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Criteria for Oscillation of Half-Linear Functional Differential Equations of Second-Order

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Abstract: The present article aims to establish more effective criteria for testing the oscillation of a class of functional differential equations with delay arguments. In the non-canonical case, we deduce some improved monotonic and asymptotic properties of the class of decreasing positive solutions of the studied equation. Depending on both the new properties and the linear representation of the studied equation, we obtain new oscillation criteria. Moreover, we test the effectiveness of the new criteria by applying them to some special cases of the studied equation.

Keywords: second-order; delay differential equations; oscillation; non-canonical case

1. Introduction

It is known that the study of the oscillatory behavior of solutions of differential equations is one of the issues of qualitative theory, which is generally concerned with studying the qualitative properties of solutions of differential equations. In the last decade, there has been a great development in the study of the oscillatory properties of differential equations, see [1–5]. This is because studying the oscillatory and asymptotic behavior of mathematical models facilitates the understanding of these models and helps to study the phenomena described by these models, see [6–10]. In addition, the oscillation theory is rich in interesting theoretical problems that need the tools of mathematical analysis.

Delay differential equations are a type of functional differential equations that take into account the memory of phenomena. Delay differential equations have many physical and engineering applications, for examples of these applications, electrical networks containing lossless transmission lines include soil settlement, elasticity issues, and structure deflection, see [11,12]. Developing a study of the oscillatory behavior of solutions of delay differential equations contributes to understanding and interpreting the behavior of these solutions. In the study of p-Laplace equations, non-Newtonian fluid theory, porous media, and other fields, half-linear equations have many uses, see [13–15].

One of the main goals of oscillation theory is to find sufficient conditions to ensure that all solutions of the differential equation oscillate. One of the first monographs dealing with oscillation theory was Ladas et al. [16], which covered the results until 1984. The primary focus of this book is on how deviating arguments affect the oscillation of solutions, but it has not touched upon equations with neutral delay. Among the important works in the theory of oscillation is the book of Gyori and Ladas [17], which made great contributions to the development of linearized oscillation theory and the connection between the oscillation of all solutions and the distribution of the roots of characteristic equations.

Finding criteria for the existence of solutions with specific asymptotic features and estimating the distance between the zeros of oscillatory solutions are additional subjects of importance to the theory of oscillation, and are discussed in [18]. For more results,

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techniques, and references, monographs [19–24] covered and summarized many of the results known in the literature up to the past decade.

In recent years, the development of oscillation theory has emerged significantly through many interesting works. Drábek et al. [25] and Džurina and Jadlovská [26,27] discussed improved criteria for testing the oscillation of delay equations. By introducing a generalized Riccati substitution, Agarwal et al. [28] and Bohner et al. [29] provided criteria for oscillation of neutral equations in the non-canonical case. Grace et al. [30] created the test criteria for oscillation, similar to [29], but in the canonical case. Their method was based on establishing sharper estimates associating a non-oscillatory solution with its derivatives. Moreover, Hindi [31] and Moaaz et al. [32,33] introduced some improved conditions to ensure that all solutions of neutral equations oscillate.

Many problems in the real world where the rate of development depends on both the present and the future can be modeled using advanced differential equations. Hassan [34] investigated the oscillation properties of second-order advanced dynamic equations on time scales. Later, by obtaining the results of the Kamenev type without needing additional conditions, Agarwal et al. [35] improved the results of Hassan [34]. In [36,37], Chatzarakis et al. used a different approach from used in [34,35] and established verifiable and efficient criteria for oscillation of the advanced equation.

In this article, we consider the half-linear delay differential equation

$$\left(r(\ell)\left(u'(\ell)\right)^{\kappa}\right)' + q(\ell)u^{\kappa}(g(\ell)) = 0, \tag{1}$$

where $\ell \in [\ell_0, \infty)$, κ is a ratio of odd natural numbers, r, q, and g are continuous real functions on $[\ell_0, \infty)$, r is a positive and differentiable function, q is non-negative, g is a delay function, i.e., $g(\ell) \leq \ell$, and $\lim_{\ell\to\infty} g(\ell) = \infty$.

By a solution of Equation (1), we mean a real function $u \in C^1([\ell_x, \infty))$ for some $\ell_x \geq \ell_0$, which has the property $r \cdot (u')^{\kappa} \in C^1([\ell_x, \infty))$ and u satisfies Equation (1) on $[\ell_x, \infty)$. Only solutions that satisfy the condition $\sup\{|u(\ell)| : \ell \geq \ell_*\} > 0$, for all $\ell_* \geq \ell_x$, will receive our attention. A solution of Equation (1) is called *non-oscillatory* if it is eventually positive or eventually negative; otherwise, it is called *oscillatory*.

In 2003, Dzurina and Stavroulakis [38] tested the oscillation of the differential equation

$$\left(r|u'|^{\kappa-1}u'\right)' + q|u \circ g|^{\kappa-1}(u \circ g) = 0, \tag{2}$$

where $(u \circ g)(\ell) = u(g(\ell))$, by the criterion

$$\int^{\infty}\left(R^{\kappa}(g(\mathfrak{b}))q(\mathfrak{b}) - \frac{1}{4\lambda}\frac{\kappa g'(\mathfrak{b})}{R(g(\mathfrak{b}))r^{1/\kappa}(g(\mathfrak{b}))}\right)\mathrm{d}\mathfrak{b} = \infty,$$

where $\kappa \geq 1$ is a real number, $\lambda \in (0,1)$, and

$$R(v) := \int_{\ell_0}^{v} \frac{1}{r^{1/\kappa}(\mathfrak{b})}\mathrm{d}\mathfrak{b} \to \infty \quad \text{as } v \to \infty.$$

Sun and Meng [39] improved the results in [38], and used the criterion

$$\int^{\infty}\left(R^{\kappa}(g(\mathfrak{b}))q(\mathfrak{b}) - \left(\frac{\kappa}{\kappa+1}\right)^{\kappa+1}\frac{g'(\mathfrak{b})}{R(g(\mathfrak{b}))r^{1/\kappa}(g(\mathfrak{b}))}\right)\mathrm{d}\mathfrak{b} = \infty, \tag{3}$$

to check the oscillation of Equation (2).

Consider the delay equation of Euler type

$$\left(|u'|^{\kappa-1}u'\right)' + \frac{q_0}{\ell^{\kappa+1}}|u(\mu\ell)|^{\kappa-1}u(\mu\ell) = 0, \tag{4}$$

where $\ell \geq 1$, $q_0 > 0$ and $\mu \in (0,1)$. Using the results in [38,39], Equation (4) is oscillatory if $q_0\mu > \kappa/4$ and $q_0\mu > (\kappa/(\kappa+1))^{\kappa+1}$, respectively. In the case where $\kappa = 1$, the two criteria are congruent. However, if $\kappa > 1$, then the results in [39] provide a sharper criterion.

For

$$\int_{\ell_0}^{\infty} r^{-1/\kappa}(\mathfrak{b})\mathrm{d}\mathfrak{b} < \infty, \quad \text{(non-canonical case)}$$

Ye and Xu [40] presented criteria for oscillation of neutral equation of second-order. Theorem 2.4 in [40] proved that Equation (2) is oscillatory under conditions (3) and

$$\int_{\ell_0}^{\infty} q(\mathfrak{b})\left(\int_{\mathfrak{b}}^{\infty} r^{-1/\kappa}(\eta)\mathrm{d}\eta\right)^{\kappa+1}\mathrm{d}\mathfrak{b} = \infty.$$

Džurina and Jadlovská [26] developed a criterion with only one condition that guarantees the oscillation of Equation (1). They proved that, if

$$\limsup_{\ell\to\infty}\left(\int_{\ell}^{\infty} r^{-1/\kappa}(\eta)\mathrm{d}\eta\right)^{\kappa}\int_{\ell_0}^{\ell} q(\mathfrak{b})\mathrm{d}\mathfrak{b} > 1, \tag{5}$$

then Equation (1) is oscillatory.

Consider the equation of Euler type

$$\left(\ell^{\kappa+1}(u'(\ell))^{\kappa}\right)' + q_0 u^{\kappa}(\mu\ell) = 0, \tag{6}$$

where $\ell \geq 1$, $\kappa \geq 1$, $q_0 > 0$ and $\mu \in (0,1]$. The results in [40] cannot be applied on (6), while Theorem 3 in [26] indicated that (6) oscillates if $q_0 > 1$.

In this article, we begin by deducing some monotonic properties of the decreasing positive solutions of (1). Next, we use these new properties to pair the behavior of Equation (1) with a linear inequality. Based on this linear inequality, we introduce a new criterion for testing the oscillation of all solutions of Equation (1). The new criterion improves (5) and takes into account the impact of both κ and the delay argument g.

2. Preliminary Lemmas

In the first lemma, we classify the positive solutions of Equation (1) based on the sign of the derivatives. Then, we put a condition that ensures that the positive solutions are decreasing and also converge to zero. After that, we deduce a set of new monotonically properties for the positive solutions of Equation (1). During the results, we will need the following notations and operators:

$$\begin{aligned} A(v) &:= \int_{v}^{\infty} r^{-1/\kappa}(\mathfrak{b})\mathrm{d}\mathfrak{b}, \\ \phi(v) &:= q(v)r^{1/\kappa}(v)A^{\kappa+1}(v), \end{aligned}$$

and

$$\mathcal{L}[G; u, v] := \int_{u}^{v} G(\mathfrak{b})\mathrm{d}\mathfrak{b}.$$

Moreover, we use Ω^{+} to represent the set of all eventually positive solutions of (1).

Finaly, we need the following hypothesis to prove the main results:

(H) There is a positive constant c such that $\phi(\ell) \geq \kappa c^{\kappa}$.

Lemma 1. *Eventually, positive solutions to Equation (1) are monotonic, meaning that they are either increasing or decreasing.*

Proof. Assuming that $u \in \Omega^{+}$ leads directly to $u \circ g$ is also ultimately positive. Through Equation (1), we also deduce that $\left(r(u')^{\kappa}\right)' \leq 0$. Then, $r(u')^{\kappa}$ is of fixed sign, and so u is a monotonic function, i.e., $u' > 0$ or $u' < 0$, eventually. □

Lemma 2. *If (H) holds, then every eventually positive solution to Equation (1) is decreasing and converges to zero.*

Proof. Assume that $u \in \Omega^{+}$. Suppose the contrary, that $u'(\ell) > 0$ for $\ell \geq \ell_1 \geq \ell_0$. Then, there is a $\varrho_0 > 0$ such that $u(\ell) \geq \varrho_0$, for $\ell \geq \ell_1$. Applying $\mathcal{L}[\,\cdot\,;\ell_1,\infty]$ on Equation (1), we obtain

$$\begin{aligned} r(\ell_1)\big(u'(\ell_1)\big)^{\kappa} &\geq \mathcal{L}[q\cdot(u^{\kappa}\circ g);\ell_1,\infty] \\ &\geq \varrho_0^{\kappa}\,\mathcal{L}[q;\ell_1,\infty], \end{aligned}$$

which, with the fact that $\phi(\ell) \geq \kappa c^{\kappa}$, gives

$$\begin{aligned} r(\ell_1)\big(u'(\ell_1)\big)^{\kappa} &\geq \kappa c^{\kappa}\varrho_0^{\kappa}\,\mathcal{L}\left[\frac{1}{r^{1/\kappa}A^{\kappa+1}};\ell_1,\infty\right] \\ &\geq c^{\kappa}\varrho_0^{\kappa}\left(\lim_{s\to\infty}\frac{1}{A^{\kappa}(s)}-\frac{1}{A^{\kappa}(\ell_1)}\right), \end{aligned}$$

which tends to ∞, which is a contradiction.

Now, we have that u is positive and decreasing. Then, $\lim_{\ell\to\infty} u(\ell) = \varrho_1 \geq 0$. Assume that $\varrho_1 > 0$. Therefore, there is a $\ell_1 \geq \ell_0$ such that $u(\ell) \geq \varrho_1$ for $\ell \geq \ell_1$. Applying $\mathcal{L}[\,\cdot\,;\ell_1,\ell]$ on Equation (1), we arrive at

$$\begin{aligned} r(\ell)\big(u'(\ell)\big)^{\kappa} &\leq -\mathcal{L}[q\cdot(u^{\kappa}\circ g);\ell_1,\ell] \\ &\leq -\varrho_1^{\kappa}\,\mathcal{L}[q;\ell_1,\ell], \end{aligned}$$

and then

$$\begin{aligned} u'(\ell) &\leq -\kappa c^{\kappa}\varrho_1^{\kappa}\frac{1}{r^{1/\kappa}(\ell)}\,\mathcal{L}^{1/\kappa}\left[\frac{1}{r^{1/\kappa}A^{\kappa+1}};\ell_1,\ell\right] \\ &\leq -\kappa c^{\kappa}\varrho_1^{\kappa}\frac{1}{r^{1/\kappa}(\ell)}\left(\frac{1}{A^{\kappa}(\ell)}-\frac{1}{A^{\kappa}(\ell_1)}\right)^{1/\kappa}. \end{aligned} \tag{7}$$

Since $\lim_{\ell\to\infty} A(\ell) = 0$, then we obtain

$$A^{-\kappa}(\ell) - A^{-\kappa}(\ell_1) \geq \lambda A^{-\kappa}(\ell), \text{ for } \lambda \in (0,1). \tag{8}$$

Applying $\mathcal{L}[\,\cdot\,;\ell_1,\infty]$ on inequality (7) and using (8), we get

$$\begin{aligned} u(\ell_1) &\geq \kappa c^{\kappa}\varrho_1^{\kappa}\lambda^{1/\kappa}\,\mathcal{L}\left[\frac{1}{r^{1/\kappa}A};\ell_1,\infty\right] \\ &\geq \kappa c^{\kappa}\varrho_1^{\kappa}\lambda^{1/\kappa}\ln\frac{A(\ell_1)}{A(\ell)}, \end{aligned}$$

which tends to ∞ as $\ell \to \infty$. This contradiction leads to $\varrho_1 = 0$. □

Lemma 3. *If $u \in \Omega^{+}$ and (H) holds, then the functions u/A and u/A^{c} are increasing and decreasing, respectively.*

Proof. Assume that $u \in \Omega^{+}$. By using Lemma 2, we have that u is decreasing and converges to zero. Since

$$\begin{aligned} u(\ell) &\geq \mathcal{L}[-u';\ell,\infty] = -\mathcal{L}\left[\frac{1}{r^{1/\kappa}}r^{1/\kappa}u';\ell,\infty\right] \geq -r^{1/\kappa}(\ell)u'(\ell)\mathcal{L}\left[\frac{1}{r^{1/\kappa}};\ell,\infty\right] \\ &= -r^{1/\kappa}(\ell)u'(\ell)A(\ell), \end{aligned}$$

We obtain $A^2(u/A)' = A\,u' + r^{-1/\kappa}\,u \geq 0$. Then, u/A is an increasing function.

Next, applying $\mathcal{L}[\,\cdot\,;\ell_1,\ell]$ on Equation (1), we find

$$\begin{aligned} r(\ell)\big(u'(\ell)\big)^{\kappa} &\leq r(\ell_1)\big(u'(\ell_1)\big)^{\kappa} - \mathcal{L}[q\cdot(u^{\kappa}\circ g);\ell_1,\ell] \\ &\leq r(\ell_1)\big(u'(\ell_1)\big)^{\kappa} - u^{\kappa}(\ell)\,\mathcal{L}[q;\ell_1,\ell], \end{aligned}$$

which, with the fact that $\phi(\ell)\geq\kappa c^{\kappa}$, gives

$$\begin{aligned} r(\ell)\big(u'(\ell)\big)^{\kappa} &\leq r(\ell_1)\big(u'(\ell_1)\big)^{\kappa} - \kappa c^{\kappa}u^{\kappa}(\ell)\,\mathcal{L}\left[\frac{1}{r^{1/\kappa}A^{\kappa+1}};\ell_1,\ell\right] \\ &\leq r(\ell_1)\big(u'(\ell_1)\big)^{\kappa} - c^{\kappa}u^{\kappa}(\ell)\left(\frac{1}{A^{\kappa}(\ell)}-\frac{1}{A^{\kappa}(\ell_1)}\right). \end{aligned} \tag{9}$$

Since u converges to zero, we have that $r(\ell_1)(u'(\ell_1))^{\kappa}+c^{\kappa}(u(\ell)/A^{\kappa}(\ell_1))\leq 0$. Hence, (9) reduces to

$$r^{1/\kappa}(\ell)u'(\ell)\leq -c\frac{u(\ell)}{A(\ell)}.$$

Therefore, $A^{-1-c}(u/A^{c})' = A\,u' + cr^{-1/\kappa}\,u \leq 0$. Then, u/A^{c} is a decreasing function. □

We recast Equation (1) as a linear inequality in the next lemma.

Lemma 4. *If $u\in\Omega^{+}$ and (H) holds, then*

$$\left(r^{1/\kappa}(\ell)u'(\ell)\right)' + Q(\ell)u(g(\ell))\leq 0, \tag{10}$$

where

$$Q(\ell) := \begin{cases} \frac{1}{\kappa}q(\ell)A^{\kappa-1}(\ell), & \text{if } \kappa\leq 1; \\ \frac{1}{\kappa}c^{1-\kappa}q(\ell)A^{\kappa-1}(g(\ell)), & \text{if } \kappa>1. \end{cases}$$

Proof. Assume that $u\in\Omega^{+}$. By using Lemma 2 we have that u is decreasing and converges to zero. It is easy to notice that

$$\begin{aligned} \left(r(u')^{\kappa}\right)' &= \left(\left(r^{1/\kappa}u'\right)^{\kappa}\right)' \\ &= \kappa\left(r^{1/\kappa}u'\right)^{\kappa-1}\left(r^{1/\kappa}u'\right)'. \end{aligned} \tag{11}$$

First, suppose that $\kappa\leq 1$. By using Lemma 3, we have that $(u/A)'\geq 0$, and so

$$-r^{1/\kappa}u'\leq\frac{u}{A}\leq\frac{(u\circ g)}{A},$$

which implies

$$\left(r^{1/\kappa}u'\right)^{\kappa-1}\geq\left(\frac{(u\circ g)}{A}\right)^{\kappa-1}. \tag{12}$$

Combining (1), (11), and (12), we get

$$-(u\circ g)^{\kappa}q = \left(r(u')^{\kappa}\right)'\geq\kappa\left(\frac{(u\circ g)}{A}\right)^{\kappa-1}\left(r^{1/\kappa}u'\right)'.$$

By a simple computation, we get that (10) holds.

Next, we assume that $\kappa>1$. From Lemma 3, we get that $(u/A)'\geq 0$ and $(u/A^{c})'\leq 0$. Hence,

$$-\,r^{1/\kappa}u'\geq c\frac{u}{A}\geq c\frac{(u\circ g)}{(A\circ g)}. \tag{13}$$

Combining (1), (11), and (13), we get

$$-(u\circ g)^{\kappa}q=\left(r\left(u'\right)^{\kappa}\right)'\geq\kappa\left(c\frac{(u\circ g)}{(A\circ g)}\right)^{\kappa-1}\left(r^{1/\kappa}u'\right)'.$$

By a simple computation, we get that (10) holds. □

3. Oscillation Theorems

Now, we use our results in the previous section to obtain the criteria of the oscillation for the solutions of (1).

Theorem 1. *Assume that (H) holds. Every solution of Equation (1) is oscillatory if*

$$\limsup_{\ell\to\infty}\left(A(g(\ell))\,\mathcal{L}[Q;\ell_1,g(\ell)]+\mathcal{L}[A\cdot Q;g(\ell),\ell]+\frac{1}{A(g(\ell))}\mathcal{L}[A\cdot Q\cdot(A\circ g);\ell,\infty]\right)>1. \tag{14}$$

Proof. Assume the contrary that $u\in\Omega^{+}$. By using Lemma 4, we have

$$\left(r^{1/\kappa}u'\right)'+Q(u\circ g)\leq 0, \tag{15}$$

which is equivalent to

$$\left(Ar^{1/\kappa}u'+u\right)'+AQ(u\circ g)\leq 0. \tag{16}$$

Applying $\mathcal{L}[\,\cdot\,;\ell_1,\ell]$ on inequality (15), we get

$$-\,r^{1/\kappa}(\ell)u'(\ell)\geq\mathcal{L}[Q\cdot(u\circ g);\ell_1,\ell]. \tag{17}$$

On the other hand, from Lemma 3, we obtain u/A is increasing, and so $A\,u'\geq -r^{-1/\kappa}\,u$. Then, the function $Ar^{1/\kappa}u'+u$ is positive. Moreover, from (16), we note that $Ar^{1/\kappa}u'+u$ is decreasing. Applying $\mathcal{L}[\,\cdot\,;\ell,\infty]$ on inequality (16), we have

$$A(\ell)r^{1/\kappa}(\ell)u'(\ell)+u(\ell)\geq\mathcal{L}[A\cdot Q\cdot(u\circ g);\ell,\infty]. \tag{18}$$

From (17) and (18), we find

$$u(\ell)\geq A(\ell)\,\mathcal{L}[Q\cdot(u\circ g);\ell_1,\ell]+\mathcal{L}[A\cdot Q\cdot(u\circ g);\ell,\infty].$$

Therefore,

$$u(g(\ell))\geq A(g(\ell))\,\mathcal{L}[Q\cdot(u\circ g);\ell_1,g(\ell)]+\mathcal{L}[A\cdot Q\cdot(u\circ g);g(\ell),\infty],$$

which is equivalent to

$$\begin{aligned}u(g(\ell))\quad\geq\quad & A(g(\ell))\,\mathcal{L}[Q\cdot(u\circ g);\ell_1,g(\ell)]+\mathcal{L}[A\cdot Q\cdot(u\circ g);g(\ell),\ell]\\ & +\mathcal{L}[A\cdot Q\cdot(u\circ g);\ell,\infty].\end{aligned} \tag{19}$$

Using the facts that u/A and u are increasing and decreasing, respectively, we conclude that

$$\begin{aligned}u(g(s))\quad&\geq\quad u(g(\ell))\ \text{ for } s\leq g(\ell)\leq\ell,\\ &\geq\quad\frac{A(g(s))}{A(g(\ell))}u(g(\ell))\ \text{ for }\ \ell\leq s,\end{aligned}$$

which, with (19), gives

$$A(g(\ell))\,\mathcal{L}[Q;\ell_1,g(\ell)]+\mathcal{L}[A\cdot Q;g(\ell),\ell]+\frac{1}{A(g(\ell))}\mathcal{L}[A\cdot Q\cdot(A\circ g);\ell,\infty]\leq 1. \tag{20}$$

Taking $\limsup_{\ell\to\infty}$ of (20), we have a contradiction with (14). $\square$

Theorem 2. *Assume that (H) holds. Every solution of Equation (1) is oscillatory if*

$$\limsup_{\ell\to\infty}\Bigg(A^{1-c}(g(\ell))\,\mathcal{L}[Q\cdot(A^c\circ g);\ell_1,g(\ell)]+\frac{1}{A^c(g(\ell))}\mathcal{L}[A\cdot Q\cdot(A^c\circ g);g(\ell),\ell]$$
$$+\frac{1}{A(g(\ell))}\mathcal{L}[A\cdot Q\cdot(A\circ g);\ell,\infty]\Bigg)>1. \tag{21}$$

Proof. Proceeding as in the proof of Theorem 1, we arrive at (19). Using the facts that u/A and u/A^c are increasing and decreasing, respectively, we conclude that

$$\begin{aligned} u(g(s)) &\geq \frac{A^c(g(s))}{A^c(g(\ell))}u(g(\ell)) \text{ for } s\leq g(\ell)\leq \ell,\\ u(g(s)) &\geq \frac{A(g(s))}{A(g(\ell))}u(g(\ell)) \text{ for } \ell\leq s, \end{aligned}$$

which, with (19), gives

$$1 \geq A^{1-c}(g(\ell))\,\mathcal{L}[Q\cdot(A^c\circ g);\ell_1,g(\ell)]+\frac{1}{A^c(g(\ell))}\mathcal{L}[A\cdot Q\cdot(A^c\circ g);g(\ell),\ell]$$
$$+\frac{1}{A(g(\ell))}\mathcal{L}[A\cdot Q\cdot(A\circ g);\ell,\infty]. \tag{22}$$

Taking $\limsup_{\ell\to\infty}$ of (22), we have a contradiction with (21). $\square$

Example 1. *Consider the delay differential equation*

$$(e^{\kappa\ell}(u'(\ell))^{\kappa})'+q_0e^{\kappa\ell}u^{\kappa}(\ell-\delta)=0, \tag{23}$$

where $\kappa>0$, $q_0>0$ *and* $\delta>0$. *Note that,*

$$g(\ell)=\ell-\delta,\ r(\ell)=e^{\kappa\ell},\ \text{and } q(\ell)=q_0e^{\kappa\ell}.$$

Hence, we get $A(\ell)=e^{-\ell}$, *and so* $A(\ell_0)<\infty$ *(the non-canonical case). It is easy to conclude that* $\phi(\ell)=q_0$, $c=(q_0/\kappa)^{1/\kappa}$, *and*

$$Q(\ell):=\begin{cases}\frac{1}{\kappa}q_0e^{\ell} & \text{if } \kappa\leq 1;\\ \left(\frac{q_0}{\kappa}\right)^{1/\kappa}e^{\ell+\delta(\kappa-1)}, & \text{if } \kappa>1.\end{cases}$$

Proceduring some substitutions and computations, condition (14) reduces to

$$\begin{aligned} & e^{\delta(\kappa-1)}\left(\frac{q_0}{\kappa}\right)^{1/\kappa}\limsup_{\ell\to\infty}\left(e^{-\ell+\delta}\int_{\ell_1}^{\ell-\delta}e^{\mathfrak{b}}d\mathfrak{b}+\int_{\ell-\delta}^{\ell}d\mathfrak{b}+\frac{1}{e^{-\ell+\delta}}\int_{\ell}^{\infty}e^{-\mathfrak{b}+\delta}d\mathfrak{b}\right)\\ = {} & e^{\delta(\kappa-1)}\left(\frac{q_0}{\kappa}\right)^{1/\kappa}\limsup_{\ell\to\infty}\left(\delta+e^{-\ell+\delta}\left(e^{\ell-\delta}-e^{\ell_1}\right)+1\right)\\ = {} & e^{\delta(\kappa-1)}\left(\frac{q_0}{\kappa}\right)^{1/\kappa}\limsup_{\ell\to\infty}\left(\delta+2-e^{-\ell+\delta+\ell_1}\right)\\ = {} & e^{\delta(\kappa-1)}(\delta+2)\left(\frac{q_0}{\kappa}\right)^{1/\kappa}\\ > {} & 1,\ \textit{if } \kappa>1, \end{aligned}$$

and

$$
\begin{aligned}
& \frac{1}{\kappa} q_0 \limsup_{\ell\to\infty}\left(e^{-\ell+\delta}\int_{\ell_1}^{\ell-\delta} \mathrm{e}^{\mathfrak{b}} \mathrm{d}\mathfrak{b} + \int_{\ell-\delta}^{\ell} \mathrm{d}\mathfrak{b} + \frac{1}{e^{-\ell+\delta}}\int_{\ell}^{\infty} \mathrm{e}^{-\mathfrak{b}+\delta} \mathrm{d}\mathfrak{b}\right) \\
= {} & \frac{1}{\kappa} q_0 \limsup_{\ell\to\infty}\left(e^{-\ell+\delta}\left(e^{\ell-\delta} - e^{\ell_1}\right) + \delta + 1\right) \\
= {} & \frac{1}{\kappa}(\delta+2) q_0 \\
> {} & 1, \text{ if } \kappa \le 1.
\end{aligned}
$$

By using Theorem 1, we have that (23) is oscillatory if

$$
q_0 > \begin{cases} \frac{\kappa}{(\delta+2)^{\kappa}} \mathrm{e}^{\kappa(1-\kappa)\delta}, & \text{if } \kappa > 1; \\ \frac{\kappa}{(\delta+2)}, & \text{if } \kappa \le 1. \end{cases} \tag{24}
$$

Now, procedurring some substitutions and computations, condition (21) reduces to

$$
\begin{aligned}
& \mathrm{e}^{\delta(\kappa-1)}\left(\frac{q_0}{\kappa}\right)^{1/\kappa} \limsup_{\ell\to\infty}\left[\mathrm{e}^{(-\ell+\delta)(1-c)}\int_{\ell_1}^{\ell-\delta} \mathrm{e}^{\mathfrak{b}} \mathrm{e}^{(-\mathfrak{b}+\delta)c} \mathrm{d}\mathfrak{b}\right. \\
& \qquad \left. + \frac{1}{\mathrm{e}^{(-\ell+\delta)c}}\int_{\ell-\delta}^{\ell} \mathrm{e}^{(-\mathfrak{b}+\delta)c} \mathrm{d}\mathfrak{b} + \frac{1}{\mathrm{e}^{(-\ell+\delta)}}\int_{\ell}^{\infty} \mathrm{e}^{(-\mathfrak{b}+\delta)} \mathrm{d}\mathfrak{b}\right] \\
& = \mathrm{e}^{\delta(\kappa-1)}\left(\frac{q_0}{\kappa}\right)^{1/\kappa} \limsup_{\ell\to\infty}\left(\left[\frac{\mathrm{e}^{c\delta}}{(1-c)} - \frac{\mathrm{e}^{(-\ell+\delta)(1-c)+\ell_1(1-c)+c\delta}}{(1-c)}\right] + \left[\frac{1}{-c} - \frac{\mathrm{e}^{\delta c}}{-c}\right] + 1\right) \\
& = \mathrm{e}^{\delta(\kappa-1)}\left(\frac{q_0}{\kappa}\right)^{1/\kappa}\left(\left(\frac{\mathrm{e}^{c\delta}}{(1-c)}\right) + \left(\frac{1}{-c} - \frac{\mathrm{e}^{\delta c}}{-c}\right) + 1\right) \\
& = \mathrm{e}^{\delta(\kappa-1)}\left(\frac{q_0}{\kappa}\right)^{1/\kappa}\left(\left[\frac{\mathrm{e}^{\delta c}}{(1-c)c}\right] + 1 - \frac{1}{c}\right) \\
& > 1, \text{ if } \kappa > 1.
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{\kappa} q_0 \limsup_{\ell\to\infty}(\mathrm{e}^{(-\ell+\delta)(1-c)}\int_{\ell_1}^{\ell-\delta} \mathrm{e}^{\mathfrak{b}} \mathrm{e}^{(-\mathfrak{b}+\delta)c} \mathrm{d}\mathfrak{b} + \frac{1}{\mathrm{e}^{(-\ell+\delta)c}}\int_{\ell-\delta}^{\ell} \mathrm{e}^{(-\mathfrak{b}+\delta)c} \mathrm{d}\mathfrak{b} \\
+ {} & \frac{1}{\mathrm{e}^{(-\ell+\delta)}}\int_{\ell}^{\infty} \mathrm{e}^{(-\mathfrak{b}+\delta)} \mathrm{d}\mathfrak{b}) \\
= {} & \frac{1}{\kappa} q_0 \limsup_{\ell\to\infty}\left(\left[\frac{\mathrm{e}^{\delta c}}{(1-c)} - \frac{\mathrm{e}^{(-\ell+\delta)(1-c)+\ell_1(1-c)+\delta c}}{(1-c)}\right] + \left[\frac{1}{-c} - \frac{\mathrm{e}^{\delta c}}{-c}\right] + 1\right) \\
= {} & \frac{1}{\kappa} q_0\left(\left[\frac{\mathrm{e}^{\delta c}}{(1-c)}\right] + \left[\frac{1}{-c} - \frac{\mathrm{e}^{\delta c}}{-c}\right] + 1\right) \\
= {} & \frac{1}{\kappa} q_0\left(\left[\frac{\mathrm{e}^{\delta c}}{(1-c)c}\right] + 1 - \frac{1}{c}\right) \\
> {} & 1, \text{ if } \kappa \le 1.
\end{aligned}
$$

By using Theorem 2, we have that (23) is oscillatory if

$$
q_0 > \begin{cases} \frac{\kappa}{\left(\left[\frac{\mathrm{e}^{\delta c}}{(1-c)c}\right] + 1 - \frac{1}{c}\right)^{\kappa}} \mathrm{e}^{\kappa\delta(1-\kappa)}, & \text{if } \kappa > 1; \\ \frac{\kappa}{\left(\left[\frac{\mathrm{e}^{\delta c}}{(1-c)c}\right] + 1 - \frac{1}{c}\right)}, & \text{if } \kappa \le 1. \end{cases} \tag{25}
$$

In the case where $\kappa = 1$ and $\delta = 0.5$, conditions (24) and (25) reduce to $q_0 > 0.4$ and $q_0 > 0.3095$, respectively.

Example 2. *Consider the equation of Euler type (6) where $\ell \geq 1$, $q_0 > 0$, and $\mu \in (0,1]$. Note that,*

$$r(\ell) = \ell^{\kappa+1},\ g(\ell) = \mu\ell,\ \textit{and}\ q(\ell) = q_0.$$

Hence, we get $A(\ell) = \kappa/\ell^{1/\kappa}$, and so $A(\ell_0) < \infty$ (the non-canonical case). It is easy to conclude that $\phi(\ell) = \kappa^{\kappa+1}q_0$, $c = \kappa q_0^{1/\kappa}$, and

$$Q(\ell) := \begin{cases} q_0 \frac{\kappa^{\kappa-2}}{\ell^{1-1/\kappa}}, & \textit{if } \kappa \leq 1; \\ \frac{1}{\kappa} q_0^{1/\kappa} \frac{1}{(\mu\ell)^{1-1/\kappa}}, & \textit{if } \kappa > 1. \end{cases}$$

Proceduring some substitutions and computations, condition (14) reduces to

$$\begin{aligned} & \frac{1}{\mu} q_0^{1/\kappa} \limsup_{\ell\to\infty} \left(\frac{\kappa}{\ell^{1/\kappa}} \int_{\ell_1}^{\mu\ell} \frac{1}{\mathfrak{b}^{1-1/\kappa}} \mathrm{d}\mathfrak{b} + \mu^{1/\kappa} \int_{\mu\ell}^{\ell} \frac{1}{\mathfrak{b}} \mathrm{d}\mathfrak{b} + \mu^{1/\kappa}\ell^{1/\kappa} \int_{\ell}^{\infty} \frac{1}{\mathfrak{b}^{1+1/\kappa}} \mathrm{d}\mathfrak{b} \right) \\ = & \frac{1}{\mu} q_0^{1/\kappa} \limsup_{\ell\to\infty} \left(\frac{\kappa}{\ell^{1/\kappa}} \left(\mu^{1/\kappa}\ell^{1/\kappa} - \ell_1^{1/\kappa} \right) + \mu^{1/\kappa} \ln \frac{1}{\mu} + \kappa\mu^{1/\kappa} \right) \\ = & \mu^{1/\kappa-1} q_0^{1/\kappa} \left(2\kappa + \ln \frac{1}{\mu} \right) \\ > & 1, \ \textit{if } \kappa > 1, \end{aligned}$$

and

$$\begin{aligned} & \kappa^{\kappa-1} q_0 \limsup_{\ell\to\infty} \left(\frac{1}{\mu^{1/\kappa}\ell^{1/\kappa}} \int_{\ell_1}^{\mu\ell} \frac{1}{\mathfrak{b}^{1-1/\kappa}} \mathrm{d}\mathfrak{b} + \int_{\mu\ell}^{\ell} \frac{1}{\mathfrak{b}} \mathrm{d}\mathfrak{b} + \ell^{1/\kappa} \int_{\ell}^{\infty} \frac{1}{\mathfrak{b}^{1+1/\kappa}} \mathrm{d}\mathfrak{b} \right) \\ = & \kappa^{\kappa-1} q_0 \limsup_{\ell\to\infty} \left(\frac{\kappa}{\mu^{1/\kappa}\ell^{1/\kappa}} \left(\mu^{1/\kappa}\ell^{1/\kappa} - \ell_1^{1/\kappa} \right) + \ln \frac{1}{\mu} + \kappa \right) \\ = & \kappa^{\kappa-1} q_0 \left(2\kappa + \ln \frac{1}{\mu} \right) \\ > & 1, \ \textit{if } \kappa \leq 1. \end{aligned}$$

By using Theorem 1, we have that (23) is oscillatory if

$$q_0 > \begin{cases} \frac{\mu^{\kappa-1}}{\left(2\kappa + \ln \frac{1}{\mu}\right)^{\kappa}}, & \textit{if } \kappa > 1; \\ \frac{\kappa^{1-\kappa}}{\left(2\kappa + \ln \frac{1}{\mu}\right)}, & \textit{if } \kappa \leq 1. \end{cases}$$

4. Conclusions

It is easy to note the great development in the study of oscillatory behavior and asymptotic properties of solutions of differential equations. This development and interest is due not only to the importance of such studies in many applications in different sciences, but also to the theoretical and analytical importance. In this article, we introduce new oscillation criteria that guarantee the oscillation of all solutions of a class of second-order half-linear delay differential equations. The focus of the study was on the non-canonical case. We obtained new monotonic properties and then used these properties to obtain improved oscillation criteria. It would be interesting to extend the results of this article to the neutral case.

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