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# Global Well-Posedness of the Dissipative Quasi-Geostrophic Equation with Dispersive Forcing

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**Abstract:** The dissipative quasi-geostrophic equation with dispersive forcing is considered. By striking new balances between the dispersive effects of the dispersive forcing and the smoothing effects of the viscous dissipation, we obtain the global well-posedness for Cauchy problem of the dissipative quasi-geostrophic equation with dispersive forcing for arbitrary initial data, provided that the dispersive parameter is large enough.

**Keywords:** dissipative quasi-geostrophic equation; dispersive forcing; global well-posedness

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## 1. Introduction

The dissipative quasi-geostrophic equation with dispersive forcing has been proposed as a simple model describing the evolution of a surface buoyancy involved with investigating wave-turbulence interactions, see [1–3]. In this paper, we investigate global well-posedness to Cauchy problem of the dissipative quasi-geostrophic equation with dispersive forcing:

$$\begin{cases} \partial_t\theta + (u\cdot\nabla)\theta + \kappa\Lambda^\alpha\theta + \Omega u_2 = 0, & \text{in } \mathbb{R}^2\times(0,\infty), \\ u = \mathcal{R}^\perp\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), & \text{in } \mathbb{R}^2\times(0,\infty), \\ \theta|_{t=0} = \theta_0, & \text{in } \mathbb{R}^2, \end{cases} \tag{1}$$

where the unknown real-value scalar function $\theta$ represents the potential temperature, and can also be interpreted as a buoyancy field; $u = (u_1, u_2)$ denotes velocity field, which is determined by $\theta$; $\alpha \in (0,2]$, $\kappa > 0$ is a dissipation coefficient, $\Omega \in \mathbb{R}\setminus\{0\}$ is an amplitude parameter; $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the Zygmund operator, and $\mathcal{R}_i = -\frac{\partial_i}{\Lambda}(i = 1,2)$ is the $i$-th Riesz transform. For the sake of simplicity, we will set $\kappa = 1$ throughout the paper.

When $\Omega = 0$, the system (1) is the classical quasi-geostrophic equation, which is a special case of the general quasi-geostrophic approximation for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers. We may refer to [4] for more details about its background in geophysics. According to the scaling transform and the $L^\infty$-maximum principle of [5], the cases $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$ are called subcritical, critical and supercritical, respectively. However, in the geophysical fluid dynamics, the perturbations of buoyancy generally give rise to the dispersive waves (see [6]), and it is quite common to encounter problems, which involve the presence of both advection and dispersion (see for example, Chapter 5 in the text by Majda [7] and Chapter 5 in the text by Chemin et al. [8]). Therefore, it is justified to consider the dissipative quasi-geostrophic equation with an environmental horizontal buoyancy gradient term $\Omega u_2$.

When $\Omega = 0$, the study of global regularity or global well-posedness issue of problem (1) for the subcritical and critical cases ($\alpha \geq 1$) is in a satisfactory state; see [9,10] for the subcritical case and [11,12] for the critical case. Whereas, it is still a challenging open problem for the supercritical case ($\alpha < 1$), and we may refer the interested reader to [13–17] and the references therein for the conditions of global well-posednes and the conditional global regularity.

When $\Omega \neq 0$, the global well-posedness or global regularity results are quite few. By applying the modulus of continuity method developed in [12], Kiselev and Nazarnov [2] proved the global regularity of problem (1) with $\alpha = 1$ for the arbitrary smooth periodic initial data. We would like to point out that it is too hard to generalize this result to the whole space case, since the dispersive term $\Omega u_2$ plays a negative role. Relying on the Strichartz-type estimates for the corresponding linear dispersive problem, Cannone, Miao and Xue [18] established the global regulatity of problem (1) with the initial data in $H^{2-\alpha}(\mathbb{R}^2)$ for $\alpha < 1$, provided that parameter $\Omega$ is large enough. Based on the more detailed analysis, Wan and Chen [19] obtained the global existence of the smooth solution of problem (1) with $\alpha < 1$ under large dispersive parameter and small dissipation coefficient. Recently, Angulo-Castillo, Ferreira and Kosloff [20] obtained the long-time solvability for the 2D inviscid dispersive quasi-geostrophic equation with improved regularity. Fujii [21] studied the long-time existence and asymptotic behavior of solutions for the 2D inviscid quasi-geostrophic equation with large dispersive forcing.

Inspired by the above literatures, in this paper, by applying the sharp dispersive estimates established in [22] for the linear operator group $\{e^{\mathcal{R}_1 t}\}_{t\in\mathbb{R}}$ related to dispersive term $\Omega u_2$, and by striking some new balances between the dispersive effects of term $\Omega u_2$ and the smoothing effects of term $\Lambda^\alpha\theta$, we obtain the following global well-posedness for problem (1) with subcritical dissipation.

## 2. Main Results

**Theorem 1.** *Let $\alpha \in (1,2]$, $r \in [1,\infty]$ and $\max\{\frac{3}{2}, \frac{4}{1+\alpha}\} < p < 2$. Let $s \in \mathbb{R}$ satisfy*

$$1 + \frac{2}{p} - \alpha < s < 1 + \frac{2}{p} - 2\alpha(1 - \frac{1}{p}) \quad \textit{and} \quad \frac{4}{p} - 2 < s < 2 - \frac{2}{p},$$

*and $\delta \in [2,\infty)$ satisfy*

$$0 < \frac{1}{\delta} < \frac{1}{p} - \frac{1}{2} \quad \textit{and} \quad \frac{1}{\alpha}(s - 1 - \frac{2}{p} + \alpha) + \frac{1}{2} - \frac{1}{p} < \frac{1}{\delta} < \frac{1}{2\alpha}(s - 1 - \frac{2}{p} + \alpha).$$

*Then there exists a positive constant C such that for $\Omega \in \mathbb{R} \setminus \{0\}$ and $\theta_0 \in \dot{B}^s_{p,r}(\mathbb{R}^2)$ satisfying*

$$\|\theta_0\|_{\dot{B}^s_{p,r}} \leq C|\Omega|^{\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)}, \tag{2}$$

*the problem (1) possesses a unique mild solution*

$$\theta \in C([0,\infty); \dot{B}^s_{p,r}(\mathbb{R}^2)) \bigcap \tilde{L}^\delta(0,\infty; \dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r}(\mathbb{R}^2)),$$

*where $\frac{1}{p'} + \frac{1}{p} = 1$.*

By applying the $TT^*$ argument, we further obtain the global well-posedness result of problem (1) for the endpoint case $p = 2$, as follows.

**Theorem 2.** *Let $\alpha \in (1,2]$, $r \in [1,\infty]$ and $s \in (2-\alpha, \frac{16-5\alpha}{8+3\alpha})$. Let $q \in (2,3]$ satisfy*

$$\max\{\frac{1}{4}(s+1), \frac{1}{2}(1-s)\} < \frac{1}{q} < \frac{1}{2} - \frac{2}{3\alpha}(s - 2 + \alpha),$$

*and $\delta \in [2,\infty)$ satisfy*

$$0 < \frac{1}{\delta} < \frac{1}{2}(\frac{1}{2} - \frac{1}{q}) \quad \textit{and} \quad \frac{1}{\alpha}(s - 2 + \alpha) + \frac{1}{q} - \frac{1}{2} < \frac{1}{\delta} < \frac{1}{2\alpha}(s - 2 + \alpha).$$

*Then there exists a positive constant C such that for* $\Omega \in \backslash\{0\}$ *and* $\theta_0 \in \dot{B}^s_{2,r}(\mathbb{R}^2)$ *satisfying*

$$\|\theta_0\|_{\dot{B}^s_{2,r}} \leq C|\Omega|^{\frac{1}{\alpha}(s-2+\alpha)}, \tag{3}$$

*the problem* (1) *possesses a unique mild solution*

$$\theta \in C\big([0,\infty);\dot{B}^s_{2,r}(\mathbb{R}^2)\big) \bigcap \tilde{L}^\delta\big(0,\infty;\dot{B}^{s+\frac{2}{q}-1}_{q,r}(\mathbb{R}^2)\big).$$

**Remark 1.** *In [18], Cannone, Miao and Xue verified the existence of a positive constant* $\Omega_0$ *for each given initial data, and proved the global regularity of problem* (1) *if* $|\Omega| > \Omega_0$*. It is worth pointing out that by fully exploiting the dispersive effects originated from the term* $\Omega u_2$*, Theorems 1 and 2 give the specific characterizations of the relationship between initial data* $\theta_0$ *and parameter* $\Omega$*, i.e.,* (2) *and* (3)*, for ensuring the global well-posedness of problem* (1)*.*

**Remark 2.** *Note that system* (1) *is invariant under the scaling*

$$\theta_\lambda(t,x) = \lambda^{\alpha-1}\theta(\lambda^\alpha t, \lambda x), \qquad \Omega_\lambda = \lambda^\alpha \Omega \tag{4}$$

*for* $\lambda > 0$*. It is easy to check that the size conditions* (2) *and* (3) *are invariant under the scaling* (4)*.*

The rest of this paper is organized as follows. In Section 3, we collect some basic facts on the Littlewood-Paley theory, Besov spaces and some basic estimates. In Section 4, we derive some linear estimates related to the linear problem. In Section 5, we present the proofs of Theorems 1 and 2.

Throughout the paper, $C$ stands for a harmless constant. In particular, $C = C(\cdot, \cdots, \cdot)$ means that this constant depends only on the quantities appearing in the parentheses. For any scaler function space $X$, we shall use the same notation $X$ to denote its 2-vector counterpart to simplify the notation. Both $\mathcal{F}g$ and $\hat{g}$ stand for Fourier transform of $g$ with respect to space variables, while $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. In some places of the paper, we may use $L^p$ and $\dot{B}^s_{p,r}$ to stand for $L^p(\mathbb{R}^2)$ and $\dot{B}^s_{p,r}(\mathbb{R}^2)$, respectively.

## 3. Preliminaries

Let $\mathscr{S}(\mathbb{R}^2)$ be the Schwartz space of smooth functions over $\mathbb{R}^2$, and let $\mathscr{S}'(\mathbb{R}^2)$ be the space of tempered distributions. First, we recall the homogeneous Littlewood-Paley decomposition. Let $\varphi, \psi \in \mathscr{S}(\mathbb{R}^2)$ be two radial functions such that their Fourier transforms $\hat{\varphi}$ and $\hat{\psi}$ satisfy the following properties:

$$\text{supp } \hat{\varphi} \subset \mathcal{B} := \{\xi \in \mathbb{R}^2 : |\xi| \leq \frac{4}{3}\},$$

$$\text{supp } \hat{\psi} \subset \mathcal{C} := \{\xi \in \mathbb{R}^2 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$$

and

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^2 \setminus \{0\}.$$

Let $\psi_j(x) := 2^{2j}\psi(2^j x)$ for $j \in \mathbb{Z}$. We define by $\Delta_j$, the following frequency localization operator in $\mathscr{S}'(\mathbb{R}^2)$:

$$\Delta_j f := \psi_j * f = 2^{2j}\int_{\mathbb{R}^2} \psi(2^j(x-y))f(y)dy \qquad \text{for } j \in \mathbb{Z} \text{ and } f \in \mathscr{S}'(\mathbb{R}^2). \tag{5}$$

Define $\mathscr{S}'_h(\mathbb{R}^2) := \mathscr{S}'(\mathbb{R}^2)/\mathcal{P}[\mathbb{R}^2]$, where $\mathcal{P}[\mathbb{R}^2]$ denotes the linear space of polynomials on $\mathbb{R}^2$. It is known that the operator $\Delta_j$ maps $L^p$ into $L^p$ with norm independent of $j$ and $p$, see [23].

Now, we introduce the definitions of the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^2)$ and the Chemin-Lerner type space $\tilde{L}^\delta(0,\infty;\dot{B}^s_{p,r}(\mathbb{R}^2))$, and recall some basic facts on the Littlewood-Paley theory and Besov spaces.

**Definition 1** ([23]). *Let $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, and let $u \in \mathscr{S}'_h(\mathbb{R}^2)$, we set (with the usual convection if $r = \infty$)*

$$\|u\|_{\dot{B}^s_{p,r}} := \Big(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\Delta_j u\|^r_{L^p}\Big)^{\frac{1}{r}}.$$

• *For $s < \frac{2}{p}$ (or $s = \frac{2}{p}$, if $r = 1$), we define*

$$\dot{B}^s_{p,r}(\mathbb{R}^2) := \{u \in \mathscr{S}'_h(\mathbb{R}^2) | \|u\|_{\dot{B}^s_{p,r}} < \infty\};$$

• *If $k \in \mathbb{N}$, $\frac{2}{p} + k \le s < \frac{2}{p} + k + 1$ (or $s = \frac{2}{p} + k + 1$, if $r = 1$), then $\dot{B}^s_{p,r}(\mathbb{R}^2)$ is defined as the subset of distributions $u \in \mathscr{S}'_h(\mathbb{R}^2)$ such that $\partial^\delta u \in \dot{B}^{s-k}_{p,r}$ whenever $|\delta| = k$.*

**Definition 2** ([23]). *For $s \in \mathbb{R}$ and $1 \le r, \delta \le \infty$, we set (with the usual convection of $r = \infty$)*

$$\|u\|_{\tilde{L}^\delta(0,\infty;\dot{B}^s_{p,r})} := \Big(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\Delta_j u\|^r_{L^\delta(0,\infty;L^p)}\Big)^{\frac{1}{r}}.$$

*We then define the space $\tilde{L}^\delta(0,\infty;\dot{B}^s_{p,r}(\mathbb{R}^2))$ as the set of temperate distributions $u$ over $(0,\infty)\times\mathbb{R}^2$ such that $\lim_{j\to-\infty} S_j u = 0$ in $\mathscr{S}'(0,\infty\times\mathbb{R}^2)$ and $\|u\|_{\tilde{L}^\delta(0,\infty;\dot{B}^s_{p,r})} < \infty$.*

**Lemma 1** (Bernstein, [23]). *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring centred at the origin of $\mathbb{R}^2$. A constant $C$ exists such that for any positive real number $\lambda$, any non-negative integer $k$, and any couple of real numbers $(p,q)$ with $q \ge p \ge 1$, there hold*

• *$Supp\hat{u} \subset \lambda\mathcal{B} \Longrightarrow \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^q} \le C^{k+1}\lambda^{k+2(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}$;*

• *$Supp\hat{u} \subset \lambda\mathcal{C} \Longrightarrow C^{-(k+1)}\lambda^k\|u\|_{L^p} \le \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^p} \le C^{k+1}\lambda^k\|u\|_{L^p}$,*

*where $Supp\hat{u} = \{\xi | \hat{u}(\xi) \neq 0\}$.*

**Lemma 2** (Product laws, [24]). *Let $(p_0, p_1, p_2, r, \lambda_1, \lambda_2) \in [1,\infty]^6$ such that $\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \le \lambda_2$, $p_2 \le \lambda_1$, $\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{\lambda_1} \le 1$ and $\frac{1}{p_0} \le \frac{1}{p_2} + \frac{1}{\lambda_2} \le 1$. If $s_1 + s_2 + 2\inf\{0, 1 - \frac{1}{p_1} - \frac{1}{p_2}\} > 0$, $s_1 + \frac{2}{\lambda_2} < \frac{2}{p_1}$ and $s_2 + \frac{2}{\lambda_1} < \frac{2}{p_2}$, then*

$$\|uv\|_{\dot{B}^{s_1+s_2-2(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0})}_{p_0,r}} \le C\|u\|_{\dot{B}^{s_1}_{p_1,r}}\|v\|_{\dot{B}^{s_2}_{p_2,\infty}}.$$

**Remark 3.** *It is easy to generalize Lemma 2 to the spaces $\tilde{L}^\delta(0,\infty;\dot{B}^s_{p,r}(\mathbb{R}^2))$. The general principle is that the indices $s, p, r$ behave just as in the stationary case whereas the time exponent $\delta$ behaves according to Hölder's inequality, one may check [23] for more details.*

To study the problem (1), we shall consider the following equivalent integral equation

$$\theta(t) = e^{\Omega\mathcal{R}_1 t}e^{\Lambda^\alpha t}\theta_0 - \int_0^t e^{\Omega\mathcal{R}_1(t-\tau)}e^{\Lambda^\alpha(t-\tau)}(\mathcal{R}^\perp\theta(\tau)\cdot\nabla\theta(\tau))d\tau, \quad \text{for } t > 0. \tag{6}$$

The following smoothing effects of semigroup $\{e^{\Lambda^\alpha t}\}_{t>0}$ and dispersive effects of semigroup $\{e^{\Omega\mathcal{R}_1 t}\}_{t>0}$ are the keys to obtain our global well-posedness results.

**Lemma 3** ([25])**.** *Let $\alpha > 0$, $-\infty < s_1 \le s_2 < \infty$, $1 \le p_1 \le p_2 \le \infty$ and $1 \le r \le \infty$. There exists a positive constant $C = C(s_1, s_2, p)$, such that*

$$\|\Delta_j e^{-\Lambda^\alpha t} f\|_{L^{p_2}} \le C 2^{-(s_2-s_1)j} t^{-\frac{1}{\alpha}(s_2-s_1)-\frac{2}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})} \|\Delta_j f\|_{L^{p_1}},$$

*for all $t > 0$, $j \in \mathbb{Z}$ and $f \in \mathscr{S}'(\mathbb{R}^2)$. Moreover,*

$$\|e^{-\Lambda^\alpha t} f\|_{\dot{B}^{s_2}_{p_2,r}} \le C t^{-\frac{1}{\alpha}(s_2-s_1)-\frac{2}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})} \|f\|_{\dot{B}^{s_1}_{p_1,r}}$$

*for all $t > 0$ and $f \in \dot{B}^{s_1}_{p_1,r}$.*

**Lemma 4** ([22])**.** *Let $1 \le p \le 2$. There exists a constant $C = C(p) > 0$ such that*

$$\|e^{\mathcal{R}_1 t} \Delta_j f\|_{L^{p'}} \le C(1+|t|)^{-(\frac{1}{p}-\frac{1}{2})} 2^{2j(\frac{2}{p}-1)} \|\Delta_j f\|_{L^p}$$

*for all $t \in \mathbb{R}$, $j \in \mathbb{Z}$ and $f \in \mathscr{S}'(\mathbb{R}^2)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\Delta_j$ is frequency localization operator defined in* (5)*. Moreover,*

$$\|e^{\mathcal{R}_1 t} f\|_{\dot{B}^s_{p',r}} \le C(1+|t|)^{-(\frac{1}{p}-\frac{1}{2})} \|f\|_{\dot{B}^{s+2(\frac{2}{p}-1)}_{p,r}}$$

*for all $t \in \mathbb{R}$ and $f \in \dot{B}^{s+2(\frac{2}{p}-1)}_{p,r}(\mathbb{R}^2)$ with $s \in \mathbb{R}$ and $1 \le r \le \infty$, where $\frac{1}{p'} + \frac{1}{p} = 1$.*

Moreover, a direct application of Mihlin's theorem implies the following $L^p - L^p$ type estimate for the semigroup $\{e^{\pm\mathcal{R}_1 t}\}_{t>0}$.

**Lemma 5.** *For $1 < p < \infty$, there exists a positive constant $C = C(p)$ such that*

$$\|e^{\mathcal{R}_1 t} f\|_{L^p} \le C(1+|t|)^2 \|f\|_{L^p}$$

*for all $t \in \mathbb{R}$ and $f \in L^p$. Moreover,*

$$\|e^{\mathcal{R}_1 t} f\|_{\dot{B}^s_{p,r}} \le C(1+|t|)^2 \|f\|_{\dot{B}^s_{p,r}}$$

*for all $t \in \mathbb{R}$ and $f \in \dot{B}^s_{p,r}$ with $s \in \mathbb{R}$, $p \in (1, \infty)$ and $r \in [1, \infty]$.*

## 4. Linear Estimates

We firstly establish the linear estimates for the semigroup $\{e^{\Omega\mathcal{R}_1 t} e^{\Lambda^\alpha t}\}_{t\ge 0}$.

**Lemma 6.** *For $\alpha \in (0, 2]$, $s \in \mathbb{R}$, $p \in [1, 2)$ and $r \in [1, \infty]$, let $\delta \in [1, \infty]$ satisfy*

$$0 < \frac{1}{\delta} < \frac{1}{p} - \frac{1}{2}.$$

*There exists a positive constant $C = C(p, \delta)$ such that*

$$\|e^{\Omega\mathcal{R}_1 t} e^{-\Lambda^\alpha t} f\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \le C|\Omega|^{-\frac{1}{\delta}} \|f\|_{\dot{B}^s_{p,r}} \tag{7}$$

*for $\Omega \in \mathbb{R}\backslash\{0\}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

**Proof.** By Definition 2, we have

$$\|e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}f\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} = \Big(\sum_{j\in\mathbb{Z}} 2^{(s+\frac{2}{p'}-\frac{2}{p})jr}\|\Delta_j e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}f\|^r_{L^\delta(0,\infty;L^{p'})}\Big)^{\frac{1}{r}}.$$

Therefore, it suffices to show that

$$\|\Delta_j e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}f\|_{L^\delta(0,\infty;L^{p'})} \le C|\Omega|^{-\frac{1}{\delta}}2^{j(\frac{2}{p}-\frac{2}{p'})}\|\Delta_j f\|_{L^p}$$

for all $t>0$ and $\Omega\in\mathbb{R}\backslash\{0\}$.

In fact, by Lemmas 3 and 4, we see

$$\|e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}\Delta_j f\|_{L^{p'}} \le C(1+|\Omega|t)^{-(\frac{1}{2}-\frac{1}{p'})}2^{j(\frac{2}{p}-\frac{2}{p'})}\|\Delta_j f\|_{L^p}.$$

Due to $0<\frac{1}{\delta}<\frac{1}{p}-\frac{1}{2}$, there exists a positive constant $C=C(p,\delta)$ such that

$$\Big(\int_0^\infty (1+|\Omega|t)^{-(\frac{1}{2}-\frac{1}{p'})\delta}dt\Big)^{\frac{1}{\delta}} \le C|\Omega|^{-\frac{1}{\delta}} \quad \text{for } \Omega\in\mathbb{R}\backslash\{0\}$$

and

$$\|\|\Delta_j e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}f\|_{L^{p'}}\|_{L^\delta(0,\infty)} \le C|\Omega|^{-\frac{1}{\delta}}2^{j(\frac{2}{p}-\frac{2}{p'})}\|\Delta_j f\|_{L^p}$$

for $\Omega\in\mathbb{R}\backslash\{0\}$. This completes the proof. □

The following lemma is concerned with the endpoint case $p=2$, in which we may use the $TT^\star$ argument.

**Lemma 7.** *For $\alpha\in(0,2], s\in\mathbb{R}, q\in(2,\infty)$ and $r\in[1,\infty]$, let $\delta\in[2,\infty)$ satisfy*

$$0<\frac{1}{\delta}<\frac{1}{2}(\frac{1}{2}-\frac{1}{q}).$$

*There exists a positive constant $C=C(s,q,\delta)$ such that*

$$\|e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}f\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{q}-1}_{q,r})} \le C|\Omega|^{-\frac{1}{\delta}}\|f\|_{\dot{B}^s_{2,r}}$$

*for $\Omega\in\mathbb{R}\backslash\{0\}$.*

**Proof.** By Definition 2, we observe

$$\|e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}f\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{q}-1}_{q,r})} = \Big(\sum_{j\in\mathbb{Z}} 2^{(s+\frac{2}{q}-\frac{2}{2})jr}\|\Delta_j e^{\Omega\mathcal{R}_1 t}e^{-\kappa\Lambda^\alpha t}f\|^r_{L^\delta(0,\infty;L^q)}\Big)^{\frac{1}{r}}.$$

We claim for $q\in(2,\infty)$ and $0<\frac{1}{\delta}<\frac{1}{2}(\frac{1}{2}-\frac{1}{q})$ that

$$\|\Delta_j e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}f\|_{L^\delta(0,\infty;L^q)} \le C2^{(1-\frac{2}{q})j}|\Omega|^{-\frac{1}{\delta}}\|\Delta_j f\|_{L^2}, \quad j\in\mathbb{Z}. \tag{8}$$

The proof of (8) is based on the usual $TT^\star$ argument, which goes back to Tomas [26] (see also Strichartz [27]). Indeed, by duality, it suffices to prove that

$$\Big|\int_0^\infty\int_{\mathbb{R}^2} e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}\Delta_j f(x)\overline{\varphi(t,x)}dxdt\Big| \le C2^{j(1-\frac{2}{q})}|\Omega|^{-\frac{1}{\delta}}\|\Delta_j f\|_{L^2}\|\varphi\|_{L^{\delta'}(0,\infty;L^{q'})}$$

for $\varphi\in C_0^\infty((0,\infty)\times\mathbb{R}^2)$, where $\frac{1}{q'}+\frac{1}{q}=1$ and $\frac{1}{\delta}+\frac{1}{\delta'}=1$.

Here, we introduce a new Littlewood-Paley operator $\tilde{\Delta}_j$, defined by

$$\tilde{\Delta}_j f := (\psi_{j-1} + \psi_j + \psi_{j+1}) * f, \quad j \in \mathbb{Z}, \ \forall f \in \mathscr{S}'(\mathbb{R}^2).$$

It is easy to check that $\tilde{\Delta}_j \Delta_j f = \Delta_j f$ for all $j \in \mathbb{Z}$ and $f \in \mathscr{S}'(\mathbb{R}^2)$, and $\tilde{\Delta}_j$ is also bounded in $L^p$.

By Hölder's inequality, we deduce

$$\begin{aligned} \Big| \int_0^\infty \int_{\mathbb{R}^2} e^{\Omega \mathcal{R}_1 t} e^{-\Lambda^\alpha t} \Delta_j f(x) \overline{\varphi(t,x)} dx dt \Big| &= \Big| \int_0^\infty \int_{\mathbb{R}^2} \Delta_j f(x) \overline{e^{-\Omega \mathcal{R}_1 t} e^{-\Lambda^\alpha t} \tilde{\Delta}_j \varphi(t,x)} dx dt \Big| \\ &\leq \|\Delta_j f\|_{L^2} \Big\| \int_0^\infty e^{-\Omega \mathcal{R}_1 t} e^{-\Lambda^\alpha t} \tilde{\Delta}_j \varphi(t) dt \Big\|_{L^2}. \end{aligned} \tag{9}$$

Moreover, it follows from Parseval's formula and Hölder's inequality that

$$\begin{aligned} &\Big\| \int_0^\infty e^{-\Omega \mathcal{R}_1 t} e^{-\Lambda^\alpha t} \tilde{\Delta}_j \varphi(t) dt \Big\|_{L^2}^2 \\ &= \int_{\mathbb{R}^2} \int_0^\infty \int_0^\infty e^{-\Omega \mathcal{R}_1 t} e^{-\Lambda^\alpha t} \tilde{\Delta}_j \varphi(t,x) \overline{e^{-\Omega \mathcal{R}_1 \tau} e^{-\Lambda^\alpha \tau} \tilde{\Delta}_j \varphi(\tau,x)} dt d\tau dx \\ &\leq \int_0^\infty \int_0^\infty \|\varphi(t)\|_{L^{q'}} \Big\| e^{\Omega \mathcal{R}_1 (\tau - t)} e^{-\Lambda^\alpha (t+\tau)} \tilde{\Delta}_j \varphi(\tau) \Big\|_{L^q} dt d\tau. \end{aligned} \tag{10}$$

Applying Lemma 3 and Lemma 4 yields

$$\Big\| e^{\Omega \mathcal{R}_1 (\tau - t)} e^{-\Lambda^\alpha (t+\tau)} \tilde{\Delta}_j \varphi(\tau) \Big\|_{L^q} \leq C (1 + |\Omega||t-\tau|)^{-(\frac{1}{2} - \frac{1}{q})} 2^{2j(1-\frac{2}{q})} \|\tilde{\Delta}_j \varphi(\tau)\|_{L^{q'}}. \tag{11}$$

Substituting (11) into (10), together with Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned} \Big\| \int_0^\infty e^{-\Omega \mathcal{R}_1 t} e^{-\Lambda^\alpha t} \tilde{\Delta}_j \varphi(t) dt \Big\|_{L^2}^2 &\leq C 2^{2j(1-\frac{2}{q})} \|\varphi\|_{L^{\delta'}(0,\infty;L^{q'})} \Big\| \int_0^\infty K_\Omega(t-\tau) \|\varphi(\tau)\|_{L^{q'}} d\tau \Big\|_{L^\delta(0,\infty)} \\ &\leq C 2^{2j(1-\frac{2}{q})} \|\varphi\|_{L^{\delta'}(0,\infty;L^{q'})}^2 \|K_\Omega\|_{L^{\frac{\delta}{2}}(0,\infty)}, \end{aligned} \tag{12}$$

where $K_\Omega(t) := (1 + |\Omega||t|)^{-(\frac{1}{2} - \frac{1}{q})}$. Due to $0 < \frac{1}{\delta} < \frac{1}{2}(\frac{1}{2} - \frac{1}{q})$, there exists a positive constant $C = C(\delta, q)$ such that

$$\|K_\Omega\|_{L^{\frac{\delta}{2}}(0,\infty)} = |\Omega|^{-\frac{2}{\delta}} \left( \int_0^\infty (1+\tau)^{-\frac{\delta}{2}(\frac{1}{2} - \frac{1}{q})} d\tau \right)^{\frac{2}{\delta}} \leq C |\Omega|^{-\frac{2}{\delta}},$$

which implies

$$\Big\| \int_0^\infty e^{-\Omega \mathcal{R}_1 t} e^{-\Lambda^\alpha t} \tilde{\Delta}_j \varphi(t) dt \Big\|_{L^2}^2 \leq C 2^{2j(1-\frac{2}{q})} |\Omega|^{-\frac{2}{\delta}} \|\varphi\|_{L^{\delta'}(0,\infty;L^{q'})}^2. \tag{13}$$

Substituting (13) into (9) yields the desired estimate. This completes the proof. □

The following lemmas will be used to deal with Duhamel's term in the equivalent integral Equation (6).

**Lemma 8.** *For $\alpha \in (0,2]$, $p \in (1,2)$ and $r \in [1,\infty]$, let $s \in \mathbb{R}$ satisfy*

$$1 + \frac{2}{p} - \alpha < s < 1 + \frac{2}{p} - \alpha(1 - \frac{1}{p}),$$

*and $\delta \in [2,\infty)$ satisfy*

$$\frac{1}{\alpha}(s - 1 - \frac{2}{p} + \alpha) + \frac{1}{2} - \frac{1}{p} < \frac{1}{\delta} < \frac{1}{\alpha}(s - 1 - \frac{2}{p} + \alpha).$$

*There exists a positive constant $C = C(s, p, \delta)$ such that*

$$\Big\| \int_0^t e^{\Omega \mathcal{R}_1(t-\tau)} e^{-\Lambda^\alpha (t-\tau)} \nabla f(\tau) d\tau \Big\|_{\tilde{L}^\delta(0,\infty; \dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \leq C|\Omega|^{-[\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)-\frac{1}{\delta}]} \|f\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;\dot{B}^{2s-\frac{2}{p}}_{p,r})}$$

*for $\Omega \in \mathbb{R}\backslash\{0\}$, where $\frac{1}{p'} + \frac{1}{p} = 1$.*

**Proof.** By Definition 2, one sees

$$\begin{aligned}
&\Big\| \int_0^t e^{\Omega \mathcal{R}_1(t-\tau)} e^{-\Lambda^\alpha (t-\tau)} \nabla f(\tau) d\tau \Big\|_{\tilde{L}^\delta(0,\infty; \dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \\
&= \Big\| \Big\{ 2^{j(s+\frac{2}{p'}-\frac{2}{p})} \Big\| \Delta_j \int_0^t e^{\Omega \mathcal{R}_1(t-\tau)} e^{-\Lambda^\alpha (t-\tau)} \nabla f(\tau) d\tau \Big\|_{L^\delta(0,\infty;L^q)} \Big\}_{j\in\mathbb{Z}} \Big\|_{\ell^r(\mathbb{Z})}.
\end{aligned}$$

It thus suffices to show that

$$\Big\| \int_0^t \| e^{\Omega \mathcal{R}_1(t-\tau)} e^{-\Lambda^\alpha (t-\tau)} \nabla \Delta_j f(\tau) \|_{L^{p'}} d\tau \Big\|_{L^\delta(0,\infty)} \leq C|\Omega|^{-[\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)-\frac{1}{\delta}]} 2^{j(s-\frac{2}{p'})} \|\Delta_j f\|_{L^{\frac{\delta}{2}}(0,\infty;L^p)}.$$

In fact, by Lemma 1, 3 and 4, as well as Young's inequality, one deduces

$$\begin{aligned}
\Big\| \int_0^t \| e^{\Omega \mathcal{R}_1(t-\tau)} e^{-\Lambda^\alpha (t-\tau)} \nabla \Delta_j f(\tau) \|_{L^{p'}} d\tau \Big\|_{L^\delta(0,\infty)} &\leq C 2^{j(s-\frac{2}{p'})} \Big\| \int_0^t K_\Omega(t-\tau) \|\Delta_j f(\tau)\|_{L^p} d\tau \Big\|_{L^\delta(0,\infty)} \\
&\leq C 2^{j(s-\frac{2}{p'})} \|K_\Omega\|_{L^{\delta'}(0,\infty)} \|\Delta_j f\|_{L^{\frac{\delta}{2}}(0,\infty;L^p)},
\end{aligned}$$

where $K_\Omega(t) := (1+|\Omega|t)^{-(\frac{1}{2}-\frac{1}{p'})} t^{-\frac{1}{\alpha}(1+\frac{2}{p}-s)}$ and $\frac{1}{\delta} + \frac{1}{\delta'} = 1$. Since $\delta \in [2,\infty)$ satisfies

$$\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha) + \frac{1}{p'} - \frac{1}{2} < \frac{1}{\delta} < \frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha),$$

there exists a positive constant $C = C(s, p, \delta)$ such that

$$\begin{aligned}
\|K_\Omega\|_{L^{\delta'}(0,\infty)} &= |\Omega|^{-[\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)-\frac{1}{\delta}]} \Big( \int_0^\infty (1+\tau)^{-(\frac{1}{2}-\frac{1}{p'})\delta'} \tau^{-\frac{1}{\alpha}(1+\frac{2}{p}-s)\delta'} d\tau \Big)^{\frac{1}{\delta'}} \\
&\leq C|\Omega|^{-[\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)-\frac{1}{\delta}]}
\end{aligned}$$

for $\Omega \in \mathbb{R} \setminus \{0\}$. This completes the proof. □

**Lemma 9.** *For $\alpha \in (0,2], r \in [1,\infty]$ and $s \in (2-\alpha, 2)$, let $q \in (2,\infty)$ satisfy $\frac{1}{q} < \frac{1}{\alpha}(2-s)$ and $\delta \in [2,\infty)$ satisfy*

$$\frac{1}{\alpha}(s-2+\alpha) + \frac{1}{q} - \frac{1}{2} < \frac{1}{\delta} < \frac{1}{\alpha}(s-2+\alpha).$$

*Then there exists a positive constant $C = C(s, q, \delta)$ such that*

$$\Big\| \int_0^t e^{\Omega \mathcal{R}_1(t-\tau)} e^{\Lambda^\alpha (t-\tau)} \nabla f(\tau) d\tau \Big\|_{\tilde{L}^\delta(0,\infty; \dot{B}^{s+\frac{2}{q}-1}_{q,r})} \leq C|\Omega|^{-[\frac{1}{\alpha}(s-2+\alpha)-\frac{1}{\delta}]} \|f\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;\dot{B}^{2s-2+\frac{2}{q'}}_{q',r})}$$

*for $\Omega \in \mathbb{R}\backslash\{0\}$, where $\frac{1}{q'} + \frac{1}{q} = 1$.*

**Proof.** By Definition 2, one sees

$$\begin{aligned}
&\Big\| \int_0^t e^{\Omega\mathcal{R}_1(t-\tau)} e^{\Lambda^\alpha(t-\tau)} \nabla f(\tau) d\tau \Big\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{q}-1}_{q,r})} \\
&= \Big\| \Big\{ 2^{j(s+\frac{2}{q}-1)} \Big\| \Delta_j \int_0^t e^{\Omega\mathcal{R}_1(t-\tau)} e^{\Lambda^\alpha(t-\tau)} \nabla f(\tau) d\tau \Big\|_{L^\delta(0,\infty;L^q)} \Big\}_{j\in\mathbb{Z}} \Big\|_{\ell^r(\mathbb{Z})}.
\end{aligned}$$

Then, it suffices to show that

$$\Big\| \int_0^t \| e^{\Omega\mathcal{R}_1(t-\tau)} e^{\Lambda^\alpha(t-\tau)} \nabla \Delta_j f(\tau) \|_{L^q} d\tau \Big\|_{L^\delta(0,\infty)} \le C|\Omega|^{-[\frac{1}{2}(s-\frac{1}{2})-\frac{1}{\delta}]} 2^{j(s+\frac{2}{q'}-\frac{2}{q}-1)} \| \Delta_j f \|_{L^{\frac{\delta}{2}}(0,\infty;L^{q'})}.$$

In fact, by Lemma 1, Lemma 3 and Lemma 4, as well as Young's inequality, one then obtains

$$\begin{aligned}
&\Big\| \int_0^t \| e^{\Omega\mathcal{R}_1(t-\tau)} e^{\Lambda^\alpha(t-\tau)} \nabla \Delta_j f(\tau) \|_{L^q} d\tau \Big\|_{L^\delta(0,\infty)} \\
&\le C 2^{j(s+\frac{2}{q'}-\frac{2}{q}-1)} \Big\| \int_0^t K_\Omega(t-\tau) \| \Delta_j f(\tau) \|_{L^{q'}} d\tau \Big\|_{L^\delta(0,\infty)} \\
&\le C 2^{j(s+\frac{2}{q'}-\frac{2}{q}-1)} \| K_\Omega \|_{L^{\delta'}(0,\infty)} \| \Delta_j f \|_{L^{\frac{\delta}{2}}(0,\infty;L^{q'})},
\end{aligned} \tag{14}$$

where $K_\Omega(t) := (1+|\Omega|t)^{-(\frac{1}{2}-\frac{1}{q})} t^{-\frac{1}{\alpha}(2-s)}$ and $\frac{1}{\delta} + \frac{1}{\delta'} = 1$. Due to

$$\frac{1}{\alpha}(s-2+\alpha) - \frac{1}{2} + \frac{1}{q} < \frac{1}{\delta} < \frac{1}{\alpha}(s-2+\alpha),$$

there exists a positive constant $C = C(s,q,\delta)$ such that

$$\begin{aligned}
\| K_\Omega \|_{L^{\delta'}(0,\infty)} &= |\Omega|^{-[\frac{1}{\alpha}(s-2+\alpha)-\frac{1}{\delta}]} \left( \int_0^\infty (1+\tau)^{-\delta'(\frac{1}{2}-\frac{1}{q})} \tau^{-\frac{\delta'}{\alpha}(2-s)} d\tau \right)^{\frac{1}{\delta'}} \\
&\le C|\Omega|^{-[\frac{1}{\alpha}(s-2+\alpha)-\frac{1}{\delta}]}
\end{aligned} \tag{15}$$

for all $\Omega \in \mathbb{R} \setminus \{0\}$. Plugging (15) into (14), one concludes the proof. □

## 5. Proofs of Theorems 1 and 2

In this section, we are devoted to giving the proofs of Theorems 1 and 2.

**Proof of Theorem 1.** For $\Omega \in \mathbb{R} \setminus \{0\}$ and $\theta_0 \in \dot{B}^s_{p,r}(\mathbb{R}^2)$, it follows from Lemma 6 that there exists a positive constant $C_0$ such that

$$\| e^{\Omega\mathcal{R}_1 t} e^{-\Lambda^\alpha t} \theta_0 \|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \le C_0 |\Omega|^{-\frac{1}{\delta}} \| \theta_0 \|_{\dot{B}^s_{p,r}}. \tag{16}$$

Define the mapping $\mathscr{B}$ and the solution space $Y$ by

$$\mathscr{B}(\theta)(t) := e^{\Omega R_1 t} e^{-\Lambda^\alpha t} \theta_0 + \int_0^t e^{\Omega\mathcal{R}_1(t-\tau)} e^{\Lambda^\alpha(t-\tau)} \nabla \cdot \big( \theta(\tau) \cdot \mathcal{R}^\perp \theta(\tau) \big) d\tau$$

and

$$Y := \left\{ \theta \in \tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r}(\mathbb{R}^2)) : \| \theta \|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \le 2C_0 |\Omega|^{-\frac{1}{\delta}} \| \theta_0 \|_{\dot{B}^s_{p,r}} \right\}.$$

Applying Lemma 8 gives that there exists a positive constant $C_1$, such that

$$\begin{aligned}&\Big\| \int_0^t e^{\Omega \mathcal{R}_1 (t-\tau)} e^{\Lambda^\alpha (t-\tau)} \nabla \cdot \big(\theta(\tau) \cdot \mathcal{R}^\perp \theta(\tau)\big) d\tau \Big\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \\ &\leq C_1 |\Omega|^{-[\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)-\frac{1}{\delta}]} \|\theta(\tau) \cdot \mathcal{R}^\perp \theta(\tau)\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;\dot{B}^{2s-\frac{2}{p}}_{p,r})}.\end{aligned} \tag{17}$$

Moreover, since $\frac{4}{p} - 2 < s < 2 - \frac{2}{p}$ and $\frac{1}{p} \leq \frac{2}{3}$, by taking $s_1 = s_2 = s + \frac{2}{p'} - \frac{2}{p}$, $p_0 = p$, $p_1 = p_2 = p'$ and $\lambda_1 = \lambda_2 = \frac{p'}{p'-2}$, Lemma 2 and Remark 3, together with the fact that the $\mathcal{R}$ is bounded in Besov spaces, imply that there exists a positive constant $C_2$ such that

$$\|\theta(\tau) \cdot \mathcal{R}^\perp \theta(\tau)\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;\dot{B}^{2s-\frac{2}{p}}_{p,r})} \leq C_2 \|\theta\|^2_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})}. \tag{18}$$

Hence, from (16)–(18), we observe for all $\theta \in Y$ that

$$\begin{aligned}\|\mathscr{B}(\theta)\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} &\leq C_0 |\Omega|^{-\frac{1}{\delta}} \|u_0\|_{\dot{B}^s_{p,r}} + C_1 C_2 |\Omega|^{-[\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)-\frac{1}{\delta}]} \|\theta\|^2_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \\ &\leq C_0 |\Omega|^{-\frac{1}{\delta}} \|\theta_0\|_{\dot{B}^s_{p,r}} + 4C_0^2 C_1 C_2 |\Omega|^{-\frac{2}{\delta}} |\Omega|^{-[\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)-\frac{1}{\delta}]} \|\theta_0\|^2_{\dot{B}^s_{p,r}} \\ &\leq C_0 |\Omega|^{-\frac{1}{\delta}} \|\theta_0\|_{\dot{B}^s_{p,r}} \Big\{ 1 + 4C_0 C_1 C_2 |\Omega|^{-\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)} \|\theta_0\|_{\dot{B}^s_{p,r}} \Big\}.\end{aligned} \tag{19}$$

and similarly, for all $\theta, \sigma \in Y$, we see

$$\begin{aligned}&\|\mathscr{B}(\theta) - \mathscr{B}(\sigma)\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \\ &= \Big\| \int_0^t e^{\Omega \mathcal{R}_1 (t-\tau)} e^{\Lambda^\alpha (t-\tau)} \nabla \cdot \big[ [\theta(\tau) - \sigma(\tau)] \cdot \mathcal{R}^\perp \theta(\tau) + \sigma(\tau) \cdot \mathcal{R}^\perp [\theta(\tau) - \sigma(\tau)] \big] d\tau \Big\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \\ &\leq C_1 C_2 |\Omega|^{-[\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)-\frac{1}{\delta}]} \Big( \|\theta\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} + \|\sigma\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \Big) \|\theta - \sigma\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \\ &\leq 4C_0 C_1 C_2 |\Omega|^{-\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)} \|\theta_0\|_{\dot{B}^s_{p,r}} \|u - v\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})}.\end{aligned} \tag{20}$$

Now, let us assume $\theta_0 \in \dot{B}^s_{p,r}(\mathbb{R}^2)$ satisfing

$$\|\theta_0\|_{\dot{B}^s_{p,r}} \leq \frac{1}{16 C_0 C_1 C_2} |\Omega|^{\frac{1}{\alpha}(s-1-\frac{2}{p}+\alpha)},$$

(19) and (20) immediately thus imply for every $\theta, \sigma \in Y$ that

$$\|\mathscr{B}(\theta)\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} \leq 2C_0 |\Omega|^{-\frac{1}{\delta}} \|\theta_0\|_{\dot{B}^s_{p,r}}$$

and

$$\|\mathscr{B}(\theta) - \mathscr{B}(\sigma)\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})} < \frac{1}{2} \|\theta - \sigma\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})}.$$

Therefore, by the contraction mapping principle, there exists a unique solution $\theta \in Y$ satisfying (6) for all $t > 0$.

It remains to demonstrate that the solution $\theta \in Y$ also belongs to $C([0,\infty); \dot{B}^s_{p,r}(\mathbb{R}^2))$. By the definition of Besov spaces and applying Minkowski's inequality, we have

$$\begin{aligned}\|\theta(t)\|_{\dot{B}^s_{p,r}} \leq\, & C\Big\|\Big\{2^{sj}\Big\|e^{\Omega\mathcal{R}_1 t}e^{-\Lambda^\alpha t}\Delta_j\theta_0\Big\|_{L^p}\Big\}_{j\in\mathbb{Z}}\Big\|_{\ell^r} \\ & + C\Big\|\Big\{2^{js}\int_0^t\Big\|\Delta_j e^{\Omega\mathcal{R}_1(t-\tau)}e^{-\Lambda^\alpha(t-\tau)}\nabla\cdot\big(\theta(\tau)\cdot\mathcal{R}^\perp\theta(\tau)\big)\Big\|_{L^p}d\tau\Big\}_{j\in\mathbb{Z}}\Big\|_{\ell^r}\end{aligned}$$

Furthermore, by applying Lemmas 1 and 3, together with (18) and Young's inequality, we obtain

$$\begin{aligned}\|\theta(t)\|_{\dot{B}^s_{p,r}} &\leq C(1+|\Omega|t)^2\|u_0\|_{\dot{B}^s_{p,r}} + C\Big\|\Big\{2^{j(2s-\frac{2}{p})}\int_0^t K_\Omega(t-\tau)\Big\|\Delta_j\big(\theta(\tau)\cdot\mathcal{R}^\perp\theta(\tau)\big)\Big\|_{L^p}d\tau\Big\}_{j\in\mathbb{Z}}\Big\|_{\ell^r} \\ &\leq C(1+|\Omega|t)^2\|u_0\|_{\dot{B}^s_{p,r}} + C\|K_\Omega\|_{L^{(\frac{\delta}{2})'}(0,t)}\|\theta\cdot\mathcal{R}^\perp\theta\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;\dot{B}^{2s-\frac{2}{p}}_{p,r})}, \\ &\leq C(1+|\Omega|t)^2\|\theta_0\|_{\dot{B}^s_{p,r}} + C\|K_\Omega\|_{L^{(\frac{\delta}{2})'}(0,t)}\|\theta\|^2_{\tilde{L}^{\delta}(0,\infty;\dot{B}^{s+\frac{2}{p'}-\frac{2}{p}}_{p',r})},\end{aligned} \tag{21}$$

where $K_\Omega(t) := (1+|\Omega|t)^2 t^{-\frac{1}{\alpha}(1+\frac{2}{p}-s)}$ and $\frac{1}{(\frac{\delta}{2})'}+\frac{1}{(\frac{\delta}{2})}=1$. Due to $\frac{1}{\delta}<\frac{1}{2\alpha}(s-1-\frac{2}{p}+\alpha)$, it is easy to check that there exists a continuous function $C(t)>0$ defined on $[0,\infty)$ such that

$$\int_0^t(1+|\Omega|\tau)^{2(\frac{\delta}{2})'}\tau^{-\frac{1}{\alpha}(1+\frac{2}{p}-s)(\frac{\delta}{2})'}d\tau \leq C(t). \tag{22}$$

Substituting (22) into (21) implies that $\theta(t)\in\dot{B}^s_{p,r}(\mathbb{R}^2)$ for $t\geq 0$. The standard argument immediately implies $\theta\in C([0,\infty);\dot{B}^s_{p,r}(\mathbb{R}^2))$. This completes the proof of Theorem 1. □

**Proof of Theorem 2.** The proof of Theorem 2 is quite similar to that of Theorem 1. By replacing the solution space $Y$ with

$$Y := \Big\{\theta\in\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{q}-1}_{q,r}(\mathbb{R}^2)) : \|\theta\|_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{q}-1}_{q,r}(\mathbb{R}^3))} \leq 2C_0|\Omega|^{-\frac{1}{\delta}}\|\theta_0\|_{\dot{B}^s_{2,r}}\Big\},$$

and using Lemmas 7, 9 and 2, we shall verify that there exists a unique solution $\theta\in Y$ satisfying (6) for all $t>0$. Here, we would like to point out that by the similar argument of (18), the conditions $q\in(2,3]$, $\frac{1}{4}(s+1)<\frac{1}{q}$ and $\frac{1}{2}(1-s)<\frac{1}{q}$ ensure that

$$\|\theta\cdot\mathcal{R}^\perp\theta\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;\dot{B}^{2s-2+\frac{2}{q'}}_{q',r})} \leq C\|\theta\|^2_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{q}-1}_{q,r})}.$$

It remains to demonstrate that the solution $\theta\in Y$ also belongs to $C([0,\infty);\dot{B}^s_{2,r}(\mathbb{R}^2))$. Similarly, we obtain

$$\begin{aligned}\|\theta(t)\|_{\dot{B}^s_{2,r}} &\leq \|e^{\Omega R_1 t}e^{-\Lambda^\alpha t}\theta_0\|_{\dot{B}^s_{2,r}} + \Big\|\Big\{2^{js}\int_0^t\Big\|\Delta_j e^{\Omega R_1(t-\tau)}e^{-\Lambda^\alpha(t-\tau)}\nabla\cdot(R^\perp\theta(\tau)\otimes\theta(\tau))\Big\|_{L^2}d\tau\Big\}_{j\in\mathbb{Z}}\Big\|_{\ell^r} \\ &\leq C\|\theta_0\|_{\dot{B}^s_{2,r}} + C\Big\|\Big\{2^{j(2s+\frac{2}{q'}-2)}\int_0^t(t-\tau)^{-\frac{1}{\alpha}(2-s)}\Big\|\Delta_j(R^\perp\theta(\tau)\otimes\theta(\tau))\Big\|_{L^{q'}}d\tau\Big\}_{j\in\mathbb{Z}}\Big\|_{\ell^r} \\ &\leq C\|\theta_0\|_{\dot{B}^s_{2,r}} + C\Big(\int_0^t\tau^{-\frac{1}{\alpha}(2-s)(\frac{\delta}{2})'}d\tau\Big)^{\frac{1}{(\frac{\delta}{2})'}}\|R^\perp\theta(\tau)\otimes\theta(\tau)\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;\dot{B}^{2s+\frac{2}{q'}-2}_{q',r})} \\ &\leq C\|\theta_0\|_{\dot{B}^s_{2,r}} + C\Big(\int_0^t\tau^{-\frac{1}{\alpha}(2-s)(\frac{\delta}{2})'}d\tau\Big)^{\frac{1}{(\frac{\delta}{2})'}}\|\theta\|^2_{\tilde{L}^\delta(0,\infty;\dot{B}^{s+\frac{2}{q}-1}_{q,r})}.\end{aligned}$$

Thanks to $\frac{1}{\delta} < \frac{1}{2\alpha}(s-2+\alpha)$, it is easy to check that

$$\left(\int_0^t \tau^{-\frac{1}{\alpha}(2-s)(\frac{\delta}{2})'} d\tau\right)^{\frac{1}{(\frac{\delta}{2})'}} \leq Ct^{\frac{1}{\alpha}(s-2+\alpha)-\frac{2}{\delta}},$$

which implies that $\theta(t) \in \dot{B}^s_{2,r}(\mathbb{R}^2)$ for all $t \geq 0$. Similarly, we observe that $\theta \in C([0,\infty); \dot{B}^s_{2,r}(\mathbb{R}^2))$. This completes the proof of Theorem 2. $\square$

## 6. Conclusions

Theorems 1 and 2 show that for any given $\theta_0 \in \dot{B}^s_{p,r}(\mathbb{R}^2)$ with prescribed $s, p, r$, there exists a positive parameter $\Omega_0$, and if $|\Omega| \geq \Omega_0$, then problem (1) with $\alpha > 1$ is globally well-posed. It is worth mentioning that compared with [18], we obtain the explicit expression of $\Omega_0$:

$$\Omega_0 := (C^{-1}\|\theta_0\|_{\dot{B}^s_{p,r}})^{\frac{\alpha}{s-1-\frac{2}{p}+\alpha}}.$$

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