_{26}^{(2)} & D_{27}^{(1)} & X_{28}^{(1)} \\ -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi & -(D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)})_\phi & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{36}^{(1)} - X_{36}^{(2)} & D_{37}^{(1)} & X_{38}^{(1)} \\ -(D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)})_\phi & -(D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)})_\phi & -(D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)})_\phi & D_{44}^{(1)} - D_{44}^{(2)} + X_{44}^{(3)} & D_{45}^{(1)} - D_{45}^{(2)} & D_{46}^{(1)} - X_{46}^{(2)} & D_{47}^{(1)} & X_{48}^{(1)} \\ -(D_{15}^{(1)} - D_{15}^{(2)})_\phi & -(D_{25}^{(1)} - D_{25}^{(2)})_\phi & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi & -(D_{45}^{(1)} - D_{45}^{(2)})_\phi & D_{55}^{(1)} - D_{55}^{(2)} & D_{56}^{(1)} - X_{56}^{(2)} & D_{57}^{(1)} & X_{58}^{(1)} \\ -(D_{16}^{(1)} - X_{16}^{(2)})_\phi & -(D_{26}^{(1)} - X_{26}^{(2)})_\phi & -(D_{36}^{(1)} - X_{36}^{(2)})_\phi & -(D_{46}^{(1)} - X_{46}^{(2)})_\phi & -(D_{56}^{(1)} - X_{56}^{(2)})_\phi & D_{66}^{(1)} - X_{66}^{(2)} & D_{67}^{(1)} & X_{68}^{(1)} \\ -(D_{17}^{(1)})_\phi & -(D_{27}^{(1)})_\phi & -(D_{37}^{(1)})_\phi & -(D_{47}^{(1)})_\phi & -(D_{57}^{(1)})_\phi & -(D_{67}^{(1)})_\phi & D_{77}^{(1)} & X_{78}^{(1)} \\ -(X_{18}^{(1)})_\phi & -(X_{28}^{(1)})_\phi & -(X_{38}^{(1)})_\phi & -(X_{48}^{(1)})_\phi & -(X_{58}^{(1)})_\phi & -(X_{68}^{(1)})_\phi & -(X_{78}^{(1)})_\phi & X_{88}^{(1)} \end{pmatrix}, \quad (7)$$

$$\widehat{X}_3 = \left(\begin{matrix} D_{11}^{(3)} - D_{11}^{(4)} & D_{12}^{(3)} - X_{12}^{(4)} & D_{13}^{(3)} & X_{14}^{(3)} & D_{14}^{(3)} - D_{12}^{(4)} & D_{15}^{(3)} - X_{14}^{(4)} & D_{16}^{(3)} \\ -(D_{12}^{(3)} - X_{12}^{(4)})_\phi & D_{22}^{(3)} - X_{22}^{(4)} & D_{23}^{(3)} & X_{24}^{(3)} & D_{24}^{(3)} - X_{23}^{(4)} & D_{25}^{(3)} - X_{24}^{(4)} & D_{26}^{(3)} \\ -(D_{13}^{(3)})_\phi & -(D_{23}^{(3)})_\phi & D_{33}^{(3)} & X_{34}^{(3)} & D_{34}^{(3)} & D_{35}^{(3)} & D_{36}^{(3)} \\ -(X_{14}^{(3)})_\phi & -(X_{24}^{(3)})_\phi & -(X_{34}^{(3)})_\phi & X_{44}^{(3)} & X_{45}^{(3)} & X_{46}^{(3)} & X_{47}^{(3)} \\ -(D_{14}^{(3)} - D_{12}^{(4)})_\phi & -(D_{24}^{(3)} - X_{23}^{(4)})_\phi & -(D_{34}^{(3)})_\phi & -(X_{45}^{(3)})_\phi & D_{44}^{(3)} - D_{22}^{(4)} & D_{45}^{(3)} - X_{34}^{(4)} & D_{46}^{(3)} \\ -(D_{15}^{(3)} - X_{14}^{(4)})_\phi & -(D_{25}^{(3)} - X_{24}^{(4)})_\phi & -(D_{35}^{(3)})_\phi & -(X_{46}^{(3)})_\phi & -(D_{45}^{(3)} - X_{34}^{(4)})_\phi & D_{55}^{(3)} - X_{44}^{(4)} & D_{56}^{(3)} \\ -(D_{16}^{(3)})_\phi & -(D_{26}^{(3)})_\phi & -(D_{36}^{(3)})_\phi & -(X_{47}^{(3)})_\phi & -(D_{46}^{(3)})_\phi & -(D_{56}^{(3)})_\phi & D_{66}^{(3)} \\ -(X_{18}^{(3)})_\phi & -(X_{28}^{(3)})_\phi & -(X_{38}^{(3)})_\phi & -(X_{48}^{(3)})_\phi & -(X_{58}^{(3)})_\phi & -(X_{68}^{(3)})_\phi & -(X_{78}^{(3)})_\phi \\ -(D_{17}^{(3)} - D_{13}^{(4)})_\phi & -(D_{27}^{(3)} - X_{25}^{(4)})_\phi & -(D_{37}^{(3)})_\phi & -(X_{49}^{(3)})_\phi & -(D_{47}^{(3)} - D_{23}^{(4)})_\phi & -(D_{57}^{(3)} - X_{45}^{(4)})_\phi & -(D_{67}^{(3)})_\phi \\ -(D_{18}^{(3)} - X_{16}^{(4)})_\phi & -(D_{28}^{(3)} - X_{26}^{(4)})_\phi & -(D_{38}^{(3)})_\phi & -(X_{4,10}^{(3)})_\phi & -(D_{48}^{(3)} - X_{36}^{(4)})_\phi & -(D_{58}^{(3)} - X_{46}^{(4)})_\phi & -(D_{68}^{(3)})_\phi \\ -(D_{19}^{(3)})_\phi & -(D_{29}^{(3)})_\phi & -(D_{39}^{(3)})_\phi & -(X_{4,11}^{(3)})_\phi & -(D_{49}^{(3)})_\phi & -(D_{59}^{(3)})_\phi & -(D_{69}^{(3)})_\phi \\ -(X_{1,12}^{(3)})_\phi & -(X_{2,12}^{(3)})_\phi & -(X_{3,12}^{(3)})_\phi & -(X_{4,12}^{(3)})_\phi & -(X_{5,12}^{(3)})_\phi & -(X_{6,12}^{(3)})_\phi & -(X_{7,12}^{(3)})_\phi \\ -(D_{1,10}^{(3)} - D_{14}^{(4)})_\phi & -(D_{2,10}^{(3)} - X_{27}^{(4)})_\phi & -(D_{3,10}^{(3)})_\phi & -(D_{4,16}^{(2)})_\phi & -(D_{4,10}^{(3)} - D_{24}^{(4)})_\phi & -(D_{5,10}^{(3)} - X_{47}^{(4)})_\phi & -(D_{6,10}^{(3)})_\phi \\ -(D_{1,11}^{(3)} - X_{18}^{(4)})_\phi & -(D_{2,11}^{(3)} - X_{28}^{(4)})_\phi & -(D_{3,11}^{(3)})_\phi & -(D_{4,17}^{(2)})_\phi & -(D_{4,11}^{(3)} - X_{38}^{(4)})_\phi & -(D_{5,11}^{(3)} - X_{48}^{(4)})_\phi & -(D_{6,11}^{(3)})_\phi \\ -(D_{1,12}^{(3)})_\phi & -(D_{2,12}^{(3)})_\phi & -(D_{3,12}^{(3)})_\phi & -(D_{4,18}^{(2)})_\phi & -(D_{4,12}^{(3)})_\phi & -(D_{5,12}^{(3)})_\phi & -(D_{6,12}^{(3)})_\phi \\ -(D_{1,19}^{(2)})_\phi & -(D_{2,19}^{(2)})_\phi & -(D_{3,19}^{(2)})_\phi & -(D_{4,19}^{(2)})_\phi & -(D_{6,19}^{(2)})_\phi & -(D_{7,19}^{(2)})_\phi & -(D_{8,19}^{(2)})_\phi \\ -(D_{1,13}^{(3)} - D_{15}^{(4)})_\phi & -(D_{2,13}^{(3)} - X_{29}^{(4)})_\phi & -(D_{3,13}^{(3)})_\phi & -(X_{4,17}^{(3)})_\phi & -(D_{4,13}^{(3)} - D_{25}^{(4)})_\phi & -(D_{5,13}^{(3)} - X_{49}^{(4)})_\phi & -(D_{6,13}^{(3)})_\phi \\ -(D_{1,14}^{(3)} - X_{1,10}^{(4)})_\phi & -(D_{2,14}^{(3)} - X_{2,10}^{(4)})_\phi & -(D_{3,14}^{(3)})_\phi & -(X_{4,18}^{(3)})_\phi & -(D_{4,14}^{(3)} - X_{3,10}^{(4)})_\phi & -(D_{5,14}^{(3)} - X_{4,10}^{(4)})_\phi & -(D_{6,14}^{(3)})_\phi \\ -(D_{1,15}^{(3)})_\phi & -(D_{2,15}^{(3)})_\phi & -(D_{3,15}^{(3)})_\phi & -(X_{4,19}^{(3)})_\phi & -(D_{4,15}^{(3)})_\phi & -(D_{5,15}^{(3)})_\phi & -(D_{6,15}^{(3)})_\phi \\ -(X_{1,20}^{(3)})_\phi & -(X_{2,20}^{(3)})_\phi & -(X_{3,20}^{(3)})_\phi & -(X_{4,20}^{(3)})_\phi & -(X_{5,20}^{(3)})_\phi & -(X_{6,20}^{(3)})_\phi & -(X_{7,20}^{(3)})_\phi \end{matrix} \right.$$

$$
\begin{array}{ccccccc}
X^{(3)}_{18} & D^{(3)}_{17}-D^{(4)}_{13} & D^{(3)}_{18}-X^{(4)}_{16} & D^{(3)}_{19} & X^{(3)}_{1,12} & D^{(3)}_{1,10}-D^{(4)}_{14} & D^{(3)}_{1,11}-X^{(4)}_{18} \\
X^{(3)}_{28} & D^{(3)}_{27}-X^{(4)}_{25} & D^{(3)}_{28}-X^{(4)}_{26} & D^{(3)}_{29} & X^{(3)}_{2,12} & D^{(3)}_{2,10}-X^{(4)}_{27} & D^{(3)}_{2,11}-X^{(4)}_{28} \\
X^{(3)}_{38} & D^{(3)}_{37} & D^{(3)}_{38} & D^{(3)}_{39} & X^{(3)}_{3,12} & D^{(3)}_{3,10} & D^{(3)}_{3,11} \\
X^{(3)}_{48} & X^{(3)}_{49} & X^{(3)}_{4,10} & X^{(3)}_{4,11} & X^{(3)}_{4,12} & D^{(2)}_{4,16} & D^{(2)}_{4,17} \\
X^{(3)}_{58} & D^{(3)}_{47}-D^{(4)}_{23} & D^{(3)}_{48}-X^{(4)}_{36} & D^{(3)}_{49} & X^{(3)}_{5,12} & D^{(3)}_{4,10}-D^{(4)}_{24} & D^{(3)}_{4,11}-X^{(4)}_{38} \\
X^{(3)}_{68} & D^{(3)}_{57}-X^{(4)}_{45} & D^{(3)}_{58}-X^{(4)}_{46} & D^{(3)}_{59} & X^{(3)}_{6,12} & D^{(3)}_{5,10}-X^{(4)}_{47} & D^{(3)}_{5,11}-X^{(4)}_{48} \\
X^{(3)}_{78} & D^{(3)}_{67} & D^{(3)}_{68} & D^{(3)}_{69} & X^{(3)}_{7,12} & D^{(3)}_{6,10} & D^{(3)}_{6,11} \\
X^{(3)}_{88} & X^{(3)}_{89} & X^{(3)}_{8,10} & X^{(3)}_{8,11} & X^{(3)}_{8,12} & D^{(2)}_{9,16} & D^{(2)}_{9,17} \\
-(X^{(3)}_{89})_\phi & D^{(3)}_{77}-D^{(4)}_{33} & D^{(3)}_{78}-X^{(4)}_{56} & D^{(3)}_{79} & X^{(3)}_{9,12} & D^{(3)}_{7,10}-D^{(4)}_{34} & D^{(3)}_{7,11}-X^{(4)}_{58} \\
-(X^{(3)}_{8,10})_\phi & -(D^{(3)}_{78}-X^{(4)}_{56})_\phi & D^{(3)}_{88}-X^{(4)}_{66} & D^{(3)}_{89} & X^{(3)}_{10,12} & D^{(3)}_{8,10}-X^{(4)}_{67} & D^{(3)}_{8,11}-X^{(4)}_{68} \\
-(X^{(3)}_{8,11})_\phi & -(D^{(3)}_{79})_\phi & -(D^{(3)}_{89})_\phi & D^{(3)}_{99} & X^{(3)}_{11,12} & D^{(3)}_{9,10} & D^{(3)}_{9,11} \\
-(X^{(3)}_{8,12})_\phi & -(X^{(3)}_{9,12})_\phi & -(X^{(3)}_{10,12})_\phi & -(X^{(3)}_{11,12})_\phi & X^{(3)}_{12,12} & D^{(2)}_{14,16} & D^{(2)}_{14,17} \\
-(D^{(2)}_{9,16})_\phi & -(D^{(3)}_{7,10}-D^{(4)}_{34})_\phi & -(D^{(3)}_{8,10}-X^{(4)}_{67})_\phi & -(D^{(3)}_{9,10})_\phi & -(D^{(2)}_{14,16})_\phi & D^{(3)}_{10,10}-D^{(4)}_{44} & D^{(3)}_{10,11}-X^{(4)}_{78} \\
-(D^{(2)}_{9,17})_\phi & -(D^{(3)}_{7,11}-X^{(4)}_{58})_\phi & -(D^{(3)}_{8,11}-X^{(4)}_{68})_\phi & -(D^{(3)}_{9,11})_\phi & -(D^{(2)}_{14,17})_\phi & -(D^{(3)}_{10,11}-X^{(4)}_{78})_\phi & D^{(3)}_{11,11}-X^{(4)}_{88} \\
-(D^{(2)}_{9,18})_\phi & -(D^{(3)}_{7,12})_\phi & -(D^{(3)}_{8,12})_\phi & -(D^{(3)}_{9,12})_\phi & -(D^{(2)}_{14,18})_\phi & -(D^{(3)}_{10,12})_\phi & -(D^{(3)}_{11,12})_\phi \\
-(D^{(2)}_{9,19})_\phi & -(D^{(2)}_{11,19})_\phi & -(D^{(2)}_{12,19})_\phi & -(D^{(2)}_{13,19})_\phi & -(D^{(2)}_{14,19})_\phi & -(D^{(2)}_{16,19})_\phi & -(D^{(2)}_{17,19})_\phi \\
-(X^{(3)}_{8,17})_\phi & -(D^{(3)}_{7,13}-D^{(4)}_{35})_\phi & -(D^{(3)}_{8,13}-X^{(4)}_{69})_\phi & -(D^{(3)}_{9,13})_\phi & -(X^{(3)}_{12,17})_\phi & -(D^{(3)}_{10,13}-D^{(4)}_{45})_\phi & -(D^{(3)}_{11,13}-X^{(4)}_{89})_\phi \\
-(X^{(3)}_{8,18})_\phi & -(D^{(3)}_{7,14}-X^{(4)}_{5,10})_\phi & -(D^{(3)}_{8,14}-X^{(4)}_{6,10})_\phi & -(D^{(3)}_{9,14})_\phi & -(X^{(3)}_{12,18})_\phi & -(D^{(3)}_{10,14}-X^{(4)}_{7,10})_\phi & -(D^{(3)}_{11,14}-X^{(4)}_{8,10})_\phi \\
-(X^{(3)}_{8,19})_\phi & -(D^{(3)}_{7,15})_\phi & -(D^{(3)}_{8,15})_\phi & -(D^{(3)}_{9,15})_\phi & -(X^{(3)}_{12,19})_\phi & -(D^{(3)}_{10,15})_\phi & -(D^{(3)}_{11,15})_\phi \\
-(X^{(3)}_{8,20})_\phi & -(X^{(3)}_{9,20})_\phi & -(X^{(3)}_{10,20})_\phi & -(X^{(3)}_{11,20})_\phi & -(X^{(3)}_{12,20})_\phi & -(X^{(3)}_{13,20})_\phi & -(X^{(3)}_{14,20})_\phi
\end{array}
$$

$$
\left.\begin{array}{cccccc}
D^{(3)}_{1,12} & D^{(2)}_{1,19} & D^{(3)}_{1,13}-D^{(4)}_{15} & D^{(3)}_{1,14}-X^{(4)}_{1,10} & D^{(3)}_{1,15} & X^{(3)}_{1,20} \\
D^{(3)}_{2,12} & D^{(2)}_{2,19} & D^{(3)}_{2,13}-X^{(4)}_{29} & D^{(3)}_{2,14}-X^{(4)}_{2,10} & D^{(3)}_{2,15} & X^{(3)}_{2,20} \\
D^{(3)}_{3,12} & D^{(2)}_{3,19} & D^{(3)}_{3,13} & D^{(3)}_{3,14} & D^{(3)}_{3,15} & X^{(3)}_{3,20} \\
D^{(2)}_{4,18} & D^{(2)}_{4,19} & X^{(3)}_{4,17} & X^{(3)}_{4,18} & X^{(3)}_{4,19} & X^{(3)}_{4,20} \\
D^{(3)}_{4,12} & D^{(2)}_{6,19} & D^{(3)}_{4,13}-D^{(4)}_{25} & D^{(3)}_{4,14}-X^{(4)}_{3,10} & D^{(3)}_{4,15} & X^{(3)}_{5,20} \\
D^{(3)}_{5,12} & D^{(2)}_{7,19} & D^{(3)}_{5,13}-X^{(4)}_{49} & D^{(3)}_{5,14}-X^{(4)}_{4,10} & D^{(3)}_{5,15} & X^{(3)}_{6,20} \\
D^{(3)}_{6,12} & D^{(2)}_{8,19} & D^{(3)}_{6,13} & D^{(3)}_{6,14} & D^{(3)}_{6,15} & X^{(3)}_{7,20} \\
D^{(2)}_{9,18} & D^{(2)}_{9,19} & X^{(3)}_{8,17} & X^{(3)}_{8,18} & X^{(3)}_{8,19} & X^{(3)}_{8,20} \\
D^{(3)}_{7,12} & D^{(2)}_{11,19} & D^{(3)}_{7,13}-D^{(4)}_{35} & D^{(3)}_{7,14}-X^{(4)}_{5,10} & D^{(3)}_{7,15} & X^{(3)}_{9,20} \\
D^{(3)}_{8,12} & D^{(2)}_{12,19} & D^{(3)}_{8,13}-X^{(4)}_{69} & D^{(3)}_{8,14}-X^{(4)}_{6,10} & D^{(3)}_{8,15} & X^{(3)}_{10,20} \\
D^{(3)}_{9,12} & D^{(2)}_{13,19} & D^{(3)}_{9,13} & D^{(3)}_{9,14} & D^{(3)}_{9,15} & X^{(3)}_{11,20} \\
D^{(2)}_{14,18} & D^{(2)}_{14,19} & X^{(3)}_{12,17} & X^{(3)}_{12,18} & X^{(3)}_{12,19} & X^{(3)}_{12,20} \\
D^{(3)}_{10,12} & D^{(2)}_{16,19} & D^{(3)}_{10,13}-D^{(4)}_{45} & D^{(3)}_{10,14}-X^{(4)}_{7,10} & D^{(3)}_{10,15} & X^{(3)}_{13,20} \\
D^{(3)}_{11,12} & D^{(2)}_{17,19} & D^{(3)}_{11,13}-X^{(4)}_{89} & D^{(3)}_{11,14}-X^{(4)}_{8,10} & D^{(3)}_{11,15} & X^{(3)}_{14,20} \\
D^{(3)}_{12,12} & D^{(2)}_{18,19} & D^{(3)}_{12,13} & D^{(3)}_{12,14} & D^{(3)}_{12,15} & X^{(3)}_{15,20} \\
-(D^{(2)}_{18,19})_\phi & D^{(2)}_{19,19} & X^{(3)}_{16,17} & X^{(3)}_{16,18} & X^{(3)}_{16,19} & X^{(3)}_{16,20} \\
-(D^{(3)}_{12,13})_\phi & -(X^{(3)}_{16,17})_\phi & D^{(3)}_{13,13}-D^{(4)}_{55} & D^{(3)}_{13,14}-X^{(4)}_{9,10} & D^{(3)}_{13,15} & X^{(3)}_{17,20} \\
-(D^{(3)}_{12,14})_\phi & -(X^{(3)}_{16,18})_\phi & -(D^{(3)}_{13,14}-X^{(4)}_{9,10})_\phi & D^{(3)}_{14,14}-X^{(4)}_{10,10} & D^{(3)}_{14,15} & X^{(3)}_{18,20} \\
-(D^{(3)}_{12,15})_\phi & -(X^{(3)}_{16,19})_\phi & -(D^{(3)}_{13,15})_\phi & -(D^{(3)}_{14,15})_\phi & D^{(3)}_{15,15} & X^{(3)}_{19,20} \\
-(X^{(3)}_{15,20})_\phi & -(X^{(3)}_{16,20})_\phi & -(X^{(3)}_{17,20})_\phi & -(X^{(3)}_{18,20})_\phi & -(X^{(3)}_{19,20})_\phi & X^{(3)}_{20,20}
\end{array}\right), \qquad (8)
$$

$$\widehat{X}_4 = \begin{pmatrix}
D^{(4)}_{11} & X^{(4)}_{12} & D^{(4)}_{12} & X^{(4)}_{14} & D^{(4)}_{13} & X^{(4)}_{16} & D^{(4)}_{14} & X^{(4)}_{18} & D^{(4)}_{15} & X^{(4)}_{1,10} & D^{(4)}_{16} & D^{(3)}_{1,17} & D^{(4)}_{17} & X^{(4)}_{1,14} \\
-(X^{(4)}_{12})_\phi & X^{(4)}_{22} & X^{(4)}_{23} & X^{(4)}_{24} & X^{(4)}_{25} & X^{(4)}_{26} & X^{(4)}_{27} & X^{(4)}_{28} & X^{(4)}_{29} & X^{(4)}_{2,10} & D^{(3)}_{2,16} & D^{(3)}_{2,17} & X^{(4)}_{2,13} & X^{(4)}_{2,14} \\
-(D^{(4)}_{12})_\phi & -(X^{(4)}_{23})_\phi & D^{(4)}_{22} & X^{(4)}_{34} & D^{(4)}_{23} & X^{(4)}_{36} & D^{(4)}_{24} & X^{(4)}_{38} & D^{(4)}_{25} & X^{(4)}_{3,10} & D^{(4)}_{26} & D^{(3)}_{4,17} & D^{(4)}_{27} & X^{(4)}_{3,14} \\
-(X^{(4)}_{14})_\phi & -(X^{(4)}_{24})_\phi & -(X^{(4)}_{34})_\phi & X^{(4)}_{44} & X^{(4)}_{45} & X^{(4)}_{46} & X^{(4)}_{47} & X^{(4)}_{48} & X^{(4)}_{49} & X^{(4)}_{4,10} & D^{(3)}_{5,16} & D^{(3)}_{5,17} & X^{(4)}_{4,13} & X^{(4)}_{4,14} \\
-(D^{(4)}_{13})_\phi & -(X^{(4)}_{25})_\phi & -(D^{(4)}_{23})_\phi & -(X^{(4)}_{45})_\phi & D^{(4)}_{33} & X^{(4)}_{56} & D^{(4)}_{34} & X^{(4)}_{58} & D^{(4)}_{35} & X^{(4)}_{5,10} & D^{(4)}_{36} & D^{(3)}_{7,17} & D^{(4)}_{37} & X^{(4)}_{5,14} \\
-(X^{(4)}_{16})_\phi & -(X^{(4)}_{26})_\phi & -(X^{(4)}_{36})_\phi & -(X^{(4)}_{46})_\phi & -(X^{(4)}_{56})_\phi & X^{(4)}_{66} & X^{(4)}_{67} & X^{(4)}_{68} & X^{(4)}_{69} & X^{(4)}_{6,10} & D^{(3)}_{8,16} & D^{(3)}_{8,17} & X^{(4)}_{6,13} & X^{(4)}_{6,14} \\
-(D^{(4)}_{14})_\phi & -(X^{(4)}_{27})_\phi & -(D^{(4)}_{24})_\phi & -(X^{(4)}_{47})_\phi & -(D^{(4)}_{34})_\phi & -(X^{(4)}_{67})_\phi & D^{(4)}_{44} & X^{(4)}_{78} & D^{(4)}_{45} & X^{(4)}_{7,10} & D^{(4)}_{46} & D^{(3)}_{10,17} & D^{(4)}_{47} & X^{(4)}_{7,14} \\
-(X^{(4)}_{18})_\phi & -(X^{(4)}_{28})_\phi & -(X^{(4)}_{38})_\phi & -(X^{(4)}_{48})_\phi & -(X^{(4)}_{58})_\phi & -(X^{(4)}_{68})_\phi & -(X^{(4)}_{78})_\phi & X^{(4)}_{88} & X^{(4)}_{89} & X^{(4)}_{8,10} & D^{(3)}_{11,16} & D^{(3)}_{11,17} & X^{(4)}_{8,13} & X^{(4)}_{8,14} \\
-(D^{(4)}_{15})_\phi & -(X^{(4)}_{29})_\phi & -(D^{(4)}_{25})_\phi & -(X^{(4)}_{49})_\phi & -(D^{(4)}_{35})_\phi & -(X^{(4)}_{69})_\phi & -(D^{(4)}_{45})_\phi & -(X^{(4)}_{89})_\phi & D^{(4)}_{55} & X^{(4)}_{9,10} & D^{(4)}_{56} & D^{(3)}_{13,17} & D^{(4)}_{57} & X^{(4)}_{9,14} \\
-(X^{(4)}_{1,10})_\phi & -(X^{(4)}_{2,10})_\phi & -(X^{(4)}_{3,10})_\phi & -(X^{(4)}_{4,10})_\phi & -(X^{(4)}_{5,10})_\phi & -(X^{(4)}_{6,10})_\phi & -(X^{(4)}_{7,10})_\phi & -(X^{(4)}_{8,10})_\phi & -(X^{(4)}_{9,10})_\phi & X^{(4)}_{10,10} & D^{(3)}_{14,16} & D^{(3)}_{14,17} & X^{(4)}_{10,13} & X^{(4)}_{10,14} \\
-(D^{(4)}_{16})_\phi & -(D^{(3)}_{2,16})_\phi & -(D^{(4)}_{26})_\phi & -(D^{(3)}_{5,16})_\phi & -(D^{(4)}_{36})_\phi & -(D^{(3)}_{8,16})_\phi & -(D^{(4)}_{46})_\phi & -(D^{(3)}_{11,16})_\phi & -(D^{(4)}_{56})_\phi & -(D^{(3)}_{14,16})_\phi & D^{(4)}_{66} & D^{(3)}_{16,17} & D^{(4)}_{67} & X^{(4)}_{11,14} \\
-(D^{(3)}_{1,17})_\phi & -(D^{(3)}_{2,17})_\phi & -(D^{(3)}_{4,17})_\phi & -(D^{(3)}_{5,17})_\phi & -(D^{(3)}_{7,17})_\phi & -(D^{(3)}_{8,17})_\phi & -(D^{(3)}_{10,17})_\phi & -(D^{(3)}_{11,17})_\phi & -(D^{(3)}_{13,17})_\phi & -(D^{(3)}_{14,17})_\phi & -(D^{(3)}_{16,17})_\phi & D^{(3)}_{17,17} & X^{(4)}_{12,13} & X^{(4)}_{12,14} \\
-(D^{(4)}_{17})_\phi & -(X^{(4)}_{2,13})_\phi & -(D^{(4)}_{27})_\phi & -(X^{(4)}_{4,13})_\phi & -(D^{(4)}_{37})_\phi & -(X^{(4)}_{6,13})_\phi & -(D^{(4)}_{47})_\phi & -(X^{(4)}_{8,13})_\phi & -(D^{(4)}_{57})_\phi & -(X^{(4)}_{10,13})_\phi & -(D^{(4)}_{67})_\phi & -(X^{(4)}_{12,13})_\phi & D^{(4)}_{77} & X^{(4)}_{13,14} \\
-(X^{(4)}_{1,14})_\phi & -(X^{(4)}_{2,14})_\phi & -(X^{(4)}_{3,14})_\phi & -(X^{(4)}_{4,14})_\phi & -(X^{(4)}_{5,14})_\phi & -(X^{(4)}_{6,14})_\phi & -(X^{(4)}_{7,14})_\phi & -(X^{(4)}_{8,14})_\phi & -(X^{(4)}_{9,14})_\phi & -(X^{(4)}_{10,14})_\phi & -(X^{(4)}_{11,14})_\phi & -(X^{(4)}_{12,14})_\phi & -(X^{(4)}_{13,14})_\phi & X^{(4)}_{14,14}
\end{pmatrix}, \tag{9}$$

where $D^{(i)}_{kj} = \widehat{P}_i C_i (\widehat{P}_i)_\phi$, $(i = 1,2,3,4)$ *are defined in [20], and the remaining* $X^{(1)}_{l_1 m_1}$, $X^{(2)}_{l_2 m_2}$, $X^{(3)}_{l_3 m_3}$, $X^{(4)}_{l_4 m_4}$ *are arbitrary matrices over* $\mathbb{H}$ *with appropriate sizes.*

Proof. According to the idea of [20], we assume $\widehat{X}_1$, $\widehat{X}_2$, $\widehat{X}_3$, $\widehat{X}_4$ have the following form:

$$\widehat{X}_1 = -(\widehat{X}_1)_\phi = \begin{pmatrix}
X^{(1)}_{11} & X^{(1)}_{12} & \cdots & X^{(1)}_{18} \\
-(X^{(1)}_{12})_\phi & X^{(1)}_{22} & \cdots & X^{(1)}_{28} \\
\vdots & \vdots & \cdots & \vdots \\
-(X^{(1)}_{18})_\phi & -(X^{(1)}_{28})_\phi & \cdots & X^{(1)}_{88}
\end{pmatrix},$$

$$\widehat{X}_2 = -(\widehat{X}_2)_\phi = \begin{pmatrix} X_{11}^{(2)} & X_{12}^{(2)} & \cdots & X_{1,18}^{(2)} \\ -(X_{12}^{(2)})_\phi & X_{22}^{(2)} & \cdots & X_{2,18}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ -(X_{1,18}^{(2)})_\phi & -(X_{2,18}^{(2)})_\phi & \cdots & X_{18,18}^{(2)} \end{pmatrix},$$

$$\widehat{X}_3 = -(\widehat{X}_3)_\phi = \begin{pmatrix} X_{11}^{(3)} & X_{12}^{(3)} & \cdots & X_{1,20}^{(3)} \\ -(X_{12}^{(3)})_\phi & X_{22}^{(3)} & \cdots & X_{2,20}^{(3)} \\ \vdots & \vdots & \cdots & \vdots \\ -(X_{1,20}^{(3)})_\phi & -(X_{2,20}^{(3)})_\phi & \cdots & X_{20,20}^{(3)} \end{pmatrix},$$

$$\widehat{X}_4 = -(\widehat{X}_4)_\phi = \begin{pmatrix} X_{11}^{(4)} & X_{12}^{(4)} & \cdots & X_{1,14}^{(4)} \\ -(X_{12}^{(4)})_\phi & X_{22}^{(4)} & \cdots & X_{2,14}^{(4)} \\ \vdots & \vdots & \cdots & \vdots \\ -(X_{1,14}^{(4)})_\phi & -(X_{2,14}^{(4)})_\phi & \cdots & X_{14,14}^{(4)} \end{pmatrix}.$$

Putting $\widehat{X}_1$, $\widehat{X}_2$, $\widehat{X}_3$, $\widehat{X}_4$ into the Equation (4) yields

$$(D_{ij}^{(1)})_{14\times 14} = (D_1^{(1)}, D_2^{(1)}), \tag{10}$$

$$(D_{ij}^{(2)})_{20\times 20} = (D_1^{(2)}, D_2^{(2)}, D_3^{(2)}), \tag{11}$$

$$(D_{ij}^{(3)})_{18\times 18} = (D_1^{(3)}, D_2^{(3)}, D_3^{(3)}), \tag{12}$$

$$(D_{ij}^{(4)})_{8\times 8} = \begin{pmatrix} X_{11}^{(4)} & X_{13}^{(4)} & X_{15}^{(4)} & X_{17}^{(4)} & X_{19}^{(4)} & X_{1,11}^{(4)} & X_{1,13}^{(4)} & 0 \\ -(X_{13}^{(4)})_\phi & X_{33}^{(4)} & X_{35}^{(4)} & X_{37}^{(4)} & X_{39}^{(4)} & X_{3,11}^{(4)} & X_{3,13}^{(4)} & 0 \\ -(X_{15}^{(4)})_\phi & -(X_{35}^{(4)})_\phi & X_{55}^{(4)} & X_{57}^{(4)} & X_{59}^{(4)} & X_{5,11}^{(4)} & X_{5,13}^{(4)} & 0 \\ -(X_{17}^{(4)})_\phi & -(X_{37}^{(4)})_\phi & -(X_{57}^{(4)})_\phi & X_{77}^{(4)} & X_{79}^{(4)} & X_{7,11}^{(4)} & X_{7,13}^{(4)} & 0 \\ -(X_{19}^{(4)})_\phi & -(X_{39}^{(4)})_\phi & -(X_{59}^{(4)})_\phi & -(X_{79}^{(4)})_\phi & X_{99}^{(4)} & X_{9,11}^{(4)} & X_{9,13}^{(4)} & 0 \\ -(X_{1,11}^{(4)})_\phi & -(X_{3,11}^{(4)})_\phi & -(X_{5,11}^{(4)})_\phi & -(X_{7,11}^{(4)})_\phi & -(X_{9,11}^{(4)})_\phi & X_{11,11}^{(4)} & X_{11,13}^{(4)} & 0 \\ -(X_{1,13}^{(4)})_\phi & -(X_{3,13}^{(4)})_\phi & -(X_{5,13}^{(4)})_\phi & -(X_{7,13}^{(4)})_\phi & -(X_{9,13}^{(4)})_\phi & -(X_{11,13}^{(4)})_\phi & X_{13,13}^{(4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{13}$$

where

$$D_1^{(1)} = \begin{pmatrix} X_{11}^{(1)} + X_{11}^{(2)} & X_{12}^{(1)} + X_{12}^{(2)} & X_{13}^{(1)} + X_{13}^{(2)} & X_{14}^{(1)} + X_{14}^{(2)} & X_{15}^{(1)} + X_{15}^{(2)} \\ -(X_{12}^{(1)} + X_{12}^{(2)})_\phi & X_{22}^{(1)} + X_{22}^{(2)} & X_{23}^{(1)} + X_{23}^{(2)} & X_{24}^{(1)} + X_{24}^{(2)} & X_{25}^{(1)} + X_{25}^{(2)} \\ -(X_{13}^{(1)} + X_{13}^{(2)})_\phi & -(X_{23}^{(1)} + X_{23}^{(2)})_\phi & X_{33}^{(1)} + X_{33}^{(2)} & X_{34}^{(1)} + X_{34}^{(2)} & X_{35}^{(1)} + X_{35}^{(2)} \\ -(X_{14}^{(1)} + X_{14}^{(2)})_\phi & -(X_{24}^{(1)} + X_{24}^{(2)})_\phi & -(X_{34}^{(1)} + X_{34}^{(2)})_\phi & X_{44}^{(1)} + X_{44}^{(2)} & X_{45}^{(1)} + X_{45}^{(2)} \\ -(X_{15}^{(1)} + X_{15}^{(2)})_\phi & -(X_{25}^{(1)} + X_{25}^{(2)})_\phi & -(X_{35}^{(1)} + X_{35}^{(2)})_\phi & -(X_{45}^{(1)} + X_{45}^{(2)})_\phi & X_{55}^{(1)} + X_{55}^{(2)} \\ -(X_{16}^{(1)} + X_{16}^{(2)})_\phi & -(X_{26}^{(1)} + X_{26}^{(2)})_\phi & -(X_{36}^{(1)} + X_{36}^{(2)})_\phi & -(X_{46}^{(1)} + X_{46}^{(2)})_\phi & -(X_{56}^{(1)} + X_{56}^{(2)})_\phi \\ -(X_{17}^{(1)})_\phi & -(X_{27}^{(1)})_\phi & -(X_{37}^{(1)})_\phi & -(X_{47}^{(1)})_\phi & -(X_{57}^{(1)})_\phi \\ -(X_{17}^{(2)})_\phi & -(X_{27}^{(2)})_\phi & -(X_{37}^{(2)})_\phi & -(X_{47}^{(2)})_\phi & -(X_{57}^{(2)})_\phi \\ -(X_{18}^{(2)})_\phi & -(X_{28}^{(2)})_\phi & -(X_{38}^{(2)})_\phi & -(X_{48}^{(2)})_\phi & -(X_{58}^{(2)})_\phi \\ -(X_{19}^{(2)})_\phi & -(X_{29}^{(2)})_\phi & -(X_{39}^{(2)})_\phi & -(X_{49}^{(2)})_\phi & -(X_{59}^{(2)})_\phi \\ -(X_{1,10}^{(2)})_\phi & -(X_{2,10}^{(2)})_\phi & -(X_{3,10}^{(2)})_\phi & -(X_{4,10}^{(2)})_\phi & -(X_{5,10}^{(2)})_\phi \\ -(X_{1,11}^{(2)})_\phi & -(X_{2,11}^{(2)})_\phi & -(X_{3,11}^{(2)})_\phi & -(X_{4,11}^{(2)})_\phi & -(X_{5,11}^{(2)})_\phi \\ -(X_{1,12}^{(2)})_\phi & -(X_{2,12}^{(2)})_\phi & -(X_{3,12}^{(2)})_\phi & -(X_{4,12}^{(2)})_\phi & -(X_{5,12}^{(2)})_\phi \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_2^{(1)} = \begin{pmatrix}
X_{16}^{(1)} + X_{16}^{(2)} & X_{17}^{(1)} & X_{17}^{(2)} & X_{18}^{(2)} & X_{19}^{(2)} & X_{1,10}^{(2)} & X_{1,11}^{(2)} & X_{1,12}^{(2)} & 0 \\
X_{26}^{(1)} + X_{26}^{(2)} & X_{27}^{(1)} & X_{27}^{(2)} & X_{28}^{(2)} & X_{29}^{(2)} & X_{2,10}^{(2)} & X_{2,11}^{(2)} & X_{2,12}^{(2)} & 0 \\
X_{36}^{(1)} + X_{36}^{(2)} & X_{37}^{(1)} & X_{37}^{(2)} & X_{38}^{(2)} & X_{39}^{(2)} & X_{3,10}^{(2)} & X_{3,11}^{(2)} & X_{3,12}^{(2)} & 0 \\
X_{46}^{(1)} + X_{46}^{(2)} & X_{47}^{(1)} & X_{47}^{(2)} & X_{48}^{(2)} & X_{49}^{(2)} & X_{4,10}^{(2)} & X_{4,11}^{(2)} & X_{4,12}^{(2)} & 0 \\
X_{56}^{(1)} + X_{56}^{(2)} & X_{57}^{(1)} & X_{57}^{(2)} & X_{58}^{(2)} & X_{59}^{(2)} & X_{5,10}^{(2)} & X_{5,11}^{(2)} & X_{5,12}^{(2)} & 0 \\
X_{66}^{(1)} + X_{66}^{(2)} & X_{67}^{(1)} & X_{67}^{(2)} & X_{68}^{(2)} & X_{69}^{(2)} & X_{6,10}^{(2)} & X_{6,11}^{(2)} & X_{6,12}^{(2)} & 0 \\
-(X_{67}^{(1)})_\phi & X_{77}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-(X_{67}^{(2)})_\phi & 0 & X_{77}^{(2)} & X_{78}^{(2)} & X_{79}^{(2)} & X_{7,10}^{(2)} & X_{7,11}^{(2)} & X_{7,12}^{(2)} & 0 \\
-(X_{68}^{(2)})_\phi & 0 & -(X_{78}^{(2)})_\phi & X_{88}^{(2)} & X_{89}^{(2)} & X_{8,10}^{(2)} & X_{8,11}^{(2)} & X_{8,12}^{(2)} & 0 \\
-(X_{69}^{(2)})_\phi & 0 & -(X_{79}^{(2)})_\phi & -(X_{89}^{(2)})_\phi & X_{9,9}^{(2)} & X_{9,10}^{(2)} & X_{9,11}^{(2)} & X_{9,12}^{(2)} & 0 \\
-(X_{6,10}^{(2)})_\phi & 0 & -(X_{7,10}^{(2)})_\phi & -(X_{8,10}^{(2)})_\phi & -(X_{9,10}^{(2)})_\phi & X_{10,10}^{(2)} & X_{10,11}^{(2)} & X_{10,12}^{(2)} & 0 \\
-(X_{6,11}^{(2)})_\phi & 0 & -(X_{7,11}^{(2)})_\phi & -(X_{8,11}^{(2)})_\phi & -(X_{9,11}^{(2)})_\phi & -(X_{10,11}^{(2)})_\phi & X_{11,11}^{(2)} & X_{11,12}^{(2)} & 0 \\
-(X_{6,12}^{(2)})_\phi & 0 & -(X_{7,12}^{(2)})_\phi & -(X_{8,12}^{(2)})_\phi & -(X_{9,12}^{(2)})_\phi & -(X_{10,12}^{(2)})_\phi & -(X_{11,12}^{(2)})_\phi & X_{12,12}^{(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$D_1^{(2)} = \begin{pmatrix}
X_{11}^{(2)} + X_{11}^{(3)} & X_{12}^{(2)} + X_{12}^{(3)} & X_{13}^{(2)} + X_{13}^{(3)} & X_{14}^{(2)} + X_{14}^{(3)} & X_{15}^{(2)} & X_{17}^{(2)} + X_{15}^{(3)} \\
-(X_{12}^{(2)} + X_{12}^{(3)})_\phi & X_{22}^{(2)} + X_{22}^{(3)} & X_{23}^{(2)} + X_{23}^{(3)} & X_{24}^{(2)} + X_{24}^{(3)} & X_{25}^{(2)} & X_{27}^{(2)} + X_{25}^{(3)} \\
-(X_{13}^{(2)} + X_{13}^{(3)})_\phi & -(X_{23}^{(2)} + X_{23}^{(3)})_\phi & X_{33}^{(2)} + X_{33}^{(3)} & X_{34}^{(2)} + X_{34}^{(3)} & X_{35}^{(2)} & X_{37}^{(2)} + X_{35}^{(3)} \\
-(X_{14}^{(2)} + X_{14}^{(3)})_\phi & -(X_{24}^{(2)} + X_{24}^{(3)})_\phi & -(X_{34}^{(2)} + X_{34}^{(3)})_\phi & X_{44}^{(2)} + X_{44}^{(3)} & X_{45}^{(2)} & X_{47}^{(2)} + X_{45}^{(3)} \\
-(X_{15}^{(2)})_\phi & -(X_{25}^{(2)})_\phi & -(X_{35}^{(2)})_\phi & -(X_{45}^{(2)})_\phi & X_{55}^{(2)} & X_{57}^{(2)} \\
-(X_{17}^{(2)} + X_{15}^{(3)})_\phi & -(X_{27}^{(2)} + X_{25}^{(3)})_\phi & -(X_{37}^{(2)} + X_{35}^{(3)})_\phi & -(X_{47}^{(2)} + X_{45}^{(3)})_\phi & -(X_{57}^{(2)})_\phi & X_{77}^{(2)} + X_{55}^{(3)} \\
-(X_{18}^{(2)} + X_{16}^{(3)})_\phi & -(X_{28}^{(2)} + X_{26}^{(3)})_\phi & -(X_{38}^{(2)} + X_{36}^{(3)})_\phi & -(X_{48}^{(2)} + X_{46}^{(3)})_\phi & -(X_{58}^{(2)})_\phi & -(X_{78}^{(2)} + X_{56}^{(3)})_\phi \\
-(X_{19}^{(2)} + X_{17}^{(3)})_\phi & -(X_{29}^{(2)} + X_{27}^{(3)})_\phi & -(X_{39}^{(2)} + X_{37}^{(3)})_\phi & -(X_{49}^{(2)} + X_{47}^{(3)})_\phi & -(X_{59}^{(2)})_\phi & -(X_{79}^{(2)} + X_{57}^{(3)})_\phi \\
-(X_{1,10}^{(2)} + X_{18}^{(3)})_\phi & -(X_{2,10}^{(2)} + X_{28}^{(3)})_\phi & -(X_{3,10}^{(2)} + X_{38}^{(3)})_\phi & -(X_{4,10}^{(2)} + X_{48}^{(3)})_\phi & -(X_{5,10}^{(2)})_\phi & -(X_{7,10}^{(2)} + X_{58}^{(3)})_\phi \\
-(X_{1,11}^{(2)})_\phi & -(X_{2,11}^{(2)})_\phi & -(X_{3,11}^{(2)})_\phi & -(X_{4,11}^{(2)})_\phi & -(X_{5,11}^{(2)})_\phi & -(X_{7,11}^{(2)})_\phi \\
-(X_{1,13}^{(2)} + X_{19}^{(3)})_\phi & -(X_{2,13}^{(2)} + X_{29}^{(3)})_\phi & -(X_{3,13}^{(2)} + X_{39}^{(3)})_\phi & -(X_{4,13}^{(2)} + X_{49}^{(3)})_\phi & -(X_{5,13}^{(2)})_\phi & -(X_{7,13}^{(2)} + X_{59}^{(3)})_\phi \\
-(X_{1,14}^{(2)} + X_{1,10}^{(3)})_\phi & -(X_{2,14}^{(2)} + X_{2,10}^{(3)})_\phi & -(X_{3,14}^{(2)} + X_{3,10}^{(3)})_\phi & -(X_{4,14}^{(2)} + X_{4,10}^{(3)})_\phi & -(X_{5,14}^{(2)})_\phi & -(X_{7,14}^{(2)} + X_{5,10}^{(3)})_\phi \\
-(X_{1,15}^{(2)} + X_{1,11}^{(3)})_\phi & -(X_{2,15}^{(2)} + X_{2,11}^{(3)})_\phi & -(X_{3,15}^{(2)} + X_{3,11}^{(3)})_\phi & -(X_{4,15}^{(2)} + X_{4,11}^{(3)})_\phi & -(X_{5,15}^{(2)})_\phi & -(X_{7,15}^{(2)} + X_{5,11}^{(3)})_\phi \\
-(X_{1,16}^{(2)} + X_{1,12}^{(3)})_\phi & -(X_{2,16}^{(2)} + X_{2,12}^{(3)})_\phi & -(X_{3,16}^{(2)} + X_{3,12}^{(3)})_\phi & -(X_{4,16}^{(2)} + X_{4,12}^{(3)})_\phi & -(X_{5,16}^{(2)})_\phi & -(X_{7,16}^{(2)} + X_{5,12}^{(3)})_\phi \\
-(X_{1,17}^{(2)})_\phi & -(X_{2,17}^{(2)})_\phi & -(X_{3,17}^{(2)})_\phi & -(X_{4,17}^{(2)})_\phi & -(X_{5,17}^{(2)})_\phi & -(X_{7,17}^{(2)})_\phi \\
-(X_{1,13}^{(3)})_\phi & -(X_{2,13}^{(3)})_\phi & -(X_{3,13}^{(3)})_\phi & -(X_{4,13}^{(3)})_\phi & 0 & -(X_{5,13}^{(3)})_\phi \\
-(X_{1,14}^{(3)})_\phi & -(X_{2,14}^{(3)})_\phi & -(X_{3,14}^{(3)})_\phi & -(X_{4,14}^{(3)})_\phi & 0 & -(X_{5,14}^{(3)})_\phi \\
-(X_{1,15}^{(3)})_\phi & -(X_{2,15}^{(3)})_\phi & -(X_{3,15}^{(3)})_\phi & -(X_{4,15}^{(3)})_\phi & 0 & -(X_{5,15}^{(3)})_\phi \\
-(X_{1,16}^{(3)})_\phi & -(X_{2,16}^{(3)})_\phi & -(X_{3,16}^{(3)})_\phi & -(X_{4,16}^{(3)})_\phi & 0 & -(X_{5,16}^{(3)})_\phi \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$D_2^{(2)} = \left(\begin{array}{cccccc}
X_{18}^{(2)} + X_{16}^{(3)} & X_{19}^{(2)} + X_{17}^{(3)} & X_{1,10}^{(2)} + X_{18}^{(3)} & X_{1,11}^{(2)} & X_{1,13}^{(2)} + X_{19}^{(3)} & X_{1,14}^{(2)} + X_{1,10}^{(3)} \\
X_{28}^{(2)} + X_{26}^{(3)} & X_{29}^{(2)} + X_{27}^{(3)} & X_{2,10}^{(2)} + X_{28}^{(3)} & X_{2,11}^{(2)} & X_{2,13}^{(2)} + X_{29}^{(3)} & X_{2,14}^{(2)} + X_{2,10}^{(3)} \\
X_{38}^{(2)} + X_{36}^{(3)} & X_{39}^{(2)} + X_{37}^{(3)} & X_{3,10}^{(2)} + X_{38}^{(3)} & X_{3,11}^{(2)} & X_{3,13}^{(2)} + X_{39}^{(3)} & X_{3,14}^{(2)} + X_{3,10}^{(3)} \\
X_{48}^{(2)} + X_{46}^{(3)} & X_{49}^{(2)} + X_{47}^{(3)} & X_{4,10}^{(2)} + X_{48}^{(3)} & X_{4,11}^{(2)} & X_{4,13}^{(2)} + X_{49}^{(3)} & X_{4,14}^{(2)} + X_{4,10}^{(3)} \\
X_{58}^{(2)} & X_{59}^{(2)} & X_{5,10}^{(2)} & X_{5,11}^{(2)} & X_{5,13}^{(2)} & X_{5,14}^{(2)} \\
X_{78}^{(2)} + X_{56}^{(3)} & X_{79}^{(2)} + X_{57}^{(3)} & X_{7,10}^{(2)} + X_{58}^{(3)} & X_{7,11}^{(2)} & X_{7,13}^{(2)} + X_{59}^{(3)} & X_{7,14}^{(2)} + X_{5,10}^{(3)} \\
X_{88}^{(2)} + X_{66}^{(3)} & X_{89}^{(2)} + X_{67}^{(3)} & X_{8,10}^{(2)} + X_{68}^{(3)} & X_{8,11}^{(2)} & X_{8,13}^{(2)} + X_{69}^{(3)} & X_{8,14}^{(2)} + X_{6,10}^{(3)} \\
-(X_{89}^{(2)} + X_{67}^{(3)})_\phi & X_{99}^{(2)} + X_{77}^{(3)} & X_{9,10}^{(2)} + X_{78}^{(3)} & X_{9,11}^{(2)} & X_{9,13}^{(2)} + X_{79}^{(3)} & X_{9,14}^{(2)} + X_{7,10}^{(3)} \\
-(X_{8,10}^{(2)} + X_{68}^{(3)})_\phi & -(X_{9,10}^{(2)} + X_{78}^{(3)})_\phi & X_{10,10}^{(2)} + X_{88}^{(3)} & X_{10,11}^{(2)} & X_{10,13}^{(2)} + X_{89}^{(3)} & X_{10,14}^{(2)} + X_{8,10}^{(3)} \\
-(X_{8,11}^{(2)})_\phi & -(X_{9,11}^{(2)})_\phi & -(X_{10,11}^{(2)})_\phi & X_{11,11}^{(2)} & X_{11,13}^{(2)} & X_{11,14}^{(2)} \\
-(X_{8,13}^{(2)} + X_{69}^{(3)})_\phi & -(X_{9,13}^{(2)} + X_{79}^{(3)})_\phi & -(X_{10,13}^{(2)} + X_{89}^{(3)})_\phi & -(X_{11,13}^{(2)})_\phi & X_{13,13}^{(2)} + X_{99}^{(3)} & X_{13,14}^{(2)} + X_{9,10}^{(3)} \\
-(X_{8,14}^{(2)} + X_{6,10}^{(3)})_\phi & -(X_{9,14}^{(2)} + X_{7,10}^{(3)})_\phi & -(X_{10,14}^{(2)} + X_{8,10}^{(3)})_\phi & -(X_{11,14}^{(2)})_\phi & -(X_{13,14}^{(2)} + X_{9,10}^{(3)})_\phi & X_{14,14}^{(2)} + X_{10,10}^{(3)} \\
-(X_{8,15}^{(2)} + X_{6,11}^{(3)})_\phi & -(X_{9,15}^{(2)} + X_{7,11}^{(3)})_\phi & -(X_{10,15}^{(2)} + X_{8,11}^{(3)})_\phi & -(X_{11,15}^{(2)})_\phi & -(X_{13,15}^{(2)} + X_{9,11}^{(3)})_\phi & -(X_{14,15}^{(2)} + X_{10,11}^{(3)})_\phi \\
-(X_{8,16}^{(2)} + X_{6,12}^{(3)})_\phi & -(X_{9,16}^{(2)} + X_{7,12}^{(3)})_\phi & -(X_{10,16}^{(2)} + X_{8,12}^{(3)})_\phi & -(X_{11,16}^{(2)})_\phi & -(X_{13,16}^{(2)} + X_{9,12}^{(3)})_\phi & -(X_{14,16}^{(2)} + X_{10,12}^{(3)})_\phi \\
-(X_{8,17}^{(2)})_\phi & -(X_{9,17}^{(2)})_\phi & -(X_{10,17}^{(2)})_\phi & -(X_{11,17}^{(2)})_\phi & -(X_{13,17}^{(2)})_\phi & -(X_{14,17}^{(2)})_\phi \\
-(X_{6,13}^{(3)})_\phi & -(X_{7,13}^{(3)})_\phi & -(X_{8,13}^{(3)})_\phi & 0 & -(X_{9,13}^{(3)})_\phi & -(X_{10,13}^{(3)})_\phi \\
-(X_{6,14}^{(3)})_\phi & -(X_{7,14}^{(3)})_\phi & -(X_{8,14}^{(3)})_\phi & 0 & -(X_{9,14}^{(3)})_\phi & -(X_{10,14}^{(3)})_\phi \\
-(X_{6,15}^{(3)})_\phi & -(X_{7,15}^{(3)})_\phi & -(X_{8,15}^{(3)})_\phi & 0 & -(X_{9,15}^{(3)})_\phi & -(X_{10,15}^{(3)})_\phi \\
-(X_{6,16}^{(3)})_\phi & -(X_{7,16}^{(3)})_\phi & -(X_{8,16}^{(3)})_\phi & 0 & -(X_{9,16}^{(3)})_\phi & -(X_{10,16}^{(3)})_\phi \\
0 & 0 & 0 & 0 & 0 & 0
\end{array} \right),$$

$$D_3^{(2)} = \left(\begin{array}{cccccccc}
X_{1,15}^{(2)} + X_{1,11}^{(3)} & X_{1,16}^{(2)} + X_{1,12}^{(3)} & X_{1,17}^{(2)} & X_{1,13}^{(3)} & X_{1,14}^{(3)} & X_{1,15}^{(3)} & X_{1,16}^{(3)} & 0 \\
X_{2,15}^{(2)} + X_{2,11}^{(3)} & X_{2,16}^{(2)} + X_{2,12}^{(3)} & X_{2,17}^{(2)} & X_{2,13}^{(3)} & X_{2,14}^{(3)} & X_{2,15}^{(3)} & X_{2,16}^{(3)} & 0 \\
X_{3,15}^{(2)} + X_{3,11}^{(3)} & X_{3,16}^{(2)} + X_{3,12}^{(3)} & X_{3,17}^{(2)} & X_{3,13}^{(3)} & X_{3,14}^{(3)} & X_{3,15}^{(3)} & X_{3,16}^{(3)} & 0 \\
X_{4,15}^{(2)} + X_{4,11}^{(3)} & X_{4,16}^{(2)} + X_{4,12}^{(3)} & X_{4,17}^{(2)} & X_{4,13}^{(3)} & X_{4,14}^{(3)} & X_{4,15}^{(3)} & X_{4,16}^{(3)} & 0 \\
X_{5,15}^{(2)} & X_{5,16}^{(2)} & X_{5,17}^{(2)} & 0 & 0 & 0 & 0 & 0 \\
X_{7,15}^{(2)} + X_{5,11}^{(3)} & X_{7,16}^{(2)} + X_{5,12}^{(3)} & X_{7,17}^{(2)} & X_{5,13}^{(3)} & X_{5,14}^{(3)} & X_{5,15}^{(3)} & X_{5,16}^{(3)} & 0 \\
X_{8,15}^{(2)} + X_{6,11}^{(3)} & X_{8,16}^{(2)} + X_{6,12}^{(3)} & X_{8,17}^{(2)} & X_{6,13}^{(3)} & X_{6,14}^{(3)} & X_{6,15}^{(3)} & X_{6,16}^{(3)} & 0 \\
X_{9,15}^{(2)} + X_{7,11}^{(3)} & X_{9,16}^{(2)} + X_{7,12}^{(3)} & X_{9,17}^{(2)} & X_{7,13}^{(3)} & X_{7,14}^{(3)} & X_{7,15}^{(3)} & X_{7,16}^{(3)} & 0 \\
X_{10,15}^{(2)} + X_{8,11}^{(3)} & X_{10,16}^{(2)} + X_{8,12}^{(3)} & X_{10,17}^{(2)} & X_{8,13}^{(3)} & X_{8,14}^{(3)} & X_{8,15}^{(3)} & X_{8,16}^{(3)} & 0 \\
X_{11,15}^{(2)} & X_{11,16}^{(2)} & X_{11,17}^{(2)} & 0 & 0 & 0 & 0 & 0 \\
X_{13,15}^{(2)} + X_{9,11}^{(3)} & X_{13,16}^{(2)} + X_{9,12}^{(3)} & X_{13,17}^{(2)} & X_{9,13}^{(3)} & X_{9,14}^{(3)} & X_{9,15}^{(3)} & X_{9,16}^{(3)} & 0 \\
X_{14,15}^{(2)} + X_{10,11}^{(3)} & X_{14,16}^{(2)} + X_{10,12}^{(3)} & X_{14,17}^{(2)} & X_{10,13}^{(3)} & X_{10,14}^{(3)} & X_{10,15}^{(3)} & X_{10,16}^{(3)} & 0 \\
X_{15,15}^{(2)} + X_{11,11}^{(3)} & X_{15,16}^{(2)} + X_{11,12}^{(3)} & X_{15,17}^{(2)} & X_{11,13}^{(3)} & X_{11,14}^{(3)} & X_{11,15}^{(3)} & X_{11,16}^{(3)} & 0 \\
-(X_{15,16}^{(2)} + X_{11,12}^{(3)})_\phi & X_{16,16}^{(2)} + X_{12,12}^{(3)} & X_{16,17}^{(2)} & X_{12,13}^{(3)} & X_{12,14}^{(3)} & X_{12,15}^{(3)} & X_{12,16}^{(3)} & 0 \\
-(X_{15,17}^{(2)})_\phi & -(X_{16,17}^{(2)})_\phi & X_{17,17}^{(2)} & 0 & 0 & 0 & 0 & 0 \\
-(X_{11,13}^{(3)})_\phi & -(X_{12,13}^{(3)})_\phi & 0 & X_{13,13}^{(3)} & X_{13,14}^{(3)} & X_{13,15}^{(3)} & X_{13,16}^{(3)} & 0 \\
-(X_{11,14}^{(3)})_\phi & -(X_{12,14}^{(3)})_\phi & 0 & -(X_{13,14}^{(3)})_\phi & X_{14,14}^{(3)} & X_{14,15}^{(3)} & X_{14,16}^{(3)} & 0 \\
-(X_{11,15}^{(3)})_\phi & -(X_{12,15}^{(3)})_\phi & 0 & -(X_{13,15}^{(3)})_\phi & -(X_{14,15}^{(3)})_\phi & X_{15,15}^{(3)} & X_{15,16}^{(3)} & 0 \\
-(X_{11,16}^{(3)})_\phi & -(X_{12,16}^{(3)})_\phi & 0 & -(X_{13,16}^{(3)})_\phi & -(X_{14,16}^{(3)})_\phi & -(X_{15,16}^{(3)})_\phi & X_{16,16}^{(3)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right),$$

$$D_1^{(3)} = \begin{pmatrix}
X_{11}^{(3)} + X_{11}^{(4)} & X_{12}^{(3)} + X_{12}^{(4)} & X_{13}^{(3)} & X_{15}^{(3)} + X_{13}^{(4)} & X_{16}^{(3)} + X_{14}^{(4)} & X_{17}^{(3)} \\
-(X_{12}^{(3)} + X_{12}^{(4)})_\phi & X_{22}^{(3)} + X_{22}^{(4)} & X_{23}^{(3)} & X_{25}^{(3)} + X_{23}^{(4)} & X_{26}^{(3)} + X_{24}^{(4)} & X_{27}^{(3)} \\
-(X_{13}^{(3)})_\phi & -(X_{23}^{(3)})_\phi & X_{33}^{(3)} & X_{35}^{(3)} & X_{36}^{(3)} & X_{37}^{(3)} \\
-(X_{15}^{(3)} + X_{13}^{(4)})_\phi & -(X_{25}^{(3)} + X_{23}^{(4)})_\phi & -(X_{35}^{(3)})_\phi & X_{55}^{(3)} + X_{33}^{(4)} & X_{56}^{(3)} + X_{34}^{(4)} & X_{57}^{(3)} \\
-(X_{16}^{(3)} + X_{14}^{(4)})_\phi & -(X_{26}^{(3)} + X_{24}^{(4)})_\phi & -(X_{36}^{(3)})_\phi & -(X_{56}^{(3)} + X_{34}^{(4)})_\phi & X_{66}^{(3)} + X_{44}^{(4)} & X_{67}^{(3)} \\
-(X_{17}^{(3)})_\phi & -(X_{27}^{(3)})_\phi & -(X_{37}^{(3)})_\phi & -(X_{57}^{(3)})_\phi & -(X_{67}^{(3)})_\phi & X_{77}^{(3)} \\
-(X_{19}^{(3)} + X_{15}^{(4)})_\phi & -(X_{29}^{(3)} + X_{25}^{(4)})_\phi & -(X_{39}^{(3)})_\phi & -(X_{59}^{(3)} + X_{35}^{(4)})_\phi & -(X_{69}^{(3)} + X_{45}^{(4)})_\phi & -(X_{79}^{(3)})_\phi \\
-(X_{1,10}^{(3)} + X_{16}^{(4)})_\phi & -(X_{2,10}^{(3)} + X_{26}^{(4)})_\phi & -(X_{3,10}^{(3)})_\phi & -(X_{5,10}^{(3)} + X_{36}^{(4)})_\phi & -(X_{6,10}^{(3)} + X_{46}^{(4)})_\phi & -(X_{7,10}^{(3)})_\phi \\
-(X_{1,11}^{(3)})_\phi & -(X_{2,11}^{(3)})_\phi & -(X_{3,11}^{(3)})_\phi & -(X_{5,11}^{(3)})_\phi & -(X_{6,11}^{(3)})_\phi & -(X_{7,11}^{(3)})_\phi \\
-(X_{1,13}^{(3)} + X_{17}^{(4)})_\phi & -(X_{2,13}^{(3)} + X_{27}^{(4)})_\phi & -(X_{3,13}^{(3)})_\phi & -(X_{5,13}^{(3)} + X_{37}^{(4)})_\phi & -(X_{6,13}^{(3)} + X_{47}^{(4)})_\phi & -(X_{7,13}^{(3)})_\phi \\
-(X_{1,14}^{(3)} + X_{18}^{(4)})_\phi & -(X_{2,14}^{(3)} + X_{28}^{(4)})_\phi & -(X_{3,14}^{(3)})_\phi & -(X_{5,14}^{(3)} + X_{38}^{(4)})_\phi & -(X_{6,14}^{(3)} + X_{48}^{(4)})_\phi & -(X_{7,14}^{(3)})_\phi \\
-(X_{1,15}^{(3)})_\phi & -(X_{2,15}^{(3)})_\phi & -(X_{3,15}^{(3)})_\phi & -(X_{5,15}^{(3)})_\phi & -(X_{6,15}^{(3)})_\phi & -(X_{7,15}^{(3)})_\phi \\
-(X_{1,17}^{(3)} + X_{19}^{(4)})_\phi & -(X_{2,17}^{(3)} + X_{29}^{(4)})_\phi & -(X_{3,17}^{(3)})_\phi & -(X_{5,17}^{(3)} + X_{39}^{(4)})_\phi & -(X_{6,17}^{(3)} + X_{49}^{(4)})_\phi & -(X_{7,17}^{(3)})_\phi \\
-(X_{1,18}^{(3)} + X_{1,10}^{(4)})_\phi & -(X_{2,18}^{(3)} + X_{2,10}^{(4)})_\phi & -(X_{3,18}^{(3)})_\phi & -(X_{5,18}^{(3)} + X_{3,10}^{(4)})_\phi & -(X_{6,18}^{(3)} + X_{4,10}^{(4)})_\phi & -(X_{7,18}^{(3)})_\phi \\
-(X_{1,19}^{(3)})_\phi & -(X_{2,19}^{(3)})_\phi & -(X_{3,19}^{(3)})_\phi & -(X_{5,19}^{(3)})_\phi & -(X_{6,19}^{(3)})_\phi & -(X_{7,19}^{(3)})_\phi \\
-(X_{1,11}^{(4)})_\phi & -(X_{2,11}^{(4)})_\phi & 0 & -(X_{3,11}^{(4)})_\phi & -(X_{4,11}^{(4)})_\phi & 0 \\
-(X_{1,12}^{(4)})_\phi & -(X_{2,12}^{(4)})_\phi & 0 & -(X_{3,12}^{(4)})_\phi & -(X_{4,12}^{(4)})_\phi & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$D_2^{(3)} = \begin{pmatrix}
X_{19}^{(3)} + X_{15}^{(4)} & X_{1,10}^{(3)} + X_{16}^{(4)} & X_{1,11}^{(3)} & X_{1,13}^{(3)} + X_{17}^{(4)} & X_{1,14}^{(3)} + X_{18}^{(4)} & X_{1,15}^{(3)} \\
X_{29}^{(3)} + X_{25}^{(4)} & X_{2,10}^{(3)} + X_{26}^{(4)} & X_{2,11}^{(3)} & X_{2,13}^{(3)} + X_{27}^{(4)} & X_{2,14}^{(3)} + X_{28}^{(4)} & X_{2,15}^{(3)} \\
X_{39}^{(3)} & X_{3,10}^{(3)} & X_{3,11}^{(3)} & X_{3,13}^{(3)} & X_{3,14}^{(3)} & X_{3,15}^{(3)} \\
X_{59}^{(3)} + X_{35}^{(4)} & X_{5,10}^{(3)} + X_{36}^{(4)} & X_{5,11}^{(3)} & X_{5,13}^{(3)} + X_{37}^{(4)} & X_{5,14}^{(3)} + X_{38}^{(4)} & X_{5,15}^{(3)} \\
X_{69}^{(3)} + X_{45}^{(4)} & X_{6,10}^{(3)} + X_{46}^{(4)} & X_{6,11}^{(3)} & X_{6,13}^{(3)} + X_{47}^{(4)} & X_{6,14}^{(3)} + X_{48}^{(4)} & X_{6,15}^{(3)} \\
X_{79}^{(3)} & X_{7,10}^{(3)} & X_{7,11}^{(3)} & X_{7,13}^{(3)} & X_{7,14}^{(3)} & X_{7,15}^{(3)} \\
X_{99}^{(3)} + X_{55}^{(4)} & X_{9,10}^{(3)} + X_{56}^{(4)} & X_{9,11}^{(3)} & X_{9,13}^{(3)} + X_{57}^{(4)} & X_{9,14}^{(3)} + X_{58}^{(4)} & X_{9,15}^{(3)} \\
-(X_{9,10}^{(3)} + X_{56}^{(4)})_\phi & X_{10,10}^{(3)} + X_{66}^{(4)} & X_{10,11}^{(3)} & X_{10,13}^{(3)} + X_{67}^{(4)} & X_{10,14}^{(3)} + X_{68}^{(4)} & X_{10,15}^{(3)} \\
-(X_{9,11}^{(3)})_\phi & -(X_{10,11}^{(3)})_\phi & X_{11,11}^{(3)} & X_{11,13}^{(3)} & X_{11,14}^{(3)} & X_{11,15}^{(3)} \\
-(X_{9,13}^{(3)} + X_{57}^{(4)})_\phi & -(X_{10,13}^{(3)} + X_{67}^{(4)})_\phi & -(X_{11,13}^{(3)})_\phi & X_{13,13}^{(3)} + X_{77}^{(4)} & X_{13,14}^{(3)} + X_{78}^{(4)} & X_{13,15}^{(3)} \\
-(X_{9,14}^{(3)} + X_{58}^{(4)})_\phi & -(X_{10,14}^{(3)} + X_{68}^{(4)})_\phi & -(X_{11,14}^{(3)})_\phi & -(X_{13,14}^{(3)} + X_{78}^{(4)})_\phi & X_{14,14}^{(3)} + X_{88}^{(4)} & X_{14,15}^{(3)} \\
-(X_{9,15}^{(3)})_\phi & -(X_{10,15}^{(3)})_\phi & -(X_{11,15}^{(3)})_\phi & -(X_{13,15}^{(3)})_\phi & -(X_{14,15}^{(3)})_\phi & X_{15,15}^{(3)} \\
-(X_{9,17}^{(3)} + X_{59}^{(4)})_\phi & -(X_{10,17}^{(3)} + X_{69}^{(4)})_\phi & -(X_{11,17}^{(3)})_\phi & -(X_{13,17}^{(3)} + X_{79}^{(4)})_\phi & -(X_{14,17}^{(3)} + X_{89}^{(4)})_\phi & -(X_{15,17}^{(3)})_\phi \\
-(X_{9,18}^{(3)} + X_{5,10}^{(4)})_\phi & -(X_{10,18}^{(3)} + X_{6,10}^{(4)})_\phi & -(X_{11,18}^{(3)})_\phi & -(X_{13,18}^{(3)} + X_{7,10}^{(4)})_\phi & -(X_{14,18}^{(3)} + X_{8,10}^{(4)})_\phi & -(X_{15,18}^{(3)})_\phi \\
-(X_{9,19}^{(3)})_\phi & -(X_{10,19}^{(3)})_\phi & -(X_{11,19}^{(3)})_\phi & -(X_{13,19}^{(3)})_\phi & -(X_{14,19}^{(3)})_\phi & -(X_{15,19}^{(3)})_\phi \\
-(X_{5,11}^{(4)})_\phi & -(X_{6,11}^{(4)})_\phi & 0 & -(X_{7,11}^{(4)})_\phi & -(X_{8,11}^{(4)})_\phi & 0 \\
-(X_{5,12}^{(4)})_\phi & -(X_{6,12}^{(4)})_\phi & 0 & -(X_{7,12}^{(4)})_\phi & -(X_{8,12}^{(4)})_\phi & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$D_3^{(3)} = \begin{pmatrix}
X^{(3)}_{1,17} + X^{(4)}_{19} & X^{(3)}_{1,18} + X^{(4)}_{1,10} & X^{(3)}_{1,19} & X^{(4)}_{1,11} & X^{(4)}_{1,12} & 0 \\
X^{(3)}_{2,17} + X^{(4)}_{29} & X^{(3)}_{2,18} + X^{(4)}_{2,10} & X^{(3)}_{2,19} & X^{(4)}_{2,11} & X^{(4)}_{2,12} & 0 \\
X^{(3)}_{3,17} & X^{(3)}_{3,18} & X^{(3)}_{3,19} & 0 & 0 & 0 \\
X^{(3)}_{5,17} + X^{(4)}_{39} & X^{(3)}_{5,18} + X^{(4)}_{3,10} & X^{(3)}_{5,19} & X^{(4)}_{3,11} & X^{(4)}_{3,12} & 0 \\
X^{(3)}_{6,17} + X^{(4)}_{49} & X^{(3)}_{6,18} + X^{(4)}_{4,10} & X^{(3)}_{6,19} & X^{(4)}_{4,11} & X^{(4)}_{4,12} & 0 \\
X^{(3)}_{7,17} & X^{(3)}_{7,18} & X^{(3)}_{7,19} & 0 & 0 & 0 \\
X^{(3)}_{9,17} + X^{(4)}_{59} & X^{(3)}_{9,18} + X^{(4)}_{5,10} & X^{(3)}_{9,19} & X^{(4)}_{5,11} & X^{(4)}_{5,12} & 0 \\
X^{(3)}_{10,17} + X^{(4)}_{69} & X^{(3)}_{10,18} + X^{(4)}_{6,10} & X^{(3)}_{10,19} & X^{(4)}_{6,11} & X^{(4)}_{6,12} & 0 \\
X^{(3)}_{11,17} & X^{(3)}_{11,18} & X^{(3)}_{11,19} & 0 & 0 & 0 \\
X^{(3)}_{13,17} + X^{(4)}_{79} & X^{(3)}_{13,18} + X^{(4)}_{7,10} & X^{(3)}_{13,19} & X^{(4)}_{7,11} & X^{(4)}_{7,12} & 0 \\
X^{(3)}_{14,17} + X^{(4)}_{89} & X^{(3)}_{14,18} + X^{(4)}_{8,10} & X^{(3)}_{14,19} & X^{(4)}_{8,11} & X^{(4)}_{8,12} & 0 \\
X^{(3)}_{15,17} & X^{(3)}_{15,18} & X^{(3)}_{15,19} & 0 & 0 & 0 \\
X^{(3)}_{17,17} + X^{(4)}_{99} & X^{(3)}_{17,18} + X^{(4)}_{9,10} & X^{(3)}_{17,19} & X^{(4)}_{9,11} & X^{(4)}_{9,12} & 0 \\
-(X^{(3)}_{17,18} + X^{(4)}_{9,10})_\phi & X^{(3)}_{18,18} + X^{(4)}_{10,10} & X^{(3)}_{18,19} & X^{(4)}_{10,11} & X^{(4)}_{10,12} & 0 \\
-(X^{(3)}_{17,19})_\phi & -(X^{(3)}_{18,19})_\phi & X^{(3)}_{19,19} & 0 & 0 & 0 \\
-(X^{(4)}_{9,11})_\phi & -(X^{(4)}_{10,11})_\phi & 0 & X^{(4)}_{11,11} & X^{(4)}_{11,12} & 0 \\
-(X^{(4)}_{9,12})_\phi & -(X^{(4)}_{10,12})_\phi & 0 & -(X^{(4)}_{11,12})_\phi & X^{(4)}_{12,12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Observe that the matrices (10)–(13) are all defined in [20].

It follows from (10)–(13) that the general ϕ-skew-Hermitian solution to the system (4) is provided in the form (7)–(9). □

Example 1. *Given a system (1). We consider the general ϕ-skew-Hermitian solution to this system, where $\phi(\boldsymbol{a}) = \boldsymbol{a}^{\mathbf{j}*} = -\mathbf{j}\boldsymbol{a}^*\mathbf{j}$ for $\boldsymbol{a} \in \mathbb{H}$. The quaternion matrices A_i, B_i, (i = 1, 2, 3), A_4 are given:*

$$A_1 = \begin{pmatrix} \mathbf{i} & 0 & 1+\mathbf{k} \\ 0 & \mathbf{j}-\mathbf{k} & \mathbf{j}+\mathbf{k} \\ 1 & 0 & \mathbf{k} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mathbf{j} & 2\mathbf{i} & 0 \\ 3\mathbf{k} & 0 & \mathbf{i}+\mathbf{j} \\ 5+\mathbf{j} & 6 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \mathbf{i} & 0 & 0 \\ 0 & \mathbf{i}+\mathbf{j} & 2\mathbf{i} \\ 2\mathbf{i}+\mathbf{k} & \mathbf{k} & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & \mathbf{i}+\mathbf{j} & 0 \\ \mathbf{k} & 0 & \mathbf{i}+\mathbf{j}-\mathbf{k} \\ 2\mathbf{j} & 0 & \mathbf{i}+3\mathbf{k} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1+\mathbf{i} & 3 & 0 \\ 0 & \mathbf{i}+\mathbf{k} & \mathbf{j}-\mathbf{k} \\ \mathbf{j} & 0 & 2+\mathbf{k} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \mathbf{k} & 2\mathbf{j} \\ 3\mathbf{i} & \mathbf{i}-\mathbf{j} & 0 \\ 5\mathbf{k} & 0 & \mathbf{j}+\mathbf{k} \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & \mathbf{k} & \mathbf{i}+\mathbf{j} \\ 0 & 2\mathbf{i} & 0 \\ 3\mathbf{j} & 0 & 5\mathbf{k} \end{pmatrix}.$$

The ϕ-skew-Hermitian matrices C_i, (i = 1,2,3,4) are given:

$$C_1 = \begin{pmatrix} 4\mathbf{j} & 6+11\mathbf{i}+3\mathbf{j} & -9+8\mathbf{i}+16\mathbf{j}+23\mathbf{k} \\ -6-11\mathbf{i}+3\mathbf{j} & 18\mathbf{j} & -24-10\mathbf{i}-12\mathbf{j}+2\mathbf{k} \\ 9-8\mathbf{i}+16\mathbf{j}-23\mathbf{k} & 24+10\mathbf{i}-12\mathbf{j}-2\mathbf{k} & 15\mathbf{j} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -15\mathbf{j} & 6+2\mathbf{i}+20\mathbf{j}+3k & -10-10\mathbf{i}+3\mathbf{j}-14\mathbf{k} \\ -6-2\mathbf{i}+20\mathbf{j}-3\mathbf{k} & 45\mathbf{j} & -36-50\mathbf{i}+62\mathbf{j}+39\mathbf{k} \\ 10+10\mathbf{i}+3\mathbf{j}+14\mathbf{k} & 36+50\mathbf{i}+62\mathbf{j}-39\mathbf{k} & 43\mathbf{j} \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 4\mathbf{j} & -2-3\mathbf{i}+\mathbf{j} & -12-4\mathbf{j}-4\mathbf{k} \\ 2+3\mathbf{i}+\mathbf{j} & -14\mathbf{j} & 6+18\mathbf{i}-4\mathbf{j} \\ 12-4\mathbf{j}+4\mathbf{k} & -6-18\mathbf{i}-4\mathbf{j} & 39\mathbf{j} \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -2\mathbf{j} & -2+2\mathbf{i}+2\mathbf{j} & 4+4\mathbf{i}+4\mathbf{k} \\ 2-2\mathbf{i}+2\mathbf{j} & 9\mathbf{j} & -6-3\mathbf{i}-2\mathbf{j}-5\mathbf{k} \\ -4-4\mathbf{i}-4\mathbf{k} & 6+3\mathbf{i}-2\mathbf{j}+5\mathbf{k} & 12\mathbf{j} \end{pmatrix}.$$

According to Theorem 1, the following ϕ-skew-Hermitian matrices satisfy the system

$$X_1 = -(X_1)_\phi = \begin{pmatrix} \mathbf{j} & 2+7\mathbf{k} & 3\mathbf{i}+\mathbf{j}-\mathbf{k} \\ -2-7\mathbf{k} & 5\mathbf{j} & \mathbf{i}+\mathbf{k} \\ -3\mathbf{i}+\mathbf{j}+\mathbf{k} & -\mathbf{i}-\mathbf{k} & 3\mathbf{j} \end{pmatrix},$$

$$X_2 = -(X_2)_\phi = \begin{pmatrix} 2\mathbf{j} & 1+2\mathbf{i} & 0 \\ -1-2\mathbf{i} & -\mathbf{j} & 2\mathbf{j}+3\mathbf{k} \\ 0 & 2\mathbf{j}-3\mathbf{k} & 3\mathbf{j} \end{pmatrix},$$

$$X_3 = -(X_3)_\phi = \begin{pmatrix} \mathbf{j} & \mathbf{j} & 2\mathbf{i} \\ \mathbf{j} & 3\mathbf{j} & \mathbf{j}+3\mathbf{k} \\ -2\mathbf{i} & \mathbf{j}-3\mathbf{k} & -5\mathbf{j} \end{pmatrix}, \quad X_4 = -(X_4)_\phi = \begin{pmatrix} \mathbf{j} & \mathbf{i}+\mathbf{j}+2\mathbf{k} & \mathbf{k} \\ -\mathbf{i}+\mathbf{j}-2\mathbf{k} & -\mathbf{j} & 0 \\ -\mathbf{k} & 0 & 2\mathbf{j} \end{pmatrix}.$$

4. The $\beta(\phi)$-Signature Bounds of the Solution X_1 to the System (1)

In this section, we investigate the property of the solution X_1 to the system (1). Firstly, we consider the $\beta(\phi)$-signature bounds of the ϕ-skew-Hermitian solution X_1 to the system (1). The following Lemmas provide the $\beta(\phi)$-signature bounds and minimum rank of block matrices.

Lemma 2 ([19])**.** *Let M be a ϕ-skew-Hermitian block matrix*

$$M = \begin{pmatrix} X & A \\ -A_\phi & B \end{pmatrix}, \tag{14}$$

where $A \in \mathbb{H}^{n\times m}$ and $B = -B_\phi \in \mathbb{H}^{m\times m}$ are given, $X \in \mathbb{H}^{n\times n}$ is a variable ϕ-skew-Hermitian matrix. Then,

$$\max_{X=-X_\phi} \ln_\pm(M) = n + \ln_\pm(B), \quad \min_{X=-X_\phi} \ln_\pm(M) = r\begin{pmatrix} A \\ B \end{pmatrix} - \ln_\mp(B).$$

Lemma 3 ([26–29])**.** *Let N be a block matrix*

$$N = \begin{pmatrix} A & B \\ D & Y \end{pmatrix}, \tag{15}$$

where A, B and D are given quaternion matrices, $Y \in \mathbb{H}^{n\times m}$ is a variable matrix. Then,

$$\min_Y r(N) = r\begin{pmatrix} A & B \end{pmatrix} + r\begin{pmatrix} A \\ D \end{pmatrix} - r(A).$$

The following Theorem derives the $\beta(\phi)$-signature bounds of the solution X_1 to the system (1).

Theorem 2. *Assume that the system (1) has a ϕ-skew-Hermitian solution $(X_1, X_2, X_3, X_4) \in \mathbb{H}^{t_1\times t_1} \times \mathbb{H}^{t_2\times t_2} \times \mathbb{H}^{t_3\times t_3} \times \mathbb{H}^{t_4\times t_4}$. We denote*

$$S = \{X_1 = -(X_1)_\phi \in \mathbb{H}^{t_1\times t_1} \mid A_i X_i (A_i)_\phi + B_i X_{i+1} (B_i)_\phi = C_i,\ (i = 1,2,3),\ A_4 X_4 (A_4)_\phi = C_4\}.$$

Then, we can obtain

$$\max \ln_{\pm}(X_1) = t_1 - r(A_1) - r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} - r\begin{pmatrix} A_4 & 0 & 0 \\ B_3 & A_3 & 0 \\ 0 & B_2 & A_2 \\ 0 & 0 & B_1 \end{pmatrix}$$

$$+r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix} + \ln_{\pm}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}, \quad (16)$$

$$\min \ln_{\pm}(X_1) = r\begin{pmatrix} (B_1)_\phi \\ C_1 \end{pmatrix} - r\begin{pmatrix} A_2 & 0 \\ 0 & (B_1)_\phi \\ B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} (B_2)_\phi & 0 & 0 \\ -C_2 & A_2 & 0 \\ (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & B_1 & C_1 \end{pmatrix}$$

$$-r\begin{pmatrix} A_3 & 0 & 0 & 0 \\ 0 & (B_2)_\phi & 0 & 0 \\ B_2 & -C_2 & A_2 & 0 \\ 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} (B_3)_\phi & 0 & 0 & 0 & 0 \\ C_3 & A_3 & 0 & 0 & 0 \\ (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$-r\begin{pmatrix} A_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix} + \ln_{\pm}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}. \quad (17)$$

Proof. According to Theorem 1, the ϕ-skew-Hermitian solution X_1 can be written as

$$X_1 = \widehat{T}_1 \widehat{X}_1 (\widehat{T}_1)_\phi,$$

we have

$$\ln_{\pm}_{X_1 \in S}(X_1) = \ln_{\pm}(\widehat{T}_1 \widehat{X}_1 (\widehat{T}_1)_\phi) = \ln_{\pm}(\widehat{X}_1).$$

Thus, in order to study the $\beta(\phi)$-signature bounds of X_1 under S, we just have to consider the $\beta(\phi)$-signature bounds of $\widehat{X}_1$. Assume $\widehat{X}_1 = (\widehat{X}_1^{(1)},\ \widehat{X}_1^{(2)})$, where

$$\widehat{X}_1^{(1)} = \begin{array}{c} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \\ t_1 - r_{a_1} \end{array} \overset{\begin{array}{ccc} n_1 & n_2 & n_3 \end{array}}{\begin{pmatrix} D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} & D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} \\ -(D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)})_\phi & D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)} & D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} \\ -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi & -(D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)})_\phi & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} \\ -(D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)})_\phi & -(D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)})_\phi & -(D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)})_\phi \\ -(D_{15}^{(1)} - D_{15}^{(2)})_\phi & -(D_{25}^{(1)} - D_{25}^{(2)})_\phi & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi \\ -(D_{16}^{(1)} - X_{16}^{(2)})_\phi & -(D_{26}^{(1)} - X_{26}^{(2)})_\phi & -(D_{36}^{(1)} - X_{36}^{(2)})_\phi \\ -(D_{17}^{(1)})_\phi & -(D_{27}^{(1)})_\phi & -(D_{37}^{(1)})_\phi \\ -(X_{18}^{(1)})_\phi & -(X_{28}^{(1)})_\phi & -(X_{38}^{(1)})_\phi \end{pmatrix}},$$

$$\widehat{X}_1^{(2)} = \begin{array}{c} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \\ t_1 - r_{a_1} \end{array} \overset{\begin{array}{ccccc} n_4 & n_5 & n_6 & n_7 & t_1 - r_{a_1} \end{array}}{\begin{pmatrix} D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{16}^{(1)} - X_{16}^{(2)} & D_{17}^{(1)} & X_{18}^{(1)} \\ D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)} & D_{25}^{(1)} - D_{25}^{(2)} & D_{26}^{(1)} - X_{26}^{(2)} & D_{27}^{(1)} & X_{28}^{(1)} \\ D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{36}^{(1)} - X_{36}^{(2)} & D_{37}^{(1)} & X_{38}^{(1)} \\ D_{44}^{(1)} - D_{44}^{(2)} + X_{44}^{(3)} & D_{45}^{(1)} - D_{45}^{(2)} & D_{46}^{(1)} - X_{46}^{(2)} & D_{47}^{(1)} & X_{48}^{(1)} \\ -(D_{45}^{(1)} - D_{45}^{(2)})_\phi & D_{55}^{(1)} - D_{55}^{(2)} & D_{56}^{(1)} - X_{56}^{(2)} & D_{57}^{(1)} & X_{58}^{(1)} \\ -(D_{46}^{(1)} - X_{46}^{(2)})_\phi & -(D_{56}^{(1)} - X_{56}^{(2)})_\phi & D_{66}^{(1)} - X_{66}^{(2)} & D_{67}^{(1)} & X_{68}^{(1)} \\ -(D_{47}^{(1)})_\phi & -(D_{57}^{(1)})_\phi & -(D_{67}^{(1)})_\phi & D_{77}^{(1)} & X_{78}^{(1)} \\ -(X_{48}^{(1)})_\phi & -(X_{58}^{(1)})_\phi & -(X_{68}^{(1)})_\phi & -(X_{78}^{(1)})_\phi & X_{88}^{(1)} \end{pmatrix}}.$$

We treat matrix $\widehat{X}_1$ as a block matrix, using the Lemma 2 and Lemma 3 to get the $\beta(\phi)$-signature bounds of the ϕ-skew-Hermitian matrix $\widehat{X}_1$, which is equivalent to the $\beta(\phi)$-signature bounds of the ϕ-skew-Hermitian solution X_1. The specific steps are as follows.

Step 1. We treat the variable ϕ-skew-Hermitian matrix $X_{88}^{(1)}$ of $\widehat{X}_1$ as the matrix block X in (14). According to Lemma 2, we derive

$$\max_{X_{88}^{(1)}} \ln_\pm(\widehat{X}_1) = t_1 - r_{a_1} + \ln_\pm(\Psi_1), \ \min_{X_{88}^{(1)}} \ln_\pm(\widehat{X}_1) = r(\Psi) - \ln_\mp(\Psi_1),$$

assume $\Psi = (\Psi^{(1)}, \Psi^{(2)}), \Psi_1 = (\Psi_1^{(1)}, \Psi_1^{(2)})$, where

$$
\Psi^{(1)} = \begin{array}{c} \\ n_7 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ t_1 - r_{a_1} \end{array}
\begin{array}{c} \begin{array}{ccc} n_7 & n_1 & n_2 \end{array} \\
\begin{pmatrix}
D_{77}^{(1)} & -(D_{17}^{(1)})_\phi & -(D_{27}^{(1)})_\phi \\
D_{17}^{(1)} & D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} & D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)} \\
D_{27}^{(1)} & -(D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)})_\phi & D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)} \\
D_{37}^{(1)} & -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi & -(D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)})_\phi \\
D_{47}^{(1)} & -(D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)})_\phi & -(D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)})_\phi \\
D_{57}^{(1)} & -(D_{15}^{(1)} - D_{15}^{(2)})_\phi & -(D_{25}^{(1)} - D_{25}^{(2)})_\phi \\
D_{67}^{(1)} & -(D_{16}^{(1)} - X_{16}^{(2)})_\phi & -(D_{26}^{(1)} - X_{26}^{(2)})_\phi \\
(X_{78}^{(1)})_\phi & (X_{18}^{(1)})_\phi & (X_{28}^{(1)})_\phi
\end{pmatrix} \end{array},
$$

$$
\Psi^{(2)} = \begin{array}{c} \\ n_7 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ t_1 - r_{a_1} \end{array}
\begin{array}{c} \begin{array}{cccc} n_3 & n_4 & n_5 & n_6 \end{array} \\
\begin{pmatrix}
-(D_{37}^{(1)})_\phi & -(D_{47}^{(1)})_\phi & -(D_{57}^{(1)})_\phi & -(D_{67}^{(1)})_\phi \\
D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{16}^{(1)} - X_{16}^{(2)} \\
D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} & D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)} & D_{25}^{(1)} - D_{25}^{(2)} & D_{26}^{(1)} - X_{26}^{(2)} \\
D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{36}^{(1)} - X_{36}^{(2)} \\
-(D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)})_\phi & D_{44}^{(1)} - D_{44}^{(2)} + X_{44}^{(3)} & D_{45}^{(1)} - D_{45}^{(2)} & D_{46}^{(1)} - X_{46}^{(2)} \\
-(D_{35}^{(1)} - D_{35}^{(2)})_\phi & -(D_{45}^{(1)} - D_{45}^{(2)})_\phi & D_{55}^{(1)} - D_{55}^{(2)} & D_{56}^{(1)} - X_{56}^{(2)} \\
-(D_{36}^{(1)} - X_{36}^{(2)})_\phi & -(D_{46}^{(1)} - X_{46}^{(2)})_\phi & -(D_{56}^{(1)} - X_{56}^{(2)})_\phi & D_{66}^{(1)} - X_{66}^{(2)} \\
(X_{38}^{(1)})_\phi & (X_{48}^{(1)})_\phi & (X_{58}^{(1)})_\phi & (X_{68}^{(1)})_\phi
\end{pmatrix} \end{array},
$$

$$
\Psi_1^{(1)} = \begin{array}{c} \\ n_7 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{array}
\begin{array}{c} \begin{array}{ccc} n_7 & n_1 & n_2 \end{array} \\
\begin{pmatrix}
D_{77}^{(1)} & -(D_{17}^{(1)})_\phi & -(D_{27}^{(1)})_\phi \\
D_{17}^{(1)} & D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} & D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)} \\
D_{27}^{(1)} & -(D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)})_\phi & D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)} \\
D_{37}^{(1)} & -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi & -(D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)})_\phi \\
D_{47}^{(1)} & -(D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)})_\phi & -(D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)})_\phi \\
D_{57}^{(1)} & -(D_{15}^{(1)} - D_{15}^{(2)})_\phi & -(D_{25}^{(1)} - D_{25}^{(2)})_\phi \\
D_{67}^{(1)} & -(D_{16}^{(1)} - X_{16}^{(2)})_\phi & -(D_{26}^{(1)} - X_{26}^{(2)})_\phi
\end{pmatrix} \end{array},
$$

$$\Psi_1^{(2)} = \begin{array}{c} \\ n_7 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cccc} n_3 & n_4 & n_5 & n_6 \end{array} \\ \begin{pmatrix} -(D_{37}^{(1)})_\phi & -(D_{47}^{(1)})_\phi & -(D_{57}^{(1)})_\phi & -(D_{67}^{(1)})_\phi \\ D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{16}^{(1)} - X_{16}^{(2)} \\ D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} & D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)} & D_{25}^{(1)} - D_{25}^{(2)} & D_{26}^{(1)} - X_{26}^{(2)} \\ D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{36}^{(1)} - X_{36}^{(2)} \\ -(D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)})_\phi & D_{44}^{(1)} - D_{44}^{(2)} + X_{44}^{(3)} & D_{45}^{(1)} - D_{45}^{(2)} & D_{46}^{(1)} - X_{46}^{(2)} \\ -(D_{35}^{(1)} - D_{35}^{(2)})_\phi & -(D_{45}^{(1)} - D_{45}^{(2)})_\phi & D_{55}^{(1)} - D_{55}^{(2)} & D_{56}^{(1)} - X_{56}^{(2)} \\ -(D_{36}^{(1)} - X_{36}^{(2)})_\phi & -(D_{46}^{(1)} - X_{46}^{(2)})_\phi & -(D_{56}^{(1)} - X_{56}^{(2)})_\phi & D_{66}^{(1)} - X_{66}^{(2)} \end{pmatrix} \end{array}.$$

Then, we treat the matrix Ψ as a block matrix. According to Lemma 3, we have

$$\min_{(X_{78}^{(1)}, X_{18}^{(1)}, X_{28}^{(1)}, X_{38}^{(1)}, X_{48}^{(1)}, X_{58}^{(1)}, X_{68}^{(1)})} r(\Psi) = r(\Psi_1).$$

Thus, we can obtain

$$\max_{X_{88}^{(1)}, \begin{pmatrix} X_{78}^{(1)} \\ X_{18}^{(1)} \\ X_{28}^{(1)} \\ X_{38}^{(1)} \\ X_{48}^{(1)} \\ X_{58}^{(1)} \\ X_{68}^{(1)} \end{pmatrix}} \ln_{\pm}(\widehat{X}_1) = t_1 - r_{a_1} + \ln_{\pm}(\Psi_1), \qquad \min_{X_{88}^{(1)}, \begin{pmatrix} X_{78}^{(1)} \\ X_{18}^{(1)} \\ X_{28}^{(1)} \\ X_{38}^{(1)} \\ X_{48}^{(1)} \\ X_{58}^{(1)} \\ X_{68}^{(1)} \end{pmatrix}} \ln_{\pm}(\widehat{X}_1) = \ln_{\pm}(\Psi_1).$$

Step 2. We treating the variable ϕ-skew-Hermitian matrix $D_{66}^{(1)} - X_{66}^{(2)}$ of Ψ_1 as the matrix block X in (14). According to Lemma 2, we provide

$$\max_{D_{66}^{(1)} - X_{66}^{(2)}} \ln_{\pm}(\Psi_1) = n_6 + \ln_{\pm}(\Psi_3), \quad \min_{D_{66}^{(1)} - X_{66}^{(2)}} \ln_{\pm}(\Psi_1) = r(\Psi_2) - \ln_{\mp}(\Psi_3),$$

assume $\Psi_2 = (\Psi_2^{(1)}, \ \Psi_2^{(2)})$, $\Psi_3 = (\Psi_3^{(1)}, \ \Psi_3^{(2)})$, where

$$\Psi_2^{(1)} = \begin{array}{c} \\ n_7 \\ n_1 \\ n_2 \\ n_3 \\ n_5 \\ n_4 \\ n_6 \end{array} \begin{array}{c} \begin{array}{ccc} n_7 & n_1 & n_2 \end{array} \\ \begin{pmatrix} D_{77}^{(1)} & -(D_{17}^{(1)})_\phi & -(D_{27}^{(1)})_\phi \\ D_{17}^{(1)} & D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} & D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)} \\ D_{27}^{(1)} & -(D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)})_\phi & D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)} \\ D_{37}^{(1)} & -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi & -(D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)})_\phi \\ D_{57}^{(1)} & -(D_{15}^{(1)} - D_{15}^{(2)})_\phi & -(D_{25}^{(1)} - D_{25}^{(2)})_\phi \\ D_{47}^{(1)} & -(D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)})_\phi & -(D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)})_\phi \\ D_{67}^{(1)} & -(D_{16}^{(1)} - X_{16}^{(2)})_\phi & -(D_{26}^{(1)} - X_{26}^{(2)})_\phi \end{pmatrix} \end{array},$$

$$\Psi_2^{(2)} = \begin{array}{c} \\ n_7 \\ n_1 \\ n_2 \\ n_3 \\ n_5 \\ n_4 \\ n_6 \end{array} \begin{array}{c} \begin{array}{ccc} n_3 & n_5 & n_4 \end{array} \\ \begin{pmatrix} -(D_{37}^{(1)})_\phi & -(D_{57}^{(1)})_\phi & -(D_{47}^{(1)})_\phi \\ D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)} \\ D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} & D_{25}^{(1)} - D_{25}^{(2)} & D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)} \\ D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)} \\ -(D_{35}^{(1)} - D_{35}^{(2)})_\phi & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{45}^{(1)} - D_{45}^{(2)})_\phi \\ -(D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)})_\phi & D_{45}^{(1)} - D_{45}^{(2)} & D_{44}^{(1)} - D_{44}^{(2)} + X_{44}^{(3)} \\ -(D_{36}^{(1)} - X_{36}^{(2)})_\phi & -(D_{56}^{(1)} - X_{56}^{(2)})_\phi & -(D_{46}^{(1)} - X_{46}^{(2)})_\phi \end{pmatrix} \end{array},$$

$$\Psi_3^{(1)} = \begin{array}{c} \\ n_7 \\ n_1 \\ n_2 \\ n_3 \\ n_5 \\ n_4 \end{array} \begin{array}{c} \begin{array}{ccc} n_7 & n_1 & n_2 \end{array} \\ \begin{pmatrix} D_{77}^{(1)} & -(D_{17}^{(1)})_\phi & -(D_{27}^{(1)})_\phi \\ D_{17}^{(1)} & D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} & D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)} \\ D_{27}^{(1)} & -(D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)})_\phi & D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)} \\ D_{37}^{(1)} & -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi & -(D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)})_\phi \\ D_{57}^{(1)} & -(D_{15}^{(1)} - D_{15}^{(2)})_\phi & -(D_{25}^{(1)} - D_{25}^{(2)})_\phi \\ D_{47}^{(1)} & -(D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)})_\phi & -(D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)})_\phi \end{pmatrix} \end{array},$$

$$\Psi_3^{(2)} = \begin{array}{c} \\ n_7 \\ n_1 \\ n_2 \\ n_3 \\ n_5 \\ n_4 \end{array} \begin{array}{c} \begin{array}{ccc} n_3 & n_5 & n_4 \end{array} \\ \begin{pmatrix} -(D_{37}^{(1)})_\phi & -(D_{57}^{(1)})_\phi & -(D_{47}^{(1)})_\phi \\ D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)} \\ D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} & D_{25}^{(1)} - D_{25}^{(2)} & D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)} \\ D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)} \\ -(D_{35}^{(1)} - D_{35}^{(2)})_\phi & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{45}^{(1)} - D_{45}^{(2)})_\phi \\ -(D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)})_\phi & D_{45}^{(1)} - D_{45}^{(2)} & D_{44}^{(1)} - D_{44}^{(2)} + X_{44}^{(3)} \end{pmatrix} \end{array}.$$

Then, we treat the matrix Ψ_2 as a block matrix. According to Lemma 3, we derive

$$\min_{(D_{16}^{(1)} - X_{16}^{(2)}, D_{26}^{(1)} - X_{26}^{(2)}, D_{36}^{(1)} - X_{36}^{(2)}, D_{56}^{(1)} - X_{56}^{(2)}, D_{46}^{(1)} - X_{46}^{(2)})} r(\Psi_2) = r(\Psi_3) + r\begin{pmatrix} D_{77}^{(1)} \\ D_{17}^{(1)} \\ D_{27}^{(1)} \\ D_{37}^{(1)} \\ D_{57}^{(1)} \\ D_{47}^{(1)} \\ D_{67}^{(1)} \end{pmatrix} - r\begin{pmatrix} D_{77}^{(1)} \\ D_{17}^{(1)} \\ D_{27}^{(1)} \\ D_{37}^{(1)} \\ D_{57}^{(1)} \\ D_{47}^{(1)} \end{pmatrix}.$$

Thus, we can provide

$$\max_{D_{66}^{(1)}-X_{66}^{(2)},\begin{pmatrix} D_{16}^{(1)}-X_{16}^{(2)} \\ D_{26}^{(1)}-X_{26}^{(2)} \\ D_{36}^{(1)}-X_{36}^{(2)} \\ D_{56}^{(1)}-X_{56}^{(2)} \\ D_{46}^{(1)}-X_{46}^{(2)} \end{pmatrix}} \ln_{\pm}(\Psi_1) = n_6 + \ln_{\pm}(\Psi_3),$$

$$\min_{D_{66}^{(1)}-X_{66}^{(2)},\begin{pmatrix} D_{16}^{(1)}-X_{16}^{(2)} \\ D_{26}^{(1)}-X_{26}^{(2)} \\ D_{36}^{(1)}-X_{36}^{(2)} \\ D_{56}^{(1)}-X_{56}^{(2)} \\ D_{46}^{(1)}-X_{46}^{(2)} \end{pmatrix}} \ln_{\pm}(\Psi_1) = \ln_{\pm}(\Psi_3) + r\begin{pmatrix} D_{77}^{(1)} \\ D_{17}^{(1)} \\ D_{27}^{(1)} \\ D_{37}^{(1)} \\ D_{57}^{(1)} \\ D_{47}^{(1)} \\ D_{67}^{(1)} \end{pmatrix} - r\begin{pmatrix} D_{77}^{(1)} \\ D_{17}^{(1)} \\ D_{27}^{(1)} \\ D_{37}^{(1)} \\ D_{57}^{(1)} \\ D_{47}^{(1)} \end{pmatrix}.$$

Step 3. Using Lemma 2, Lemma 3 and the similar methods in **Step 1** and **Step 2** (the specific proof process please see Appendix A), we establish

$$\max \ln_{\pm}(\widehat{X}_1) = t_1 - r_{a_1} + n_6 + n_4 + n_2$$
$$+ \ln_{\pm}\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_{\phi} & -(D_{37}^{(1)})_{\phi} & -(D_{17}^{(1)})_{\phi} \\ D_{57}^{(1)} & D_{55}^{(1)}-D_{55}^{(2)} & -(D_{35}^{(1)}-D_{35}^{(2)})_{\phi} & -(D_{15}^{(1)}-D_{15}^{(2)})_{\phi} \\ D_{37}^{(1)} & D_{35}^{(1)}-D_{35}^{(2)} & D_{33}^{(1)}-D_{33}^{(2)}+D_{33}^{(3)} & -(D_{13}^{(1)}-D_{13}^{(2)}+D_{13}^{(3)})_{\phi} \\ D_{17}^{(1)} & D_{15}^{(1)}-D_{15}^{(2)} & D_{13}^{(1)}-D_{13}^{(2)}+D_{13}^{(3)} & D_{11}^{(1)}-D_{11}^{(2)}+D_{11}^{(3)}-D_{11}^{(4)} \end{pmatrix}, \quad (18)$$

$$\min \ln_{\pm}(\widehat{X}_1) = r\begin{pmatrix} D_{77}^{(1)} \\ D_{17}^{(1)} \\ D_{27}^{(1)} \\ D_{37}^{(1)} \\ D_{57}^{(1)} \\ D_{47}^{(1)} \\ D_{67}^{(1)} \end{pmatrix} - r\begin{pmatrix} D_{77}^{(1)} \\ D_{17}^{(1)} \\ D_{27}^{(1)} \\ D_{37}^{(1)} \\ D_{57}^{(1)} \\ D_{47}^{(1)} \end{pmatrix} + r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_{\phi} \\ D_{57}^{(1)} & D_{55}^{(1)}-D_{55}^{(2)} \\ D_{17}^{(1)} & D_{15}^{(1)}-D_{15}^{(2)} \\ D_{37}^{(1)} & D_{35}^{(1)}-D_{35}^{(2)} \\ D_{27}^{(1)} & D_{25}^{(1)}-D_{25}^{(2)} \\ D_{47}^{(1)} & D_{45}^{(1)}-D_{45}^{(2)} \end{pmatrix} - r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_{\phi} \\ D_{57}^{(1)} & D_{55}^{(1)}-D_{55}^{(2)} \\ D_{17}^{(1)} & D_{15}^{(1)}-D_{15}^{(2)} \\ D_{37}^{(1)} & D_{35}^{(1)}-D_{35}^{(2)} \\ D_{27}^{(1)} & D_{25}^{(1)}-D_{25}^{(2)} \end{pmatrix}$$
$$+r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_{\phi} & -(D_{37}^{(1)})_{\phi} \\ D_{57}^{(1)} & D_{55}^{(1)}-D_{55}^{(2)} & -(D_{35}^{(1)}-D_{35}^{(2)})_{\phi} \\ D_{37}^{(1)} & D_{35}^{(1)}-D_{35}^{(2)} & D_{33}^{(1)}-D_{33}^{(2)}+D_{33}^{(3)} \\ D_{17}^{(1)} & D_{15}^{(1)}-D_{15}^{(2)} & D_{13}^{(1)}-D_{13}^{(2)}+D_{13}^{(3)} \\ D_{27}^{(1)} & D_{25}^{(1)}-D_{25}^{(2)} & D_{23}^{(1)}-D_{23}^{(2)}+D_{23}^{(3)} \end{pmatrix} - r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_{\phi} & -(D_{37}^{(1)})_{\phi} \\ D_{57}^{(1)} & D_{55}^{(1)}-D_{55}^{(2)} & -(D_{35}^{(1)}-D_{35}^{(2)})_{\phi} \\ D_{37}^{(1)} & D_{35}^{(1)}-D_{35}^{(2)} & D_{33}^{(1)}-D_{33}^{(2)}+D_{33}^{(3)} \\ D_{17}^{(1)} & D_{15}^{(1)}-D_{15}^{(2)} & D_{13}^{(1)}-D_{13}^{(2)}+D_{13}^{(3)} \end{pmatrix}$$
$$+ \ln_{\pm}\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_{\phi} & -(D_{37}^{(1)})_{\phi} & -(D_{17}^{(1)})_{\phi} \\ D_{57}^{(1)} & D_{55}^{(1)}-D_{55}^{(2)} & -(D_{35}^{(1)}-D_{35}^{(2)})_{\phi} & -(D_{15}^{(1)}-D_{15}^{(2)})_{\phi} \\ D_{37}^{(1)} & D_{35}^{(1)}-D_{35}^{(2)} & D_{33}^{(1)}-D_{33}^{(2)}+D_{33}^{(3)} & -(D_{13}^{(1)}-D_{13}^{(2)}+D_{13}^{(3)})_{\phi} \\ D_{17}^{(1)} & D_{15}^{(1)}-D_{15}^{(2)} & D_{13}^{(1)}-D_{13}^{(2)}+D_{13}^{(3)} & D_{11}^{(1)}-D_{11}^{(2)}+D_{11}^{(3)}-D_{11}^{(4)} \end{pmatrix}. \quad (19)$$

Step 4. The ranks and $\beta(\phi)$-signature of matrices in (4.5) and (4.6) can be represented by the following expression

$$r\begin{pmatrix}(B_1)_\phi\\C_1\end{pmatrix}=r(B_1)+r\begin{pmatrix}D_{77}^{(1)}\\D_{17}^{(1)}\\D_{27}^{(1)}\\D_{37}^{(1)}\\D_{57}^{(1)}\\D_{47}^{(1)}\\D_{67}^{(1)}\end{pmatrix},\ r\begin{pmatrix}A_2&0\\0&(B_1)_\phi\\B_1&C_1\end{pmatrix}=r\begin{pmatrix}A_2\\B_1\end{pmatrix}+r(B_1)+r\begin{pmatrix}D_{77}^{(1)}\\D_{17}^{(1)}\\D_{27}^{(1)}\\D_{37}^{(1)}\\D_{57}^{(1)}\\D_{47}^{(1)}\end{pmatrix},$$

$$r\begin{pmatrix}(B_2)_\phi&0&0\\-C_2&A_2&0\\(A_2)_\phi&0&(B_1)_\phi\\0&B_1&C_1\end{pmatrix}=r\begin{pmatrix}A_2\\B_1\end{pmatrix}+r\begin{pmatrix}B_2&A_2\\0&B_1\end{pmatrix}+r\begin{pmatrix}D_{77}^{(1)}&-(D_{57}^{(1)})_\phi\\D_{57}^{(1)}&D_{55}^{(1)}-D_{55}^{(2)}\\D_{17}^{(1)}&D_{15}^{(1)}-D_{15}^{(2)}\\D_{37}^{(1)}&D_{35}^{(1)}-D_{35}^{(2)}\\D_{27}^{(1)}&D_{25}^{(1)}-D_{25}^{(2)}\\D_{47}^{(1)}&D_{45}^{(1)}-D_{45}^{(2)}\end{pmatrix},$$

$$r\begin{pmatrix}A_3&0&0&0\\0&(B_2)_\phi&0&0\\B_2&-C_2&A_2&0\\0&(A_2)_\phi&0&(B_1)_\phi\\0&0&B_1&C_1\end{pmatrix}=r\begin{pmatrix}A_3&0\\B_2&A_2\\0&B_1\end{pmatrix}+r\begin{pmatrix}B_2&A_2\\0&B_1\end{pmatrix}+r\begin{pmatrix}D_{77}^{(1)}&-(D_{57}^{(1)})_\phi\\D_{57}^{(1)}&D_{55}^{(1)}-D_{55}^{(2)}\\D_{17}^{(1)}&D_{15}^{(1)}-D_{15}^{(2)}\\D_{37}^{(1)}&D_{35}^{(1)}-D_{35}^{(2)}\\D_{27}^{(1)}&D_{25}^{(1)}-D_{25}^{(2)}\end{pmatrix},$$

$$r\begin{pmatrix}(B_3)_\phi&0&0&0&0\\C_3&A_3&0&0&0\\(A_3)_\phi&0&(B_2)_\phi&0&0\\0&B_2&-C_2&A_2&0\\0&0&(A_2)_\phi&0&(B_1)_\phi\\0&0&0&B_1&C_1\end{pmatrix}=r\begin{pmatrix}A_3&0\\B_2&A_2\\0&B_1\end{pmatrix}+r\begin{pmatrix}B_3&A_3&0\\0&B_2&A_2\\0&0&B_1\end{pmatrix}+r\begin{pmatrix}D_{77}^{(1)}&-(D_{57}^{(1)})_\phi&-(D_{37}^{(1)})_\phi\\D_{57}^{(1)}&D_{55}^{(1)}-D_{55}^{(2)}&-(D_{35}^{(1)}-D_{35}^{(2)})_\phi\\D_{37}^{(1)}&D_{35}^{(1)}-D_{35}^{(2)}&D_{33}^{(1)}-D_{33}^{(2)}+D_{33}^{(3)}\\D_{17}^{(1)}&D_{15}^{(1)}-D_{15}^{(2)}&D_{13}^{(1)}-D_{13}^{(2)}+D_{13}^{(3)}\\D_{27}^{(1)}&D_{25}^{(1)}-D_{25}^{(2)}&D_{23}^{(1)}-D_{23}^{(2)}+D_{23}^{(3)}\end{pmatrix},$$

$$r\begin{pmatrix}A_4&0&0&0&0&0\\0&(B_3)_\phi&0&0&0&0\\B_3&C_3&A_3&0&0&0\\0&(A_3)_\phi&0&(B_2)_\phi&0&0\\0&0&B_2&-C_2&A_2&0\\0&0&0&(A_2)_\phi&0&(B_1)_\phi\\0&0&0&0&B_1&C_1\end{pmatrix}$$

$$=r\begin{pmatrix}B_3&A_3&0\\0&B_2&A_2\\0&0&B_1\end{pmatrix}+r\begin{pmatrix}A_4&0&0\\B_3&A_3&0\\0&B_2&A_2\\0&0&B_1\end{pmatrix}+r\begin{pmatrix}D_{77}^{(1)}&-(D_{57}^{(1)})_\phi&-(D_{37}^{(1)})_\phi\\D_{57}^{(1)}&D_{55}^{(1)}-D_{55}^{(2)}&-(D_{35}^{(1)}-D_{35}^{(2)})_\phi\\D_{37}^{(1)}&D_{35}^{(1)}-D_{35}^{(2)}&D_{33}^{(1)}-D_{33}^{(2)}+D_{33}^{(3)}\\D_{17}^{(1)}&D_{15}^{(1)}-D_{15}^{(2)}&D_{13}^{(1)}-D_{13}^{(2)}+D_{13}^{(3)}\end{pmatrix},$$

$$\ln_\pm\begin{pmatrix}-C_4&A_4&0&0&0&0&0\\(A_4)_\phi&0&(B_3)_\phi&0&0&0&0\\0&B_3&C_3&A_3&0&0&0\\0&0&(A_3)_\phi&0&(B_2)_\phi&0&0\\0&0&0&B_2&-C_2&A_2&0\\0&0&0&0&(A_2)_\phi&0&(B_1)_\phi\\0&0&0&0&0&B_1&C_1\end{pmatrix}$$

$$= r\begin{pmatrix} A_4 & 0 & 0 \\ B_3 & A_3 & 0 \\ 0 & B_2 & A_2 \\ 0 & 0 & B_1 \end{pmatrix} + \ln_{\pm}\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi & -(D_{37}^{(1)})_\phi & -(D_{17}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi & -(D_{15}^{(1)} - D_{15}^{(2)})_\phi \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} \end{pmatrix},$$

$$n_2 = r\begin{pmatrix} B_1 & 0 \\ A_2 & B_2 \end{pmatrix} + r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix} - r\begin{pmatrix} A_1 & B_1 & 0 \\ 0 & A_2 & B_2 \end{pmatrix} - r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix},$$

$$n_4 = r(B_1) + r\begin{pmatrix} A_1 & B_1 & 0 \\ 0 & A_2 & B_2 \end{pmatrix} - r\begin{pmatrix} A_1 & B_1 \end{pmatrix} - r\begin{pmatrix} B_1 & 0 \\ A_2 & B_2 \end{pmatrix},$$

$$n_6 = r\begin{pmatrix} A_1 & B_1 \end{pmatrix} - r(B_1),\ r_{a_1} = r(A_1).$$

The above results indicate that the $\beta(\phi)$-signature bounds of the ϕ-skew-Hermitian matrix $\widehat{X}_1$ can be expressed as

$$\max \ln_{\pm}(\widehat{X}_1) = t_1 - r(A_1) - r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} - r\begin{pmatrix} A_4 & 0 & 0 \\ B_3 & A_3 & 0 \\ 0 & B_2 & A_2 \\ 0 & 0 & B_1 \end{pmatrix}$$

$$+ r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix} + \ln_{\pm}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix},$$

$$\min \ln_{\pm}(\widehat{X}_1) = r\begin{pmatrix} (B_1)_\phi \\ C_1 \end{pmatrix} - r\begin{pmatrix} A_2 & 0 \\ 0 & (B_1)_\phi \\ B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} (B_2)_\phi & 0 & 0 \\ -C_2 & A_2 & 0 \\ (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & B_1 & C_1 \end{pmatrix}$$

$$- r\begin{pmatrix} A_3 & 0 & 0 & 0 \\ 0 & (B_2)_\phi & 0 & 0 \\ B_2 & -C_2 & A_2 & 0 \\ 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} (B_3)_\phi & 0 & 0 & 0 & 0 \\ C_3 & A_3 & 0 & 0 & 0 \\ (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$- r\begin{pmatrix} A_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix} + \ln_{\pm}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}.$$

In conclusion, Theorem 2 is proved. □

Based on Theorem 2, we derive some necessary and sufficient conditions for the system (1) to have $\beta(\phi)$-positive definite, $\beta(\phi)$-positive semidefinite, $\beta(\phi)$-negative definite and $\beta(\phi)$-negative semidefinite solutions.

Theorem 3. *Assume that the system (1) has a ϕ-skew-Hermitian solution $(X_1, X_2, X_3, X_4) \in \mathbb{H}^{t_1\times t_1} \times \mathbb{H}^{t_2\times t_2} \times \mathbb{H}^{t_3\times t_3} \times \mathbb{H}^{t_4\times t_4}$. We can derive the following conclusions.*

(*a*) *There is a* $\beta(\phi)$*-positive definite solution* X_1 *if and only if*

$$\ln_{+}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$= r(A_1) + r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} + r\begin{pmatrix} A_4 & 0 & 0 \\ B_3 & A_3 & 0 \\ 0 & B_2 & A_2 \\ 0 & 0 & B_1 \end{pmatrix} - r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix}.$$

(*b*) *There is a* $\beta(\phi)$*-negative definite solution* X_1 *if and only if*

$$\ln_{-}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$= r(A_1) + r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} + r\begin{pmatrix} A_4 & 0 & 0 \\ B_3 & A_3 & 0 \\ 0 & B_2 & A_2 \\ 0 & 0 & B_1 \end{pmatrix} - r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix}.$$

(*c*) *There is a* $\beta(\phi)$*-positive semidefinite solution* X_1 *if and only if*

$$\ln_{+}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$= r\begin{pmatrix} A_2 & 0 \\ 0 & (B_1)_\phi \\ B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} A_3 & 0 & 0 & 0 \\ 0 & (B_2)_\phi & 0 & 0 \\ B_2 & -C_2 & A_2 & 0 \\ 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} A_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$-r\begin{pmatrix} (B_1)_\phi \\ C_1 \end{pmatrix} - r\begin{pmatrix} (B_2)_\phi & 0 & 0 \\ -C_2 & A_2 & 0 \\ (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & B_1 & C_1 \end{pmatrix} - r\begin{pmatrix} (B_3)_\phi & 0 & 0 & 0 & 0 \\ C_3 & A_3 & 0 & 0 & 0 \\ (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}.$$

(*d*) *There is a* $\beta(\phi)$*-negative semidefinite solution* X_1 *if and only if*

$$\ln_{-}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$= r\begin{pmatrix} A_2 & 0 \\ 0 & (B_1)_\phi \\ B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} A_3 & 0 & 0 & 0 \\ 0 & (B_2)_\phi & 0 & 0 \\ B_2 & -C_2 & A_2 & 0 \\ 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} A_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$-r\begin{pmatrix} (B_1)_\phi \\ C_1 \end{pmatrix} - r\begin{pmatrix} (B_2)_\phi & 0 & 0 \\ -C_2 & A_2 & 0 \\ (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & B_1 & C_1 \end{pmatrix} - r\begin{pmatrix} (B_3)_\phi & 0 & 0 & 0 & 0 \\ C_3 & A_3 & 0 & 0 & 0 \\ (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}.$$

(e) All the solutions X_1 are $\beta(\phi)$-positive definite if and only if

$$r\begin{pmatrix} (B_1)_\phi \\ C_1 \end{pmatrix} - r\begin{pmatrix} A_2 & 0 \\ 0 & (B_1)_\phi \\ B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} (B_2)_\phi & 0 & 0 \\ -C_2 & A_2 & 0 \\ (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & B_1 & C_1 \end{pmatrix}$$

$$-r\begin{pmatrix} A_3 & 0 & 0 & 0 \\ 0 & (B_2)_\phi & 0 & 0 \\ B_2 & -C_2 & A_2 & 0 \\ 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} (B_3)_\phi & 0 & 0 & 0 & 0 \\ C_3 & A_3 & 0 & 0 & 0 \\ (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$-r\begin{pmatrix} A_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix} + \ln_{-}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix} = t_1.$$

(f) All the solutions X_1 are $\beta(\phi)$-negative definite if and only if

$$r\begin{pmatrix} (B_1)_\phi \\ C_1 \end{pmatrix} - r\begin{pmatrix} A_2 & 0 \\ 0 & (B_1)_\phi \\ B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} (B_2)_\phi & 0 & 0 \\ -C_2 & A_2 & 0 \\ (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & B_1 & C_1 \end{pmatrix}$$

$$-r\begin{pmatrix} A_3 & 0 & 0 & 0 \\ 0 & (B_2)_\phi & 0 & 0 \\ B_2 & -C_2 & A_2 & 0 \\ 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & B_1 & C_1 \end{pmatrix} + r\begin{pmatrix} (B_3)_\phi & 0 & 0 & 0 & 0 \\ C_3 & A_3 & 0 & 0 & 0 \\ (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$-r\begin{pmatrix} A_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix} + \ln_+\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix} = t_1.$$

(g) *All the solutions X_1 are $\beta(\phi)$-positive semidefinite if and only if*

$$t_1 + \ln_-\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$= r(A_1) + r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} + r\begin{pmatrix} A_4 & 0 & 0 \\ B_3 & A_3 & 0 \\ 0 & B_2 & A_2 \\ 0 & 0 & B_1 \end{pmatrix} - r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix}.$$

(h) *All the solutions X_1 are $\beta(\phi)$-negative semidefinite if and only if*

$$t_1 + \ln_+\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$= r(A_1) + r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} + r\begin{pmatrix} A_4 & 0 & 0 \\ B_3 & A_3 & 0 \\ 0 & B_2 & A_2 \\ 0 & 0 & B_1 \end{pmatrix} - r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix}.$$

Proof. According to Theorem 2, the system (1) has a $\beta(\phi)$-positive definite solution X_1 if and only if

$$\max \ln_+(X_1) = t_1.$$

It follows from Theorem 2 that

$$\max \ln_\pm(X_1) = t_1 - r(A_1) - r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} - r\begin{pmatrix} A_4 & 0 & 0 \\ B_3 & A_3 & 0 \\ 0 & B_2 & A_2 \\ 0 & 0 & B_1 \end{pmatrix}$$

$$+r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix} + \ln_{\pm}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix} = t_1.$$

Then, we can obtain that there is a $\beta(\phi)$-positive definite solution X_1 if and only if

$$\ln_{+}\begin{pmatrix} -C_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\ (A_4)_\phi & 0 & (B_3)_\phi & 0 & 0 & 0 & 0 \\ 0 & B_3 & C_3 & A_3 & 0 & 0 & 0 \\ 0 & 0 & (A_3)_\phi & 0 & (B_2)_\phi & 0 & 0 \\ 0 & 0 & 0 & B_2 & -C_2 & A_2 & 0 \\ 0 & 0 & 0 & 0 & (A_2)_\phi & 0 & (B_1)_\phi \\ 0 & 0 & 0 & 0 & 0 & B_1 & C_1 \end{pmatrix}$$

$$= r(A_1) + r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} + r\begin{pmatrix} A_4 & 0 & 0 \\ B_3 & A_3 & 0 \\ 0 & B_2 & A_2 \\ 0 & 0 & B_1 \end{pmatrix} - r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix}.$$

Hence, we can prove the statements (a). In a similar way, we can obtain (b)–(h). □

5. Conclusions

We have provided the general solution to the system (1). Furthermore, we have given the $\beta(\phi)$-signature bounds of the ϕ-skew-Hermitian solution to the system (1). Finally, we have presented some necessary and sufficient conditions for the system (1) to have $\beta(\phi)$-positive definite, $\beta(\phi)$-positive semidefinite, $\beta(\phi)$-negative definite and $\beta(\phi)$-negative semidefinite solutions.

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Appendix A. The Specific Proof Process for Theorem Step 3

Step 3 (1). Treating the variable ϕ-skew-Hermitian matrix $D_{44}^{(1)} - D_{44}^{(2)} + X_{44}^{(3)}$ of Ψ_3 is the matrix block X in (14), using Lemma 2 we have

$$\max_{D_{44}^{(1)}-D_{44}^{(2)}+X_{44}^{(3)}} \ln_{\pm}(\Psi_3) = n_4 + \ln_{\pm}(\Psi_5), \quad \min_{D_{44}^{(1)}-D_{44}^{(2)}+X_{44}^{(3)}} \ln_{\pm}(\Psi_3) = r(\Psi_4) - \ln_{\mp}(\Psi_5),$$

where

$$\Psi_4 = \begin{array}{c} \\ n_7 \\ n_5 \\ n_1 \\ n_3 \\ n_2 \\ n_4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} n_7 & n_5 & n_1 & n_3 & n_2 \end{array} \\ \begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi & -(D_{17}^{(1)})_\phi & -(D_{37}^{(1)})_\phi & -(D_{27}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{15}^{(1)} - D_{15}^{(2)})_\phi & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi & -(D_{25}^{(1)} - D_{25}^{(2)})_\phi \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)} \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} & -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & -(D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)})_\phi \\ D_{27}^{(1)} & D_{25}^{(1)} - D_{25}^{(2)} & -(D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)})_\phi & D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} & D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)} \\ D_{47}^{(1)} & D_{45}^{(1)} - D_{45}^{(2)} & -(D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)})_\phi & -(D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)})_\phi & -(D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)})_\phi \end{pmatrix} \end{array},$$

$$\Psi_5 = \begin{array}{c} \\ n_7 \\ n_5 \\ n_1 \\ n_3 \\ n_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} n_7 & n_5 & n_1 & n_3 & n_2 \end{array} \\ \begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi & -(D_{17}^{(1)})_\phi & -(D_{37}^{(1)})_\phi & -(D_{27}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{15}^{(1)} - D_{15}^{(2)})_\phi & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi & -(D_{25}^{(1)} - D_{25}^{(2)})_\phi \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)} \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} & -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & -(D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)})_\phi \\ D_{27}^{(1)} & D_{25}^{(1)} - D_{25}^{(2)} & -(D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)})_\phi & D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} & D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)} \end{pmatrix} \end{array}.$$

Then, we treat the matrix Ψ_4 as a block matrix, using Lemma 3 we have

$$\min_{(D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)}, D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)}, D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)})} r(\Psi_4) = r(\Psi_5) + r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} \\ D_{27}^{(1)} & D_{25}^{(1)} - D_{25}^{(2)} \\ D_{47}^{(1)} & D_{45}^{(1)} - D_{45}^{(2)} \end{pmatrix} - r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} \\ D_{27}^{(1)} & D_{25}^{(1)} - D_{25}^{(2)} \end{pmatrix}.$$

Thus, we can get

$$\max_{D_{44}^{(1)} - D_{44}^{(2)} + X_{44}^{(3)}, \begin{pmatrix} D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)} \\ D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)} \\ D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)} \end{pmatrix}} \ln_\pm(\Psi_3) = n_4 + \ln_\pm(\Psi_5),$$

$$\min_{D_{44}^{(1)} - D_{44}^{(2)} + X_{44}^{(3)}, \begin{pmatrix} D_{14}^{(1)} - D_{14}^{(2)} + X_{14}^{(3)} \\ D_{34}^{(1)} - D_{34}^{(2)} + X_{34}^{(3)} \\ D_{24}^{(1)} - D_{24}^{(2)} + X_{24}^{(3)} \end{pmatrix}} \ln_\pm(\Psi_3) = \ln_\pm(\Psi_5) + r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} \\ D_{27}^{(1)} & D_{25}^{(1)} - D_{25}^{(2)} \\ D_{47}^{(1)} & D_{45}^{(1)} - D_{45}^{(2)} \end{pmatrix} - r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} \\ D_{27}^{(1)} & D_{25}^{(1)} - D_{25}^{(2)} \end{pmatrix}.$$

Step 3 (2). Treating the variable ϕ-skew-Hermitian matrix $D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)}$ of Ψ_5 is the matrix block X in (14), using Lemma 2 we have

$$\max_{D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)}} \ln_\pm(\Psi_5) = n_2 + \ln_\pm(\Psi_7), \quad \min_{D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)}} \ln_\pm(\Psi_5) = r(\Psi_6) - \ln_\mp(\Psi_7),$$

where

$$\Psi_6 = \begin{array}{c} \\ n_7 \\ n_5 \\ n_3 \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cccc} n_7 & n_5 & n_3 & n_1 \end{array} \\ \begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi & -(D_{37}^{(1)})_\phi & -(D_{17}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi & -(D_{15}^{(1)} - D_{15}^{(2)})_\phi \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} \\ D_{27}^{(1)} & D_{25}^{(1)} - D_{25}^{(2)} & D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} & -(D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)})_\phi \end{pmatrix} \end{array},$$

$$\Psi_7 = \begin{array}{c} \\ n_7 \\ n_5 \\ n_3 \\ n_1 \end{array} \begin{array}{c} \begin{array}{cccc} n_7 & n_5 & n_3 & n_1 \end{array} \\ \begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi & -(D_{37}^{(1)})_\phi & -(D_{17}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi & -(D_{15}^{(1)} - D_{15}^{(2)})_\phi \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} & -(D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)})_\phi \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} & D_{11}^{(1)} - D_{11}^{(2)} + D_{11}^{(3)} - D_{11}^{(4)} \end{pmatrix} \end{array}.$$

Then, we treat the matrix Ψ_6 as a block matrix, using Lemma 3 we have

$$\min_{D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)}} r(\Psi_6) = r(\Psi_7) + r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi & -(D_{37}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} \\ D_{27}^{(1)} & D_{25}^{(1)} - D_{25}^{(2)} & D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} \end{pmatrix}$$

$$-r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi & -(D_{37}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} \end{pmatrix}.$$

Thus, we can get

$$\max_{D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)}, D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)}} \ln_\pm(\Psi_5) = n_2 + \ln_\pm(\Psi_7),$$

$$\min_{D_{22}^{(1)} - D_{22}^{(2)} + D_{22}^{(3)} - X_{22}^{(4)}, D_{12}^{(1)} - D_{12}^{(2)} + D_{12}^{(3)} - X_{12}^{(4)}} \ln_\pm(\Psi_5)$$

$$= \ln_\pm(\Psi_7) + r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi & -(D_{37}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} \\ D_{27}^{(1)} & D_{25}^{(1)} - D_{25}^{(2)} & D_{23}^{(1)} - D_{23}^{(2)} + D_{23}^{(3)} \end{pmatrix} - r\begin{pmatrix} D_{77}^{(1)} & -(D_{57}^{(1)})_\phi & -(D_{37}^{(1)})_\phi \\ D_{57}^{(1)} & D_{55}^{(1)} - D_{55}^{(2)} & -(D_{35}^{(1)} - D_{35}^{(2)})_\phi \\ D_{37}^{(1)} & D_{35}^{(1)} - D_{35}^{(2)} & D_{33}^{(1)} - D_{33}^{(2)} + D_{33}^{(3)} \\ D_{17}^{(1)} & D_{15}^{(1)} - D_{15}^{(2)} & D_{13}^{(1)} - D_{13}^{(2)} + D_{13}^{(3)} \end{pmatrix}.$$

According to **Step 3 (1)** and **Step 3 (2)**, the results in the paper (**Step 3**) can be obtained.

References

1. Stephen, L.; Adler. Scattering and decay theory for quaternionic quantum mechanics, and the structure of induced *T* nonconservation *Phys. Rev. D* **1988**, *37*, 3654–3662.
2. Ghouti, L. Robust perceptual color image hashing using quaternion singular value decomposition. In Proceedings of the 2014 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Florence, Italy, 4–9 May 2014.

3. Jia, Z.; Ng, M.K.; Song, G.J. Robust quaternion matrix completion with applications to image inpainting. *Numer. Linear Algebra Appl.* **2019**, *26*, e2245. [CrossRef]
4. Sangwine, S.J. Fourier transforms of colour images using quaternion or hypercomplex, numbers. *Electron. Lett.* **1996**, *32*, 1979–1980. [CrossRef]
5. Bihan, N.L.; Mars J. Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing. *Signal Process.* **2004**, *84*, 1177–1199. [CrossRef]
6. Song, C.; Chen, G.; Zhang, X. An itertive solution to coupled quaternion matrix equations. *Filomat* **2012**, *26*, 809–826. [CrossRef]
7. Wang, Q.W.; Wang, X.; Zhang, Y. A constraint system of coupled two-sided Sylvester-like quaternion tensor equations. *Comput. Appl. Math.* **2020**, *39*, 317. [CrossRef]
8. Wang, Q.W.; Yang, X.X.; Yuan, S.F. The least square solution with the least norm to a system of quaternion matrix equations. *Iran J. Sci. Technol. Trans. Sci.* **2018**, *42*, 1317–1325. [CrossRef]
9. Yuan, S.F.; Wang, Q.W. Two special kinds of least squares solutions for the quaternion matrix equation $AXB + CXD = E$. *Electron. J. Linear Algebra* **2012**, *23*, 257–274. [CrossRef]
10. Rodman, L. *Topics in Quaternion Linear Algebra*; Princeton University Press: Princeton, NJ, USA, 2014.
11. Aghamollaei, G.; Rahjoo, M. On quaternionic numerical ranges with respect to nonstandard involutions. *Linear Algebra Appl.* **2018**, *540*, 11–25. [CrossRef]
12. Rahjoo, M.; Aghamollaei, G.; Momenaee Kermani, H. On the boundedness of quaternionic numerical ranges with respect to nonstandard involutions. *Linear Algebra Appl.* **2021**, *610*, 59–72. [CrossRef]
13. He, Z.H.; Liu, J.Z.; Tam, T.Y. The general ϕ-Hermitian solution to mixed pairs of quaternion matrix Sylvester equations. *Electron. J. Linear Algebra* **2017**, *32*, 475–499. [CrossRef]
14. Wang, Q.W.; Jiang, J. Extreme ranks of $(skew-)$ Hermitian solutions to a quaternion matrix equation. *Electron. J. Linear Algebra* **2010**, *20*, 537–522. [CrossRef]
15. Enshaeifar, S.; Took, C.C.; Sanei, S.; Mandic, D.P. Novel quaternion matrix factorisations. In Proceedings of the 2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Shanghai, China, 20–25 March 2016.
16. Took, C.C.; Mandic, D.P. Augmented second-order statistics of quaternion random signals. *Signal Process.* **2011**, *91*, 214–224. [CrossRef]
17. Took, C.C.; Mandic, D.P.; Zhang F. On the unitary diagonalisation of a special class of quaternion matrices. *Appl. Math. Lett.* **2011**, *24*, 1806–1809. [CrossRef]
18. Took, C.C.; Mandic, D.P. Quaternion-Valued Stochastic Gradient-Based Adaptive IIR Filtering. *IEEE Trans. Signal Process.* **2010**, *58*, 3895–3901. [CrossRef]
19. He, Z.H. Structure, properties and applications of some simultaneous decompositions for quaternion matrices involving ϕ-skew-Hermicity. *Adv. Appl. Clifford Algebr.* **2019**, *29*, 6. [CrossRef]
20. He, Z.H.; Zhang, X.N.; Zhao, Y.F.; Yu, S.W. The solvability of a system of quaternion matrix equations involving ϕ-skew-Hermiticy. *Symmetry* **2022**, *14*, 1273. [CrossRef]
21. Dai H.; Lancaster P. Linear matrix equations from an inverse problem of vibration theory. *Linear Algebra Appl.* **1996**, *246*, 31–47.
22. Groß J. Nonnegative-definite and positive-definite solutions to the matrix equation $AXA^* = B$ revisited. *Linear Algebra Appl.* **2000**, *321*, 123–129. [CrossRef]
23. Berman, A. Mass Matrix Correction Using an Incomplete Set of Measured Modes. *AIAA J.* **1979**, *17*, 1147–1148. [CrossRef]
24. Berman, A.; Nagy, E.J. Improvement of a large analytical model using test data. *AZAA J.* **1983**, *21*, 1168–1173. [CrossRef]
25. Zhang, F. Quaternions and matrices of quaternions. *Linear Algebra Appl.* **1997**, *251*, 21–57. [CrossRef]
26. Cohen, N.; Johnson, C.R.; Rodman, L.; Woerdeman, H.J. Ranks of completions of partial matrices. *Oper. Theory Adv. Appl.* **1989**, *40*, 165–185.
27. Woerdeman, H.J. The lower order of triangular operators and minimal rank extensions. *Integral Equ. Oper. Theory* **1987**, *10*, 859–879. [CrossRef]
28. Woerdeman, H.J. Minimal rank completions for block matrices. *Linear Algebra Appl.* **1989**, *121*, 105–122. [CrossRef]
29. Woerdeman, H.J. Minimal rank completions of partial banded matrices. *Linear Multilinear Algebra* **1993**, *36*, 59–69. [CrossRef]