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Parameterized Quantum Fractional Integral Inequalities Defined by Using n-Polynomial Convex Functions

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Abstract: Convexity performs the appropriate role in the theoretical study of inequalities according to the nature and behaviour. There is a strong relation between symmetry and convexity. In this article, we consider a new parameterized quantum fractional integral identity. Following that, our main results are established, which consist of some integral inequalities of Ostrowski and midpoint type pertaining to n-polynomial convex functions. From our main results, we discuss in detail several special cases. Finally, an example and an application to special means of positive real numbers are presented to support our theoretical results.

Keywords: Ostrowski inequality; Riemann–Liouville $\dot{q}$-fractional integrals; $\dot{q}$-Hölder's inequality; $\dot{q}$-power mean inequality; n-polynomial convex functions; special means

MSC: 26A51; 26A33; 26D07; 26D10; 26D15; 26E60

1. Introduction

Integral inequalities are very useful tools for finding estimations. They can be applied in different fields of mathematics such as fractional calculus and discrete fractional calculus, etc. (see [1–4]).

Convexity study is crucial regarding theoretical behavior of mathematical inequalities. For some other theoretical studies of inequalities on different types of convex functions, see, e.g., GA-convex [5], MT-convex [6], (α, m)-convex [7], a generalized class of convexity [8], and many other types can be found in [9].

Definition 1 ([9]). *A function $\Theta : \mathtt{I} \subseteq \mathtt{R} \to \mathtt{R}$ is said to be convex if*

$$\Theta(\upsilon\omega_1 + (1-\upsilon)\omega_2) \leq \upsilon\Theta(\omega_1) + (1-\upsilon)\Theta(\omega_2)$$

holds for all $\omega_1, \omega_2 \in \mathtt{I}$ and $\upsilon \in [0,1]$. Likewise, Θ is concave if $-\Theta$ is convex.

Definition 2 ([10,11])**.** *Let $n \in \mathtt{N}$. A nonnegative function $\Theta : \mathtt{I} \subseteq \mathtt{R} \to \mathtt{R}$ is called n-polynomial convex if*

$$\Theta(v\omega_1 + (1-v)\omega_2) \leq \frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-v)^{\ell}\right)\Theta(\omega_1) + \frac{1}{n}\sum_{\ell=1}^{n}\left(1-v^{\ell}\right)\Theta(\omega_2)$$

holds for all $\omega_1, \omega_2 \in \mathtt{I}$ and $v \in [0,1]$.

Remark 1. *From [10], every nonnegative convex function is also an n-polynomial convex function. Moreover, we get the convex function by taking $n = 1$ in Definition 2.*

Symmetry has a significant role in integral inequality models with convexity. Furthermore, for convex functions and their types, many basic inequalities are found, such as Hermite–Hadamard type [12], Hermite–Hadamard–Fejér type [13], Ostrowski type [14–16], Simpson type [17,18], Hardy type [19], and Olsen type [20].

In this paper, we are focused in Ostrowski and midpoint type inequalities as follows.

Theorem 1 ([21])**.** *(Ostrowski type inequality) Assume that $\Theta : \mathtt{I} \subseteq \mathtt{R} \to \mathtt{R}$ is a differentiable function on $\mathtt{I}$ and let $\omega_1, \omega_2 \in \mathtt{I}^{\circ}$ (the interior of $\mathtt{I}$) with $\omega_1 < \omega_2$. If $|\Theta'(\xi)| \leq \mathcal{M}$ for all $\xi \in [\omega_1, \omega_2]$, then the following inequality holds true:*

$$\left|\Theta(\xi) - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\Theta(v)dv\right| \leq \mathcal{M}(\omega_2-\omega_1)\left[\frac{1}{4} + \frac{\left(\xi - \frac{\omega_1+\omega_2}{2}\right)^2}{(\omega_2-\omega_1)^2}\right]. \tag{1}$$

Choosing $\xi = \frac{\omega_1+\omega_2}{2}$ in (1), we get the following midpoint type inequality:

$$\left|\Theta\left(\frac{\omega_1+\omega_2}{2}\right) - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\Theta(v)dv\right| \leq \frac{\mathcal{M}(\omega_2-\omega_1)}{4}. \tag{2}$$

Let us recall some published papers about above inequalities using quantum calculus that inspired us.

The ꝗ-analogue of the trapezium's inequality was discovered by Tariboon and Ntouyas [22] using the concepts of quantum calculus, also known as calculus without limits, on the finite intervals. See [23] for more information on how to get classical calculus by taking $\dot{q} \to 1^{-}$. An updated version of the ꝗ-analogue of the inequality of the trapezium was discovered by Alp et al. [24]. Meanwhile, ꝗ-analogues of trapezium-like inequalities involving first order ꝗ-differentiable convex functions were deduced by Sudsutad et al. [25] and Noor et al. [26]. These analogues were created by Liu and Zhuang [27] by using twice ꝗ-differentiable convex functions. Budak et al. [28] are able to further develop certain quantum Hermite–Hadamard-type inequality. Ali et al. [29] presented some quantum Ostrowski-type inequalities for twice quantum differentiable functions. Convexity was used by Butt et al. [30] to generate some new quantum Simpson-Newton-like estimates in the frame of Mercer type inequalities. Aljinović et al. [31] established Ostrowski inequality for quantum calculus. Wang et al. [32] developed new Ostrowski-type inequalities via ꝗ-fractional integrals involving s-convex functions. To the best of our knowledge, the recently published paper from Wang et al. [32] is the first one working in this new direction that mixed together the fractional calculus with quantum calculus. Inspired by it, we will try to give some new quantum fractional integral inequalities of Ostrowski and midpoint type.

The article is set up as follows. The purpose of the Section 2 is to review several earlier findings of fractional calculus and ꝗ-calculus to provide the main interpretation of this article. In Section 3, we look at proving a new parameterized quantum fractional integral identity and demonstrate some integral inequalities of Ostrowski and midpoint type via n-polynomial convex functions. From our main results, we will discuss in detail several special cases. In Section 4, we offer an example and an application to special means of positive real numbers in order to show the efficiency of our theoretical results. The conclusion and future research will be given in Section 5.

2. Preliminaries

Let us denote, respectively, $\mathcal{L}[\omega_1, \omega_2]$ the set of all Lebesgue integrable functions on $[\omega_1, \omega_2]$ and $\mathcal{C}[\omega_1, \omega_2]$ the set of all differentiable continuous functions on $[\omega_1, \omega_2]$.

2.1. Fractional Calculus

Definition 3. *Let $\alpha > 0$, $0 \leq \omega_1 < \omega_2$ and $\Theta \in \mathcal{L}[\omega_1, \omega_2]$. Then the Riemann–Liouville fractional integral operators of order α are defined by*

$$J^{\alpha}_{\omega_1^+}\Theta(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\omega_1}^{\xi}(\xi - v)^{\alpha-1}\Theta(v)dv, \quad \omega_1 < \xi \tag{3}$$

and

$$J^{\alpha}_{\omega_2^-}\Theta(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\xi}^{\omega_2}(v - \xi)^{\alpha-1}\Theta(v)dv, \quad \xi < \omega_2,$$

where $\Gamma(\cdot)$ is gamma function, defined by

$$\Gamma(\alpha) = \int_0^{\infty} v^{\alpha-1}e^{-v}dv, \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

For $\alpha = 1$, we get the classical Riemann integrals.

2.2. Quantum Calculus

Throughout the remaining paper, let us consider $0 < \dot{q} < 1$ as a constant.

Definition 4 ([23])**.** *For $\Theta \in \mathcal{C}[\omega_1, \omega_2]$, the left $\dot{q}$-derivative of Θ at $\xi \in [\omega_1, \omega_2]$ is given by*

$$_{\omega_1}D_{\dot{q}}\Theta(\xi) = \frac{\Theta(\xi) - \Theta(\dot{q}\xi + (1-\dot{q})\omega_1)}{(1-\dot{q})(\xi - \omega_1)}, \quad \xi \neq \omega_1. \tag{4}$$

The function Θ is said to be $\dot{q}$-differentiable on $[\omega_1, \omega_2]$ if $_{\omega_1}D_{\dot{q}}\Theta(\xi)$ exists for all $\xi \in [\omega_1, \omega_2]$. If we choose $\omega_1 = 0$, then we will used the notation $_{\omega_1}D_{\dot{q}}\Theta(\xi) = D_{\dot{q}}\Theta(\xi)$, which is the $\dot{q}$-Jackson derivative [23,24,33] for more details.

The $\dot{q}$-integer is expressed as follows:

$$[n]_{\dot{q}} := \frac{\dot{q}^n - 1}{\dot{q} - 1} = 1 + \dot{q} + \dot{q}^2 + \cdots + \dot{q}^{n-1}, \quad n \in \mathbb{N}, \dot{q} \in (0,1).$$

The following $\dot{q}$-integral along with its properties can be studied in [24].

Definition 5. *Suppose that $\Theta \in \mathcal{C}[\omega_1, \omega_2]$. Then $\dot{q}$-definite integral for $\xi \in [\omega_1, \omega_2]$ is defined as*

$$\int_{\omega_1}^{\xi}\Theta(v)_{\omega_1}d_{\dot{q}}v = (1-\dot{q})(\xi - \omega_1)\sum_{r=0}^{\infty}\dot{q}^r\Theta(\dot{q}^r\xi + (1-\dot{q}^r)\omega_1). \tag{5}$$

Choosing $\omega_1 = 0$ in (5), we have

$$\int_0^{\xi}\Theta(v)d_{\dot{q}}v = (1-\dot{q})\xi\sum_{r=0}^{\infty}\dot{q}^r\Theta(\dot{q}^r\xi),$$

which gives

$$\int_0^1 v^{\alpha+s} d_{\dot{q}} v = \frac{1}{[\alpha+s+1]_{\dot{q}}}, \quad \int_0^1 v^{\alpha}(1-v)^s d_{\dot{q}} v = \frac{\Gamma_{\dot{q}}(\alpha+1)\Gamma_{\dot{q}}(s+1)}{\Gamma_{\dot{q}}(\alpha+s+2)},$$

where the $\dot{q}$-gamma function for $\xi > 0$ is defined by

$$\Gamma_{\dot{q}}(\xi) = \int_0^\infty v^{\xi-1} \mathcal{E}_{\dot{q}}^{-\dot{q}v} d_{\dot{q}} v,$$
$$\Gamma_{\dot{q}}(\xi+1) = [\xi]_{\dot{q}} \Gamma_{\dot{q}}(\xi)$$

and the $\dot{q}$-exponential function is given as

$$\mathcal{E}_{\dot{q}}^{v} = \sum_{r=0}^{\infty} \dot{q}^{\frac{r(r-1)}{2}} \frac{v^r}{[r]_{\dot{q}}!}.$$

The following $\dot{q}$-fractional integrals can be studied in [34].

Definition 6. *Let $\alpha > 0$, $0 \leq \omega_1 < \omega_2$ and $\Theta \in \mathcal{L}[\omega_1, \omega_2]$. Then the Riemann–Liouville $\dot{q}$-fractional integrals of order α are defined by*

$$\mathrm{J}^{\alpha}_{\dot{q},\omega_1^+}\Theta(\xi) = \frac{1}{\Gamma_{\dot{q}}(\alpha)} \int_{\omega_1}^{\xi} (\xi - \dot{q}v)^{\alpha-1} \Theta(v) \,{}_{\omega_1}d_{\dot{q}} v, \quad \omega_1 < \xi \tag{6}$$

and

$$\mathrm{J}^{\alpha}_{\dot{q},\omega_2^-}\Theta(\xi) = \frac{1}{\Gamma_{\dot{q}}(\alpha)} \int_{\dot{q}\xi}^{\omega_2} (v - \dot{q}\xi)^{\alpha-1} \Theta(v) d_{\dot{q}} v, \quad \xi < \omega_2,$$

where $\Gamma_{\dot{q}}(\cdot)$ is $\dot{q}$-gamma function. For $\dot{q} \to 1^-$, we get the Riemann–Liouville fractional integral operators.

Theorem 2 ([22]). *($\dot{q}$-integration by parts) Let Θ_1, $\Theta_2 \in \mathcal{C}[\omega_1, \omega_2]$, then for all $\xi \in [\omega_1, \omega_2]$, and we have*

$$\int_{\omega_1}^{\xi} \Theta_1(v) \,{}_{\omega_1}\mathrm{D}_{\dot{q}}\Theta_2(v) \,{}_{\omega_1}d_{\dot{q}} v = \Theta_1(\xi)\Theta_2(\xi) - \Theta_1(\omega_1)\Theta_2(\omega_1)$$
$$- \int_{\omega_1}^{\xi} \Theta_2(\dot{q}v + (1-\dot{q})\omega_1) \,{}_{\omega_1}\mathrm{D}_{\dot{q}}\Theta_1(v) \,{}_{\omega_1}d_{\dot{q}} v. \tag{7}$$

Theorem 3 ([35]). *($\dot{q}$-Hölder's inequality) Let Θ_1, Θ_2 be two $\dot{q}$-integrable functions on $[\omega_1, \omega_2]$ such that p, $\dot{q}_* > 1$, and $\frac{1}{p} + \frac{1}{\dot{q}_*} = 1$, and then we have*

$$\int_{\omega_1}^{\omega_2} |\Theta_1(v)\Theta_2(v)| \,{}_{\omega_1}d_{\dot{q}} v \leq \left(\int_{\omega_1}^{\omega_2} |\Theta_1(v)|^p \,{}_{\omega_1}d_{\dot{q}} v \right)^{\frac{1}{p}} \left(\int_{\omega_1}^{\omega_2} |\Theta_2(v)|^{\dot{q}_*} \,{}_{\omega_1}d_{\dot{q}} v \right)^{\frac{1}{\dot{q}_*}}. \tag{8}$$

Theorem 4 ([35]). *($\dot{q}$-power mean inequality) Let Θ_1, Θ_2 be two $\dot{q}$-integrable functions on $[\omega_1, \omega_2]$ such that $\dot{q}_* \geq 1$, and then we have*

$$\int_{\omega_1}^{\omega_2} |\Theta_1(v)\Theta_2(v)| \,{}_{\omega_1}d_{\dot{q}} v \leq \left(\int_{\omega_1}^{\omega_2} |\Theta_1(v)| \,{}_{\omega_1}d_{\dot{q}} v \right)^{1-\frac{1}{\dot{q}_*}} \left(\int_{\omega_1}^{\omega_2} |\Theta_1(v)||\Theta_2(v)|^{\dot{q}_*} \,{}_{\omega_1}d_{\dot{q}} v \right)^{\frac{1}{\dot{q}_*}}. \tag{9}$$

3. Main Results

For the simplicity of notations, let

$$\delta(\xi, \alpha) := \int_0^1 |\xi - v^{\alpha}| dv, \quad \rho(\xi, p, \alpha) := \int_0^1 |\xi - v^{\alpha}|^p dv.$$

Let us recall the well-known beta and hypergeometric functions below:

$$\beta(x,y) := \int_0^1 v^{x-1}(1-v)^{y-1}dv, \quad x, y > 0$$

and

$$_2\mathcal{F}_1(\omega_1,\omega_2;\tau;z) := \frac{1}{\beta(\omega_2,\tau-\omega_2)}\int_0^1 v^{\omega_2-1}(1-v)^{\tau-\omega_2-1}(1-zv)^{-\omega_1}dv$$

for $\Re(\tau) > \Re(\omega_2) > 0$, and $|z| \leq 1$.

To establish our main results, we need the following lemmas.

Lemma 1. *For $\alpha > 0$ and $0 \leq \xi \leq 1$, we have*

$$\delta(\xi,\alpha) := \begin{cases} \dfrac{1}{\alpha+1}, & \text{for } \xi = 0; \\ \dfrac{2\alpha\xi^{1+\frac{1}{\alpha}}+1}{\alpha+1} - \xi, & \text{for } 0 < \xi < 1; \\ \dfrac{\alpha}{\alpha+1}, & \text{for } \xi = 1. \end{cases}$$

Proof. The proof is evident. □

Lemma 2. *For $\alpha > 0$, $p \geq 1$ and $0 \leq \xi \leq 1$, we have*

$$\rho(\xi,p,\alpha) := \begin{cases} \dfrac{1}{p\alpha+1}, & \text{for } \xi = 0; \\ \dfrac{\xi^{p+\frac{1}{\alpha}}}{\alpha}\beta\left(\dfrac{1}{\alpha},p+1\right) + \dfrac{(1-\xi)^{p+1}}{\alpha(p+1)}{}_2\mathcal{F}_1\left(1-\dfrac{1}{\alpha},1;p+2;1-\xi\right), & \text{for } 0 < \xi < 1; \\ \dfrac{1}{\alpha}\beta\left(\dfrac{1}{\alpha},p+1\right), & \text{for } \xi = 1. \end{cases}$$

Proof. The proof is a straightforward computations. We omit here their details. □

Lemma 3. *Suppose $\Theta : [\omega_1,\omega_2] \to \mathtt{R}$ be a $\dot{\mathtt{q}}$-differentiable mapping, where $0 < \dot{\mathtt{q}} < 1$, such that $0 \leq \omega_1 < \omega_2$. If $\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x) \in \mathcal{L}[\omega_1,\omega_2]$ for all $x \in [\omega_1,\omega_2]$, $\theta,\varpi \in \mathtt{R}$ and $\alpha \in \mathtt{N}$, and then we have*

$$\begin{aligned} &\frac{\theta(\omega_2-x)^\alpha(\Theta(\omega_2)-\Theta(x)) - \varpi(x-\omega_1)^\alpha(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1} + \left(\frac{(x-\omega_1)^\alpha+(\omega_2-x)^\alpha}{\omega_2-\omega_1}\right)\Theta(x) \\ &\quad + \frac{[\alpha]_{\dot{\mathtt{q}}}(1-\dot{\mathtt{q}})}{\dot{\mathtt{q}}^\alpha(\omega_2-\omega_1)}((x-\omega_1)^\alpha+(\omega_2-x)^\alpha)\Theta(x) - \frac{\Gamma_{\dot{\mathtt{q}}}(\alpha+1)}{\dot{\mathtt{q}}^\alpha(\omega_2-\omega_1)}\left[\mathtt{J}^\alpha_{\dot{\mathtt{q}},x^+}\Theta(\omega_2) + \mathtt{J}^\alpha_{\dot{\mathtt{q}},x^-}\Theta(\omega_1)\right] \\ &= \frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}\int_0^1(\theta-v^\alpha)\mathtt{D}_{\dot{\mathtt{q}}}\Theta(vx+(1-v)\omega_2)d_{\dot{\mathtt{q}}}v \\ &\quad - \frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}\int_0^1(\varpi-v^\alpha)\mathtt{D}_{\dot{\mathtt{q}}}\Theta((1-v)\omega_1+vx)d_{\dot{\mathtt{q}}}v. \end{aligned} \tag{10}$$

Proof. Let us denote, respectively,

$$\mathtt{I}_1 := \int_0^1(\theta-v^\alpha)\mathtt{D}_{\dot{\mathtt{q}}}\Theta(vx+(1-v)\omega_2)d_{\dot{\mathtt{q}}}v$$

and

$$\mathtt{I}_2 := \int_0^1(\varpi-v^\alpha)\mathtt{D}_{\dot{\mathtt{q}}}\Theta((1-v)\omega_1+vx)d_{\dot{\mathtt{q}}}v.$$

With the help of $\dot{q}$-integration by parts, we have

$$\begin{aligned}
\mathtt{I}_1 &= \theta \int_0^1 \mathtt{D}_{\dot{q}}\Theta(vx+(1-v)\omega_2)d_{\dot{q}}v - \int_0^1 v^{\alpha}\mathtt{D}_{\dot{q}}\Theta(vx+(1-v)\omega_2)d_{\dot{q}}v \\
&= \frac{\theta}{\omega_2 - x}\int_x^{\omega_2} {}_x\mathtt{D}_{\dot{q}}\Theta(v)\,{}_x d_{\dot{q}}v - v^{\alpha}\frac{\Theta(vx+(1-v)\omega_2)}{x-\omega_2}\Bigg|_0^1 \\
&\quad + \frac{[\alpha]_{\dot{q}}}{x-\omega_2}\int_0^1 v^{\alpha-1}\Theta(\dot{q}vx+(1-\dot{q}v)\omega_2)d_{\dot{q}}v \\
&= \frac{\theta}{\omega_2-x}(\Theta(\omega_2)-\Theta(x)) + \frac{\Theta(x)}{\omega_2-x} + \frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^{\alpha}(\omega_2-x)}\Theta(x) \\
&\quad - \frac{[\alpha]_{\dot{q}}\Gamma_{\dot{q}}(\alpha)}{\dot{q}^{\alpha}(\omega_2-x)^{\alpha+1}}\frac{1}{\Gamma_{\dot{q}}(\alpha)}\int_x^{\omega_2}(\omega_2-\dot{q}v)^{\alpha-1}\Theta(v)d_{\dot{q}}v \\
&= \frac{\theta}{\omega_2-x}(\Theta(\omega_2)-\Theta(x)) + \frac{\Theta(x)}{\omega_2-x} + \frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^{\alpha}(\omega_2-x)}\Theta(x) \\
&\quad - \frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^{\alpha}(\omega_2-x)^{\alpha+1}}\mathtt{J}^{\alpha}_{\dot{q},x^+}\Theta(\omega_2).
\end{aligned} \tag{11}$$

Similarly, we get

$$\begin{aligned}
\mathtt{I}_2 &= \varpi \int_0^1 \mathtt{D}_{\dot{q}}\Theta((1-v)\omega_1+vx)d_{\dot{q}}v - \int_0^1 v^{\alpha}\mathtt{D}_{\dot{q}}\Theta((1-v)\omega_1+vx)d_{\dot{q}}v \\
&= \frac{\varpi}{x-\omega_1}(\Theta(x)-\Theta(\omega_1)) - \frac{\Theta(x)}{x-\omega_1} - \frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^{\alpha}(x-\omega_1)}\Theta(x) \\
&\quad + \frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^{\alpha}(x-\omega_1)^{\alpha+1}}\mathtt{J}^{\alpha}_{\dot{q},x^-}\Theta(\omega_1).
\end{aligned} \tag{12}$$

By multiplying (11) by $\frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}$ and (12) with $\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}$ and subtracting them, we acquire (10). □

Remark 2. *Considering $\dot{q} \to 1^-$ in Lemma 3, we have*

$$\begin{aligned}
&\frac{\theta(\omega_2-x)^{\alpha}(\Theta(\omega_2)-\Theta(x)) - \varpi(x-\omega_1)^{\alpha}(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1} + \left(\frac{(x-\omega_1)^{\alpha}+(\omega_2-x)^{\alpha}}{\omega_2-\omega_1}\right)\Theta(x) \\
&\quad - \frac{\Gamma(\alpha+1)}{\omega_2-\omega_1}\left[\mathtt{J}^{\alpha}_{x^+}\Theta(\omega_2)+\mathtt{J}^{\alpha}_{x^-}\Theta(\omega_1)\right] \\
&= \frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}\int_0^1(\theta-v^{\alpha})\Theta'(vx+(1-v)\omega_2)dv \\
&\quad - \frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}\int_0^1(\varpi-v^{\alpha})\Theta'((1-v)\omega_1+vx)dv.
\end{aligned} \tag{13}$$

Remark 3. *Choosing $\alpha = 1$ in Lemma 3, we get the following $\dot{q}$-identity:*

$$\begin{aligned}
&\frac{\theta(\omega_2-x)(\Theta(\omega_2)-\Theta(x)) - \varpi(x-\omega_1)(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1} + \frac{\Theta(x)}{\dot{q}} - \frac{1}{\dot{q}(\omega_2-\omega_1)}\int_{\dot{q}\omega_1}^{\omega_2}\Theta(v)\,{}_{\omega_1}d_{\dot{q}}v \\
&= \frac{(\omega_2-x)^2}{\omega_2-\omega_1}\int_0^1(\theta-v)\mathtt{D}_{\dot{q}}\Theta(vx+(1-v)\omega_2)d_{\dot{q}}v \\
&\quad - \frac{(x-\omega_1)^2}{\omega_2-\omega_1}\int_0^1(\varpi-v)\mathtt{D}_{\dot{q}}\Theta((1-v)\omega_1+vx)d_{\dot{q}}v.
\end{aligned} \tag{14}$$

By using Lemmas 1–3, we established the following $\dot{q}$-fractional integral inequalities.

Theorem 5. *Let $\Theta : [\omega_1, \omega_2] \to \mathtt{R}$ be a $\dot{\mathtt{q}}$-differentiable mapping, where $0 < \dot{\mathtt{q}} < 1$, such that $0 \le \omega_1 < \omega_2$ and $\theta, \varpi \in [0,1]$. If $\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x) \in \mathcal{L}[\omega_1, \omega_2]$ for every $x \in [\omega_1, \omega_2]$ and $|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x)|^{\dot{\mathtt{q}}_*}$ is n-polynomial convex function for all $n \in \mathtt{N}$, such that $p, \dot{\mathtt{q}}_* > 1$, and $\frac{1}{p} + \frac{1}{\dot{\mathtt{q}}_*} = 1$, and then for $\alpha \in \mathtt{N}$, the following $\dot{\mathtt{q}}$-fractional integral inequality holds true:*

$$\begin{aligned}
&\Bigg| \frac{\theta(\omega_2 - x)^\alpha(\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha(\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x-\omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1}\right)\Theta(x) \\
&+ \frac{[\alpha]_{\dot{\mathtt{q}}}(1-\dot{\mathtt{q}})}{\dot{\mathtt{q}}^\alpha(\omega_2 - \omega_1)}((x-\omega_1)^\alpha + (\omega_2 - x)^\alpha)\Theta(x) - \frac{\Gamma_{\dot{\mathtt{q}}}(\alpha+1)}{\dot{\mathtt{q}}^\alpha(\omega_2 - \omega_1)}\left[\mathrm{J}^\alpha_{\dot{\mathtt{q}},x^+}\Theta(\omega_2) + \mathrm{J}^\alpha_{\dot{\mathtt{q}},x^-}\Theta(\omega_1)\right]\Bigg| \\
&\le \frac{1}{\omega_2 - \omega_1}\left(1 - \frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\dot{\mathtt{q}}}}\right)^{\frac{1}{\dot{\mathtt{q}}_*}}\Bigg\{(\omega_2 - x)^{\alpha+1}\mathrm{A}^{\frac{1}{p}}_{\dot{\mathtt{q}}}(\theta, p, \alpha)\left[|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x)|^{\dot{\mathtt{q}}_*} + |\mathtt{D}_{\dot{\mathtt{q}}}\Theta(\omega_2)|^{\dot{\mathtt{q}}_*}\right]^{\frac{1}{\dot{\mathtt{q}}_*}} \\
&+(x-\omega_1)^{\alpha+1}\mathrm{A}^{\frac{1}{p}}_{\dot{\mathtt{q}}}(\varpi, p, \alpha)\left[|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x)|^{\dot{\mathtt{q}}_*} + |\mathtt{D}_{\dot{\mathtt{q}}}\Theta(\omega_1)|^{\dot{\mathtt{q}}_*}\right]^{\frac{1}{\dot{\mathtt{q}}_*}}\Bigg\}, \qquad (15)
\end{aligned}$$

where

$$\mathrm{A}_{\dot{\mathtt{q}}}(\theta, p, \alpha) := \int_0^1 |\theta - v^\alpha|^p d_{\dot{\mathtt{q}}}v.$$

Proof. By using Lemma 3, $\dot{\mathtt{q}}$-Hölder's inequality, n-polynomial convexity of $|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x)|^{\dot{\mathtt{q}}_*}$ and properties of modulus, we have

$$\begin{aligned}
&\Bigg| \frac{\theta(\omega_2 - x)^\alpha(\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha(\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x-\omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1}\right)\Theta(x) \\
&+ \frac{[\alpha]_{\dot{\mathtt{q}}}(1-\dot{\mathtt{q}})}{\dot{\mathtt{q}}^\alpha(\omega_2 - \omega_1)}((x-\omega_1)^\alpha + (\omega_2 - x)^\alpha)\Theta(x) - \frac{\Gamma_{\dot{\mathtt{q}}}(\alpha+1)}{\dot{\mathtt{q}}^\alpha(\omega_2 - \omega_1)}\left[\mathrm{J}^\alpha_{\dot{\mathtt{q}},x^+}\Theta(\omega_2) + \mathrm{J}^\alpha_{\dot{\mathtt{q}},x^-}\Theta(\omega_1)\right]\Bigg| \\
&\le \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1}\int_0^1 |\theta - v^\alpha|\left|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(vx + (1-v)\omega_2)\right| d_{\dot{\mathtt{q}}}v \\
&+ \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1}\int_0^1 |\varpi - v^\alpha|\left|\mathtt{D}_{\dot{\mathtt{q}}}\Theta((1-v)\omega_1 + vx)\right| d_{\dot{\mathtt{q}}}v \\
&\le \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1}\left(\int_0^1 |\theta - v^\alpha|^p d_{\dot{\mathtt{q}}}v\right)^{\frac{1}{p}}\left(\int_0^1 \left|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(vx + (1-v)\omega_2)\right|^{\dot{\mathtt{q}}_*} d_{\dot{\mathtt{q}}}v\right)^{\frac{1}{\dot{\mathtt{q}}_*}} \\
&+ \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1}\left(\int_0^1 |\varpi - v^\alpha|^p d_{\dot{\mathtt{q}}}v\right)^{\frac{1}{p}}\left(\int_0^1 \left|\mathtt{D}_{\dot{\mathtt{q}}}\Theta((1-v)\omega_1 + vx)\right|^{\dot{\mathtt{q}}_*} d_{\dot{\mathtt{q}}}v\right)^{\frac{1}{\dot{\mathtt{q}}_*}} \\
&\le \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1}\mathrm{A}^{\frac{1}{p}}_{\dot{\mathtt{q}}}(\theta, p, \alpha)\left[\int_0^1 \frac{1}{n}\sum_{\ell=1}^{n}\left(1 - (1-v)^\ell\right)|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x)|^{\dot{\mathtt{q}}_*} d_{\dot{\mathtt{q}}}v + \int_0^1 \frac{1}{n}\sum_{\ell=1}^{n}\left(1 - v^\ell\right)|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(\omega_2)|^{\dot{\mathtt{q}}_*} d_{\dot{\mathtt{q}}}v\right]^{\frac{1}{\dot{\mathtt{q}}_*}} \\
&+ \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1}\mathrm{A}^{\frac{1}{p}}_{\dot{\mathtt{q}}}(\varpi, p, \alpha)\left[\int_0^1 \frac{1}{n}\sum_{\ell=1}^{n}\left(1 - (1-v)^\ell\right)|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x)|^{\dot{\mathtt{q}}_*} d_{\dot{\mathtt{q}}}v + \int_0^1 \frac{1}{n}\sum_{\ell=1}^{n}\left(1 - v^\ell\right)|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(\omega_1)|^{\dot{\mathtt{q}}_*} d_{\dot{\mathtt{q}}}v\right]^{\frac{1}{\dot{\mathtt{q}}_*}} \\
&= \frac{1}{\omega_2 - \omega_1}\left(1 - \frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\dot{\mathtt{q}}}}\right)^{\frac{1}{\dot{\mathtt{q}}_*}}\Bigg\{(\omega_2 - x)^{\alpha+1}\mathrm{A}^{\frac{1}{p}}_{\dot{\mathtt{q}}}(\theta, p, \alpha)\left[|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x)|^{\dot{\mathtt{q}}_*} + |\mathtt{D}_{\dot{\mathtt{q}}}\Theta(\omega_2)|^{\dot{\mathtt{q}}_*}\right]^{\frac{1}{\dot{\mathtt{q}}_*}} \\
&+(x-\omega_1)^{\alpha+1}\mathrm{A}^{\frac{1}{p}}_{\dot{\mathtt{q}}}(\varpi, p, \alpha)\left[|\mathtt{D}_{\dot{\mathtt{q}}}\Theta(x)|^{\dot{\mathtt{q}}_*} + |\mathtt{D}_{\dot{\mathtt{q}}}\Theta(\omega_1)|^{\dot{\mathtt{q}}_*}\right]^{\frac{1}{\dot{\mathtt{q}}_*}}\Bigg\}.
\end{aligned}$$

This concludes the desired proof. □

Corollary 1. *Suppose $\dot{\mathtt{q}} \to 1^-$ in Theorem 5. Then we have*

$$\left|\frac{\theta(\omega_2-x)^{\alpha}(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)^{\alpha}(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^{\alpha}+(\omega_2-x)^{\alpha}}{\omega_2-\omega_1}\right)\Theta(x)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{\omega_2-\omega_1}\left[\mathrm{J}^{\alpha}_{x^+}\Theta(\omega_2)+\mathrm{J}^{\alpha}_{x^-}\Theta(\omega_1)\right]\right|$$
$$\leq\frac{1}{\omega_2-\omega_1}\left(1-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{\ell+1}\right)^{\frac{1}{\dot{q}_*}}\left\{(\omega_2-x)^{\alpha+1}\rho^{\frac{1}{p}}(\theta,p,\alpha)\left[|\Theta'(x)|^{\dot{q}_*}+|\Theta'(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right. \tag{16}$$
$$\left.+(x-\omega_1)^{\alpha+1}\rho^{\frac{1}{p}}(\varpi,p,\alpha)\left[|\Theta'(x)|^{\dot{q}_*}+|\Theta'(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right\}.$$

Corollary 2. *Considering $\alpha=1$ in Theorem 5, we get*

$$\left|\frac{\theta(\omega_2-x)(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\frac{\Theta(x)}{\dot{q}}-\frac{1}{\dot{q}(\omega_2-\omega_1)}\int_{\dot{q}\omega_1}^{\omega_2}\Theta(v)\,{}_{\omega_1}d_{\dot{q}}v\right|$$
$$\leq\frac{1}{\omega_2-\omega_1}\left(1-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\dot{q}}}\right)^{\frac{1}{\dot{q}_*}}\left\{(\omega_2-x)^{2}\mathrm{A}_{\dot{q}}^{\frac{1}{p}}(\theta,p)\left[|\mathrm{D}_{\dot{q}}\Theta(x)|^{\dot{q}_*}+|\mathrm{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right. \tag{17}$$
$$\left.+(x-\omega_1)^{2}\mathrm{A}_{\dot{q}}^{\frac{1}{p}}(\varpi,p)\left[|\mathrm{D}_{\dot{q}}\Theta(x)|^{\dot{q}_*}+|\mathrm{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right\},$$

where

$$\mathrm{A}_{\dot{q}}(\theta,p):=\int_0^1|\theta-v|^p d_{\dot{q}}v.$$

Corollary 3. *Taking $|\mathrm{D}_{\dot{q}}\Theta|\leq\mathcal{K}$ in Theorem 5, we obtain*

$$\left|\frac{\theta(\omega_2-x)^{\alpha}(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)^{\alpha}(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^{\alpha}+(\omega_2-x)^{\alpha}}{\omega_2-\omega_1}\right)\Theta(x)\right.$$
$$\left.+\frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^{\alpha}(\omega_2-\omega_1)}((x-\omega_1)^{\alpha}+(\omega_2-x)^{\alpha})\Theta(x)-\frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^{\alpha}(\omega_2-\omega_1)}\left[\mathrm{J}^{\alpha}_{\dot{q},x^+}\Theta(\omega_2)+\mathrm{J}^{\alpha}_{\dot{q},x^-}\Theta(\omega_1)\right]\right| \tag{18}$$
$$\leq\frac{2^{\frac{1}{\dot{q}_*}}\mathcal{K}}{\omega_2-\omega_1}\left(1-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\dot{q}}}\right)^{\frac{1}{\dot{q}_*}}\left\{(\omega_2-x)^{\alpha+1}\mathrm{A}_{\dot{q}}^{\frac{1}{p}}(\theta,p,\alpha)+(x-\omega_1)^{\alpha+1}\mathrm{A}_{\dot{q}}^{\frac{1}{p}}(\varpi,p,\alpha)\right\}.$$

Corollary 4. *Considering $n=1$ in Theorem 5, we have*

$$\left|\frac{\theta(\omega_2-x)^{\alpha}(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)^{\alpha}(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^{\alpha}+(\omega_2-x)^{\alpha}}{\omega_2-\omega_1}\right)\Theta(x)\right.$$
$$\left.+\frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^{\alpha}(\omega_2-\omega_1)}((x-\omega_1)^{\alpha}+(\omega_2-x)^{\alpha})\Theta(x)-\frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^{\alpha}(\omega_2-\omega_1)}\left[\mathrm{J}^{\alpha}_{\dot{q},x^+}\Theta(\omega_2)+\mathrm{J}^{\alpha}_{\dot{q},x^-}\Theta(\omega_1)\right]\right|$$
$$\leq\frac{1}{\omega_2-\omega_1}\left(\frac{\dot{q}}{1+\dot{q}}\right)^{\frac{1}{\dot{q}_*}}\left\{(\omega_2-x)^{\alpha+1}\mathrm{A}_{\dot{q}}^{\frac{1}{p}}(\theta,p,\alpha)\left[|\mathrm{D}_{\dot{q}}\Theta(x)|^{\dot{q}_*}+|\mathrm{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right. \tag{19}$$
$$\left.+(x-\omega_1)^{\alpha+1}\mathrm{A}_{\dot{q}}^{\frac{1}{p}}(\varpi,p,\alpha)\left[|\mathrm{D}_{\dot{q}}\Theta(x)|^{\dot{q}_*}+|\mathrm{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right\}.$$

Corollary 5. *Taking $\theta=\varpi$ in Theorem 5, we get*

$$\begin{aligned}&\Bigg|\frac{\theta[(\omega_2-x)^\alpha(\Theta(\omega_2)-\Theta(x))-(x-\omega_1)^\alpha(\Theta(x)-\Theta(\omega_1))]}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^\alpha+(\omega_2-x)^\alpha}{\omega_2-\omega_1}\right)\Theta(x)\\&+\frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^\alpha(\omega_2-\omega_1)}((x-\omega_1)^\alpha+(\omega_2-x)^\alpha)\Theta(x)-\frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^\alpha(\omega_2-\omega_1)}\left[J^\alpha_{\dot{q},x^+}\Theta(\omega_2)+J^\alpha_{\dot{q},x^-}\Theta(\omega_1)\right]\Bigg|\\&\leq\frac{1}{\omega_2-\omega_1}\left(1-\frac{1}{n}\sum_{\ell=1}^n\frac{1}{[\ell+1]_{\dot{q}}}\right)^{\frac{1}{\dot{q}_*}}A^{\frac{1}{p}}_{\dot{q}}(\theta,p,\alpha)\Big\{(\omega_2-x)^{\alpha+1}\left[|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}+|D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\\&+(x-\omega_1)^{\alpha+1}\left[|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}+|D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\Big\}.\end{aligned}\tag{20}$$

Corollary 6. *Choosing $\theta = 0$ and $\alpha = 1$ in Corollary 5, we obtain the following $\dot{q}$-integral inequality of Ostrowski type:*

$$\begin{aligned}&\left|\Theta(x)-\frac{1}{\omega_2-\omega_1}\int_{\dot{q}\omega_1}^{\omega_2}\Theta(v)\,{}_{\omega_1}d_{\dot{q}}v\right|\\&\leq\frac{\dot{q}}{\omega_2-\omega_1}\left(1-\frac{1}{n}\sum_{\ell=1}^n\frac{1}{[\ell+1]_{\dot{q}}}\right)^{\frac{1}{\dot{q}_*}}\left(\frac{1}{[p+1]_{\dot{q}}}\right)^{\frac{1}{p}}\Big\{(\omega_2-x)^2\left[|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}+|D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\\&+(x-\omega_1)^2\left[|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}+|D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\Big\}.\end{aligned}\tag{21}$$

Corollary 7. *Taking $x = \frac{\omega_1+\omega_2}{2}$ in Corollary 6, we have the following $\dot{q}$-integral inequality of midpoint type:*

$$\begin{aligned}&\left|\Theta\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{1}{\omega_2-\omega_1}\int_{\dot{q}\omega_1}^{\omega_2}\Theta(v)\,{}_{\omega_1}d_{\dot{q}}v\right|\\&\leq\frac{\dot{q}(\omega_2-\omega_1)}{4}\left(1-\frac{1}{n}\sum_{\ell=1}^n\frac{1}{[\ell+1]_{\dot{q}}}\right)^{\frac{1}{\dot{q}_*}}\left(\frac{1}{[p+1]_{\dot{q}}}\right)^{\frac{1}{p}}\left\{\left[\left|D_{\dot{q}}\Theta\left(\frac{\omega_1+\omega_2}{2}\right)\right|^{\dot{q}_*}+|D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right.\\&\left.+\left[\left|D_{\dot{q}}\Theta\left(\frac{\omega_1+\omega_2}{2}\right)\right|^{\dot{q}_*}+|D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right\}.\end{aligned}\tag{22}$$

Theorem 6. *Suppose $\Theta : [\omega_1, \omega_2] \to R$ be a $\dot{q}$-differentiable mapping, where $0 < \dot{q} < 1$, such that $0 \leq \omega_1 < \omega_2$ and $\theta, \varpi \in [0, 1]$. If $D_{\dot{q}}\Theta(x) \in \mathcal{L}[\omega_1, \omega_2]$ for every $x \in [\omega_1, \omega_2]$ and $|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}$ is n-polynomial convex function for all $n \in N$, such that $\dot{q}_* \geq 1$, and then for $\alpha \in N$, the following $\dot{q}$-fractional integral inequality holds true:*

$$\begin{aligned}&\Bigg|\frac{\theta(\omega_2-x)^\alpha(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)^\alpha(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^\alpha+(\omega_2-x)^\alpha}{\omega_2-\omega_1}\right)\Theta(x)\\&+\frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^\alpha(\omega_2-\omega_1)}((x-\omega_1)^\alpha+(\omega_2-x)^\alpha)\Theta(x)-\frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^\alpha(\omega_2-\omega_1)}\left[J^\alpha_{\dot{q},x^+}\Theta(\omega_2)+J^\alpha_{\dot{q},x^-}\Theta(\omega_1)\right]\Bigg|\\&\leq\frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}A^{1-\frac{1}{\dot{q}_*}}_{\dot{q}}(\theta,\alpha)\Bigg[\left(A_{\dot{q}}(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^nB_{\dot{q}}(\theta,\alpha,\ell)\right)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}\\&+\left(A_{\dot{q}}(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^nC_{\dot{q}}(\theta,\alpha,\ell)\right)|D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\Bigg]^{\frac{1}{\dot{q}_*}}\\&+\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}A^{1-\frac{1}{\dot{q}_*}}_{\dot{q}}(\varpi,\alpha)\Bigg[\left(A_{\dot{q}}(\varpi,\alpha)-\frac{1}{n}\sum_{\ell=1}^nB_{\dot{q}}(\varpi,\alpha,\ell)\right)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}\\&+\left(A_{\dot{q}}(\varpi,\alpha)-\frac{1}{n}\sum_{\ell=1}^nC_{\dot{q}}(\varpi,\alpha,\ell)\right)|D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\Bigg]^{\frac{1}{\dot{q}_*}},\end{aligned}\tag{23}$$

where

$$\mathrm{A}_{\dot{q}}(\theta,\alpha) := \int_0^1 |\theta - v^{\alpha}| d_{\dot{q}}v,$$

$$\mathrm{B}_{\dot{q}}(\theta,\alpha,\ell) := \int_0^1 (1-v)^{\ell} |\theta - v^{\alpha}| d_{\dot{q}}v$$

and

$$\mathrm{C}_{\dot{q}}(\theta,\alpha,\ell) := \int_0^1 v^{\ell} |\theta - v^{\alpha}| d_{\dot{q}}v.$$

Proof. By using Lemma 3, $\dot{q}$-power mean inequality, n-polynomial convexity of $|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}$ and properties of modulus, we have

$$\begin{aligned}
&\Bigg|\frac{\theta(\omega_2-x)^{\alpha}(\Theta(\omega_2)-\Theta(x)) - \varpi(x-\omega_1)^{\alpha}(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1} + \left(\frac{(x-\omega_1)^{\alpha}+(\omega_2-x)^{\alpha}}{\omega_2-\omega_1}\right)\Theta(x) \\
&+\frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^{\alpha}(\omega_2-\omega_1)}((x-\omega_1)^{\alpha}+(\omega_2-x)^{\alpha})\Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^{\alpha}(\omega_2-\omega_1)}\left[J^{\alpha}_{\dot{q},x^+}\Theta(\omega_2) + J^{\alpha}_{\dot{q},x^-}\Theta(\omega_1)\right]\Bigg| \\
\leq{}& \frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}\int_0^1 |\theta - v^{\alpha}|\left|D_{\dot{q}}\Theta(vx+(1-v)\omega_2)\right| d_{\dot{q}}v \\
&+\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}\int_0^1 |\varpi - v^{\alpha}|\left|D_{\dot{q}}\Theta((1-v)\omega_1+vx)\right| d_{\dot{q}}v \\
\leq{}& \frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}\left(\int_0^1 |\theta-v^{\alpha}| d_{\dot{q}}v\right)^{1-\frac{1}{\dot{q}_*}}\left(\int_0^1 |\theta - v^{\alpha}|\left|D_{\dot{q}}\Theta(vx+(1-v)\omega_2)\right|^{\dot{q}_*} d_{\dot{q}}v\right)^{\frac{1}{\dot{q}_*}} \\
&+\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}\left(\int_0^1 |\varpi-v^{\alpha}| d_{\dot{q}}v\right)^{1-\frac{1}{\dot{q}_*}}\left(\int_0^1 |\varpi - v^{\alpha}|\left|D_{\dot{q}}\Theta((1-v)\omega_1+vx)\right|^{\dot{q}_*} d_{\dot{q}}v\right)^{\frac{1}{\dot{q}_*}} \\
\leq{}& \frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}\mathrm{A}_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\theta,\alpha) \\
&\times\left[\int_0^1 |\theta-v^{\alpha}|\left(\frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-v)^{\ell}\right)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*} + \frac{1}{n}\sum_{\ell=1}^{n}\left(1-v^{\ell}\right)|D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right) d_{\dot{q}}v\right]^{\frac{1}{\dot{q}_*}} \\
&+\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}\mathrm{A}_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\varpi,\alpha) \\
&\times\left[\int_0^1 |\varpi-v^{\alpha}|\left(\frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-v)^{\ell}\right)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*} + \frac{1}{n}\sum_{\ell=1}^{n}\left(1-v^{\ell}\right)|D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right) d_{\dot{q}}v\right]^{\frac{1}{\dot{q}_*}} \\
={}& \frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}\mathrm{A}_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\theta,\alpha) \\
&\times\left[\left(\mathrm{A}_{\dot{q}}(\theta,\alpha) - \frac{1}{n}\sum_{\ell=1}^{n}\mathrm{B}_{\dot{q}}(\theta,\alpha,\ell)\right)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*} + \left(\mathrm{A}_{\dot{q}}(\theta,\alpha) - \frac{1}{n}\sum_{\ell=1}^{n}\mathrm{C}_{\dot{q}}(\theta,\alpha,\ell)\right)|D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}} \\
&+\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}\mathrm{A}_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\varpi,\alpha) \\
&\times\left[\left(\mathrm{A}_{\dot{q}}(\varpi,\alpha) - \frac{1}{n}\sum_{\ell=1}^{n}\mathrm{B}_{\dot{q}}(\varpi,\alpha,\ell)\right)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*} + \left(\mathrm{A}_{\dot{q}}(\varpi,\alpha) - \frac{1}{n}\sum_{\ell=1}^{n}\mathrm{C}_{\dot{q}}(\varpi,\alpha,\ell)\right)|D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}.
\end{aligned}$$

This completes the proof. □

Corollary 8. *Taking* $\dot{q}_* = 1$ *in Theorem 6, we have*

$$\left|\frac{\theta(\omega_2-x)^\alpha(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)^\alpha(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^\alpha+(\omega_2-x)^\alpha}{\omega_2-\omega_1}\right)\Theta(x)\right.$$
$$\left.+\frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^\alpha(\omega_2-\omega_1)}((x-\omega_1)^\alpha+(\omega_2-x)^\alpha)\Theta(x)-\frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^\alpha(\omega_2-\omega_1)}\left[\mathrm{J}^\alpha_{\dot{q},x^+}\Theta(\omega_2)+\mathrm{J}^\alpha_{\dot{q},x^-}\Theta(\omega_1)\right]\right|$$
$$\leq\frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}\left[\left(\mathrm{A}_{\dot{q}}(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{B}_{\dot{q}}(\theta,\alpha,\ell)\right)|\mathrm{D}_{\dot{q}}\Theta(x)|+\left(\mathrm{A}_{\dot{q}}(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{C}_{\dot{q}}(\theta,\alpha,\ell)\right)|\mathrm{D}_{\dot{q}}\Theta(\omega_2)|\right] \quad (24)$$
$$+\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}\left[\left(\mathrm{A}_{\dot{q}}(\varpi,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{B}_{\dot{q}}(\varpi,\alpha,\ell)\right)|\mathrm{D}_{\dot{q}}\Theta(x)|+\left(\mathrm{A}_{\dot{q}}(\varpi,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{C}_{\dot{q}}(\varpi,\alpha,\ell)\right)|\mathrm{D}_{\dot{q}}\Theta(\omega_1)|\right].$$

Corollary 9. *If we choose $\dot{q}\to 1^-$ in Theorem 6, we get*

$$\left|\frac{\theta(\omega_2-x)^\alpha(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)^\alpha(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^\alpha+(\omega_2-x)^\alpha}{\omega_2-\omega_1}\right)\Theta(x)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{\omega_2-\omega_1}\left[\mathrm{J}^\alpha_{x^+}\Theta(\omega_2)+\mathrm{J}^\alpha_{x^-}\Theta(\omega_1)\right]\right|$$
$$\leq\frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}\delta^{1-\frac{1}{\dot{q}_*}}(\theta,\alpha)\left[\left(\delta(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{B}(\theta,\alpha,\ell)\right)|\Theta'(x)|^{\dot{q}_*}+\left(\delta(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{C}(\theta,\alpha,\ell)\right)|\Theta'(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}} \quad (25)$$
$$+\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}\delta^{1-\frac{1}{\dot{q}_*}}(\varpi,\alpha)\left[\left(\delta(\varpi,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{B}(\varpi,\alpha,\ell)\right)|\Theta'(x)|^{\dot{q}_*}+\left(\delta(\varpi,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{C}(\varpi,\alpha,\ell)\right)|\Theta'(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}},$$

where

$$\mathrm{B}(\theta,\alpha,\ell):=\int_0^1(1-v)^\ell|\theta-v^\alpha|dv$$

and

$$\mathrm{C}(\theta,\alpha,\ell):=\int_0^1 v^\ell|\theta-v^\alpha|dv.$$

Corollary 10. *Taking $\alpha=1$ in Theorem 6, we obtain*

$$\left|\frac{\theta(\omega_2-x)(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\frac{\Theta(x)}{\dot{q}}-\frac{1}{\dot{q}(\omega_2-\omega_1)}\int_{\dot{q}\omega_1}^{\omega_2}\Theta(v)\,{}_{\omega_1}d_{\dot{q}}v\right|$$
$$\leq\frac{(\omega_2-x)^2}{\omega_2-\omega_1}\mathrm{A}_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\theta)\left[\left(\mathrm{A}_{\dot{q}}(\theta)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{B}_{\dot{q}}(\theta,\ell)\right)|\mathrm{D}_{\dot{q}}\Theta(x)|^{\dot{q}_*}+\left(\mathrm{A}_{\dot{q}}(\theta)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{C}_{\dot{q}}(\theta,\ell)\right)|\mathrm{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}} \quad (26)$$
$$+\frac{(x-\omega_1)^2}{\omega_2-\omega_1}\mathrm{A}_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\varpi)\left[\left(\mathrm{A}_{\dot{q}}(\varpi)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{B}_{\dot{q}}(\varpi,\ell)\right)|\mathrm{D}_{\dot{q}}\Theta(x)|^{\dot{q}_*}+\left(\mathrm{A}_{\dot{q}}(\varpi)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{C}_{\dot{q}}(\varpi,\ell)\right)|\mathrm{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}},$$

where

$$\mathrm{A}_{\dot{q}}(\theta):=\int_0^1|\theta-v|d_{\dot{q}}v,$$
$$\mathrm{B}_{\dot{q}}(\theta,\ell):=\int_0^1(1-v)^\ell|\theta-v|d_{\dot{q}}v$$

and

$$\mathrm{C}_{\dot{q}}(\theta,\ell):=\int_0^1 v^\ell|\theta-v|d_{\dot{q}}v.$$

Corollary 11. *Choosing $|\mathrm{D}_{\dot{q}}\Theta|\leq\mathcal{K}$ in Theorem 6, we have*

$$\left|\frac{\theta(\omega_2-x)^\alpha(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)^\alpha(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^\alpha+(\omega_2-x)^\alpha}{\omega_2-\omega_1}\right)\Theta(x)\right.$$
$$\left.+\frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^\alpha(\omega_2-\omega_1)}((x-\omega_1)^\alpha+(\omega_2-x)^\alpha)\Theta(x)-\frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^\alpha(\omega_2-\omega_1)}\left[J^\alpha_{\dot{q},x^+}\Theta(\omega_2)+J^\alpha_{\dot{q},x^-}\Theta(\omega_1)\right]\right|$$
$$\le \mathcal{K}\frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}A_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\theta,\alpha)\left[2A_{\dot{q}}(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}(B_{\dot{q}}(\theta,\alpha,\ell)+C_{\dot{q}}(\theta,\alpha,\ell))\right]^{\frac{1}{\dot{q}_*}} \tag{27}$$
$$+\mathcal{K}\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}A_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\varpi,\alpha)\left[2A_{\dot{q}}(\varpi,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}(B_{\dot{q}}(\varpi,\alpha,\ell)+C_{\dot{q}}(\varpi,\alpha,\ell))\right]^{\frac{1}{\dot{q}_*}}.$$

Corollary 12. *Taking $n=1$ in Theorem 6, we get*

$$\left|\frac{\theta(\omega_2-x)^\alpha(\Theta(\omega_2)-\Theta(x))-\varpi(x-\omega_1)^\alpha(\Theta(x)-\Theta(\omega_1))}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^\alpha+(\omega_2-x)^\alpha}{\omega_2-\omega_1}\right)\Theta(x)\right.$$
$$\left.+\frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^\alpha(\omega_2-\omega_1)}((x-\omega_1)^\alpha+(\omega_2-x)^\alpha)\Theta(x)-\frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^\alpha(\omega_2-\omega_1)}\left[J^\alpha_{\dot{q},x^+}\Theta(\omega_2)+J^\alpha_{\dot{q},x^-}\Theta(\omega_1)\right]\right|$$
$$\le \frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}A_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\theta,\alpha)\left[C_{\dot{q}}(\theta,\alpha)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}+(A_{\dot{q}}(\theta,\alpha)-C_{\dot{q}}(\theta,\alpha))|D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}} \tag{28}$$
$$+\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}A_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\varpi,\alpha)\left[C_{\dot{q}}(\varpi,\alpha)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}+(A_{\dot{q}}(\varpi,\alpha)-C_{\dot{q}}(\varpi,\alpha))|D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}},$$

where

$$C_{\dot{q}}(\theta,\alpha):=\int_0^1 v|\theta-v^\alpha|d_{\dot{q}}v.$$

Corollary 13. *Choosing $\theta=\varpi$ in Theorem 6, we obtain*

$$\left|\frac{\theta[(\omega_2-x)^\alpha(\Theta(\omega_2)-\Theta(x))-(x-\omega_1)^\alpha(\Theta(x)-\Theta(\omega_1))]}{\omega_2-\omega_1}+\left(\frac{(x-\omega_1)^\alpha+(\omega_2-x)^\alpha}{\omega_2-\omega_1}\right)\Theta(x)\right.$$
$$\left.+\frac{[\alpha]_{\dot{q}}(1-\dot{q})}{\dot{q}^\alpha(\omega_2-\omega_1)}((x-\omega_1)^\alpha+(\omega_2-x)^\alpha)\Theta(x)-\frac{\Gamma_{\dot{q}}(\alpha+1)}{\dot{q}^\alpha(\omega_2-\omega_1)}\left[J^\alpha_{\dot{q},x^+}\Theta(\omega_2)+J^\alpha_{\dot{q},x^-}\Theta(\omega_1)\right]\right|$$
$$\le \frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}A_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\theta,\alpha)$$
$$\times\left[\left(A_{\dot{q}}(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}B_{\dot{q}}(\theta,\alpha,\ell)\right)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}+\left(A_{\dot{q}}(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}C_{\dot{q}}(\theta,\alpha,\ell)\right)|D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}} \tag{29}$$
$$+\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}A_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\theta,\alpha)$$
$$\times\left[\left(A_{\dot{q}}(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}B_{\dot{q}}(\theta,\alpha,\ell)\right)|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}+\left(A_{\dot{q}}(\theta,\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}C_{\dot{q}}(\theta,\alpha,\ell)\right)|D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}.$$

Corollary 14. *Taking $\theta=0$ and $\alpha=1$ in Corollary 13, we have the following $\dot{q}$-integral inequality of Ostrowski type:*

$$\begin{aligned}
&\left|\Theta(x)-\frac{1}{\omega_2-\omega_1}\int_{\dot{q}\omega_1}^{\omega_2}\Theta(v)\,{}_{\omega_1}d_{\dot{q}}v\right| \\
&\leq \frac{\dot{q}(\omega_2-x)^2}{\omega_2-\omega_1}\left(\frac{1}{1+\dot{q}}\right)^{1-\frac{1}{\dot{q}_*}} \\
&\times\left[\left(\frac{1}{1+\dot{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\dot{q}}[\ell+2]_{\dot{q}}}\right)|\mathtt{D}_{\dot{q}}\Theta(x)|^{\dot{q}_*}+\left(\frac{1}{1+\dot{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+2]_{\dot{q}}}\right)|\mathtt{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}} \\
&+\frac{\dot{q}(x-\omega_1)^2}{\omega_2-\omega_1}\left(\frac{1}{1+\dot{q}}\right)^{1-\frac{1}{\dot{q}_*}} \\
&\times\left[\left(\frac{1}{1+\dot{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\dot{q}}[\ell+2]_{\dot{q}}}\right)|\mathtt{D}_{\dot{q}}\Theta(x)|^{\dot{q}_*}+\left(\frac{1}{1+\dot{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+2]_{\dot{q}}}\right)|\mathtt{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}.
\end{aligned} \tag{30}$$

Corollary 15. *Choosing $x=\frac{\omega_1+\omega_2}{2}$ in Corollary 14, we get the following $\dot{q}$-integral inequality of midpoint type:*

$$\begin{aligned}
&\left|\Theta\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{1}{\omega_2-\omega_1}\int_{\dot{q}\omega_1}^{\omega_2}\Theta(v)\,{}_{\omega_1}d_{\dot{q}}v\right| \\
&\leq \frac{\dot{q}(\omega_2-\omega_1)}{4}\left(\frac{1}{1+\dot{q}}\right)^{1-\frac{1}{\dot{q}_*}} \\
&\times\left\{\left[\left(\frac{1}{1+\dot{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\dot{q}}[\ell+2]_{\dot{q}}}\right)\left|\mathtt{D}_{\dot{q}}\Theta\left(\frac{\omega_1+\omega_2}{2}\right)\right|^{\dot{q}_*}+\left(\frac{1}{1+\dot{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+2]_{\dot{q}}}\right)|\mathtt{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right. \\
&\left.+\left[\left(\frac{1}{1+\dot{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\dot{q}}[\ell+2]_{\dot{q}}}\right)\left|\mathtt{D}_{\dot{q}}\Theta\left(\frac{\omega_1+\omega_2}{2}\right)\right|^{\dot{q}_*}+\left(\frac{1}{1+\dot{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+2]_{\dot{q}}}\right)|\mathtt{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*}\right]^{\frac{1}{\dot{q}_*}}\right\}.
\end{aligned} \tag{31}$$

4. Example and Application

4.1. Example

Let $\Theta(v)=\frac{v^{\frac{r+1}{\dot{q}_*}+1}}{\left[\frac{r+1}{\dot{q}_*}+1\right]_{\dot{q}}}$, where $r\in\mathtt{N}$, $\dot{q}_*\geq 1$ and $\dot{q}\in(0,1)$. After simple calculations, we have $\left|\mathtt{D}_{\dot{q}}\Theta(v)\right|^{\dot{q}_*}=v^{r+1}$, which shows that $|\mathtt{D}_{\dot{q}}\Theta(v)|^{\dot{q}_*}$ is convex function for all $v>0$ and $r\in\mathtt{N}$. Then, by applying Corollaries 2 and 10 with Remark 1 for specific values $n=1$, $\omega_1=0$, $\omega_2=1$ such that $x,\theta,\varpi\in[0,1]$, we deduce the following $\dot{q}$-inequalities:

$$\begin{aligned}
&\left|\theta(1-x)\left(\left[\frac{r+1}{\dot{q}_*}+1\right]_{\dot{q}}-x^{\frac{r+1}{\dot{q}_*}+1}\right)-\varpi x^{\frac{r+1}{\dot{q}_*}+2}+\frac{x^{\frac{r+1}{\dot{q}_*}+1}}{\dot{q}}-\frac{1}{\dot{q}\left[\frac{r+1}{\dot{q}_*}+2\right]_{\dot{q}}}\right| \\
&\leq\left[\frac{r+1}{\dot{q}_*}+1\right]_{\dot{q}}\left(\frac{\dot{q}_*}{1+\dot{q}_*}\right)^{\frac{1}{\dot{q}_*}}\left\{(1-x)^2\left(1+x^{r+1}\right)^{\frac{1}{\dot{q}_*}}\mathrm{A}_{\dot{q}}^{\frac{1}{p}}(\theta,p)+x^{\frac{r+1}{\dot{q}_*}+2}\mathrm{A}_{\dot{q}}^{\frac{1}{p}}(\varpi,p)\right\},
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\left|\theta(1-x)\left(\left[\frac{r+1}{\dot{q}_*}+1\right]_{\dot{q}}-x^{\frac{r+1}{\dot{q}_*}+1}\right)-\varpi x^{\frac{r+1}{\dot{q}_*}+2}+\frac{x^{\frac{r+1}{\dot{q}_*}+1}}{\dot{q}}-\frac{1}{\dot{q}\left[\frac{r+1}{\dot{q}_*}+2\right]_{\dot{q}}}\right| \\
&\leq\left[\frac{r+1}{\dot{q}_*}+1\right]_{\dot{q}}\left\{(1-x)^2\mathrm{A}_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\theta)\left[\left(x^{r+1}-1\right)\mathrm{C}_{\dot{q}}(\theta)+\mathrm{A}_{\dot{q}}(\theta)\right]^{\frac{1}{\dot{q}_*}}+x^{\frac{r+1}{\dot{q}_*}+2}\mathrm{A}_{\dot{q}}^{1-\frac{1}{\dot{q}_*}}(\varpi)\mathrm{C}_{\dot{q}}^{\frac{1}{\dot{q}_*}}(\varpi)\right\},
\end{aligned} \tag{33}$$

where

$$C_{\dot{q}}(\theta) := \int_0^1 v|\theta - v| d_{\dot{q}} v.$$

4.2. Application to Special Means

We consider the following arithmetic mean for real numbers ω_1 and ω_2 such that $0 \le \omega_1 < \omega_2$:

$$\mathcal{A}(\omega_1, \omega_2) = \frac{\omega_1 + \omega_2}{2}.$$

For the simplicity of notation, let

$$\Delta_{\dot{q}}(r, \dot{q}_*; \omega_1, \omega_2) := (1 - \dot{q})(\omega_2 - \dot{q}\omega_1) \sum_{n=0}^{\infty} \dot{q}^n (\dot{q}^n \omega_2 + (1 - \dot{q}^n)\dot{q}\omega_1)^{\frac{r+1}{\dot{q}_*}+1},$$

where $r \in \mathbb{N}$, $\dot{q}_* \ge 1$, and $\dot{q} \in (0, 1)$.

Proposition 1. *Let $n, r \in \mathbb{N}$, $\dot{q} \in (0, 1)$ and $\omega_1, \omega_2 \in \mathbb{R}$, where $0 \le \omega_1 < \omega_2$. Then for $p, \dot{q}_* > 1$ and $\frac{1}{p} + \frac{1}{\dot{q}_*} = 1$, we have*

$$\begin{aligned} &\left| \mathcal{A}^{\frac{r+1}{\dot{q}_*}+1}(\omega_1, \omega_2) - \frac{\Delta_{\dot{q}}(r, \dot{q}_*; \omega_1, \omega_2)}{\omega_2 - \omega_1} \right| \\ &\le \frac{2^{\frac{1}{\dot{q}_*}} \dot{q}(\omega_2 - \omega_1)}{4} \left(1 - \frac{1}{n} \sum_{\ell=1}^{n} \frac{1}{[\ell + 1]_{\dot{q}}} \right)^{\frac{1}{\dot{q}_*}} \\ &\times \left(\frac{1}{[p+1]_{\dot{q}}} \right)^{\frac{1}{p}} \left[\frac{r+1}{\dot{q}_*} + 1 \right]_{\dot{q}} \left\{ \mathcal{A}^{\frac{1}{\dot{q}_*}} \left(\omega_1^{r+1}, \mathcal{A}^{r+1}(\omega_1, \omega_2) \right) + \mathcal{A}^{\frac{1}{\dot{q}_*}} \left(\omega_2^{r+1}, \mathcal{A}^{r+1}(\omega_1, \omega_2) \right) \right\}. \end{aligned} \tag{34}$$

Proof. By applying Corollary 7 and Remark 1 with $\Theta(v) = \frac{v^{\frac{r+1}{\dot{q}_*}+1}}{\left[\frac{r+1}{\dot{q}_*}+1\right]_{\dot{q}}}$ for all $v \in [\omega_1, \omega_2]$, then we can get the desired result (34). □

Proposition 2. *Let $n, r \in \mathbb{N}$, $\dot{q} \in (0, 1)$ and $\omega_1, \omega_2 \in \mathbb{R}$, where $0 \le \omega_1 < \omega_2$. Then for $\dot{q}_* \ge 1$, we have*

$$\begin{aligned} &\left| \mathcal{A}^{\frac{r+1}{\dot{q}_*}+1}(\omega_1, \omega_2) - \frac{\Delta_{\dot{q}}(r, \dot{q}_*; \omega_1, \omega_2)}{\omega_2 - \omega_1} \right| \\ &\le \frac{\dot{q}(\omega_2 - \omega_1)}{4} \left(\frac{1}{1+\dot{q}} \right)^{1-\frac{1}{\dot{q}_*}} \left[\frac{r+1}{\dot{q}_*} + 1 \right]_{\dot{q}} \\ &\times \left\{ \left[\left(\frac{1}{1+\dot{q}} - \frac{1}{n} \sum_{\ell=1}^{n} \frac{1}{[\ell+1]_{\dot{q}}[\ell+2]_{\dot{q}}} \right) \mathcal{A}^{r+1}(\omega_1, \omega_2) + \left(\frac{1}{1+\dot{q}} - \frac{1}{n} \sum_{\ell=1}^{n} \frac{1}{[\ell+2]_{\dot{q}}} \right) \omega_1^{r+1} \right]^{\frac{1}{\dot{q}_*}} \right. \\ &+ \left. \left[\left(\frac{1}{1+\dot{q}} - \frac{1}{n} \sum_{\ell=1}^{n} \frac{1}{[\ell+1]_{\dot{q}}[\ell+2]_{\dot{q}}} \right) \mathcal{A}^{r+1}(\omega_1, \omega_2) + \left(\frac{1}{1+\dot{q}} - \frac{1}{n} \sum_{\ell=1}^{n} \frac{1}{[\ell+2]_{\dot{q}}} \right) \omega_2^{r+1} \right]^{\frac{1}{\dot{q}_*}} \right\}. \end{aligned} \tag{35}$$

Proof. By using Corollary 15 and Remark 1 with $\Theta(v) = \frac{v^{\frac{r+1}{\dot{q}_*}+1}}{\left[\frac{r+1}{\dot{q}_*}+1\right]_{\dot{q}}}$ for all $v \in [\omega_1, \omega_2]$, then we can obtain the desired result (35). □

5. Conclusions

In this article, we considered a new parameterized quantum fractional integral identity. By using this, we have established some quantum fractional integral inequalities of Ostrowski and midpoint type via n-polynomial convex functions. Many consequences and several special cases are analyzed and an example, and an application is given. Interested readers can use q̇-deformed real numbers [36] to extend our results. Furthermore, numerical analysis and comparison with fractional calculus, and quantum calculus, separately, can be done as well. We hope that this novel idea, which mixed together fractional calculus and quantum calculus, opens many avenues for interested researchers working in these fascinating fields and that they can discover further approximations for different kinds of convexity.

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