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Subclasses of Uniformly Convex Functions with Negative Coefficients Based on Deniz–Özkan Differential Operator

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Abstract: We introduce in this paper a new family of uniformly convex functions related to the Deniz–Özkan differential operator. By using this family of functions with a negative coefficient, we obtain coefficient estimates, the radius of starlikeness, convexity, and close-to-convexity, and we find their extreme points. Moreover, the neighborhood, partial sums, and integral means of functions for this new family are studied.

Keywords: parabolic starlike; neighborhood; partial sum; coefficient estimates; uniformly convex; differential operator; univalent functions

MSC: 0C45; 30C50; 30C80

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disc $\mathcal{U} = \{t : |t| < 1\}$ given by

$$h(t) = t + \sum_{s\geq 2} a_s t^s. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathcal{U}$ and $\mathcal{T}$ be the subclass of functions $\mathcal{A}$ given by

$$h(t) = t - \sum_{s\geq 2} a_s t^s, \ \ a_s \geq 0. \tag{2}$$

The function $h \in \mathcal{A}$ is said to be in $\beta - \mathcal{SP}(\eta)$, the class of $\beta-$parabolic starlike functions of the order η, $0 \leq \eta < 1$, if h is satisfies the condition

$$Re\left\{\frac{th'(t)}{h(t)}\right\} > \beta\left|\frac{th'(t)}{h(t)} - 1\right| + \eta, \quad \beta \geq 0. \tag{3}$$

By substituting th' for h in (3), we are able to derive the condition

$$Re\left\{1 + \frac{th''(t)}{h'(t)}\right\} > \beta\left|\frac{th''(t)}{h'(t)}\right| + \eta, \quad \beta \geq 0 \tag{4}$$

required for the function h to be in the subclass $\beta - \mathcal{UCV}(\eta)$ of the $\beta-$uniformly convex functions of order η.

These classes were defined by Bharati et al. [1] and generalized by other classes. For example, the class $\beta - \mathcal{UCV}(0) = \beta - \mathcal{UCV}$ is known as a $\beta-$uniformly convex function [2].

Indeed, it follows from (3) and (4) that

$$h \in \beta - \mathcal{UCV}(\eta) \Leftrightarrow th'(t) \in \beta - \mathcal{SP}(\eta).$$

Specifically, the classes $1-\mathcal{UCV}(0)=\mathcal{UCV}$ and $1-\mathcal{SP}(0)=\mathcal{SP}$ were defined by Goodman and Rønning [3,4], respectively.

For a function h in $\mathcal{A}$, Deniz and Özkan [5] (see also [6]) introduced the following differential operator $\mathcal{D}_\lambda^m$ as follows:

Definition 1. *If $h\in\mathcal{A}$, for the parameters $\lambda\geq 0$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ the differential operator $\mathcal{D}_\lambda^m$ on $\mathcal{A}$ is defined by*

$$\begin{aligned}\mathcal{D}_\lambda^0 h(t) &= h(t)\\ \mathcal{D}_\lambda^1 h(t) &= \lambda t^3 h'''(t)+(2\lambda+1)t^2h''(t)+th'(t)\\ \mathcal{D}_\lambda^m h(t) &= \mathcal{D}(\mathcal{D}_\lambda^{m-1}h(t))\end{aligned}$$

for $t\in\mathcal{U}$.

For the function h in $\mathcal{A}$, by the definition of the operator $\mathcal{D}_\lambda^m$, we see that

$$\mathcal{D}_\lambda^m h(t)=t+\sum_{s\geq 2}s^{2m}(\lambda(s-1)+1)^m a_s t^s. \tag{5}$$

Moreover, $\mathcal{D}_\lambda^m h(t)\in\mathcal{A}$.

For $h\in\mathcal{A}$ given by (1) and $g(t)\in\mathcal{A}$ of the form $g(t)=t+\sum_{s=2}^\infty b_s t^s$, the Hadamard product (or the convolution) of h and g is defined by

$$(h*g)(t)=t+\sum_{s\geq 2}a_s b_s t^s=(g*h)(t),\ \ t\in\mathcal{U}.$$

Sălăgean derivative operator $\mathcal{S}^m$ [7] is one of the special cases of the operator $\mathcal{D}_\lambda^m$ as follows:

$$\mathcal{D}_0^m h(t)=\mathcal{S}^m h(t)*\mathcal{S}^m h(t)=\mathcal{S}^{2m}h(t)$$

and

$$\mathcal{D}_1^m h(t)=\mathcal{S}^m h(t)*\mathcal{S}^m h(t)*\mathcal{S}^m h(t)=\mathcal{S}^{3m}h(t).$$

Let Ω be the class of functions $w(t)$ analytic in $\mathcal{U}$ such that $w(0)=0$, $|w(t)|<1$. If there is an analytical function $w(t)\in\Omega$ such that $h(t)=g(w(t))$, then $h(t)$ is said to be subordinate to $g(t)$ in $\mathcal{A}$. The sign for this subordination is $h(t)\prec g(t)$.

Lemma 1 ([8])**.** *Let $\xi\in\mathbb{C}$. Then,*

$$Re\xi\geq\rho,\ \text{if and only if } |\xi-(1+\rho)|\leq|\xi+(1-\rho)|.$$

Lemma 2 ([8])**.** *Let $\xi\in\mathbb{C}$, and ρ,γ are real numbers. Then,*

$$Re\xi>\rho|\xi-1|+\gamma \text{ if and only if } Re\Big\{\xi(1+\rho e^{i\theta})-\rho e^{i\theta}\Big\}>\gamma.$$

Definition 2. *For $0\leq\eta<1$, $0\leq\delta\leq 1$, $\beta\geq 0$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and $\lambda\geq 0$, we let $\beta-\mathcal{UCV}_\lambda^m(\eta,\delta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfying the analytic criterion*

$$Re\left\{\frac{t\big(\mathcal{D}_\lambda^m h(t)\big)'+\delta t^2\big(\mathcal{D}_\lambda^m h(t)\big)''}{(1-\delta)\mathcal{D}_\lambda^m h(t)+\delta t\big(\mathcal{D}_\lambda^m h(t)\big)'}\right\}\geq\beta\left|\frac{t\big(\mathcal{D}_\lambda^m h(t)\big)'+\delta t^2\big(\mathcal{D}_\lambda^m h(t)\big)''}{(1-\delta)\mathcal{D}_\lambda^m h(t)+\delta t\big(\mathcal{D}_\lambda^m h(t)\big)'}-1\right|+\eta.$$

We observe that the subclass $\beta-\mathcal{UCV}_\lambda^m(\eta,\delta)$ reduces to a number of well-known analytic function subclasses by specializing the parameters $\eta,\beta,\lambda,\delta$, and m. These subclasses include:

$$
\begin{array}{lcl}
0-\mathcal{UCV}_{\lambda}^{0}(0,0) & \equiv & \mathcal{S}^{*} \rightarrow \text{The class of starlike functions (see [9], pp. 40–43);} \\
0-\mathcal{UCV}_{\lambda}^{0}(0,1) & \equiv & \mathcal{C} \rightarrow \text{The class of convex functions (see [9], pp. 40–43);} \\
0-\mathcal{UCV}_{\lambda}^{0}(\eta,1) & \equiv & \mathcal{C}(\eta) \rightarrow \text{The class of convex functions of the order } \eta \text{ ([10]);} \\
0-\mathcal{UCV}_{\lambda}^{0}(\eta,0) & \equiv & \mathcal{S}^{*}(\eta) \rightarrow \text{The class of starlike functions of the order } \eta \text{ ([10]);} \\
1-\mathcal{UCV}_{\lambda}^{0}(0,0) & \equiv & \mathcal{SP} \rightarrow \text{The class of parabolic starlike functions ([11]);} \\
1-\mathcal{UCV}_{\lambda}^{0}(0,1) & \equiv & \mathcal{UCV} \rightarrow \text{The class of uniformly convex functions ([3,12]);} \\
\beta-\mathcal{UCV}_{\lambda}^{0}(0,0) & \equiv & \beta-\mathcal{SP} \rightarrow \text{The class of } \beta- \text{parabolic starlike functions ([13]);} \\
\beta-\mathcal{UCV}_{\lambda}^{0}(0,1) & \equiv & \beta-\mathcal{UCV} \rightarrow \text{The class of } \beta- \text{uniformly convex functions ([14]);} \\
1-\mathcal{UCV}_{\lambda}^{0}(2\rho-1,0) & \equiv & \mathcal{PS}^{*}(\rho)\ (\frac{1}{2} \leq \rho < 1) \rightarrow \text{The class of parabolic starlike functions of the order } \rho \text{ ([15]);} \\
1-\mathcal{UCV}_{\lambda}^{0}(2\rho-1,1) & \equiv & \mathcal{UCV}(\rho)\ (\frac{1}{2} \leq \rho < 1) \rightarrow \text{The class of parabolic convex functions of the order } \rho \text{ ([15]);} \\
1-\mathcal{UCV}_{\lambda}^{0}(\eta,0) & \equiv & \mathcal{SP}(\eta) \rightarrow \text{The class of parabolic starlike functions of the order } \eta \text{ ([11]);} \\
1-\mathcal{UCV}_{\lambda}^{0}(\eta,1) & \equiv & \mathcal{UCV}(\eta) \rightarrow \text{The class of uniformly convex functions of the order } \eta \text{ ([11]);} \\
\beta-\mathcal{UCV}_{\lambda}^{0}(\eta,0) & \equiv & \beta-\mathcal{SP}(\eta) \rightarrow \text{The class of } \beta- \text{uniformly starlike functions of the order } \eta \text{ ([1]);} \\
\beta-\mathcal{UCV}_{\lambda}^{0}(\eta,1) & \equiv & \beta-\mathcal{UCV}(\eta) \rightarrow \text{The class of } \beta- \text{uniformly convex functions of the order } \eta \text{ ([1]);} \\
0-\mathcal{UCV}_{0}^{m}(\eta,0) & = & \mathcal{ST}^{m}\ (\eta) \rightarrow \text{ (see [7]);} \\
\beta-\mathcal{UCV}_{0}^{m}(0,0) & = & \beta-\mathcal{SP}^{m} \rightarrow \text{ (see [2,16]);} \\
\beta-\mathcal{UCV}_{0}^{m}(\eta,0) & = & \mathcal{TS}(m,\beta,\eta) \rightarrow \text{ (see [17,18]).}
\end{array}
$$

We note that certain subclasses for specialization of the parameters $\delta = 0$ and $\delta = 1$ in the class $\beta-\mathcal{UCV}_{\lambda}^{m}(\eta,\delta)$ were studied by Deniz and Özkan [6] and Şeker et al. [19].

We also let $\beta-\mathcal{TUCV}_{\lambda}^{m}(\eta,\delta) = \beta-\mathcal{UCV}_{\lambda}^{m}(\eta,\delta) \cap \mathcal{T}$.

2. Main Results

2.1. Coefficients' Bounds and Extreme Points

We give here the coefficient estimates and extreme points for the functions $h(t)$ in the class $\beta-\mathcal{TUCV}_{\lambda}^{m}(\eta,\delta)$.

Theorem 1. *The class $\beta-\mathcal{TUCV}_{\lambda}^{m}(\eta,\delta)$ contains the functions $h(t)$ defined by (2) if and only if*

$$\sum_{s\geq 2}(\delta(s-1)+1)[s(\beta+1)-(\eta+\beta)]s^{2m}(1+\lambda(s-1))^{m}a_{s} \leq 1-\eta. \tag{6}$$

Proof. By Lemma 2 and Definition 2 we have

$$Re\left\{\frac{t\big(\mathcal{D}_{\lambda}^{m}h(t)\big)'+\delta t^{2}\big(\mathcal{D}_{\lambda}^{m}h(t)\big)''}{(1-\delta)\mathcal{D}_{\lambda}^{m}h(t)+\delta t\big(\mathcal{D}_{\lambda}^{m}h(t)\big)'}\Big(1+\beta e^{i\theta}\Big)-\beta e^{i\theta}\right\} \geq \eta, \quad -\pi<\theta\leq\pi$$

or equivalently

$$\begin{aligned} Re\Bigg\{ & \frac{\Big(z\big(\mathcal{D}_{\lambda}^{m}h(t)\big)'+\delta t^{2}\big(\mathcal{D}_{\lambda}^{m}h(t)\big)''\Big)(1+\beta e^{i\theta})}{(1-\delta)\mathcal{D}_{\lambda}^{m}h(t)+\delta t\big(\mathcal{D}_{\lambda}^{m}h(t)\big)'} \\ & -\frac{\beta e^{i\theta}\Big((1-\delta)\mathcal{D}_{\lambda}^{m}h(t)+\delta t\big(\mathcal{D}_{\lambda}^{m}h(t)\big)'\Big)}{(1-\delta)\mathcal{D}_{\lambda}^{m}h(t)+\delta t\big(\mathcal{D}_{\lambda}^{m}h(t)\big)'}\Bigg\} \geq \eta. \end{aligned} \tag{7}$$

Let

$$F(t) = \left(1+\beta e^{i\theta}\right)\left[t(\mathcal{D}_\lambda^m h(t))' + \delta t^2 (\mathcal{D}_\lambda^m h(t))''\right] - \beta e^{i\theta}\left[\mathcal{D}_\lambda^m h(t)(1-\delta) + \delta t(\mathcal{D}_\lambda^m h(t))'\right],$$

and

$$E(t) = (1-\delta)\mathcal{D}_\lambda^m h(t) + \delta t(\mathcal{D}_\lambda^m h(t))'.$$

The inequality 7, by Lemma 1, is equivalent for $0 \le \eta < 1$, to

$$|F(t) + (1-\eta)E(t)| \ge |F(t) - (1+\eta)E(t)|.$$

For the left side of last inequality, we obtain

$$\begin{aligned}
& |(1-\eta)E(t) + F(t)| \\
= & \left|(2-\eta)t - \sum_{s\ge 2}(1+s-\eta)(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s t^s\right. \\
& \left. -\beta e^{i\theta}\sum_{s\ge 2}(s-1)(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s t^s\right| \\
\ge & (2-\eta)|t| - \sum_{s\ge 2}(1+s-\eta)(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s |t|^s \\
& -\beta\sum_{s\ge 2}(s-1)(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s |t|^s.
\end{aligned}$$

Additionally, the right side can be written as

$$\begin{aligned}
& |F(t) - (1+\eta)E(t)| \\
= & \left|-\eta t - \sum_{s\ge 2}(s-1-\eta)(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s t^s\right. \\
& \left. -\beta e^{i\theta}\sum_{s\ge 2}(s-1)(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s t^s\right| \\
\ge & \eta|t| + \sum_{s\ge 2}(s-1-\eta)(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s |t|^s \\
& +\beta\sum_{s\ge 2}(s-1)(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s |t|^s
\end{aligned}$$

and so

$$\begin{aligned}
& |F(t) + (1-\eta)E(t)| - |F(t) - (1+\eta)E(t)| \\
\ge & 2(1-\eta)|t| - 2\sum_{s\ge 2}(1+\delta(s-1))(s(1+\beta) - \beta - \eta)s^{2m}(\lambda(s-1)+1)^m a_s |t|^s \ge 0.
\end{aligned}$$

The last expression above is equivalent to

$$\sum_{s\ge 2}(1+(s-1)\delta)[s(1+\beta) - (\eta+\beta)]s^{2m}(\lambda(s-1)+1)^m a_s \le 1-\eta.$$

Conversely, suppose that (6) holds. Then, we must show that by Lemma 2

$$Re\left\{\frac{\left[\left(\mathcal{D}_\lambda^m h(t)\right)' t + \delta t^2\left(\mathcal{D}_\lambda^m h(t)\right)''\right](1+\beta e^{i\theta}) - \beta e^{i\theta}\left[(1-\delta)\mathcal{D}_\lambda^m h(t) + \delta t\left(\mathcal{D}_\lambda^m f(t)\right)'\right]}{(1-\delta)\mathcal{D}_\lambda^m h(t) + \delta t\left(\mathcal{D}_\lambda^m h(t)\right)'}\right\} \ge \eta.$$

Upon choosing the values of t on the positive real axis where $0 \leq t = r < 1$, the above inequality becomes

$$Re\left\{\frac{(1-\eta) - \sum_{s\geq 2}\left[(1+\delta(s-1))\left(s\left(1+\beta e^{i\theta}\right) - \left(\eta + \beta e^{i\theta}\right)\right)\right]s^{2m}(\lambda(s-1)+1)^m a_s r^{s-1}}{1 - \sum_{s\geq 2}(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s r^{s-1}}\right\} \geq 0.$$

Since $Re(-e^{i\theta}) \geq -\left|e^{i\theta}\right| = -1$, the above inequality reduces to

$$Re\left\{\frac{(1-\eta) - \sum_{s\geq 2}[(1+\delta(s-1))(s(1+\beta) - (\eta+\beta))]s^{2m}(\lambda(s-1)+1)^m a_s r^{s-1}}{1 - \sum_{s\geq 2}(1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m a_s r^{s-1}}\right\} \geq 0.$$

We reach the desired conclusion by letting $r \to 1$. □

Corollary 1. *If $h(t) \in \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$, then*

$$a_s \leq \frac{(1-\eta)}{(1+\delta(s-1))(s(1+\beta) - (\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}.$$

Next, we obtain the extreme points for $\beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$.

Theorem 2. *Let $h_1(t) = t$ and*

$$h_s(t) = t - \frac{(1-\eta)}{(1+\delta(s-1))(s(1+\beta) - (\eta+\beta))s^{2m}(\lambda(s-1)+1)^m} t^s.$$

Then $h(t)$ is in the class $\beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$ if and only if it can be expressed in the form

$$h(t) = \sum_{s\geq 2} \sigma_s h_s(t)$$

where ($\sigma_s \geq 0$ and $\sum_{s\geq 1} \sigma_s = 1$ or $1 = \sigma_1 + \sum_{s\geq 2} \sigma_s$).

Proof. Let $h(t) = \sum_{s\geq 1} \sigma_s h_s(t)$ where $\sum_{s\geq 1} \sigma_s = 1$ and $\sigma_s \geq 0$. Then,

$$h(t) = t - \sum_{s\geq 2} \sigma_s A_s t^s,$$

where

$$A_s = \frac{(1-\eta)}{(1+(s-1)\delta)((1+\beta)s - (\beta+\eta))s^{2m}((s-1)\lambda+1)^m}$$

and we get

$$\sum_{s\geq 2} \sigma_s A_s \frac{1}{A_s} = \sum_{s\geq 2} \sigma_s = 1 - \sigma_1 \leq 1, \quad \text{(by Theorem 1)}.$$

Thus $h(t) \in \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$ from the Theorem 1.

Conversely, we suppose that $h(t)$ of type (2) belongs to $\beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$. Then,

$$a_s \leq \frac{(1-\eta)}{(1+(s-1)\delta)(s(\beta+1) - (\beta+\eta))s^{2m}(1+(s-1)\lambda)^m}, \ s \geq 2.$$

Setting

$$\sigma_s = \frac{(1+\delta(s-1))(s(1+\beta) - (\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}{(1-\eta)} a_s$$

and

$$\sigma_1 = 1 - \sum_{s\geq 2} \sigma_s,$$

we obtain

$$h(t) = \sum_{s\geq 1} \sigma_s h_s(t) = \sigma_1 h_1(t) + \sum_{s\geq 2} \sigma_s h_s(t).$$

This completes the proof. □

2.2. Distortion and Growth Theorems

We obtain the covering property, distortion, and growth theorems for functions from the new family.

Theorem 3. *Let $h(t) \in \beta - \mathcal{TUCV}_\lambda^m(\eta, \delta)$. Then,*

$$r - \frac{1-\eta}{(\delta+1)(\beta+2-\eta)(4(\lambda+1))^m} r^2 \leq |h(t)| \leq r + \frac{1-\eta}{(\delta+1)(\beta+2-\eta)(4(\lambda+1))^m} r^2 \quad (|t| = r).$$

The result is sharp with the extremal function h given by

$$h(t) = t - \frac{1-\eta}{(1+\delta)(\beta+2-\eta)(4(1+\lambda))^m} t^2.$$

Proof. From Theorem 1, since $h \in \beta - \mathcal{TUCV}_\lambda^m(\eta, \delta)$ we obtain

$$\begin{aligned} & 4^m(1+\delta)(\lambda+1)^m(\beta-\eta+2)\sum_{s\geq 2} a_s \\ \leq \quad & \sum_{s\geq 2}(1+(s-1)\delta)s^{2m}[(\beta+1)s-(\eta+\beta)]((s-1)\lambda+1)^m a_s \leq 1-\eta \end{aligned}$$

or

$$\sum_{s\geq 2} a_s \leq \frac{1-\eta}{(1+\delta)(\beta-\eta+2)(4(\lambda+1))^m}. \tag{8}$$

Thus, by Equation (8), we obtain

$$\begin{aligned} |h(t)| &= \left| t - \sum_{s\geq 2} a_s t^s \right| \leq |t| + \sum_{s\geq 2} a_s |t|^s \\ &\leq r + r^2 \sum_{s\geq 2} a_s \leq r + \frac{1-\eta}{(1+\delta)(\beta-\eta+2)(4(\lambda+1))^m} r^2 \end{aligned}$$

and similarly,

$$\begin{aligned} |h(t)| &= \left| t - \sum_{s\geq 2} a_s t^s \right| \geq |t| - \sum_{s\geq 2} a_s |t|^s \\ &\geq r - r^2 \sum_{s\geq 2} a_s \geq r - \frac{1-\eta}{(1+\delta)(\beta-\eta+2)(4(\lambda+1))^m} r^2. \end{aligned}$$

□

Corollary 2. *If $h(t) \in \beta - \mathcal{TUCV}_\lambda^m(\eta, 1)$, then*

$$r - \frac{1-\eta}{2(2-\eta+\beta)(4(1+\lambda))^m} r^2 \leq |h(t)| \leq r + \frac{1-\eta}{2(2-\eta+\beta)(4(\lambda+1))^m} r^2 \quad (|t| = r).$$

For the extremal function h given by

$$h(t) = t - \frac{1-\eta}{2(\beta+2-\eta)(4(1+\lambda))^m} t^2,$$

the result is sharp.

Theorem 4. *The disk* $|t| < 1$ *is mapped onto a domain that contains the disk*

$$|\xi| < 1 - \frac{1-\eta}{(1+\delta)(\beta+2-\eta)(4(1+\lambda))^m}$$

by any $h \in \beta - \mathcal{TUCV}_\lambda^m(\eta, \delta)$ *and onto a domain that contains the disk*

$$|\xi| < 1 - \frac{1-\eta}{2(\beta+2-\eta)(4(1+\lambda))^m}$$

by any $h \in \beta - \mathcal{TUCV}_\lambda^m(\eta, 1)$.

Proof. The results follow upon letting $r \to 1$ in Theorem 3 and its corollary. □

Theorem 5. *Let* $h \in \beta - \mathcal{TUCV}_\lambda^m(\eta, \delta)$*. Then*

$$1 - r\frac{2(1-\eta)}{(\beta+2-\eta)(4(1+\lambda))^m} \leq |h'(t)| \leq 1 + r\frac{2(1-\eta)}{(\beta+2-\eta)(4(1+\lambda))^m} \quad (|t| = r).$$

Proof. We have

$$|h'(t)| \leq 1 + \sum_{s \geq 2} a_s s |t|^{s-1} \leq 1 + r \sum_{s \geq 2} s a_s. \tag{9}$$

Since $h \in \beta - \mathcal{TUCV}_\lambda^m(\eta, \delta)$, according to Theorem 1

$$\begin{aligned} & (1+\delta)2^{2m-1}(\beta-\eta+2)(\lambda+1)^m \sum_{s \geq 2} s a_s \\ \leq & \sum_{s \geq 2} (1+(s-1)\delta) s^{2m} (s(\beta+1) - (\beta+\eta))(\lambda(s-1)+1)^m a_s \\ \leq & \; 1 - \eta \end{aligned}$$

or

$$\sum_{s \geq 2} s a_s \leq \frac{2(1-\eta)}{(1+\delta)(\beta+2-\eta)(4(\lambda+1))^m}. \tag{10}$$

In view of inequalities (9) and (10), we obtain

$$\begin{aligned} |h'(t)| &\leq 1 + \sum_{s \geq 2} s a_s |t|^{s-1} \\ &\leq 1 + r \sum_{s \geq 2} s a_s \\ &\leq 1 + r\frac{2(1-\eta)}{(1+\delta)(\beta-\eta+2)(4(1+\lambda))^m}. \end{aligned}$$

Similarly

$$|h'(t)| \geq 1 - r\frac{2(1-\eta)}{(1+\delta)(\beta+2-\eta)(4(1+\lambda))^m}$$

and thus, the proof is completed. □

Corollary 3. *If* $h \in \beta - \mathcal{TUCV}_\lambda^m(\eta, 1)$*, then*

$$1 - \frac{(1-\eta)}{(\beta-\eta+2)(4(\lambda+1))^m} r \leq |h'(t)| \leq 1 + \frac{(1-\eta)}{(\beta-\eta+2)(4(\lambda+1))^m} r \quad (|t| = r).$$

2.3. Neighborhoods and Partial Sums

We now extend the familiar concept of neighborhoods to the analytic functions of the family $\beta - \mathcal{TUCV}_\lambda^m(\eta, \delta)$. The concept of neighborhoods of analytic functions was first introduced by Goodman [3]. Later, Ruscheweyh [20] investigated this concept for the elements of several notable subclasses of analytic functions, and Altintaş and Owa [21] considered a certain family of analytic functions with negative coefficients. Moreover, Aouf [17,22] and Deniz and co-authors [23–25] studied this concept in certain families of analytic functions.

Definition 3. *Let $\beta \geq 0$, $0 \leq \eta < 1$, $\lambda \geq 0$, $0 \leq \delta \leq 1, \sigma \geq 0$, and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define the $\sigma-$ neighborhood of a function $h \in \mathcal{A}$ and denote it by $N_\sigma(h)$ consisting of all functions*

$$g(t) = t - \sum_{s\geq 2} b_s t^s \in \mathcal{A}$$

which satisifies

$$\sum_{s\geq 2} \frac{(1+\delta(s-1))(s(1+\beta)-(\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}{(1-\eta)}|a_s - b_s| \leq \sigma.$$

Theorem 6. *Let $h \in \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$, and, for all real θ, we have $\eta\left(e^{i\theta}-1\right)-2e^{i\theta} \neq 0$. For any complex number μ with $|\mu| < \sigma \quad (0 \leq \sigma)$, if h satisfies the following condition:*

$$\frac{h(t)+\mu t}{1+\mu} \in \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta),$$

then $N_\sigma(h) \subset \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$.

Proof. It is obvious that $h \in \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$ if and only if

$$\left| \frac{t\left(\mathcal{D}_\lambda^m h(t)\right)' + \delta t^2 \left(\mathcal{D}_\lambda^m h(t)\right)''\left(1+\beta e^{i\theta}\right) - \left(1+\beta e^{i\theta}+\eta\right)\left((1-\delta)\mathcal{D}_\lambda^m h(t) + \delta t\left(\mathcal{D}_\lambda^m h(t)\right)'\right)}{\left(t\left(\mathcal{D}_\lambda^m h(t)\right)' + \delta t^2 \left(\mathcal{D}_\lambda^m h(t)\right)''\right)\left(1+\beta e^{i\theta}\right) + \left(1-\beta e^{i\theta}-\eta\right)\left((1-\delta)\mathcal{D}_\lambda^m h(t) + \delta t\left(\mathcal{D}_\lambda^m h(t)\right)'\right)} \right| < 1$$

for any complex number α with $|\alpha| = 1$, and $-\pi < \theta < \pi$, we have

$$\frac{\left(t\left(\mathcal{D}_\lambda^m h(t)\right)' + \delta t^2 \left(\mathcal{D}_\lambda^m h(t)\right)''\right)\left(1+\beta e^{i\theta}\right) - \left(1+\beta e^{i\theta}+\eta\right)\left((1-\delta)\mathcal{D}_\lambda^m h(t) + \delta t\left(\mathcal{D}_\lambda^m h(t)\right)'\right)}{\left(t\left(\mathcal{D}_\lambda^m h(t)\right)' + \delta t^2 \left(\mathcal{D}_\lambda^m h(t)\right)''\right)\left(1+\beta e^{i\theta}\right) + \left(1-\beta e^{i\theta}-\eta\right)\left((1-\delta)\mathcal{D}_\lambda^m h(t) + \delta t\left(\mathcal{D}_\lambda^m h(t)\right)'\right)} \neq \alpha.$$

In other words, we must have

$$\begin{aligned} &(1-\alpha)\left((t\mathcal{D}_\lambda^m h(t))' + \delta t^2 (\mathcal{D}_\lambda^m h(t))''\right)\left(1+\beta e^{i\theta}\right)) \\ &-\left[\beta e^{i\theta} + 1 + \eta - \alpha\left(1+\eta-\beta e^{i\theta}\right)\right]\left((1-\delta)\mathcal{D}_\lambda^m h(t) + \delta t(\mathcal{D}_\lambda^m h(t))'\right) \neq 0 \end{aligned}$$

which is equivalent to

$$t - \sum_{s=2}^{\infty} \frac{\beta e^{i\theta}(1-\alpha)(s-1) + (1-\alpha)(s-\eta) - \alpha - 1}{\eta(\alpha-1) - 2\alpha} s^{2m}(\lambda(s-1)+1)^m (1+\delta(s-1)) a_s t^s \neq 0.$$

However, $h \in \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$ if and only if

$$\frac{(h*\phi)(t)}{t} \neq 0, \quad t \in \mathcal{U} - \{0\}$$

where

$$\phi(t) = t - \sum_{s\geq 2} e_s t^s,$$

and

$$e_s = \frac{\beta e^{i\theta}(1-\alpha)(s-1)+(s-\alpha)(1-\alpha)-\alpha-1}{\eta(\alpha-1)-2\alpha} s^{2m}(1+\lambda(s-1))^m(1+\delta(s-1)).$$

We note that

$$|e_s| \leq \frac{s(1+\beta)-(\eta+\beta)}{(1-\eta)}(\delta(s-1)+1)s^{2m}(\lambda(s-1)+1)^m.$$

Since

$$\frac{h(t)+\mu t}{1+\mu} \in \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta), \text{ therefore } t^{-1}\left(\frac{h(t)+\mu t}{1+\mu} * \phi(t)\right) \neq 0,$$

which is equivalent to

$$\frac{(h*\phi)(t)}{(1+\mu)t} + \frac{\mu}{1+\mu} \neq 0. \tag{11}$$

Now suppose that

$$\left|\frac{(h*\phi)(t)}{t}\right| < \sigma.$$

Then, by (11), we must have

$$\left|\frac{(h*\phi)(t)}{t(1+\mu)} + \frac{\mu}{1+\mu}\right| \geq \frac{|\mu|}{|1+\mu|} - \frac{1}{|1+\mu|}\left|\frac{(h*\phi)(t)}{t}\right| > \frac{|\mu|-\sigma}{|1+\mu|} \geq 0,$$

which is a contradiction to $|\mu| < \sigma$. However, we have

$$\left|\frac{(h*\phi)(t)}{t}\right| \geq \sigma.$$

If

$$g(t) = t - \sum_{s\geq 2} b_s t^s \in N_\sigma(h),$$

then

$$\begin{aligned}\sigma - \left|\frac{(g*\phi)(t)}{t}\right| &\leq \left|\frac{((h-g)*\phi)(t)}{t}\right| \leq \sum_{s\geq 2}|a_s - b_s||e_s||t^s| \\ &< \sum_{s\geq 2}\frac{s(1+\beta)-(\eta+\beta)}{(1-\eta)}|a_s-b_s|(1+s\delta-\delta)s^{2m}(\lambda(s-1)+1)^m \leq \sigma.\end{aligned}$$

We now conclude that

$$\frac{(g*\phi)(t)}{t} \neq 0,$$

which implies that $g \in \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta)$. □

Theorem 7. *Let $h \in \mathcal{A}$ be given by (2). Define $h_1(t) = t$, $h_k(t) = t - \sum_{s\geq 2}^k a_s t^s$, $(k = 2,3,\dots)$ and also suppose that $\sum_{s\geq 2} a_s c_s \leq 1$, where*

$$c_s = \frac{(1+(s-1)\delta)((\beta+1)s-(\beta+\eta))s^{2m}(\lambda(s-1)+1)^m}{(1-\eta)}.$$

Then

(i)

$$h \in \beta - \mathcal{TUCV}_\lambda^m(\eta,\delta),$$

(ii)

$$1-\frac{1}{c_{k+1}}<Re\left\{\frac{h(t)}{h_k(t)}\right\}<1+\frac{1}{c_{k+1}},\ Re\left\{\frac{h_k(t)}{h(t)}\right\}>\frac{c_{k+1}}{1+c_{k+1}},\ t\in\mathcal{U},k=2,3,\ldots.$$

Proof. (i) Since

$$\frac{t+\mu t}{1+\mu}=t\in\beta-\mathcal{TUCV}_{\lambda}^{m}(\eta,\delta),|\mu|<1,$$

by Theorem 6 we have $N_1(t)\subset\beta-\mathcal{TUCV}_{\lambda}^{m}(\eta,\delta)$ ($N_1(t)$ denoting the one-neighborhood). Now, since $\sum_{s\geq 2}c_s a_s\leq 1, h\in N_1(t)$ and $h\in\beta-\mathcal{TUCV}_{\lambda}^{m}(\eta,\delta)$,

(ii) we have

$$c_s=\left[\frac{(s-1)(1+\beta)}{1-\eta}+1\right](1+\delta(s-1))s^{2m}(\lambda(s-1)+1)^m,$$

and $\{c_s\}$ is an increasing sequence. So, we obtain

$$\sum_{s=2}^{k}a_s+c_{k+1}\sum_{s\geq k+1}a_s\leq 1. \tag{12}$$

Now, by introducing $G_1(t)$ given by

$$G_1(t)=c_{k+1}\left[\frac{h(t)}{h_k(t)}-\left(1-\frac{1}{c_{k+1}}\right)\right]$$

and making use of (12), we obtain

$$\left|\frac{G_1(t)-1}{G_1(t)+1}\right|=\left|\frac{-c_{k+1}\sum_{s\geq k+1}a_s t^s}{2-c_{k+1}\sum_{s\geq k+1}a_s t^s}\right|<\frac{c_{k+1}\sum_{s\geq k+1}a_s}{2-c_{k+1}\sum_{s\geq k+1}a_s-\sum_{s\geq 1}^{k}a_s}<1.$$

Therefore, $ReG_1(t)>0$, and we obtain

$$Re\left\{\frac{h(t)}{h_k(t)}\right\}>1-\frac{1}{c_{k+1}}.$$

Now, let

$$G_2(t)=c_{k+1}\left[\frac{h(t)}{h_k(t)}-\left(1+\frac{1}{c_{k+1}}\right)\right],$$

then, we have

$$\left|\frac{G_2(t)+1}{G_2(t)-1}\right|=\left|\frac{-c_{k+1}\sum_{s\geq k+1}a_s t^s}{-c_{k+1}\sum_{s\geq k+1}a_s t^s-2}\right|<\frac{c_{k+1}\sum_{s\geq k+1}a_s}{2-c_{k+1}\sum_{s\geq k+1}a_s-\sum_{s\geq 1}^{k}a_s}<1.$$

Therefore, $ReG_2(t)<0$, and we obtain

$$Re\left\{\frac{h(t)}{h_k(t)}\right\}<1+\frac{1}{c_{k+1}}.$$

For the second inequality, we define

$$F(t)=(1+c_{k+1})\left[\frac{h_k(t)}{h(t)}-\frac{c_{k+1}}{1+c_{k+1}}\right],$$

then by using (12), we obtain

$$\begin{aligned}\left|\frac{F(t)-1}{F(t)+1}\right| &= \left|\frac{(1+c_{k+1})(h_k(t)-h(t))}{(1+c_{k+1})h_k(t)-(c_{k+1}-1)h(t)}\right| \\ &= \left|\frac{(1+c_{k+1})\sum_{s=k+1}^{\infty}a_s t^{s-1}}{2+(c_{k+1}-1)\sum_{s=k+1}^{\infty}a_s t^{s-1}-2\sum_{s=2}^{k}a_s t^{s-1}}\right| \\ &\le \frac{(1+c_{k+1})\sum_{s=k+1}^{\infty}a_s}{2-2\sum_{s\ge 2}^{k}a_s+(1-c_{k+1})\sum_{s\ge k+1}a_s} \le 1.\end{aligned}$$

This shows that $ReF(t) > 0$, and, finally,

$$Re\left\{\frac{h_k(t)}{h(t)}\right\} > \frac{c_{k+1}}{1+c_{k+1}}.$$

□

Theorem 8. *Let the conditions be as in Theorem 7. Then, for $k = 2, 3, \ldots,$ we have*

$$1-\frac{k+1}{c_{k+1}} < Re\left\{\frac{h'(t)}{h'_k(t)}\right\}, \quad Re\left\{\frac{h'_k(t)}{h'(t)}\right\} > \frac{c_{k+1}}{k+1+c_{k+1}}, \quad t\in\mathcal{U},\ k=2,3,\ldots.$$

Proof. The proof of *Theorem* 8 is similar to that of Theorem 7. □

2.4. Radius of Close-to-Convexity, Starlikeness, and Convexity

We focus on obtaining the radii of convexity, starlikeness, and close-to-convexity.

Theorem 9. *Let the function $h(t)$, defined by (2), be in the class $\beta-\mathcal{TUCV}_{\lambda}^{m}(\eta,\delta)$. Then, $h(t)$ is close-to-convex of the order ε $(0\le\varepsilon<1)$ in $|z| < r_1(\eta,\beta,\delta,\varepsilon,m)$, where*

$$r_1(\eta,\beta,\delta,\varepsilon,m) = \inf_{s\ge 2}\left\{\frac{(1-\varepsilon)(1+\delta(s-1))(s(1+\beta)-(\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}{s(1-\eta)}\right\}^{\frac{1}{s-1}}. \quad (13)$$

Proof. We must prove that

$$|h'(t)-1| \le 1-\varepsilon \text{ where } |t| < r_1(\eta,\beta,\delta,\varepsilon,m).$$

We have

$$|h'(t)-1| \le \sum_{s\ge 2} s a_s |t|^{s-1}.$$

Thus, $|h'(t)-1| \le 1-\varepsilon$ if

$$\sum_{s\ge 2}\left(\frac{s}{1-\varepsilon}\right)a_s|t|^{s-1} \le 1. \quad (14)$$

From Theorem 1 , we obtain

$$\sum_{s\ge 2}\frac{(1+\delta(s-1))[(\beta+1)s-(\eta+\beta)]s^{2m}(\lambda(s-1)+1)^m}{(1-\eta)}a_s \le 1. \quad (15)$$

Hence, (14) is true if

$$\frac{s|t|^{s-1}}{1-\varepsilon} \le \frac{(1+\delta(s-1))[s(1+\beta)-(\beta+\eta)]s^{2m}(\lambda(s-1)+1)^m}{(1-\eta)}$$

Equivalently,

$$|t| \leq \left\{ \frac{(1-\varepsilon)(1+\delta(s-1))(s(1+\beta)-(\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}{s(1-\eta)} \right\}^{\frac{1}{s-1}}, \; s \geq 2. \tag{16}$$

The theorem follows from (16). □

Theorem 10. *If $h(t)$ is of the form (2) and is in the class $\beta - \mathcal{TUCV}_{\lambda}^{m}(\eta,\delta)$, then $h(t)$ is starlike of the order ε $(0 \leq \varepsilon < 1)$ in $|t| < r_2(\eta,\beta,\delta,\varepsilon,m)$, where*

$$r_2(\eta,\beta,\delta,\varepsilon,m) = \inf_{s\geq 2}\left\{ \frac{(1-\varepsilon)((s-1)\delta+1)((\beta+1)s-(\beta+\eta))s^{2m}(1+(s-1)\lambda)^m}{(1-\eta)(s-\varepsilon)} \right\}^{\frac{1}{s-1}}. \tag{17}$$

Proof. It suffices to show that

$$\left| \frac{th'(t)}{h(t)} - 1 \right| \leq 1-\varepsilon \text{ for } |t| < r_2(\eta,\beta,\delta,\varepsilon,m).$$

We have

$$\left| \frac{th'(t)}{h(t)} - 1 \right| \leq \frac{\sum_{s\geq 2}(s-1)a_s|t|^{s-1}}{1-\sum_{s\geq 2}a_s|t|^{s-1}}.$$

Thus,

$$\left| \frac{th'(t)}{h(t)} - 1 \right| \leq 1-\varepsilon \text{ if } \frac{\sum_{s\geq 2}(s-\varepsilon)a_s|t|^{s-1}}{(1-\varepsilon)} \leq 1 \tag{18}$$

by using (15), (18) is true if

$$\frac{s-\varepsilon}{1-\varepsilon}|t|^{s-1} \leq \frac{(1+\delta(s-1))(s(1+\beta)-(\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}{(1-\eta)}$$

or, equivalently,

$$|t| \leq \left\{ \frac{(1-\varepsilon)(1+\delta(s-1))(s(1+\beta)-(\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}{(s-\varepsilon)(1-\eta)} \right\}^{\frac{1}{s-1}}, \; s \geq 2.$$

The theorem follows easily from the last expression. □

Theorem 11. *If h is of the form (2) and is in $\beta - \mathcal{TUCV}_{\lambda}^{m}(\eta,\delta)$, then $h(t)$ is convex of the order ε for $0 \leq \varepsilon < 1$ in $|t| < r_3(\eta,\beta,\delta,\varepsilon,m)$, where*

$$r_3(\eta,\beta,\delta,\varepsilon,m) = \inf_{s\geq 2}\left\{ \frac{(1-\varepsilon)(\delta(s-1)+1)(s(1+\beta)-(\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}{s(s-\varepsilon)(1-\eta)} \right\}^{\frac{1}{s-1}}.$$

Proof. We show that

$$\left| \frac{th''(t)}{h'(t)} \right| \leq 1-\varepsilon \text{ for } |t| < r_3(\eta,\beta,\delta,\varepsilon,m). \tag{19}$$

Substituting the series expansions $h''(t)$ and $h'(t)$ into the left side of (19), we obtain

$$\left| \frac{-\sum_{s\geq 2}s(s-1)a_s t^{s-1}}{1-\sum_{s\geq 2}s a_s t^{s-1}} \right| \leq \frac{\sum_{s\geq 2}s(s-1)a_s|t|^{s-1}}{1-\sum_{s\geq 2}s a_s|t|^{s-1}}.$$

The last expression above is bounded by $(1-\varepsilon)$ if

$$\sum_{s\geq 2}\frac{s(s-\varepsilon)}{1-\varepsilon}a_s|t|^{s-1}\leq 1. \tag{20}$$

In view of (19), it follows that (20) is true if

$$\frac{s(s-\varepsilon)}{1-\varepsilon}|t|^{s-1}\leq\left(\frac{(1+\delta(s-1))(s(1+\beta)-(\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}{(1-\eta)}\right)$$

or

$$|t|<\left\{\frac{(1-\varepsilon)(1+\delta(s-1))(s(1+\beta)-(\eta+\beta))s^{2m}(\lambda(s-1)+1)^m}{s(s-\varepsilon)(1-\eta)}\right\}^{\frac{1}{s-1}}$$

and this complete the proof. □

2.5. Integral Means

We will need Littlewood's [26] subordination result for the investigation that follows.

Lemma 3. *If $h(t)$ and $g(t)$ are analytic in $\mathcal{U}$ with $h(t)\prec g(t)$, then*

$$\int_0^{2\pi}\left|h(re^{i\theta})\right|^{\mu}d\theta\leq\int_0^{2\pi}\left|g(re^{i\theta})\right|^{\mu}d\theta,$$

where $\mu>0$, $t=re^{i\theta}$, and $0<r<1$.

Applying Lemma 3 to functions $h(t)$ in the classes $\beta-\mathcal{TUCV}_{\lambda}^{m}(\eta,\delta)$ gives the following result when utilizing established methods.

Theorem 12. *Let $\mu>0$. If $h(t)\in\beta-\mathcal{TUCV}_{\lambda}^{m}(\eta,\delta)$ is of the form (2), and $h_2(t)$ is defined by*

$$h_2(t)=t-\frac{1-\eta}{(\delta+1)(\beta+2-\eta)(4(\lambda+1))^m}t^2,$$

then we obtain

$$\int_0^{2\pi}|h(t)|^{\mu}d\theta\leq\int_0^{2\pi}|h_2(t)|^{\mu}d\theta, \tag{21}$$

for $t=re^{i\theta}$, $0<r<1$.

Proof. For $h(t)=t-\sum_{s\geq 2}^{\infty}a_st^s$, (21) is equivalent to proving that

$$\int_0^{2\pi}\left|1-\sum_{s\geq 2}a_st^{s-1}\right|^{\mu}d\theta\leq\int_0^{2\pi}\left|1-\frac{1-\eta}{(1+\delta)(\beta-\eta+2)(4(\lambda+1))^m}t\right|^{\mu}d\theta.$$

By Lemma 3, it suffices to show that

$$1-\sum_{s\geq 2}a_st^{s-1}\prec 1-\frac{1-\eta}{(1+\delta)(\beta-\eta+2)(4(\lambda+1))^m}t.$$

By definition of the subordination, we can write

$$1-\sum_{s\geq 2}a_st^{s-1}=1-\frac{1-\eta}{(1+\delta)(\beta-\eta+2)((1+\lambda)4)^m}w(t) \tag{22}$$

and, thus, from (22) and (6),

$$
\begin{aligned}
|w(t)| &= \left|\sum_{s\geq 2}\frac{(1+\delta)(\beta-\eta+2)(4(\lambda+1))^m}{1-\eta}a_s t^{s-1}\right| \\
&\leq |t|\sum_{s\geq 2}\frac{(1+\delta)(\beta-\eta+2)(4(\lambda+1))^m}{1-\eta}a_s \\
&\leq |t|\sum_{s\geq 2}\frac{(1+\delta(s-1))(s(\beta+1)-\beta-\eta)s^{2m}(1+\lambda(s-1))^m}{1-\eta}a_s \leq |t|,
\end{aligned}
$$

and the proof is completed. □

Similarly, we can prove the following result.

Corollary 4. *Let $\mu > 0$. If $h(t) \in \beta - \mathcal{TUCV}_\lambda^m(\eta, 1)$ is given by (2), and $h_2(t)$ is defined by*

$$h_2(t) = t - \frac{1-\eta}{2^{2m+1}(\beta-\eta+2)(\lambda+1)^m}t^2,$$

then for $t = re^{i\theta}, 0 < r < 1$, we have

$$\int_0^{2\pi}|h(t)|^\mu d\theta \leq \int_0^{2\pi}|h_2(t)|^\mu d\theta.$$

Remark 1. *By putting $\delta = \lambda = 0$ into all of the above results, we obtain the related results obtained by Rosy and Murugusundaramoorthy [18] and Aouf [17]. Moreover, if we use $\delta = m = 0$ and $\delta - 1 = m = 0$ in all the above results, we obtain the related results obtained by Bharati [1].*

3. Conclusions

This paper makes a modest effort to introduce the class $\beta - \mathcal{TUCV}_\lambda^m(\eta, \delta)$. This offers an intriguing changeover from uniformly convex functions, combining the concept of the differential operator $\mathcal{D}_\lambda^m$. We derived a coefficient formula, neighborhoods, partial sums, radii of close-to-convexity, starlikeness and convexity, covering, distortion theorems, and the integral mean inequalities for functions in our class. In special cases, our findings contain the results obtained by some of the authors cited in the references. These results will open up many new opportunities for research in this field and related fields. Using the operator $\mathcal{D}_\lambda^m$, someone can define different general subclasses of analytic functions. For these subclasses, some problems, such as subordination, inclusion, coefficients, and covering theorems of the Geometric Function Theory, can be solved.

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