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Bayesian Statistical Method Enhance the Decision-Making for Imperfect Preventive Maintenance with a Hybrid Competing Failure Mode

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Abstract: The study aims to provide a Bayesian statistical method with natural conjugate for facilities' preventive maintenance scheduling related to the hybrid competing failure mode. An effective preventive maintenance strategy not only can improve a system's health condition but also can increase a system's efficiency, and therefore a firm needs to make an appropriate strategy for increasing the utilization of a system with reasonable costs. In the last decades, preventive maintenance issues of deteriorating systems have been studied in the related literature, and hundreds of maintenance/replacement models have been created. However, few studies focused on the issue of hybrid deteriorating systems which are composed of maintainable and non-maintainable failure modes. Moreover, due to the situations of the scarcity of historical failure data, the related analyses of preventive maintenance would be difficult to perform. Based on the above two reasons, this study proposed a Bayesian statistical method to deal with such preventive maintenance problems. Non-homogeneous Poisson processes (NHPP) with power law failure intensity functions are employed to describe the system's deterioration behavior. Accordingly, the study can provide useful ways to help managers to make effective decisions for preventive maintenance. To apply the proposed models in actual cases, the study provides solution algorithms and a computerized architecture design for decision-makers to realize the computerization of decision-making.

Keywords: Bayesian statistics; non-homogeneous Poisson process; Monte Carlo integration; preventive maintenance; hybrid failure modes

MSC: 62F15; 62N02; 62N05; 62C10; 65C20

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1. Introduction

Preventive maintenance (PM) is in charge of maintaining equipment or facilities (repairable systems) in good condition. However, they might cause catastrophic damage and consequential losses when such equipment or facilities in production lines fail. As a result, it is critical to proceed with adequately preventative maintenance to avoid any damages and losses in order to keep these equipment or facilities in a healthy state. Preventive maintenance is capable of delaying system deterioration and returning systems to better condition, lowering failure rates and extending system lifetime. In other words, preventative maintenance has a significant influence on quality and cost, and therefore a firm should be concerned with developing appropriate maintenance programs in order to boost its competitiveness. There has been a lot of attention previous research works in the field of preventive maintenance modeling and optimization during the last several years.

Usually, preventive maintenance policies are mostly based on time intervals. There are two PM policies found in the literature: periodic and sequential (non-periodic). Periodic preventive maintenance policy is the manufacturer provides its maintenance work with equal time intervals. A sequential preventive maintenance policy is characterized by the search for the optimal number of maintenance actions at the optimal intervals. Therefore, the sequential preventive policy may provide an unequal sequence of intervals for minimizing the related costs. Park et al. [1] calculated the ideal duration and amount of PM activities based on a periodic PM program with minimum repair service after breakdowns. Yeh and Lo [2] demonstrated that the ideal PM interval between two consecutive PM activities is equal to the degree of PM and that providing an equivalent degree of PM is the best approach to decrease predicted warranty costs. Jung and Park [3] proposed an optimum post-warranty PM policy by reducing predicted long-run PM expenses. Seo and Bai [4] illustrated a periodic PM strategy for two scenarios where the operating time of PM might be disregarded or not. Yeh and Chang [5] determined the best failure rate and maintenance strategy for the lifetime of the equipment. Das and Sarmah [6] provided an overview of optimization models for preventative replacement with the related constraints in heavy-process industries. Yeh et al. [7] evaluated the impact of various PM cost functions on a leased product with a Weibull lifetime distribution's periodic PM policy. Bouguerra et al. [8] proposed a mathematical model for various PM plans when customers choose to purchase an extended warranty. They discovered a viable compromise to establish a win-win situation between producers and customers in terms of warranty costs. Chang and Lin [9] developed an ideal PM policy for repairable items with extended warranties. They considered that manufacturers could give a slight discount to consumers to incent their intention to purchase extended warranty contracts. Under the context of reliability-based optimization, Beaurepaire et al. [10] proposed the best model of mechanical component maintenance scheduling. Their model is different from the traditional approaches based on linear fracture mechanics. Schutz and Rezg [11] provided a methodology for determining an optimal product maintenance program to guarantee that the minimal reliability meets the consumers' requirements. Kim and Ozturkoglu [12] reduced the classic preventive maintenance issue to an integers programming problem. Khojandi et al. [13] investigated the optimal lifetime reward maintenance strategies for perfect and imperfect maintenance situations. They showed the tradeoff between the virtual age of the systems and the incentive rate for decision-makers. Yuan and Lu [14] suggested an effective way to solve a reliability-based optimization issue that combines the weighted approach with sequential approximation optimization. Lu et al. [15] proposed a joint model of sequential PM and quality improvement for deteriorating systems in manufacturing industries. The study is superior and more appropriate for maintaining machines in a production system. Wang and Djurdjanovic [16] also proposed a joint model for PM scheduling with consideration of stocks and logistics issues. They proposed an integrated policy to trigger PM for working parts. Zhou et al. [17] proposed a sequential PM model with a reducing failure factor. The model can be applied to urban bus systems' maintenance works. García and Salgado [18] presented a case study to analyze the selection of PM strategies in multistage industrial facilities and equipment. Their study utilized some individual indicators to evaluate which PM strategies are better. Diatte et al. [19] proposed a methodology for improving brake systems in automobile industries, and the methodology can make the integration of the machine's reliability, availability, and maintainability into system engineering and dependability analyses for reducing related costs and increasing system's reliability.

Some related studies regarding the issues of competing failures or risks models have been proposed or developed in the past. Generally, maintainable and non-maintainable failure modes compete to cause the system to fail. Furthermore, the three maintenance operations should be taken into account: imperfect preventative maintenance, minor repairs, and replacements. Some competing risk models were also proposed to reflect the complex systems' deterioration [20–22]. Salinas-Torres et al. [23] proposed a competing-risks model with a Bayesian statistical method to estimate a system component's survival time. They used Dirichlet multivariate processes to deduce the parameters' estimator. Yousef et al. [24] applied Bayesian and Non-Bayesian analyses to the reliability of the stress-strength system. The proposed Bayesian estimators can be achieved by the Markov chain Monte Carlo method. Wang and Miao [25] applied a semi-Markov model to optimize firms' preventive maintenance policy. Their model would be useful to any two symmetric components to avoid system unbalance. Alotaibi et al. [26] also utilized a Bayesian analysis and Monte-Carlo simulations to estimate the parameters of a mixture bivariate exponential model. They applied the proposed method to motor data analysis. Yousef et al. [27] applied a Bayesian estimation method and Monte Carlo simulation to evaluate a system reliability. They also compared the performance between maximum likelihood estimators and Bayesian estimators. Liu et al. [28] applied a Bayesian estimation method to evaluate products' reliability. This method can achieve both point and confidence interval estimation for the critical parameters. Zequeira and Berenguer [29] proposed a hybrid model in which the two failure modes (maintainable and non-maintainable) were believed to be dependent. It is intriguing to investigate the impact of the two failure models. El-Ferik and Ben-Daya [30] also proposed a hybrid model that includes adjustment variables in both the hazard rate and the effective age. In El-Ferik and Ben-Daya's model, the effect of PM is assumed to be imperfect. Kahrobaee and Asgarpoor [31] proposed a hybrid analytical-simulation approach to solve deteriorating equipment's PM works under some systems' constraints. They apply their approach to the wind turbines industry, and expected rewards and penalties are also taken into consideration. Rafiee et al. [32] proposed a condition-based PM policy considering competing risks of internal deterioration and external shocks. In their model, the external shocks arrive at random times and can be divided into two categories based on their impacts on the system: (1) fatal shocks that can cause the system to fail immediately (2) non-fatal shocks that can only damage the system by randomly. Their model can apply in micro-electro-mechanical systems. Zhou et al. [33] proposed a hybrid PM model to reflect the reliability status of leased equipment. Furthermore, it also can clearly discriminate between the impacts from external shocks and from internal deterioration. Yang et al. [34] modeled complex industrial systems involving a hybrid failure mode, degradation-based failure and sudden failure. Their proposed condition-based PM strategy can be applied in oil pipeline industries. Cao [35] also proposed a condition-based PM strategy, but he took competing failure processes into consideration. He successfully utilized the condition-based PM strategy and a genetic algorithm to make periodic inspection and maintenance plans for wind-driven generators. Liu et al. [36] proposed hybrid maintenance models to deal with the issue of competing failures. In their model, all the maintenance actions are perfect, and all the actions are instant, it implies that the time cost can be negligible. However, their model didn't take imperfect repair issues into consideration. Basílio et al. [37] proposed a review study regarding the applications of multi-criteria decision aid methods. They provided a complete overview of multi-criteria methods to comprehend the current and future development patterns of multi-criteria decision-making studies. The study can give useful research directions for applying multi-criteria techniques in industries.

Based on the above considerations, this work provided Bayesian decision models to cope with preventative maintenance with the hybrid deteriorating system. The deteriorating behavior of the repairable system can be described by a non-homogeneous Poisson process (NHPP) with a hybrid of power law failure intensity functions. Furthermore, the study investigates how different levels of preventive maintenance influence the related costs and the frequency of the system's breakdowns. Accordingly, we consider the study has the following advantages: (1) Few studies focused on the issue of hybrid deteriorating systems, which are composed of maintainable and non-maintainable failure modes. Furthermore, due to some situations of the scarcity of historical failure data, the related analyses of preventive maintenance will be difficult to perform. For the above two reasons, this study proposed Bayesian decision models to deal with such a preventive maintenance problem with the hybrid deteriorating system. (2) The study's Bayesian decision process can be divided into two phases. In the first phase, the decision maker can use the domain experts' knowledge and judgment to evaluate the deterioration using four prior distribution statistical characteristics. In the second phase, the decision maker can collect failure data to re-estimate the deterioration by using the NHPP likelihood functions and the properties of the natural conjugate. It can effectively extend the related applications of the study. (3) In order to apply the proposed models in actual cases, the study provides solution algorithms and a computerized architecture design for decision-makers to realize the computerization of decision-making. Accordingly, the study can easily be applied to various manufacturing industries that want to effectively manage their durable and high-priced equipment or facilities in the factory.

The rest of this paper is organized as follows: Section 2 presents the proposed hybrid competing mode, the estimation of a deteriorating system's failures under periodic preventive maintenance, and the estimation of the related costs. Section 3 provides a Bayesian decision process by using domain experts' knowledge with collected information. The design of a computerized information system is also presented in the section. Section 4 present the numerical application and sensitivity analysis. Finally, Section 5 presents the concluding remarks and the future study.

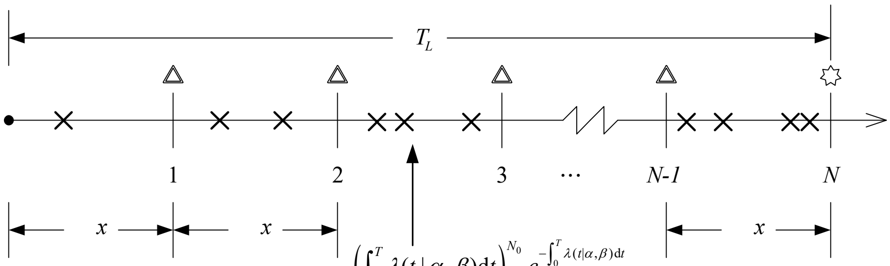
2. Preventive Maintenance for Deteriorating Systems with a Hybrid Deterioration

Both non-maintainable and maintainable failure modes may exist in a multi-component system, and therefore they will compete to cause failures. In the maintainable failure mode, a component can be replaced with a new one to restore the system to its original status. However, in the non-maintainable failure mode, a component cannot be replaced to alleviate the system's aging. From the whole system's perspective, the effect of preventive maintenance can only present imperfect recovery. Therefore, a hybrid model which is related to competing failure processes would be more adequate for measuring such deterioration model in practice. The following subsections will introduce the hybrid deterioration model with mathematical analysis.

2.1. Hybrid Competing Failure Mode

In considering system reliability and operational stability, scheduled PM works can be performed for reducing equipment or facilities' breakdowns and also to avoid potential disasters. Either periodic PM or non-periodic PM policy can be adopted for enhancing the equipment or facilities' reliability. However, a scheduled periodic PM policy is taken into consideration in this study since it may be more practically manageable to the manager. Suppose that a deteriorating system (equipment) will undergo $N-1$ PM works during its lifetime T_L, where the intervals of periodic PM are equal and designated to be x by the reliability engineering department, and the whole system is replaced at the timing of the Nth PM work. Any breakdown which occurs within an interval two PM actions would cause minimal repairs, and minimal repairs cannot reduce the system aging. Accordingly, a NHPP with a power law intensity function $\lambda(t|\alpha,\beta) = \alpha\beta t^{\beta-1}$ is for describing the process of system's deterioration, where α and β denote the scale factor and the shape factor, respectively. The domains of α and β are within the range $[0, \infty)$ according to the property of the power law intensity function. Therefore, the power law form would be more flexibility and manageability than other intensity functions.

Since PM works can partially improve the system reliability and can also extend the original system lifetime. However, it is still unable to stop the system's aging process. Ultimately, the whole system will need to be replaced after the Nth PM action (the system lifetime would be $T_L = Nx$) in the consideration of cost- effectiveness. In this study, $C(N)$ denotes the expected cost per unit time with respect to the PM number N. The expected cost includes the minimal repair cost C_{mr}, the penalty cost C_{pl}, the ith PM cost C_{pm_i}, and the replacement cost C_{rp}. In general, the cost of performing a PM action should be relatively higher than the cost of performing repairs in the initial stage since the machine's abrasion status might not be serious when the machine is in its early age. However, when the machine' abrasion status becomes more serious over time, the repair cost will more than the PM cost if the age reduction factor of PM δ is relatively low and cannot effectively reduce the exponential increase of a system's breakdowns. Figure 1 illustrates of the preventive maintenance model of an equipment or facility.



$$P\left(N_{mr}(T \mid \alpha,\beta)=N_0\right)=\frac{\left(\int_0^T \lambda(t\mid\alpha,\beta)\mathrm{d}t\right)^{N_0} e^{-\int_0^T \lambda(t\mid\alpha,\beta)\mathrm{d}t}}{N_0!}$$

Probability of a given number of failures N_0 for the NHPP model

Symbol	Description	Cost
×	Minimal repair within two PMs	C_{mr}
△	Periodical PM Schedule	C_{pm_i}
☆	Replacement at the *N*th PM	C_{rp}

Figure 1. Timeline of the preventive maintenance model of an equipment or facility.

Furthermore, a facility or equipment may include non-maintainable and maintainable components in a system, and therefore the two modes (non-maintainable mode and maintainable mode) are needed to be integrated into one for presenting the phenomenon of a system's imperfect recovery. Therefore, the two intensity functions of the system deterioration will be devised as $\lambda_o(t|\alpha_o,\beta_o) = \alpha_o\beta_o t^{\beta_o-1}$ (non-maintainable mode) and $\lambda_p(t|\alpha_p,\beta_p) = \alpha_p\beta_p t^{\beta_p-1}$ (maintainable mode) respectively. Besides, please note that the values of scale and the shape parameters may be unknown and they will be discussed as uncertain states in the Bayesian analysis.

The assumptions of the proposed model are stated as follows:

(1) The system's deterioration behaves as a non-homogeneous Poisson process (NHPP).
(2) The system's deterioration is composed of maintainable and non-maintainable failure modes.
(3) A PM cannot restore the whole system to a brand-new state; instead, it can restore the whole system to some state as better-than-now.
(4) Any breakdowns occurring within the interval between two PM actions cause a minimal repair.

The following notations are used in the analysis throughout this study (Table 1):

Table 1. The notations.

T_L: the lifetime of a equipment or facility.
t: the age of a equipment or facility.
x: the time interval between two PMs.
t_k^-: the effective age of a equipment or facility before the time point of the kth PM.
t_k^+: the effective age of a equipment or facility after the time point of the kth PM.
α_o: the scale factor of the intensity function of non-maintainable failure mode.
β_o: the shape factor of the intensity function of non-maintainable failure mode.
α_p: the scale factor of the intensity function of maintainable failure mode
β_p: the shape factor of the intensity function of maintainable failure mode.
$f(\alpha,\beta)$: the prior probability distribution of the power-law intensity function.
$g(\alpha,\beta)$: the posterior probability distribution of the power-law intensity function.
δ: the age reduction factor, where $\delta \in [0,1]$.
$\lambda_o(t|\alpha_o,\beta_o)$: the intensity function of non-maintainable failure mode of the system deterioration.
$\lambda_p(t|\alpha_p,\beta_p)$: the intensity function of maintainable failure mode of the system deterioration.
$\lambda_h(t|\alpha_o,\beta_o,\alpha_p,\beta_p)$: the intensity function of the hybrid mode of the system deterioration.

Table 1. *Cont.*

N: the number of PM action during the whole system lifetime.
$N_{mr}(\cdot)$: the expected number of performing minimal repairs of the system.
C_{mr}: the average cost to perform a minimal repair.
C_{pm_k}: the cost to perform the kth PM.
C_{rp}: the cost of the overall replacement of a equipment or facility.
$\Phi(t_r)$: the probability density function of the time for performing a minimal repair.
C_{pl}: the penalty cost if the actual repair time over the time threshold φ
φ: the time threshold for performing a minimal repair.
C_F: the base cost for a PM action, which is influenced by the degree of PM.
τ: the increasing rate of PM base cost

2.2. Estimation of a System's Failures under Preventive Maintenances

Since the interval time of PM is x which was set by maintenance engineers, and t_1^+ denotes the effective age of a system after the time point of the first PM. It can be deduced by $t_1^+ = x - \delta x = (1-\delta)x$ because t_1^+ is influenced by the age reduction factor δ under the maintainable mode. Therefore, the effective age of a system before and after the time point of the kth PM can be presented as

$$t_k^- = kx - (k-1)\delta x = (k-1)(1-\delta)x + x = ((k-1)(1-\delta)+1)x, \tag{1}$$

and

$$t_k^+ = t_k^- - \delta x = kx - (k-1)\delta x - \delta x = k(1-\delta)x \tag{2}$$

respectively.

Due to the system's deterioration which includes non-maintainable and maintainable modes, the hybrid intensity function can be written as follows:

$$\begin{aligned}\lambda_h\left(T_L \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) &= \lambda_o(kx) + \lambda_p(((k-1)(1-\delta)+1)x) \\ &= \alpha_o\beta_o(kx)^{\beta_o-1} + \alpha_p\beta_p(((k-1)(1-\delta)+1)x)^{\beta_p-1},\end{aligned} \tag{3}$$

Moreover, it means that the system belongs to perfect maintenance if the age reduction factor δ is equal to one. Thus, the expected number of failures of the equipment or facility is given by

$$\begin{aligned}N_{mr}\left(N, x, \delta \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) &= \sum_{k=1}^{N} \int_{(k-1)x}^{kx} \lambda_h\left(t, \delta \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) dt \\ &= \sum_{k=1}^{N} \int_{(k-1)x}^{kx} \lambda_o(t|\alpha_o, \beta_o) dt + \sum_{k=1}^{N} \int_{t_{k-1}^+}^{t_k^-} \lambda_p\left(t \middle| \alpha_p, \beta_p\right) dt \\ &= \sum_{k=1}^{N} \int_{(k-1)x}^{kx} \lambda_o(t) dt + \sum_{k=1}^{N} \int_{(k-1)(1-\delta)x}^{((k-1)(1-\delta)+1)x} \lambda_p(t) dt \\ &= \alpha_o(Nx)^{\beta_o} + \sum_{k=1}^{N} \alpha_p\Big((((k-1)(1-\delta)+1)x)^{\beta_p} - ((k-1)(1-\delta)x)^{\beta_p}\Big).\end{aligned} \tag{4}$$

However, if the age reduction factor δ is equal to one, the expected number of the failures of the equipment or facility can be rewritten as follows:

$$N_{mr}\left(N, x \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) = \alpha_o(Nx)^{\beta_o} + N\alpha_p x^{\beta_p} \quad (\because \delta = 1). \tag{5}$$

Since the breakdown process of a deterioration system can be modeled as a Non-Homogenous Poisson Process, the probability of the number of breakdowns N_0 in the interval $(Nx,(N+1)x)$ is thus given by

$$\begin{aligned} &Pr\{N_{mr}(N+1,x,\delta|\alpha_o,\beta_o,\alpha_p,\beta_p) - N_{mr}(N,x,\delta|\alpha_o,\beta_o,\alpha_p,\beta_p) = N_0\} \\ &= \frac{\left(N_{mr}(N+1,x,\delta|\alpha_o,\beta_o,\alpha_p,\beta_p) - N_{mr}(N,x,\delta|\alpha_o,\beta_o,\alpha_p,\beta_p)\right)^{N_0} \times e^{-(N_{mr}(N+1,x,\delta|\alpha_o,\beta_o,\alpha_p,\beta_p) - N_{mr}(N,x,\delta|\alpha_o,\beta_o,\alpha_p,\beta_p))}}{N_0!}. \end{aligned} \tag{6}$$

The reliability of a product $R(T_L = Nx)$ will decline with time, and therefore it can denote as

$$\begin{aligned} R(T_L = Nx) &= Pr\{N_{mr}(N,x,\delta|\alpha_o,\beta_o,\alpha_p,\beta_p) = 0\} \\ &= e^{-N_{mr}(N,x,\delta|\alpha_o,\beta_o,\alpha_p,\beta_p)}. \end{aligned} \tag{7}$$

Figure 2 illustrates preventive maintenance between perfect recovery and imperfect recovery under a hybrid deterioration. The difference between perfect and imperfect recoveries is that the critical component of a perfect recovery system can be maintainable or can be replaced with a new one to restore the system to its original status. As can be seen in the middle-left side of Figure 2, the maintainable component can be fully restored to its original status after preventive maintenance.

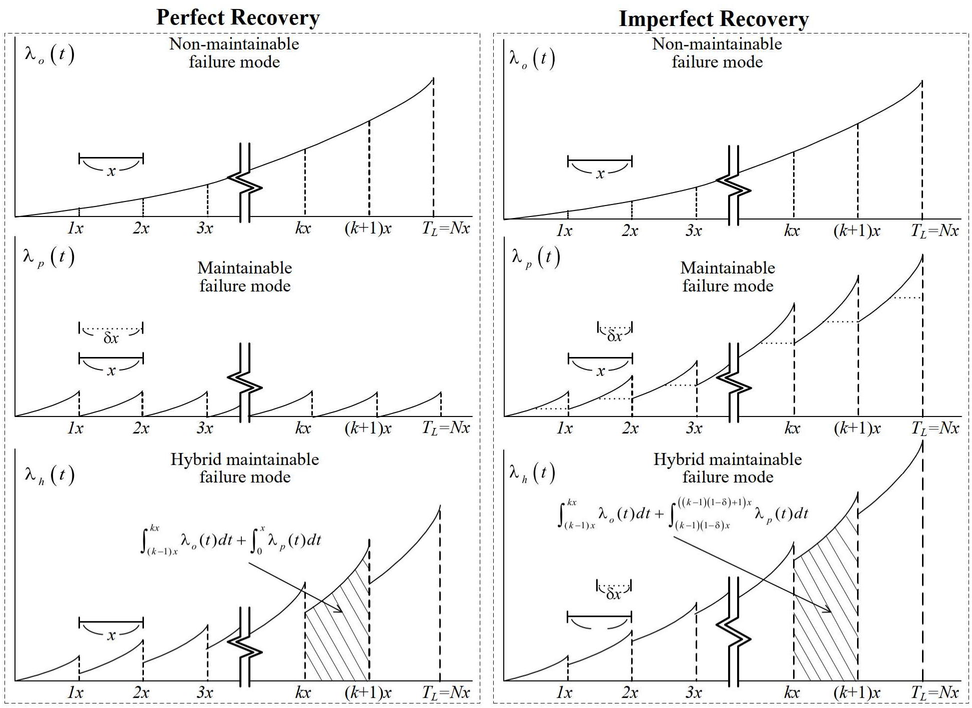


Figure 2. Preventive maintenances between perfect recovery and imperfect recovery under a hybrid deterioration.

2.3. Evaluation of Repair and Maintenance Costs of a Facility

Some expenditures will be incurred as long as equipment or facility operates during the system lifetime. The incurred expenditures are mainly from repair, penalty, replacement, and PM costs. The repair cost (C_{mr}) means the expected cost to perform a minimal repair. The penalty cost (C_{pl}) means that the cost was incurred from the loss of a production line shutdown if the actual repair time exceed the time limit (φ). Since any breakdown of the facility will be rectified by minimal repairs, the time of a minimal repair (t_r) will need to be measured. Therefore, the repair time can be regarded as a random variable, and it is assumed to follow a Gamma probability distribution. The expected repair time over the tolerable waiting time limit φ can be expressed as

$$E[t_r|\omega,\eta] = \int_{\varphi}^{\infty} t_r\left(\frac{\eta^{\omega}t_r^{\omega-1}}{\Gamma(\omega)e^{\eta t_r}}\right)dt_r = \frac{\Gamma(1+\omega)-\omega\Gamma(\omega)+\Gamma(1+\omega,\eta\varphi)}{\eta\Gamma(\omega)} \tag{8}$$

The parameters ω and η can be estimated by $\omega = \left(\frac{E(t_r)}{\sigma(t_r)}\right)^2$ and $\eta = \frac{E(t_r)}{\sigma(t_r)^2}$ under engineers' judgment or historical data of repairs. If the repair time is over the time limit φ, the penalty cost will be incurred by the owner of the equipment or facility. $\Gamma(z_0)$ denotes a Gamma function with the parameter z_0 and $\Gamma(z_1, z_2)$ denotes an upper incomplete Gamma function with the parameters z_1 and z_2.

The estimation of the equipment or facility's deterioration is critical to the owner, and therefore the manager needs to accurately evaluate the expected failure number of the equipment or facility during its lifetime. Supposed that the failure process follows an NHPP with a power-law intensity function $\lambda(t)$. Therefore, the expected number of failures during the lifetime [0, T_L] under the age reduction factor in effective age δ_{pm}^q and the interval of PM x is $N_{mr}\left(N, x, \delta_{pm}^q \middle| \alpha_o\alpha_o, \beta_o, \alpha_p, \beta_p\right)$. Accordingly, the total expected repair cost during the lifetime can be given as

$$\left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right)N_{mr}\left(N, x, \delta_{pm}^q \middle| \alpha_o\alpha_o, \beta_o, \alpha_p, \beta_p\right). \tag{9}$$

Moreover, the PM cost will get higher and higher for sequential PM activities during the lifetime due to mechanical aging of a deteriorating system. Therefore, the PM cost should be related to the ith number of PM actions with the age reduction factor. Based on this, the PM cost is defined as

$$C_{pm_k} = C_F(1 + \tau(k-1)x) \tag{10}$$

where τ denotes the periodically increasing rate of the PM cost, and C_F is the base cost of performing a PM work. Furthermore, different PM alternatives bring a different degree of the system's recovery but it also influences different PM costs to the firm. Suppose that a series of PM alternatives $M_P = \left\{M_P^1,\ M_P^2,\ \ldots M_P^q, \ldots, M_P^Q\right\}$ can be selected, and the corresponding PM cost and the expected failure numbers can be redefined as follows:

$$C_{pm}^q\left(C_F^q, \tau_q, x, T_L\right) = \sum_{k=1}^{T_L/x-1} C_{pm_k}^q = \sum_{k=1}^{T_L/x-1} C_F^q(1 + \tau_q(k-1)x). \tag{11}$$

It is important to realize the process of deterioration of the product in order to evaluate the costs of repair during the system's lifetime. Based on an assumption that the failure times can be drawn from an NHPP with a specific intensity function $\lambda_h(\cdot)$, an estimate of the number of expected failures $N_{mr}\left(N, x, \delta_{pm}^q \middle| \cdot\right)$ under the time interval of PM x as well as the age reduction factor δ_{pm}^q in the effective age of the system, the total repair cost during the system's lifetime [0, T_L] can be calculated as follows:

$$\begin{aligned} &C_{mr}N_{mr}\left(N, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) \\ &= C_{mr}\left(\alpha_o(Nx)^{\beta_1} + \textstyle\sum_{k=0}^{N-1}\alpha_p\left(\left(1 + k\left(1-\delta_{pm}^q\right)\right)x)^{\beta_2} - \left(k\left(1-\delta_{pm}^q\right)x\right)^{\beta_2}\right)\right) \end{aligned} \tag{12}$$

If the age reduction factor δ_{pm}^q is equal to one, the total repair cost can be rewritten as follows:

$$C_{mr}N_{mr}(N, x, \alpha_o, \beta_o, \alpha_p, \beta_p) = C_{mr}\left(\alpha_o(Nx)^{\beta_o} + N\alpha_p x^{\beta_p}\right). \tag{13}$$

2.4. Optimal Preventive Maintenance Schedule with Consideration of Multiple PM Alternatives

Generally, managers need to consider how to determine the optimal maintenance schedule to minimize the total expected costs associated with the project requirements. Therefore, in consideration of all the candidate PM alternatives, the total expected cost per unit time under the system lifetime T_L can be given as follows:

$$C(N|M_P^q) = \frac{\sum_{k=1}^{N-1} C_{pm_k} + \left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right) N_{mr}\left(N, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) + C_{rp}}{T_L} \\ = \frac{C_{pm}^q\left(C_F^q, \tau_p, x, T_L\right) + \left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right) N_{mr}\left(N, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) + C_{rp}}{Nx}. \tag{14}$$

The convexity of the cost function with respect to N under a specific PM alternative M_P^q can be justified if the two inequalities $C\left(N+1\middle|M_P^q\right) \geq C\left(N\middle|M_P^q\right)$ and $C\left(N\middle|M_P^q\right) < C\left(N-1\middle|M_P^q\right)$ are both held, and the optimal N^* can therefore be obtained. Proposition 1 give the proof of the convexity of $C\left(N\middle|M_P^q\right)$.

Proposition 1. *Given the intensity function*$N_{mr}(N, x, \alpha_1, \beta_1, \alpha_2, \beta_2, \delta)$ *is strictly increasing. As long as the two inequalities* $C\left(N+1\middle|M_P^q\right) \geq C\left(N\middle|M_P^q\right)$ *and* $C\left(N\middle|M_P^q\right) < C\left(N-1\middle|M_P^q\right)$ *can be both held under a specific number N, the convexity of the cost function* $C\left(N\middle|M_P^q\right)$ *with respect to N can be assured.*

Proof.

For $C\left(N+1\middle|M_P^q\right) \geq C\left(N\middle|M_P^q\right)$, we have $C\left(N+1\middle|M_P^q\right) - C\left(N\middle|M_P^q\right) \geq 0$

$$\Rightarrow \frac{\sum_{k=1}^{N} C_{pm_k} + \left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right) N_{mr}\left(N+1, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) + C_{rp}}{(N+1)x} \\ - \frac{\sum_{k=1}^{N-1} C_{pm_k} + \left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right) N_{mr}\left(N, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) + C_{rp}}{Nx} \geq 0$$

$$\Rightarrow \frac{\sum_{k=1}^{N} C_{pm_k} + \left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right) N_{mr}\left(N+1, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) + C_{rp}}{(N+1)x} \\ - \frac{\left(\frac{N+1}{N}\right)\left(\sum_{k=1}^{N-1} C_{pm_k} + \left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right)\left(N, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) + C_{rp}\right)}{(N+1)x} \geq 0$$

$$\Rightarrow \frac{\left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right) \times \left\{N_{mr}\left(N+1, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) - \left(1+\frac{1}{N}\right) N_{mr}\left(N, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right)\right\}}{(N+1)x} \\ \geq \frac{\left(\frac{1}{N}\right)C_{rp} - C_{pm_N} + \left(\frac{1}{N}\right)\sum_{k=1}^{N-1} C_{pm_k}}{(N+1)x}$$

$$\Rightarrow \left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right)\left\{N_{mr}\left(N+1, x, \delta_{pm}^q, \alpha_o, \beta_o, \alpha_p, \beta_p\right)\right. \\ \left. -\left(1+\frac{1}{N}\right) N_{mr}\left(N, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right)\right\} \geq \left(\frac{1}{N}\right)C_{rp} - C_{pm_N} + \left(\frac{1}{N}\right)\sum_{k=1}^{N-1} C_{pm_k}$$

$$\Rightarrow \left(C_{mr} + C_{pl}E[t_r|\omega,\eta]\right)\left\{(N) N_{mr}\left(N+1, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right)\right. \\ \left. -(N+1) N_{mr}\left(N, x, \delta_{pm}^q, \alpha_o, \beta_o, \alpha_p, \beta_p\right)\right\} \geq \sum_{k=1}^{N-1} C_{pm_k} - NC_{pm_N} + C_{rp}$$

$$\Rightarrow (N) N_{mr}\left(N+1, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) - (N+1) N_{mr}\left(N, x, \delta_{pm}^q \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right) \\ \geq \frac{\sum_{k=1}^{N-1} C_{pm_k} - NC_{pm_N} + C_{rp}}{C_{mr} + C_{pl}E[t_r|\omega,\eta]}. \tag{15}$$

$$\text{For } C(N\Big|M_p^q) < C(N-1\Big|M_p^q)\text{, we have } C(N\Big|M_p^q) - C(N-1\Big|M_p^q) < 0$$

$$\begin{aligned}
&\Rightarrow \frac{\sum_{k=1}^{N-1} C_{pm_k} + \left(C_{mr}+C_{pl}E[t_r|\omega,\eta]\right)N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)+C_{rp}}{Nx} \\
&\quad - \frac{\sum_{k=1}^{N-2} C_{pm_k} + \left(C_{mr}+C_{pl}E[t_r|\omega,\eta]\right)N_{mr}\left(N-1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)+C_{rp}}{(N-1)x} < 0 \\
&\Rightarrow \left(1-\tfrac{1}{N}\right)\sum_{k=1}^{N-1} C_{pm_k} + \left(1-\tfrac{1}{N}\right)\left(C_{mr}+C_{pl}E[t_r|\omega,\eta]\right)N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right) \\
&\quad + \left(1-\tfrac{1}{N}\right)C_{rp} \\
&\quad - \left(\sum_{k=1}^{N-2} C_{pm_k} + \left(C_{mr}+C_{pl}E[t_r|\omega,\eta]\right)N_{mr}\left(N-1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right) + C_{rp}\right) < 0 \\
&\Rightarrow \left(C_{mr}+C_{pl}E[t_r|\omega,\eta]\right)\left\{\left(\tfrac{N-1}{N}\right)N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right. \\
&\quad \left. -N_{mr}\left(N+1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right\} + \left(\tfrac{N-1}{N}\right)C_{pm_{N-1}} - \tfrac{1}{N}\sum_{k=1}^{N-2} C_{pm_k} - \left(\tfrac{1}{N}\right)C_{rp} \\
&\quad < 0 \\
&\Rightarrow \left(C_{mr}+C_{pl}E[t_r|\omega,\eta]\right)\left\{(N-1)N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right. \\
&\quad \left. -(N)N_{mr}\left(N-1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right\} + (N-1)C_{pm_{N-1}} - \sum_{k=1}^{N-2} C_{pm_k} - C_{rp} < 0 \\
&\Rightarrow \left(C_{mr}+C_{pl}E[t_r|\omega,\eta]\right)\left\{(N-1)N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right. \\
&\quad \left. -(N)N_{mr}\left(N-1,x,\delta_{pm}^q,\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right\} < \sum_{k=1}^{N-2} C_{pm_k} - (N-1)C_{pm_{N-1}} + C_{rp} \\
&\Rightarrow (N-1)N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right) - (N)N_{mr}\left(N-1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right) \\
&\quad < \frac{\sum_{k=1}^{N-2} C_{pm_k} - (N-1)C_{pm_{N-1}} + C_{rp}}{C_{mr}+C_{pl}E[t_r|\omega,\eta]}.
\end{aligned} \tag{16}$$

Let $L(N) = (N)N_{mr}(N+1,x,\delta_{pm}^q\Big|\alpha_o,\beta_o,\alpha_p,\beta_p) - (N+1)N_{mr}(N,x,\delta_{pm}^q\Big|\alpha_o,\beta_o,\alpha_p,\beta_p)$ for N=1, 2, ..., and $L(N) = 0$ for $N = 0$. The condition that $L(N)$ is strictly increasing with N is supported and the two inequalities (15) and (16) can hold simultaneously, and there would exist the PM number N can minimize the total expected cost per unit time under the system lifetime. Due to the fact that the failure intensity function is increasing with time for deteriorating systems, i.e., $\lambda_h((N+1)x) > \lambda_h(Nx) > \ldots > \lambda_h(0)$ for $x, 2x, \ldots, Nx, (N+1)x, \ldots$, we then have as follows:

$$\begin{aligned}
&L(N) - L(N-1) \\
&\quad = \left\{(N)N_{mr}\left(N+1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right) - (N\right. \\
&\quad \left. +1)\,N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right\} \\
&\quad -\left\{(N-1)N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right. \\
&\quad \left. -(N)N_{mr}\left(N-1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right\} > 0 \\
&= (N)\left\{N_{mr}\left(N+1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right) + N_{mr}\left(N-1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right\} \\
&\quad -(2N)N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right) > 0 \\
&\quad = \left\{N_{mr}\left(N+1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right) + N_{mr}\left(N-1,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right)\right\} \\
&\quad -2N_{mr}\left(N,x,\delta_{pm}^q\middle|\alpha_o,\beta_o,\alpha_p,\beta_p\right) > 0
\end{aligned}$$

Therefore, the convexity of the cost function $C(N|M_P^q)$ with respect to N is thus assured according to Jensen's inequality. □

The heuristic solution algorithm for obtaining the minimal cost by setting N^* and $M_P^{q^*}$ can be described in Figure 3.

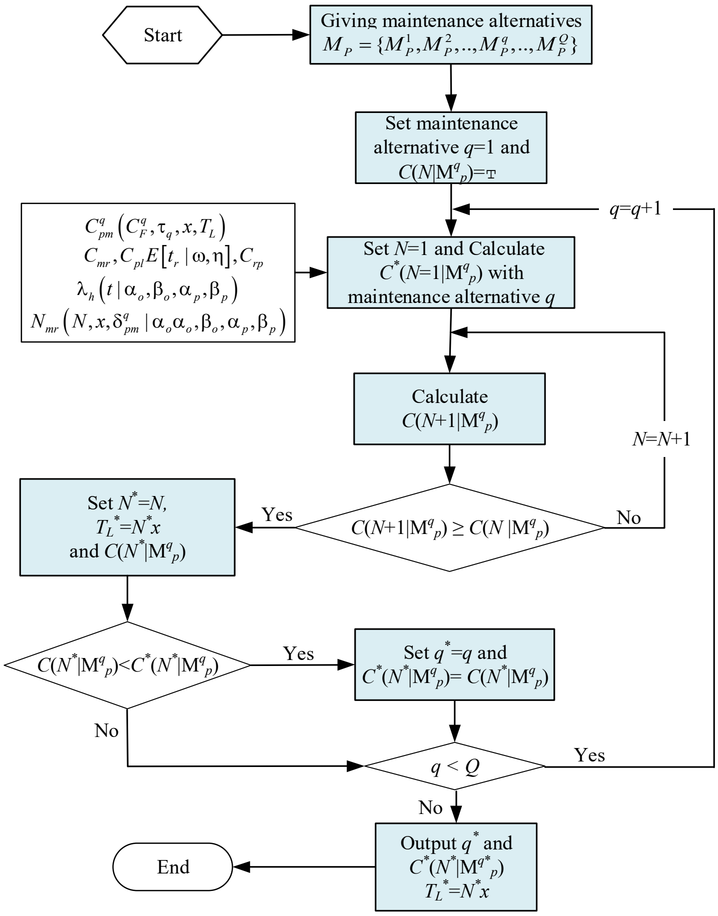


Figure 3. Heuristic algorithm for obtaining the minimal cost by setting N^* and M_P^{q*}.

3. Bayesian Decision Process by Using Domain Experts' Judgment and Collected Information

3.1. Analysis by the Natural Conjugate Probability Distribution

It might be not easy to perform Bayesian decision analysis due to the fact that numerical integration is needed to derive the prior and posterior distributions. As the state space in our case contains multiple random variables, the previous analysis would have been much more complicated. Huang and Bier [38] proposed a natural conjugate prior distribution for the power law deteriorating model for repairable systems, and the form is as follows:

$$f(\alpha,\beta) = K\alpha^{\kappa-1}\beta^{\kappa-1}\left(e^{-\varpi}\upsilon^{\kappa}\right)^{\beta-1}e^{-\alpha\psi\upsilon^{\beta}}. \tag{17}$$

In order to make sure the distribution sums up to one, K is used to be a normalizing factor. The main advantage is to using the natural conjugate prior distribution to proceed with a straightforward analysis instead of the complicated traditional calculations. In Equation (17), the joint probability distribution with the desired prior marginal moments are composed of the four parameters (ψ, ϖ, κ, and υ). By applying Equation (17) to the prior probability distribution of α and β, it will be easy to deduce the expected failure number based on the prior distribution, and its form is given by

$$\begin{aligned} \underset{Pri}{E}[N_{mr}(T_L|\alpha,\beta)] &= \int_0^\infty \int_0^\infty \int_0^{T_L} \alpha\beta t^{\beta-1} f(\alpha,\beta) dt d\beta d\alpha \\ &\cong \tfrac{\kappa\varpi^\kappa}{\psi}(\varpi + ln[v] - ln[T_L])^{-\kappa}. \end{aligned} \tag{18}$$

Here, T_L denotes the actual age of the equipment or facility. The prior analysis of the expected failure number can be performed by straightforwardly calculating $\underset{Pri}{E}[N_{mr}(T_L|\alpha,\beta)]$ in Equation (18) with the four parameters (ψ, ϖ, κ, and v) which are specified by the reliability or domain experts with their prior knowledge and judgment about the deteriorating system (μ_α, σ_α, μ_β, and σ_β). μ_α, σ_α, μ_β, and σ_β denote the mean values and standard deviations of α and β, respectively. Furthermore, in order to calculate $\underset{Pri}{E}[N_{mr}(T_L|\alpha,\beta)]$, the values of the four parameters need to be obtained first. Equations (19)–(22) can be used to obtain the four parameters values as follows:

$$\varpi = \frac{\mu_\beta}{\sigma_\beta^2} \tag{19}$$

$$\kappa = \left(\frac{\mu_\beta}{\sigma_\beta}\right)^2 \tag{20}$$

$$v = e^{\varpi(\xi^{1/\kappa}-1+\sqrt{\xi^{1/\kappa}(\xi^{1/\kappa}-1)})}\ (where\ \xi = \frac{\mu_\alpha^{-2}\sigma_\alpha^2+1}{\mu_\beta^{-2}\sigma_\beta^2+1}), \tag{21}$$

and

$$\psi = \left(\frac{\kappa}{\mu_\alpha}\right)\left(\frac{\varpi}{\varpi + ln[v]}\right)^\kappa \tag{22}$$

However, if the decision maker may not be convinced by the result of the prior analysis, he/she may collect failure datasets in practice to adjust the prior analysis. It is called the posterior analysis in a Bayesian decision process. Once the decision maker wants to proceed with the posterior analysis, he/she needs to prepare the extra budget for accelerated deterioration experiments to increase the accuracy of the prediction. If a further investigation is undertaken, the sample size should be carefully examined in consideration of the budget of the experiments. If the n breakdown times are collected from accelerated deterioration experiments $(x_1, x_2, \cdots, x_n)$, the property of natural conjugate families can be used to obtain the posterior distribution of α and β without further computation. Its form is as follows:

$$\begin{aligned} g\left(\alpha,\beta,D^{(n)}\right) &\propto L(D^{(n)}\Big|\alpha,\beta) f(\alpha,\beta) \\ &= K'\alpha^{\kappa+n-1}\beta^{\kappa+n-1}\left(e^{-\varpi}v^\kappa \prod_{i=1}^n x_i\right)^{\beta-1} e^{-\alpha(\psi v^\beta + x_n^\beta)}. \end{aligned} \tag{23}$$

$L(D^{(n)}|\alpha,\beta) = \alpha^n\beta^n(\prod_{i=1}^n x_i)^{\beta-1}e^{-\alpha x_n^\beta}$ is the likelihood function of NHPP with the power-law intensity function, and K' denotes a normalizing factor to ensure the distribution sums up to one. However, due to the math complexity of the expected failure number of the posterior analysis, the closed form expression cannot be obtained. Fortunately, numerical integration methods (e.g., Monte Carlo numerical integration) can be applied to calculate the prediction of the expected failure number based on the posterior distribution, and it can be given by

$$\begin{aligned} \underset{Pos}{E}\left[N_{mr}\left(T_L\Big|\alpha,\beta,D^{(n)}\right)\right] &= \int_0^\infty \int_0^\infty \int_0^{T_L} \alpha\beta t^{\beta-1} g\left(\alpha,\beta,D^{(n)}\right) dt d\beta d\alpha \\ &\cong \int_0^\infty \int_0^\infty \alpha T_L^\beta g\left(\alpha,\beta,D^{(n)}\right) d\beta d\alpha. \end{aligned} \tag{24}$$

Therefore, both experts' knowledge and sampling information can be considered in the posterior analysis. Moreover, in consideration of the hybrid deterioration and the number of PM with the age reduction factor, the state space can be presented as $\boldsymbol{\Theta} : \{\alpha_o, \beta_o, \alpha_p, \beta_p | \alpha_o \in (0,\infty),\ \beta_o \in (0,\infty),\ \alpha_p \in (0,\infty), \beta_p \in (0,\infty)\}$, and the expected failure numbers of the prior analysis and the posterior analysis can be given as follows:

$$\begin{aligned}
&\underset{Pri}{E}\left[N_{mr}\left(N, x, \delta_{pm}^{q} \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right)\right] \\
&\quad = \frac{\kappa_o \varpi_o{}^{\kappa_o}}{\psi_o}(\varpi_o + ln[v_o] - ln[Nx])^{-\kappa_o} \\
&\quad + \left(\frac{\kappa_p \varpi_p{}^{\kappa_p}}{\psi_p}\right)\sum_{k=0}^{N-1}\left\{\left(\varpi_p + ln[v_p] - ln\left[\left(1 + k\left(1 - \delta_{pm}^{q}\right)\right)x\right]\right)^{-\kappa_p}\right. \\
&\quad \left. - \left(\varpi_p + ln[v_p] - ln\left[k\left(1 - \delta_{pm}^{q}\right)x\right]\right)^{-\kappa_p}\right\},
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\underset{Pos}{E}\left[N_{mr}\left(N, x, \delta_{pm}^{q} \middle| \alpha_o, \beta_o, \alpha_p, \beta_p\right)\right] \\
&\quad = \int_0^\infty \int_0^\infty \alpha_o T_L^{\beta_o} g(\alpha_o, \beta_o) d\beta_o d\alpha_o \\
&\quad + \sum_{k=0}^{N-1}\left\{\int_0^\infty \int_0^\infty \alpha_p \left(\left(1 + k\left(1 - \delta_{pm}^{q}\right)\right)x\right)^{\beta_p} g(\alpha_p, \beta_p) d\beta_p d\alpha_p\right. \\
&\quad \left. - \int_0^\infty \int_0^\infty \alpha_p \left(k\left(1 - \delta_{pm}^{q}\right)x\right)^{\beta_p} g(\alpha_p, \beta_p) d\beta_p d\alpha_p\right\}.
\end{aligned} \tag{26}$$

It should be noted that the parameters of the hybrid deteriorating function (ϖ_o, κ_o, v_o, ψ_o, ϖ_p, κ_p, v_p, and ψ_p) can be calculated by applying Equations (19)–(22). After getting the parameters' value, the expected failure numbers, $\underset{Pri}{E}[N_{mr}(\cdot)]$ and $\underset{Pos}{E}[N_{mr}(\cdot)]$), can be obtained to calculate the expected costs per unit time $\underset{Pri}{E}\left[C(N|M_P^q)\right]$ and $\underset{Pos}{E}\left[C(N|M_P^q)\right]$ from Equations (27) and (28), and their form are as follows:

$$\begin{aligned}
&\underset{Pri}{E}\left[C(N|M_P^q)\right] \\
&= \frac{C_{pm}^{q}\left(C_F^q, \tau_q, x, T_L\right) + \left(C_{mr} + C_{pl}E[t_r|\omega, \eta]\right)\underset{Pri}{E}\left[N_{mr}\left(N, x, \delta_{pm}^{q}|\alpha_o, \beta_o, \alpha_p, \beta_p\right)\right] + C_{rp}}{Nx}
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\underset{Pos}{E}\left[C(N|M_P^q)\right] \\
&= \frac{C_{pm}^{q}\left(C_F^q, \tau_q, x, T_L\right) + \left(C_{mr} + C_{pl}E[t_r|\omega, \eta]\right)\underset{Pos}{E}\left[N_{mr}\left(N, x, \delta_{pm}^{q}|\alpha_o, \beta_o, \alpha_p, \beta_p\right)\right] + C_{rp}}{Nx}.
\end{aligned} \tag{28}$$

By applying the solution algorithm shown in Figure 3, the minimal expected cost of the prior analysis or the posterior analysis can be determined with the settings of the Bayesian parameters.

3.2. The Bayesian Decision Process

The decision maker should confirm whether the relevant assumptions are satisfied or not in the case before deciding to proceed with the Bayesian analysis. The reliability or domain experts can provide the judgment of the parameters' value (μ_{α_o}, σ_{α_o}, μ_{β_o}, σ_{β_o}, μ_{α_p}, σ_{α_p}, μ_{β_p}, and σ_{β_p}) according to their prior knowledge and experience. After this step, the decision makers can proceed with the prior analysis to minimize the expected cost. Suppose the decision makers consider that the prior analysis is reliable. In that case, it is not necessary to collect extra experimental data to adjust their prior judgment and just proceed with the choice of the candidate PM alternatives.

However, if the decision makers feel that the prior analysis might not be reliable or convinced, they would request more information to justify or amend the prior analysis.

However, how many experimental data will be enough? It depends on the firm's budget and possible benefit for amending the original PM alternative. Once the decision makers decide to proceed with the posterior analysis, the cost of collecting this additional information should also be considered in the decision process. The posterior analysis can utilize additional information from accelerated deterioration experiments along with domain experts' judgment to make the final decision. Nevertheless, the calculation of the posterior analysis is not easy to performance since numerical methods, Monte Carlo integration and computation engines would be needed to get the non-closed form solution. Figure 4 illustrates the analysis of the Bayesian decision process.

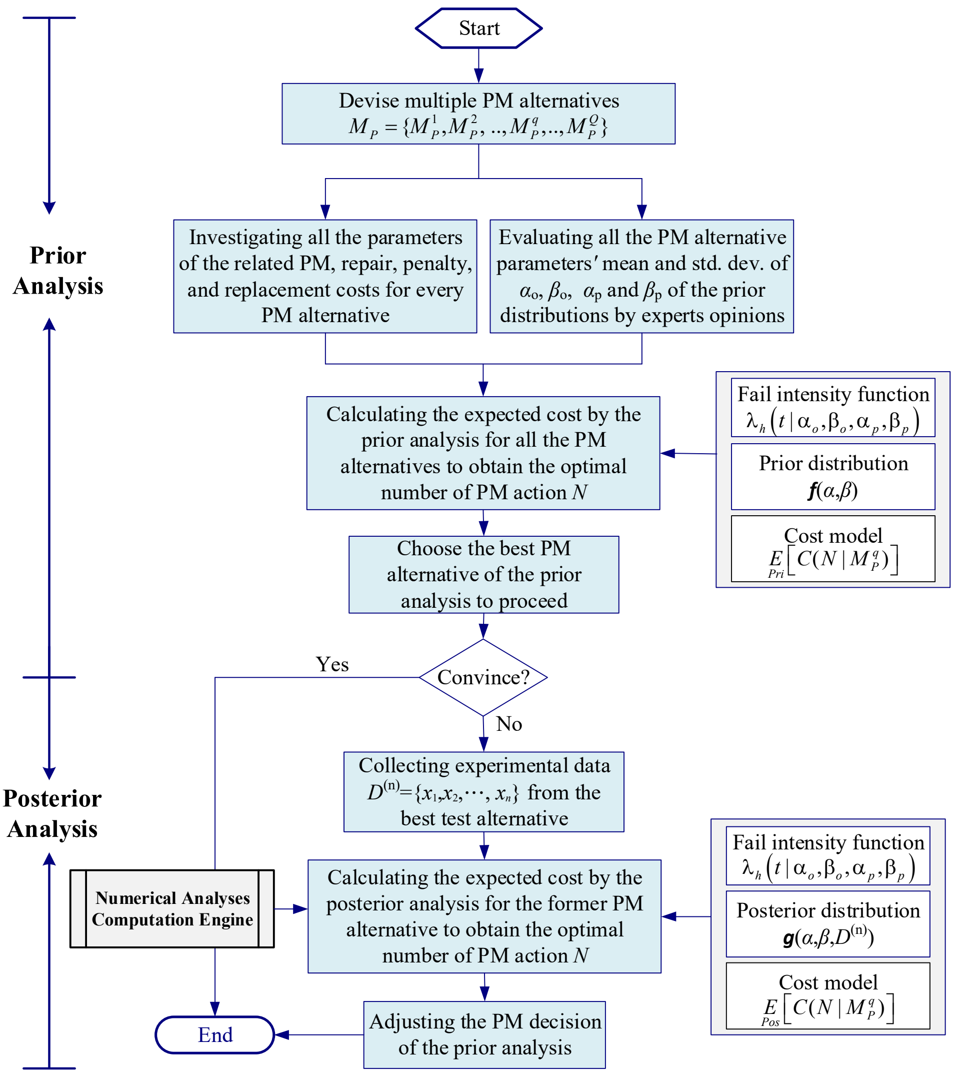


Figure 4. Flowchart for Bayesian decision process.

3.3. Computerized Information System Design

In general, the optimal decision would not be easily obtained without a computerized information system when dealing with such complicated mathematical models. The entire system can be split into two subsystems to improve manageability. Engineers and domain experts can use the model management system to update the model base and database. A decision support system is also implemented to give decision makers the knowledge they need to make informed choices. The engineers should inspect the expenses, failure intensity functions, likelihood of repair time, penalty cost, replacement cost, and related probabilities, etc. characteristics before operating the model management system. When the engineers have these data, they can use the model management system to store them in the database. Additionally, the lack of the deterioration information would make it difficult to assess how the new system is deteriorating and what the frequency of failures is within a time interval. Therefore, based on their expertise, the statistical characteristics and associated factors can be assessed by the reliability engineering domain experts. In addition, the engineers must gather failure data for the facilities from various engineering experiments if a posterior analysis is desired by the decision maker. Moreover, we can store or access data more effectively through the use of a data formalizing mechanism. This mechanism converts inconsistent data to more consistent data for storage and access to the database and model base. Additionally, computing engines could be required to handle the complexity of finding the best answer. By utilizing an application programming interface (API), system developers can move forward with all of the mathematical analyses for the decision support system. The design of the computerized information system is shown in Figure 5.

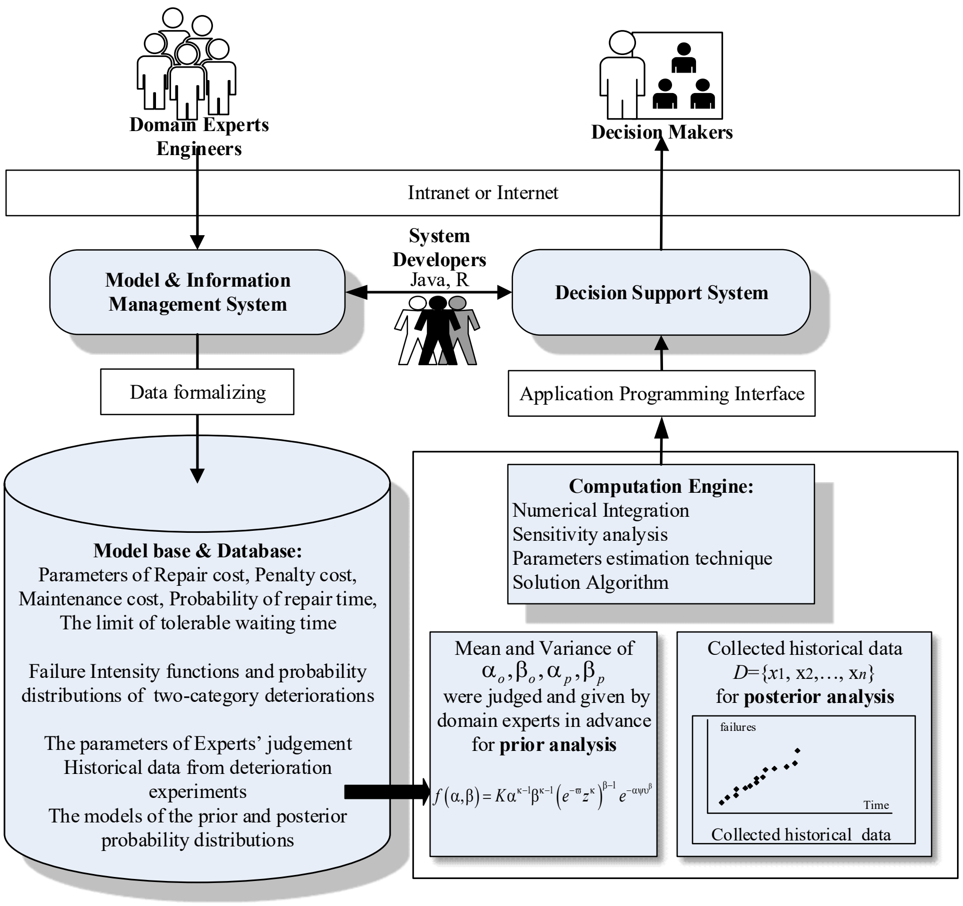


Figure 5. The design of the computerized information system.

4. Application and Sensitivity Analyses

4.1. Application of Prior and Posterior Analyses

Suppose that a firm plans to purchase a batch of new industrial equipment for manufacturing its products. After the purchase and setting of the industrial equipment up in its production lines, the firm has to make an effective preventive maintenance plan for factor management. However, due to the lack of the equipment's deterioration information, the decision makers of the firm would be hard to make an effective preventive maintenance plan to reduce unexpected equipment breakdowns for saving the related cost. Moreover, the industrial equipment's deterioration can be categorized into maintainable and non-maintainable failure modes. It means that some components can be repaired or replaced to restore the equipment to a younger status but some components cannot be restored to the system's status by repair or replacement works. Therefore, the firm needs to estimate the two modes' deterioration to handle the equipment's failures occurring. Since the new equipment did not take any accelerating deterioration experiments for some reasons, the equipment's deterioration cannot be estimated by traditional statistical methods. Due to the fact that to proceed with a complete accelerating deterioration experiment will need a lot of time and expenditure, the firm may not proceed with such experiment in time. Accordingly, in order to solve the issue of insufficient data, the decision makers would try to apply a Bayesian decision process to estimate the deterioration. The Bayesian decision process can be separated into two phases: the prior analysis, which is assessed by reliability domain experts, and the posterior analysis, which requires failure data from engineering tests. In the first phase, the decision makers will ask domain experts to evaluate the maintainable and non-maintainable parameters' statistical characteristics for the prior analysis. After the experts' evaluation, the eight prior parameters are set as μ_{α_o} = 0.6, μ_{β_o} = 0.15, σ_{α_o} = 1.25, σ_{β_o} = 0.3125; μ_{α_p} = 0.85, μ_{β_p} = 0.2125, σ_{α_p} = 1.75, and σ_{β_p} = 0.4375. Besides, the firm's engineering department proposed five candidate PM alternatives for decision makers. This information can be referred to Table 2. Besides, since different PM alternatives can bring different degrees of system's recovery, the corresponding PM costs are also different. PM alternatives 4 & 5's age reduction (δ^4_{pm} = 0.95, δ^5_{pm} = 1.0) are higher than the others (δ^1_{pm} = 0.80, δ^2_{pm} = 0.85, δ^3_{pm} = 0.90). The higher age reduction PM alternatives can effectively reduce the increase of the possible repair cost and penalty cost but they also increases the related PM cost. Therefore, it is hard to judge which PM alternative is best for the firm before the evaluation of the all PM alternatives. Moreover, the experts also evaluate the repair time's statistical characteristics and consider the time is a random variable and follows a Gamma probability distribution. The statistical characteristics of the repair time are estimated as $E(t_r)$ = 5 h and $\sigma(t_r)$ = 3 h, respectively. However, the tolerable waiting time is only 4.5 h in practice. If the repair time is over the tolerable waiting time, the production line will be halted and bring the related loss. The loss can be evaluated as the penalty cost. The parameters ω and η of the Gamma distribution can be calculated by the simultaneous equations $\omega = E(t_r)^2/\sigma(t_r)^2$ and $\eta = E(t_r)/\sigma(t_r)^2$. Based on the above mentioned, the detailed information of the five candidate PM alternatives with the firm's domain experts evaluation and the cost parameters is given as Table 2.

Table 2. The detailed information of all candidate PM alternatives.

Parameters for the two categories deterioration, which were judged by experts	$\mu_{\alpha_o} = 0.6$, $\mu_{\beta_o} = 0.15$, $\sigma_{\alpha_o} = 1.25$, $\sigma_{\beta_o} = 0.3125$; $\mu_{\alpha_p} = 0.85$, $\mu_{\beta_p} = 0.2125$, $\sigma_{\alpha_p} = 1.75$, $\sigma_{\beta_p} = 0.4375$
Interval between two PM actions	$x = 0.5$ years
PM's Base cost of the five candidate PM alternatives	$C_F^q = \{\$780, \$790, \$800, \$880, \$890\}$
Age reduction factors of the five candidate PM alternatives	$\delta_{pm}^q = \{0.8, 0.85, 0.9, 0.95, 1.0\}$
Periodically increasing rates of PM cost of the five candidate PM alternatives	$\tau_q = \{0.19, 0.195, 0.2, 0.235, 0.24\}$
Replacement cost	$C_{rp} = \$20,000$
Expected cost of performinga minimal repair	C_{mr}= \$250
Penalty cost if the repair time exceed the time limit φ	$C_{pl} = \$90$
Expected value and standard deviation of performing a minimal repair	$E(t_r) = 5$ h, $\sigma(t_r) = 3$ h
The limit of tolerable waiting time for performing a minimal repair	$\varphi = 6.5$ h

After performing the prior analysis of the Bayesian decision process based on the proposed solution algorithm, the trend of the average cost of PM alternatives 1–5 are presented in Figure 6. Table 3 provides the related results of PM alternatives 2–4 in detail. The best PM alternative is no.3, and the annual average cost, PM cost, and replacement cost are estimated to be \$6212, \$2956, and \$2105 at 9.5 years. According to the prior analysis results in Figure 6, the optimal lifetime of equipment's replacement of PM alternatives 1–5 should be set to 9, 9, 9.5, 8.5, and 9 years respectively, and the annual average cost will be \$6329, \$6268, \$6212, \$6634, and \$6585. It can be observed that the highly intensive PM alternative may not have a beneficial influence on decreasing the cost. In general, highly intensive PM alternatives (4 & 5) can lower equipment failure times to save expenditure on repairs but they need more the related PM costs. On the contrary, the lower intensive PM alternatives (1 & 2) are able to decrease the related PM costs but increase expenditure on repairs. However, although lower intensive PM alternatives may bring serious equipment' failures at the post-phase, the firm may adopt the strategy of shortening equipment's lifetime to prevent this disadvantage. According to the above mentioned, we understand that the firm cannot judge and decide the best PM alternative by qualitative analysis or without cost calculation. Moreover, it can be seen that the average cost of PM alternative 3 is lower than the other PM alternatives and the optimal equipment's lifetime is 9.5 years (marked with * in Tables 3 and 4). However, the average cost of the lower intensive PM alternatives are very close to PM alternative 3 before 8 years. It indicates that the firm may choose a medium to lower intensive PM alternative with a shorter lifetime if the firm encounter a budget or financial issue. Accordingly, the firm may adjust the present PM alternative to adapt different scenarios according to its financial health in practice.

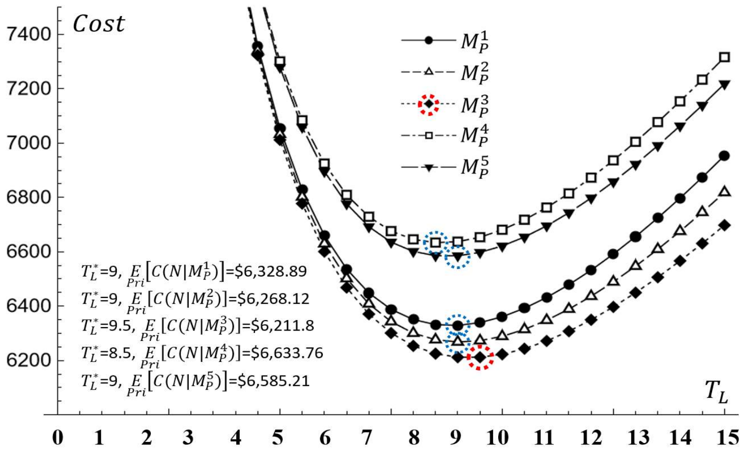


Figure 6. The overall costs per unit and year for all PM alternatives estimated by the prior analysis.

Table 3. The expected preventive cost, repair cost, penalty cost, replacement cost and overall cost per unit and year for PM Alternatives 2, 3, 4 estimated by the prior analysis.

T_L	$\frac{C_{pm}^q(C_F^q,\tau_q,x,T_L)}{T_L}$			$\frac{C_{mr}\underset{Pri}{E}[N_{mr}(\cdot)]}{T_L}$			$\frac{C_{pl}E[t_r\mid\omega,\eta]\underset{Pri}{E}[N_{mr}(\cdot)]}{T_L}$			$\frac{C_{rp}}{T_L}$	$\underset{Pri}{E}[C(N\mid M_P^q)]$		
	M_P^2	M_P^3	M_P^4	M_P^2	M_P^3	M_P^4	M_P^2	M_P^3	M_P^4		M_P^2	M_P^3	M_P^4
0.5	0	0	0	261	261	261	219	219	219	40,000	40,480	40,480	40,480
1	867	880	983	295	291	287	247	244	240	20,000	21,409	21,415	21,510
1.5	1207	1227	1380	325	317	308	272	266	258	13,333	15,138	15,142	15,280
2	1416	1440	1630	353	340	328	296	285	275	10,000	12,064	12,066	12,233
2.5	1572	1600	1822	379	363	346	318	304	290	8000	10,269	10,267	10,458
3	1702	1733	1984	404	384	364	339	322	305	6667	9111	9106	9319
3.5	1816	1851	2129	429	405	380	359	339	319	5714	8318	8310	8543
4	1922	1960	2264	452	425	397	379	356	332	5000	7753	7741	7993
4.5	2021	2062	2392	476	444	412	399	372	346	4444	7340	7323	7594
5	2115	2160	2515	499	463	428	418	389	358	4000	7032	7012	7301
5.5	2207	2255	2634	522	482	443	437	404	371	3636	6802	6777	7084
6	2296	2347	2751	544	501	457	456	420	383	3333	6630	6601	6925
6.5	2383	2437	2865	567	519	472	475	435	395	3077	6502	6468	6809
7	2468	2526	2978	589	537	486	494	450	407	2857	6408	6371	6729
7.5	2553	2613	3090	611	555	500	512	466	419	2667	6343	6301	6676
8	2637	2700	3201	633	573	514	531	480	431	2500	6300	6254	6646

Table 3. *Cont.*

T_L	$\frac{C_{pm}^q(C_F^q,\tau_q,x,T_L)}{T_L}$			$\frac{C_{mr}\underset{Pri}{E}[N_{mr}(\cdot)]}{T_L}$			$\frac{C_{pl}E[t_r \mid \omega,\eta]\underset{Pri}{E}[N_{mr}(\cdot)]}{T_L}$			$\frac{C_{rp}}{T_L}$	$\underset{Pri}{E}[C(N \mid M_P^q)]$		
	M_P^2	M_P^3	M_P^4	M_P^2	M_P^3	M_P^4	M_P^2	M_P^3	M_P^4		M_P^2	M_P^3	M_P^4
8.5	2719	2786	3311	655	591	528	549	495	442	2353	6276	6225	**6634**
9	2802	2871	3420	677	608	541	567	512	454	2222	**6268**	6214	6637
9.5 *	2883	2956	3529	699	626	555	586	525	465	2105	6273	**6212 ***	6654
10	2964	3040	3637	720	643	568	604	539	476	2000	6289	6223	6681
10.5	3045	3124	3744	742	661	581	622	554	487	1905	6314	6243	6718
11	3126	3207	3851	764	678	595	640	568	499	1818	6348	6272	6763
11.5	3206	3290	3958	786	695	608	659	583	510	1739	6389	6308	6815
12	3286	3373	4065	807	712	621	677	597	520	1667	6436	6350	6873
12.5	3365	3456	4171	829	730	634	695	612	531	1600	6489	6397	6936
13	3445	3538	4277	851	747	647	713	626	542	1538	6547	6449	7005
13.5	3524	3621	4383	872	764	660	731	640	553	1481	6609	6506	7077
14	3603	3703	4489	894	781	672	750	655	564	1429	6676	6567	7154
14.5	3682	3785	4595	916	798	685	768	669	574	1379	6745	6631	7233
15	3761	3867	4700	938	815	698	786	683	585	1333	6818	6698	7316

Table 4. The expected preventive cost, repair cost, penalty cost, replacement cost and overall cost per unit and year for PM Alternative for the prior and posterior analyses.

T_L	$\frac{C_{pm}^q(C_F^q,\tau_q,x,T_L)}{T_L}$	$\frac{C_{mr}E[N_{mr}(\cdot)]}{T_L}$		$\frac{C_{pl}E[t_r \mid \omega,\eta]E[N_{mr}(\cdot)]}{T_L}$		$\frac{C_{rp}}{T_L}$	$\underset{Pri}{E}[C(N \mid M_P^q)]$	$\underset{Pos}{E}[C(N \mid M_P^q)]$
		Prior	**Posterior**	**Prior**	**Posterior**			
0.5	0	261	252	219	212	40,000	40,480	40,464
1	880	291	287	244	240	20,000	21,415	21,407
1.5	1227	317	313	266	262	13,333	15,142	15,135
2	1440	340	334	285	280	10,000	12,066	12,055
2.5	1600	363	354	304	297	8000	10,267	10,251
3	1733	384	372	322	312	6667	9106	9083
3.5	1851	405	388	339	325	5714	8310	8279
4	1960	425	404	356	339	5000	7741	7702
4.5	2062	444	419	372	351	4444	7323	7276
5	2160	463	433	389	363	4000	7012	6956
5.5	2255	482	446	404	374	3636	6777	6712
6	2347	501	460	420	385	3333	6601	6525
6.5	2437	519	472	435	396	3077	6468	6382
7	2526	537	485	450	406	2857	6371	6274
7.5	2613	555	497	466	416	2667	6301	6193
8	2700	573	508	480	426	2500	6254	6134
8.5	2786	591	520	495	436	2353	6225	6094
9	2871	608	531	512	445	2222	6214	6069
9.5 *	2956	626	542	525	454	2105	**6212 ***	6057
10 **	3040	643	553	539	463	2000	6223	**6056 ****
10.5	3124	661	563	554	472	1905	6243	6064
11	3207	678	574	568	481	1818	6272	6080
11.5	3290	695	584	583	489	1739	6308	6103
12	3373	712	594	597	498	1667	6350	6132
12.5	3456	730	604	612	506	1600	6397	6166
13	3538	747	614	626	514	1538	6449	6205
13.5	3621	764	623	640	522	1481	6506	6248
14	3703	781	633	655	530	1429	6567	6295
14.5	3785	798	642	669	538	1379	6631	6345
15	3867	815	651	683	546	1333	6698	6398

However, if the firm didn't have enough confidence to believe the outcomes of the prior analysis results, the decision makers will request more evidences to verify the system's deterioration. Therefore, in order to adjust the previous analysis, it is necessary to gather the extra failure data by proceeding with accelerated deterioration experiments. In consideration of the limits budget and time, the firm can only gather few experimental data to revise the previous analysis result. After getting the extra experimental data to proceed with the posterior analysis, the data can integrate with the previous experts' opinions of the prior analysis, and the complete analysis results are reported in Table 4. Figure 7 shows that the overall cost per unit and year of the posterior analysis is always smaller than that of the prior analysis. The best lifetime of an equipment will be extended to 10 years (marked with ** in Table 4) and the overall cost per unit and year will be $6056. After reviewing the posterior analysis's results, it is clear that the prior analysis's conclusions may be overly pessimistic since the re-estimating deterioration in the posterior analysis is less severe than that in the prior study. In other words, the firm should moderately extend the planned lifetime of equipment to save the related costs.

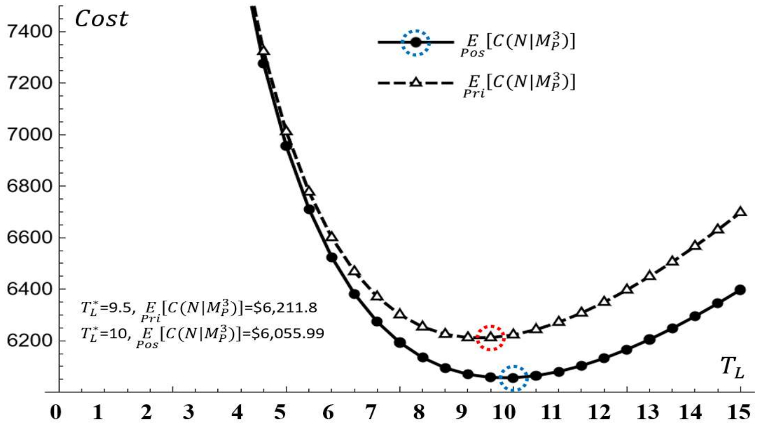


Figure 7. The overall costs per unit and year for all PM alternatives estimated by the prior and posterior analyses.

4.2. Sensitivity Analyses

Misjudging the parameters values of the maintainable and non-maintainable modes μ_{α_p}, μ_{β_p}, μ_{α_o}, and μ_{β_o} may have an influence on the predictions of the overall cost per unit and year, thus the firm should be aware of potential changes in the projections. As a result, sensitivity analysis may be undertaken to evaluate differences in the equipment's lifetime and the overall cost. It is logical to assume that if the firm underestimates these parameters' value, the related cost are also underestimated, and it will lead to poor judgments such as incorrectly extending equipment's lifetime. According to Figure 8, the firm will prolong the lifetime of equipment by taking advantage of the lower growing degradation if these parameters values are decreasing. Moreover, since the variation effect of the non-maintainable parameters is greater than that of the maintainable parameters, the firm needs carefully to evaluate the non-maintainable parameters to avoid inappropriate decisions especially for shape factors. It can be seen in that the variation of μ_{β_o} between the range (−30%, +30%) causes the change of decision about lifetime from 7 years to 10 years. The variation of the overall cost is also huge. However, the effect of misjudging the shape factor of the maintainable parameter μ_{β_p} will be smaller because the system's deterioration can be restored to a younger status after a PM action. It means that the growth of the effect age can be slowed down with time by periodic PM actions. Besides, misjudging the parameters of standard deviation σ_{α_p}, σ_{β_p}, σ_{α_o}, and σ_{β_o} will influence the range of confidence intervals of the expected overall cost but it may not result in risky decisions because the optimal decision of the equipment's lifetime will be less changed with these parameters of standard deviation.

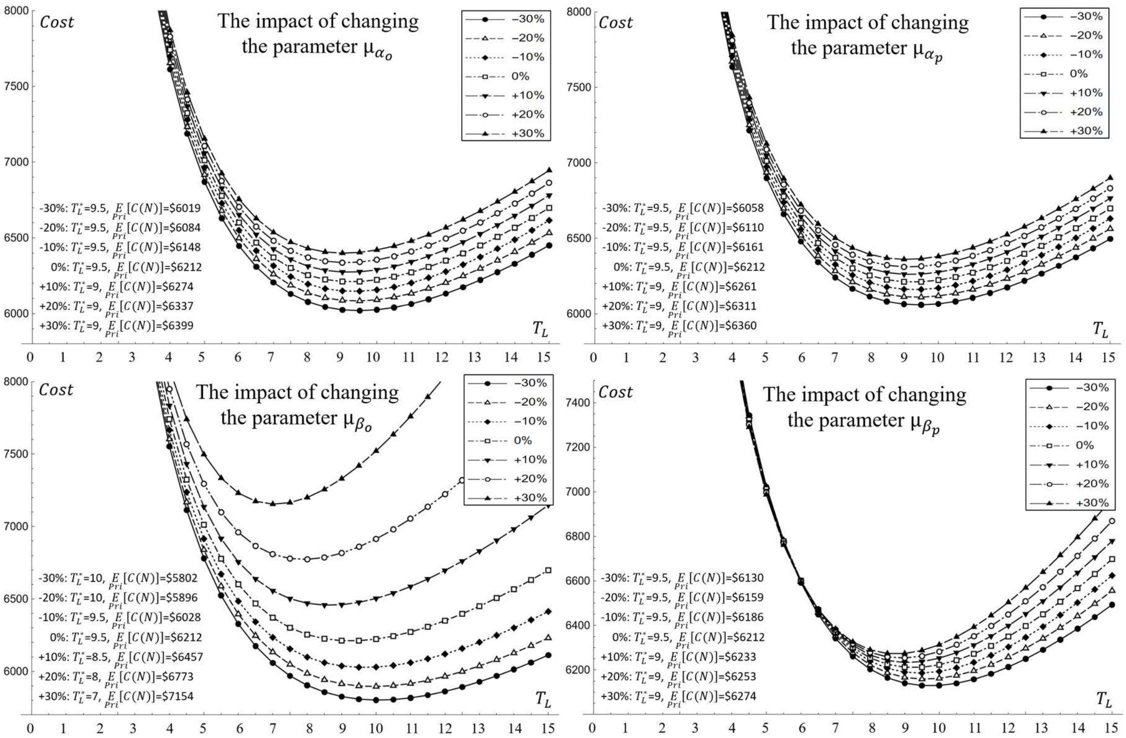


Figure 8. The impacts of μ_{α_o}, μ_{α_p}, μ_{β_o}, and μ_{β_p} on the overall cost per unit and year.

Besides, the related parameters regarding PM, repair, penalty would also influence the estimation of the overall cost. The base cost C_F^q seems the most important influencing factor in this case because the portion of this factor is almost 30–50% of the overall cost. Increasing the base cost will cause the optimal decision of shortening equipment's lifetime. It can be seen that the variation of C_F^q between the range (−30%, +30%) leads to the change of decision about lifetime from 10.5 years to 8.5 years, and the overall cost at the individual optimal lifetime will go up from $5306 to $7061. Moreover, the increaing rate τ_q is an important parameter for measuring the expected overall cost. Inflation, wage increases, or the cost of supplying components may be contributing factors to the increasing rate τ_q. As a result, if the company believes or predicts that the vital components for the replacement of PM actives will rise in the future, the company should plan to reduce the equipment's lifetime since the average replacement cost of the total equipment cannot cover the cumulative PM cost. It can be seen in Figure 9. The variation of τ_q between the range (−30%, +30%) lead to the change of the optimal lifetime T_L^* from 10.5 years to 8.5 years, and the estimated overall cost will go up from $5763 to $6609. Besides, the variations of repair cost C_{mr} and the penalty cost C_{pl} between the range (−30%, +30%) also lead to the change of the optimal lifetime from 9.5 years to 9 years. However, the effects are less than PM's parameters in this case because the equipment can adopt some high intensive PM strategies to lower the occurances of breakdowns. However, in some cases, the equipment may cannot adopt high intensive PM strategies due to the equipment's characteristics. In such cases, the repair and penalty costs will be the major portion of the overall cost, and the firm should try to develope effective repair techniques to improve the system's stability for raising the efficiency of operatons or production lines.

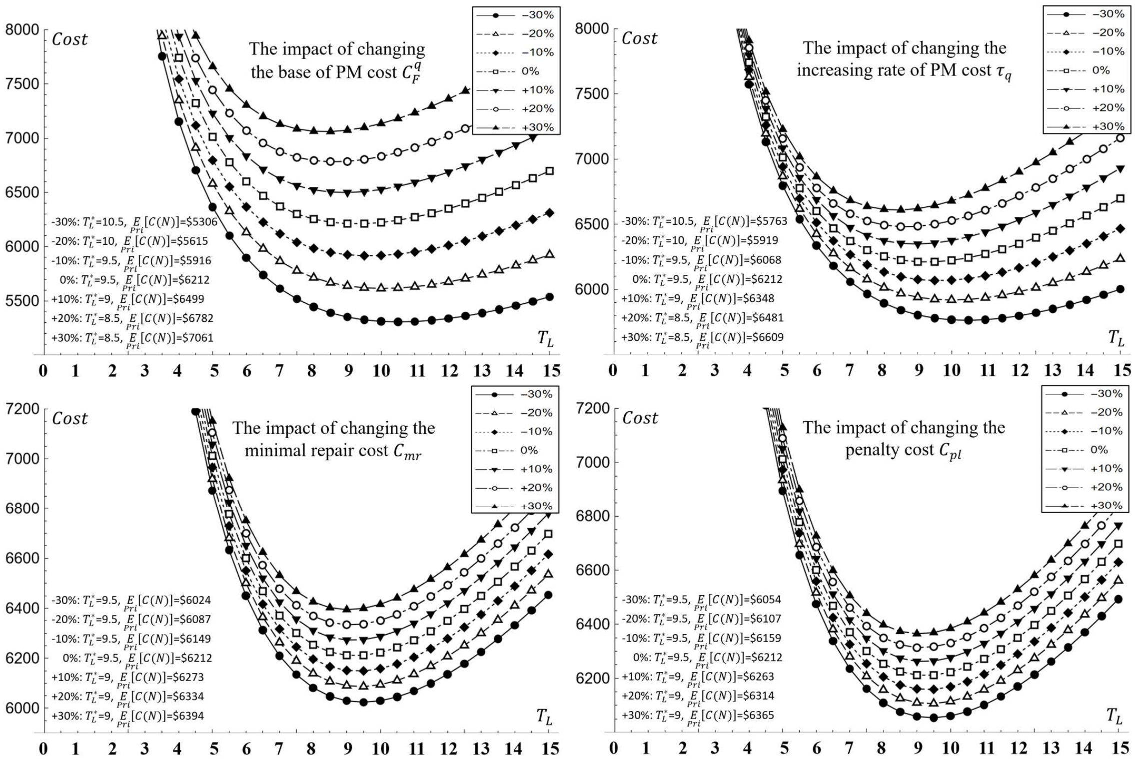


Figure 9. The Impacts of C_F^q, τ_q, C_{mr}, and C_{pl} on the overall cost per unit and year.

5. Conclusions

The study aims to provide Bayesian decision analyses for preventive maintenance related to the hybrid competing failure mode. The proposed model not only can proceed with the prior analysis under a lack of historical failure data but also can consider sampling information to proceed with the posterior analysis. It can help firms to make their best PM alternatives to achieve the advantages of lower cost and operational stability. According to the analysis results of Section 4, the managerial insights and contributions can be summarized as follows: (1) The proposed Bayesian analysis will be a feasible solution in evaluating the related repair and PM costs if the firm cannot proceed with accelerating deterioration experiments to gather enough information to estimate the model's parameters by traditional statistical methods. (2) A highly intensive PM alternative may not have a beneficial influence on reducing the related costs. It depends on whether the benefit of reducing equipment breakdowns is greater than the increment of PM cost. Therefore, the firm needs to evaluate the trade-off between the two influences before the decision-making. (3) A lower intensive PM alternative may bring serious equipment′ breakdowns during the post-phase. The firm may adopt the strategy of shortening the equipment's lifetime to prevent this disadvantage. (4) It is suggested that the firm may adopt a medium-to-lower intensive PM alternative with a shorter lifetime if the firm encounters a budget or financial issue in practice. In other words, a firm should not adopt a highly intensive PM alternative with a longer lifetime in consideration of the payback period even if the return of the PM's investment is high. (5) Misjudging the parameter values of the maintainable and non-maintainable modes will lead to inappropriate decisions. Therefore, the firm should be aware of the potential changes in the related costs and the optimal lifetime to react appropriately. (6) The influence of misjudging the maintainable parameters will be minor because the system's deterioration can be restored to a younger status after a PM action. It means that the risk of misestimating maintainable parameters is less than that of non-maintainable parameters. (7) Misjudging the standard deviation of the parameters will influence the estimation of the confidence intervals of the overall cost. However, it may not result in more risky decisions since the optimal decision will be less affected by the standard deviations.

In future works, the proposed model could be refined by taking into account the deterioration of equipment or facilities on two-dimensional failure variables. Since most equipment or facilities' deterioration depends on time and usage, considering only one of these factors could lead to inaccurate estimates. In such cases, a two-dimensional failure model would be suitable for estimating equipment or facilities' deterioration. Accordingly, the two-factor joint probability distribution (time, usage) will be constructed by using a bivariate-Weibull probability distribution. Besides, the approaches of condition-based PM can be also applied in this scenario. Therefore, decision-makers can utilize the extended approaches to improve their PM strategies.

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