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Article

Nonhomogeneous Dirichlet Problems with Unbounded Coefficient in the Principal Part

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Abstract: The main result of the paper establishes the existence of a bounded weak solution for a nonlinear Dirichlet problem exhibiting full dependence on the solution u and its gradient ∇u in the reaction term, which is driven by a p-Laplacian-type operator with a coefficient $G(u)$ that can be unbounded. Through a special Moser iteration procedure, it is shown that the solution set is uniformly bounded. A truncated problem is formulated that drops that $G(u)$ be unbounded. The existence of a bounded weak solution to the truncated problem is proven via the theory of pseudomonotone operators. It is noted that the bound of the solution for the truncated problem coincides with the uniform bound of the original problem. This estimate allows us to deduce that for an appropriate choice of truncation, one actually resolves the original problem.

Keywords: p-Laplacian with unbounded coefficient; convection term; truncated problem; uniform bound; weak solution; pseudomonotone operator

MSC: 35J70; 35J92; 47H30

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1. Introduction

In this paper, we study the following Dirichlet problem:

$$\begin{cases} -\mathrm{div}(G(u)|\nabla u|^{p-2}\nabla u) = F(x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

on a bounded domain Ω in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$. In (1) we have a continuous function $G : \mathbb{R} \to [a_0, +\infty)$, with $a_0 > 0$, a number $p \in (1, +\infty)$ with $N > p$, and a Carathéodory function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ (i.e., $F(\cdot, t, \xi)$ is measurable on Ω for each $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $F(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$). The notation ∇u stands for the gradient of u in the distributional sense. It is seen that the driving operator in Equation (1) is the p-Laplacian with a coefficient $G(u)$ depending on the solution u. The notation $G(u)$ in Equation (1) means the composition of the functions $G : \mathbb{R} \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$, that is, $G(u)(x) = G(u(x))$ for $x \in \Omega$. The main point is that $G(u)$ can be unbounded from above, which does not permit to apply any standard method. It is also worth mentioning that problem (1) is not in variational form.

The space underlying the Dirichlet problem (1) is the Banach space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall u \in W_0^{1,p}(\Omega).$$

The dual space of $W_0^{1,p}(\Omega)$ is denoted $W^{-1,p'}(\Omega)$. Since it was supposed that $N > p$, the critical Sobolev exponent is $p^* = Np/(N-p)$. Refer to [1] for the background related to the space $W_0^{1,p}(\Omega)$.

The (negative) p-Laplacian is the nonlinear operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ (linear for $p = 2$) defined by

$$\langle -\Delta_p(u), v \rangle := \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx, \quad \forall u, v \in W_0^{1,p}(\Omega). \tag{2}$$

Due to the unbounded function $G(u)$, one cannot build a definition as in (2) corresponding to the term $-\mathrm{div}(G(u)|\nabla u|^{p-2}\nabla u)$ in (1). A major tool in our arguments is the first eigenvalue λ_1 of $-\Delta_p$, which is positive and isolated in the spectrum of $-\Delta_p$, and is given by

$$\lambda_1 := \inf_{u \in W_0^{1,p}(\Omega),\, u \neq 0} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}. \tag{3}$$

For the the rest of the paper, in order to simplify the notation we make the notational convention that for any real number $r > 1$ we denote $r' := r/(r-1)$ (the Hölder conjugate of r).

The Carathéodory function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ determining the reaction term $F(x, u, \nabla u)$ is subject to the following hypotheses.

Hypothesis 1 (H1). *There exist constants $c_1 \geq 0$, $c_2 \geq 0$, $c_3 \geq 0$, and $r \in (p, p^*)$ such that*

$$|F(x,t,\xi)| \leq c_1|\xi|^{\frac{p}{r'}} + c_2|t|^{r-1} + c_3 \text{ for a.e. } x \in \Omega,\ \forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

Hypothesis 2 (H2). *There exist constants $d_1 \geq 0$ and $d_2 \geq 0$ with $d_1 + \lambda_1^{-1} d_2 < a_0$, and a function $\sigma \in L^1(\Omega)$ such that*

$$F(x,t,\xi)t \leq d_1|\xi|^p + d_2|t|^p + \sigma(x) \text{ for a.e. } x \in \Omega,\ \forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

where λ_1 denotes the first eigenvalue of $-\Delta_p$.

The main result of this paper is stated as follows.

Theorem 1. *Assume that $G : \mathbb{R} \to [a_0, +\infty)$, with $a_0 > 0$, is a continuous function and $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the conditions (H1) and (H2). Then problem (1) has at least a bounded weak solution $u \in W_0^{1,p}(\Omega)$ in the following sense:*

$$\int_\Omega G(u)|\nabla u|^{p-2}\nabla u \nabla v dx = \int_\Omega F(x, u, \nabla u) v dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{4}$$

Under hypothesis (H1), the integrals in (4) exist. The proof of Theorem 1 is presented in Section 3. In order to see the effective applicability of Theorem 1, we provide an example.

Example 1. *On a bounded domain Ω in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$, we state the Dirichlet problem*

$$\begin{cases} -\mathrm{div}(e^{u^2}|\nabla u|^{p-2}\nabla u) = b_1|u|^{p-2}u + b_2\frac{u}{u^2+1}|\nabla u|^{\frac{p(r-1)}{r}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

with constants $p \in (1, +\infty)$, $r \in (p, p^*)$, $b_1 \geq 0$, $b_2 \geq 0$, provided that $N > p$ and $1 > b_2 + \lambda_1^{-1} b_1$, where λ_1 is given by (3). We readily check that (5) fits into the framework of problem (1) taking $G(t) = e^{t^2}$ for all $t \in \mathbb{R}$ and

$$F(x,t,\xi) = b_1|t|^{p-2}t + b_2\frac{t}{t^2+1}|\xi|^{\frac{p(r-1)}{r}}, \quad \forall (x,t,\xi) \in \Omega \times \mathbb{R} \times \in \mathbb{R}^N.$$

Indeed, one has $G(t) \geq a_0 := 1$ *for all* $t \in \mathbb{R}$,

$$|F(x,t,\xi)| \leq b_1(|t|^{r-1} + 1) + b_2|\xi|^{\frac{p}{r'}}, \quad \forall (x,t,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

$$F(x,t,\xi)t \leq b_1|t|^p + b_2(|\xi|^p + 1), \quad \forall (x,t,\xi) \in \Omega \times \mathbb{R} \times \in \mathbb{R}^N.$$

Assumption (H1) is verified with $c_1 = b_2$, $c_2 = c_3 = b_1$, *while assumption (H2) holds with* $d_1 = b_2$, $d_2 = b_1$, $\sigma(x) \equiv b_2$. *Theorem 1 applies because* $a_0 > d_1 + \lambda_1^{-1} d_2$.

The inspiration for the present work comes from the recent paper [2] that deals with the Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x)g(|u|)|\nabla u|^{p-2}\nabla u) = f(x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

for a positive $a \in L^1_{loc}(\Omega)$, a continuous function $g : [0,+\infty) \to [a_0,+\infty)$, with $a_0 > 0$, and a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. The standing point in that work was to use the theory of weighted Sobolev spaces in [3] (see also [4]) with the weight $a \in L^1_{loc}(\Omega)$ requiring the condition

$$a^{-s} \in L^1(\Omega) \;\text{ for some } s \in \left(\frac{N}{p}, +\infty\right) \cap \left[\frac{1}{p-1}, +\infty\right).$$

If we consider our problem (1) as a particular case of (6) taking $a(x) \equiv 1$ on Ω and apply the result in [2], the issue is that one obtains a solution of (1) belonging to the space $W_0^{1,p_s}(\Omega)$ with

$$p_s = \frac{ps}{s+1}. \tag{7}$$

and not to the space $W_0^{1,p}(\Omega)$ as it would be natural according to the statement of (1). In this respect, by (7) we note that $p_s < p$, so $W_0^{1,p}(\Omega)$ is strictly contained in $W_0^{1,p_s}(\Omega)$. Moreover, the assumptions admitted therein for the reaction $f(x,u,\nabla u)$ in (6) are more restrictive than here because they are formulated in terms of p_s corresponding to some s and not with p as in conditions (H1)–(H2) for $F(x,u,\nabla u)$. All of this shows that the treatment in [2] does not provide the right approach to obtain Theorem 1. For this reason, we develop a direct study for problem (1) relying just on the classical Sobolev space $W_0^{1,p}(\Omega)$. The present paper is the first work studying problem (1) with unbounded coefficient $G(u)$ in the Soboleev space $W_0^{1,p}(\Omega)$. Certainly, we use some previous ideas but with substantial modifications and in a different functional setting. The technique relies on truncation, which is needed because the coefficient $G(u)$ in the principal part of Equation (1) is unbounded. Other important tools in our study are a special version of Moser iteration and the surjectivity theorem for pseudomonotone operators.

We mention a few relevant works in the area of our paper. A large amount of results in the field is based on variational smooth or nonsmooth methods for which we refer to the recent publications [5–7]. They cannot be applied to problem (1) taking into account the lack of variational structure. Nonvariational problems with convection terms have been investigated in recent years through theoretic operator techniques, sub-supersolution and approximation (see, e.g., [8–12]). The main point in these works lies in the dependence of the reaction term with respect to the gradient of the solution without weakening the ellipticity condition of the driving operator. In this connection, we also cite papers dealing with the equations and inclusions driven by the (p,q)-Laplacian operators, such as, for instance [13,14]. As an extension of this setting, the paper [15] deals with degenerate (p,q)-Laplacian problems, but without dependence on the solution u in the principal part of the equation. An advance in this direction is ref. [2], where there is dependence on solution u in the principal part of the equation of type (6) subject to a weight $a(x)$. Here, we drop the dependence on weight $a(x)$ and allow to have a unbounded coefficient $G(u)$ in problem (1).

Regarding the rest of the paper, Section 2 focuses on the bounded solutions to problem (1), and Section 3 contains the proof of Theorem 1.

2. Bounded Solutions to Problem (1)

Our first goal is to estimate the solutions in $W_0^{1,p}(\Omega)$.

Lemma 1. *Assume that condition (H2) holds. Then the set of solutions to problem (1) is bounded in $W_0^{1,p}(\Omega)$ with a bound that depends on function G only through the lower bound a_0 of G.*

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a solution of (1). Inserting $v = u$ in (4) yields

$$\int_\Omega G(u)|\nabla u|^p dx = \int_\Omega F(x,u,\nabla u)u dx.$$

Invoking hypothesis (H2) and (3), we arrive at

$$a_0\|u\|^p \le (d_1 + d_2\lambda_1^{-1})\|u\|^p + \|\sigma\|_{L^1(\Omega)}.$$

Since by hypothesis $d_1 + d_2\lambda_1^{-1} < a_0$, the stated result is true. □

We are now able to find a uniform bound for the solutions of (1).

Theorem 2. *Assume that conditions (H1) and (H2) are satisfied. Then the solution set of problem (1) is uniformly bounded, that is, there exists a constant $C > 0$ such that $\|u\|_{L^\infty(\Omega)} \le C$ for every weak solution $u \in W_0^{1,p}(\Omega)$ to problem (1). The dependence of the uniform bound C on the data in problem (1) and hypotheses (H1) and (H2) is indicated as $C = C(N, p, \Omega, a_0, c_1, c_2, c_3, d_1, d_2, \|\sigma\|_{L^1(\Omega)})$. In particular, the uniform bound C depends on G only through its lower bound a_0.*

Proof. Given a weak solution $u \in W_0^{1,p}(\Omega)$ to problem (1), we have the representation $u = u^+ - u^-$ with $u^+ = \max\{u, 0\}$ (the positive part of u) and $u^- = \max\{-u, 0\}$ (the negative part of u). We prove the uniform boundedness separately for u^+ and u^-. We only give the proof for u^+, noting that we can argue similarly in the case of u^-.

We proceed by using in (4) the test function $v = u^+u_h^{kp} \in W_0^{1,p}(\Omega)$, where $u_h := \min\{u^+, h\}$ with arbitrary constants $h > 0$ and $k > 0$. The fact that $v \in W_0^{1,p}(\Omega)$ follows from $u \in L^p(\Omega)$ and u_h is bounded, while the distributional partial derivatives

$$\frac{\partial v}{\partial x_i} = u_h^{kp}\frac{\partial u^+}{\partial x_i} + kpu_h^{kp-1}u^+\frac{\partial u_h}{\partial x_i}, \quad \forall i = 1, \cdots, N,$$

belong to $L^p(\Omega)$ because $u, \partial u_h/\partial x_i \in L^p(\Omega)$ and u_h is bounded. This gives

$$\int_\Omega G(u)|\nabla u|^{p-2}\nabla u \nabla(u^+u_h^{kp})dx = \int_\Omega F(x,u,\nabla u)u^+u_h^{kp}dx. \tag{8}$$

The left-hand side of (8) can be estimated as follows:

$$\int_\Omega G(u)|\nabla u|^{p-2}\nabla u\nabla(u^+u_h^{kp})dx$$
$$= \int_\Omega G(u)|\nabla u|^{p-2}\nabla u(u_h^{kp}\nabla(u^+) + kpu^+u_h^{kp-1}\nabla(u_h))dx \tag{9}$$
$$\geq a_0\left[\int_\Omega u_h^{kp}|\nabla(u^+)|^p dx + kp\int_{\{0<u<h\}} u_h^{kp}|\nabla(u^+)|^p dx\right].$$

For the right-hand side of (8), by hypothesis (H1), we obtain

$$\int_\Omega F(x,u,\nabla u)u^+u_h^{kp}dx \tag{10}$$
$$\leq c_1\int_\Omega |\nabla u|^{\frac{p}{r'}}u_h^{kp}u^+dx + c_2\int_\Omega |u|^{r-1}u_h^{kp}u^+dx + c_3\int_\Omega u_h^{kp}u^+dx.$$

By Young's inequality, for each $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that

$$c_1\int_\Omega |\nabla u|^{\frac{p}{r'}}u_h^{kp}u^+dx = c_1\int_\Omega (|\nabla(u^+)|^{\frac{p}{r'}}u_h^{\frac{kp}{r'}})(u_h^{\frac{kp}{r}}u^+)dx \tag{11}$$
$$\leq \varepsilon\int_\Omega u_h^{kp}|\nabla(u^+)|^p dx + c(\varepsilon)\int_\Omega u_h^{kp}(u^+)^r dx.$$

It is clear that

$$\int_\Omega |u|^{r-1}u_h^{kp}u^+dx = \int_\Omega u_h^{kp}(u^+)^r dx \tag{12}$$

and, since $r > 1$ and $u_h \leq u^+$,

$$\int_\Omega u_h^{kp}u^+dx = \int_{\{u^+\geq 1\}} u_h^{kp}u^+dx + \int_{\{u^+<1\}} u_h^{kp}u^+dx \tag{13}$$
$$\leq \int_\Omega u_h^{kp}(u^+)^r dx + |\Omega|,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

If $\varepsilon > 0$ is sufficiently small, we deduce from (8), in conjunction with (9), (10), (11), (12), and (13) that

$$\int_\Omega u_h^{kp}|\nabla(u^+)|^p dx + kp\int_{\{0<u<h\}} u_h^{kp}|\nabla(u^+)|^p dx \tag{14}$$
$$\leq b\left(\int_\Omega u_h^{kp}(u^+)^r dx + 1\right),$$

with a constant $b > 0$. The last integral exists because $r < p^*$.

On the other hand, by Bernoulli's inequality and since $u_h = u^+$ on $\{0 < u < h\}$, we derive

$$\int_\Omega u_h^{kp}|\nabla(u^+)|^p dx + kp\int_{\{0<u<h\}} u_h^{kp}|\nabla(u^+)|^p dx$$
$$= \int_{\{u\geq h\}} |\nabla(u_h^k u^+)|^p dx + \int_{\{u<h\}} |\nabla(u_h^k u^+)|^p dx + \frac{kp}{(k+1)^p}\int_{\{0<u<h\}} |\nabla(u_h^k u^+)|^p dx$$
$$= \int_{\{u\geq h\}} |\nabla(u_h^k u^+)|^p dx + \frac{kp+1}{(k+1)^p}\int_{\{0<u<h\}} |\nabla(u_h^k u^+)|^p dx \tag{15}$$
$$\geq \frac{kp+1}{(k+1)^p}\int_\Omega |\nabla(u_h^k u^+)|^p dx.$$

Combining (14) and (15) leads to

$$\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla(u_h^k u^+)|^p dx \le b\Big(\int_\Omega u_h^{kp}(u^+)^r dx + 1\Big). \tag{16}$$

At this point, we choose $q \in (p, r)$ with

$$\frac{(r-p)q}{q-p} < p^*. \tag{17}$$

The validity of such a choice holds in view of $p < r < p^*$ as postulated in condition (H1). Then (17), the Sobolev embedding theorem, Hölder's inequality with $(q-p)/q + p/q = 1$, and Lemma 1 imply

$$\int_\Omega u_h^{kp}(u^+)^r dx = \int_\Omega (u^+)^{r-p}(u_h^k u^+)^p dx$$
$$\le \left(\int_\Omega (u^+)^{\frac{(r-p)q}{q-p}} dx\right)^{\frac{q-p}{q}} \left(\int_\Omega (u_h^k u^+)^q dx\right)^{\frac{p}{q}} \le K\|u_h^k u^+\|^p_{L^q(\Omega)},$$

with a constant $K > 0$.

On the basis of the previous inequality and the Sobolev embedding theorem, we obtain from (16) that

$$c_0 \frac{kp+1}{(k+1)^p}\|u_h^k u^+\|^p_{L^{p^*}(\Omega)} \le b\Big(\|(u^+)^{k+1}\|^p_{L^q(\Omega)} + 1\Big),$$

with a constant $c_0 > 0$. Then Fatou's lemma letting $h \to +\infty$ entails

$$c_0 \frac{kp+1}{(k+1)^p}\|u^+\|^{p(k+1)}_{L^{p^*(k+1)}(\Omega)} \le b\Big(\|(u^+)^{k+1}\|^p_{L^q(\Omega)} + 1\Big).$$

By some arrangements, we obtain for a constant $C_1 > 0$ the estimate

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le C_1^{\frac{1}{k+1}}(k+1)^{\frac{1}{k+1}}\left(\|(u^+)^{k+1}\|^p_{L^q(\Omega)} + 1\right)^{\frac{1}{(k+1)p}}.$$

Noticing that the sequence $(k+1)^{\frac{1}{\sqrt{k+1}}}$ is bounded, we find a constant $C_0 > 0$ for which it holds

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le C_0^{\frac{1}{\sqrt{k+1}}}\left(\|u^+\|^{(k+1)p}_{L^{(k+1)q}(\Omega)} + 1\right)^{\frac{1}{(k+1)p}}. \tag{18}$$

We claim that there exists a constant $C > 0$ independent of the solution u to (1) such that

$$\|u^+\|_{L^d(\Omega)} \le C, \quad \forall d \ge 1. \tag{19}$$

In the case where $\|u^+\|_{L^{(k+1)q}(\Omega)} \le 1$ for infinitely many k, it is straightforward to show the validity of the claim. Therefore, we may suppose that $\|u^+\|_{L^{(k+1)q}(\Omega)} > 1$ for all $k \ge k_0$,

If $\|u^+\|_{L^{(k+1)q}(\Omega)} > 1$ for all k, we see that (18) takes the form

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le C_1^{\frac{1}{\sqrt{k+1}}}\|u^+\|_{L^{(k+1)q}(\Omega)}, \tag{20}$$

with a constant $C_1 > 0$. Through (20), we are able to carry on a Moser iteration, setting inductively $(k_n + 1)q = (k_{n-1} + 1)p^*$ with the initial step $(k_1 + 1)q = p^*$. Applying repeatedly (20), it turns out that

$$\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \leq C_1^{\sum_{1\leq i\leq n} \frac{1}{\sqrt{k_i+1}}} \|u^+\|_{L^{(k_1+1)q}(\Omega)}, \quad \forall n \geq 1. \tag{21}$$

The series $\sum_{n\geq 1} \frac{1}{\sqrt{k_n+1}}$ converges because $q < p^*$ and $k_n \to +\infty$ as $n \to \infty$. Consequently, we can obtain (19) letting $n \to \infty$ in (21).

It remains to handle the case when the number k_0 is such that $\|u^+\|_{L^{(k_0+1)q}(\Omega)} \leq 1$ and $\|u^+\|_{L^{(k+1)q}(\Omega)} > 1$ for all $k > k_0$. In this case, the Moser iteration reads as $(k_n + 1)q = (k_{n-1} + 1)p^*$ with the initial step $(k_1 + 1)q = k_0$ if $k_0 < p^*$ and $(k_1 + 1)q = p^*$ if $k_0 \geq p^*$. In any case, we are led to (21) from which (19) can be established as before.

We can pass to the limit as $d \to \infty$ in (19) obtaining $\|u^+\|_{L^\infty(\Omega)} \leq C$ for each weak solution $u \in W_0^{1,p}(\Omega)$ to problem (1). Analogously, we can prove that $\|u^-\|_{L^\infty(\Omega)} \leq C$ for all weak solutions $u \in W_0^{1,p}(\Omega)$ to problem (1). Altogether, we have the uniform bound $\|u\|_{L^\infty(\Omega)} \leq C$ for the solution set of problem (1).

A careful reading of the above proof reveals the dependence of the uniform bound C on the data in problem (1) and on the coefficients, entering assumptions (H1) and (H2). Precisely, we have to check how the constants b, q, K, c_0, C_1, and C_0 arising in the proof depend on the data given in (1), (H1), and (H2). Collecting all these renders the dependence $C = C(N, p, \Omega, a_0, c_1, c_2, c_3, d_1, d_2, \|\sigma\|_{L^1(\Omega)})$. This completes the proof. □

3. Truncation Problem and Proof of Theorem 1

The method of proof relies on the truncation of the coefficient $G(u)$ of the p-Laplacian in problem (1) to drop its unboundedness. This idea was used in [2] in the context of the degenerate p-Laplacian. Specifically, for any number $R > 0$, we introduce the truncation

$$G_R(t) = \begin{cases} G(t) & \text{if} \quad |t| \leq R \\ G(R) & \text{if} \quad t > R \\ G(-R) & \text{if} \quad t < -R. \end{cases} \tag{22}$$

By (22), we obtain a continuous function $G_R : \mathbb{R} \to [a_0, +\infty)$. We also consider the associated operator $\mathcal{A}_R : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ given by

$$\langle \mathcal{A}_R(u), v\rangle = \int_\Omega G_R(u)|\nabla u|^{p-2}\nabla u \nabla v dx, \quad \forall u, v \in W_0^{1,p}(\Omega). \tag{23}$$

The notation $G_R(u)|$ in Equation (23) means the composition of the functions $G_R : \mathbb{R} \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$, that is $G_R(u)(x) = G_R(u(x))$ for $x \in \Omega$. The next proposition discusses the properties of $\mathcal{A}_R$.

Proposition 1. *The nonlinear operator $\mathcal{A}_R$ in (23) is well defined, bounded (i.e., it maps bounded sets into bounded sets), continuous, and satisfies the S_+ property, that is, any sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle \mathcal{A}_R(u_n), u_n - u\rangle \leq 0 \tag{24}$$

fulfills $u_n \to u$ in $W_0^{1,p}(\Omega)$.

Proof. The continuity of the function G combined with (22), (23), and Hölder's inequality ensures

$$|\langle \mathcal{A}_R(u), v\rangle| \le \max_{t\in[-R,R]} G(t)\|u\|^{p-1}\|v\|.$$

for all $u, v \in W_0^{1,p}(\Omega)$. It follows that the operator $\mathcal{A}_R$ is well-defined and bounded.

In order to show the continuity of $\mathcal{A}_R$ let $u_n \to u$ in $W_0^{1,p}(\Omega)$. By the continuity of G, (22), (23), Hölder's inequality, and (2), we find

$$\begin{aligned}
&|\langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), v\rangle| \\
&\le \left|\int_\Omega G_R(u_n)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla v dx\right| \\
&+ \left|\int_\Omega (G_R(u_n) - G_R(u))|\nabla u|^{p-2}\nabla u\nabla v dx\right| \\
&\le \max_{t\in[-R,R]} G(t)|\langle -\Delta_p(u_n) - (-\Delta_p(u)), v\rangle| \\
&+ \left(\int_\Omega |G_R(u_n) - G_R(u)|^{\frac{p}{p-1}}|\nabla u|^p dx\right)^{\frac{p-1}{p}} \|v\|
\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. We infer that

$$\begin{aligned}
&\|\mathcal{A}_R(u_n) - \mathcal{A}_R(u)\|_{W^{-1,p'}(\Omega)} \\
&\le \max_{t\in[-R,R]} G(t)\| - \Delta_p(u_n) - (-\Delta_p(u))\|_{W^{-1,p'}(\Omega)} \\
&+ \left(\int_\Omega |G_R(u_n) - G_R(u)|^{\frac{p}{p-1}}|\nabla u|^p dx\right)^{\frac{p-1}{p}}.
\end{aligned}$$

The continuity of the p-Laplacian Δ_p implies that $-\Delta_p(u_n) \to -\Delta_p(u)$ in $W^{-1,p'}(\Omega)$. By Lebesgue's dominated convergence theorem, we derive

$$\lim_{n\to\infty} \int_\Omega |G_R(u_n) - G_R(u)|^{\frac{p}{p-1}}|\nabla u|^p dx = 0,$$

whence $\mathcal{A}_R(u_n) \to \mathcal{A}_R(u)$ in $W^{-1,p'}(\Omega)$, so the continuity of $\mathcal{A}_R$ is proven.

Now we show the S_+ property for the operator $\mathcal{A}_R$. Let a sequence $\{u_n\}$ satisfy $u_n \rightharpoonup u$ in $W^{1,p}(a, \Omega)$ and (24). It is seen that

$$\limsup_{n\to\infty} \langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), u_n - u\rangle \le 0. \tag{25}$$

Taking into account (23) and the monotonicity of $-\Delta_p$, we have

$$\begin{aligned}
&\langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), u_n - u\rangle \\
= &\int_\Omega G_R(u_n)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u)dx \\
+ &\int_\Omega (G_R(u_n) - G_R(u))|\nabla u|^{p-2}\nabla u\nabla(u_n - u)dx \\
\ge\ & a_0\langle -\Delta_p(u_n) - (-\Delta_p(u)), u_n - u\rangle \\
+ &\int_\Omega (G_R(u_n) - G_R(u))|\nabla u|^{p-2}\nabla u\nabla(u_n - u)dx.
\end{aligned} \tag{26}$$

We claim that

$$\lim_{n\to\infty}\int_\Omega (G_R(u_n)-G_R(u))|\nabla u|^{p-2}\nabla u\nabla(u_n-u)dx=0. \tag{27}$$

To this end, by Hölder's inequality and since the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, we find a constant $C>0$ such that

$$\begin{aligned}&\left|\int_\Omega (G_R(u_n)-G_R(u))|\nabla u|^{p-2}\nabla u\nabla(u_n-u)dx\right| \\ \leq\ & C\Big(\int_\Omega |G_R(u_n)-G_R(u)|^{\frac{p}{p-1}}|\nabla u|^p dx\Big)^{\frac{p-1}{p}}.\end{aligned} \tag{28}$$

By Lebesgue's dominated convergence theorem, it holds

$$\lim_{n\to\infty}\int_\Omega |G_R(u_n)-G_R(u)|^{\frac{p}{p-1}}|\nabla u|^p dx=0. \tag{29}$$

This is true because G_R is continuous, $u_n\to u$ in $L^p(\Omega)$ and there is the domination

$$|G_R(u_n)-G_R(u)|^{\frac{p}{p-1}}|\nabla u|^p\leq 2^{\frac{1}{p-1}}(\max_{t\in[-R,R]}G(t))^{\frac{p}{p-1}}|\nabla u|^p\in L^1(\Omega).$$

Then (25), (26), (27), (28), (29), and $u_n\rightharpoonup u$ in $W^{1,p}(\Omega)$ yield

$$\lim_{n\to\infty}\langle -\Delta_p(u_n),u_n-u\rangle=0. \tag{30}$$

Since it holds,

$$\begin{aligned}\|u_n\|^p&=\langle -\Delta_p(u_n),u\rangle+\langle -\Delta_p(u_n),u_n-u\rangle\\ &\leq \|u_n\|^{p-1}\|u\|+\langle -\Delta_p(u_n),u_n-u\rangle,\end{aligned}$$

Equation (30) results in $\limsup_{n\to\infty}\|u_n\|\leq\|u\|$. Recalling that space $W_0^{1,p}(\Omega)$ is uniformly convex, we conclude that $u_n\to u$ in $W_0^{1,p}(\Omega)$, which proves the S_+ property of the operator $\mathcal{A}_R$. The proof is thus complete. □

For any $R>0$ and the truncation G_R in (22), let us consider the auxiliary problem

$$\begin{cases}-\mathrm{div}(G_R(u)|\nabla u|^{p-2}\nabla u)=F(x,u,\nabla u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases} \tag{31}$$

The solvability and a priori estimates for problem (31) are now studied.

Theorem 3. *Assume that $G:[0,+\infty)\to[a_0,+\infty)$ is a continuous function with $a_0>0$, and that $F:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function satisfying the conditions (H1) and (H2). Then, for every $R>0$, the auxiliary problem (31) has a weak solution $u_R\in W_0^{1,p}(\Omega)$ in the sense that*

$$\int_\Omega G_R(u_R)|\nabla u_R|^{p-2}\nabla u_R\nabla v dx=\int_\Omega F(x,u_R,\nabla u_R)v dx,\quad \forall v\in W_0^{1,p}(\Omega). \tag{32}$$

Moreover, the solution u_R is uniformly bounded and fulfills the a priori estimate $\|u_R\|_{L^\infty(\Omega)}\leq C$ with the constant $C=C(N,p,\Omega,a_0,c_1,c_2,c_3,d_1,d_2,\|\sigma\|_{L^1(\Omega)})$ provided by Theorem 2.

Proof. Fix an $R > 0$. In view of (23), equality (32) reads as

$$\langle \mathcal{A}_R(u_R), v\rangle = \int_\Omega F(x, u_R, \nabla u_R) v dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{33}$$

Through hypothesis (H1) and Hólder's inequality, we find

$$\left| \int_\Omega F(x, u, \nabla u) v dx \right| \leq \int_\Omega |F(x, u, \nabla u)||v| dx$$

$$\leq \int_\Omega (c_1 |\nabla u|^{\frac{p}{r'}} |v| + c_2 |u|^{r-1} |v| + c_3 |v|) dx$$

$$\leq c_1 \|u\|^{\frac{p}{r'}} \|v\|_{L^r(\Omega)} + c_2 \|u\|^{r-1}_{L^r(\Omega)} \|v\|_{L^r(\Omega)} + c_3 |\Omega|^{\frac{1}{r'}} \|v\|_{L^r(\Omega)}$$

for all $u \in W_0^{1,p}(\Omega)$ and $v \in L^r(\Omega)$. We deduce that the mapping

$$u \in W_0^{1,p}(\Omega) \mapsto F(\cdot, u(\cdot), \nabla u(\cdot)) \in L^{r'}(\Omega) \tag{34}$$

is well-defined and bounded. Furthermore, by Krasnoselskii's theorem for Nemytskii operators, the mapping in (34) is continuous from $W_0^{1,p}(\Omega)$ to $L^{r'}(\Omega)$, so continuous from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ due to the continuous embedding $W_0^{1,p}(\Omega) \subset L^r(\Omega)$.

Let us define the mapping $\mathcal{B}_R : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by

$$\mathcal{B}_R(u) = \mathcal{A}_R(u) - F(\cdot, u(\cdot), \nabla u(\cdot)), \quad \forall u \in W_0^{1,p}(\Omega). \tag{35}$$

On account of Proposition 1 and on what was said regarding the mapping in (34), we are entitled to assert that $\mathcal{B}_R : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ introduced in (35) is well-defined, bounded and continuous.

The next step in the proof is to show that the mapping $\mathcal{B}_R : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a pseudomonotone operator, which means that if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and

$$\limsup_{n\to\infty} \langle \mathcal{B}_R(u_n), u_n - u\rangle \leq 0, \tag{36}$$

then

$$\langle \mathcal{B}_R(v), u - v\rangle \leq \liminf_{n\to\infty} \langle \mathcal{B}_R(u_n), u_n - v\rangle \text{ for all } v \in W_0^{1,p}(\Omega). \tag{37}$$

To this end, let $\{u_n\}$ be a sequence as above. By the Rellich–Kondrachov theorem, we derive from $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ that $u_n \to u$ in $L^r(\Omega)$. As noted before, the sequence $\{F(\cdot, u_n(\cdot), \nabla u_n(\cdot))\}$ is bounded in $L^{r'}(\Omega)$. Therefore, we have

$$\lim_{n\to\infty} \int_\Omega F(x, u_n(x), \nabla u_n(x))(u_n(x) - u(x)) dx = 0.$$

Then (36) entails that (24) holds true. As Proposition 1 guarantees that $\mathcal{A}_R$ has the S_+ property, we can conclude that $u_n \to u$ in $W_0^{1,p}(\Omega)$. From here, it can be readily shown (37) thanks to the continuity and boundedness properties stated in Proposition 1 and those related to (34). This amounts to saying that $\mathcal{B}_R$ is a pseudomonotone operator.

In the following, we prove that the operator $\mathcal{B}_R : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ is coercive, that is

$$\lim_{\|u\|\to\infty} \frac{\langle \mathcal{B}_R(u), u\rangle}{\|u\|} = +\infty. \tag{38}$$

Toward this we infer from (35), (33), (22), (3), Hölder's inequality and hypothesis (H2) that

$$\langle \mathcal{B}_R(u), u\rangle = \int_\Omega G_R(u)|\nabla u|^p dx - \int_\Omega F(x,u,\nabla u)u dx$$
$$\geq \quad (a_0 - d_1 - d_2\lambda_1^{-1})\|u\|^p - \|\sigma\|_{L^1(\Omega)}$$

for all $u \in W_0^{1,p}(\Omega)$. Since $p > 1$ and $a_0 - d_1 - d_2\lambda_1^{-1} > 0$ as known from hypothesis (H2), we confirm the validity of (38).

We showed on the reflexive Banach space $W_0^{1,p}(\Omega)$ that the operator $\mathcal{B}_R : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ defined in (35) is bounded, pseudomonotone and coercive. According to the main theorem for pseudomonotone operators (see, for example, [16], Th. 2.99), we can conclude that the mapping $\mathcal{B}_R$ is surjective. So, in particular, there exists $u_R \in W_0^{1,p}(\Omega)$ such that $\mathcal{B}_R(u_R) = 0$, which is exactly (32). Therefore u_R is a weak solution of auxiliary problem (31).

Let us point out that the function G and its truncation G_R take values in the same set $[a_0, +\infty)$, and function F is the same in both problems (1) and the (31). Consequently, Theorem 2 can be applied to the auxiliary problem (31) and provides the same uniform bound $C = C(N, p, \Omega, a_0, c_1, c_2, c_3, d_1, d_2, \|\sigma\|_{L^1(\Omega)})$ of the solution set as for the original problem (1). This ensures that $\|u_R\|_{L^\infty(\Omega)} \leq C$, which completes the proof. □

Relying on Theorem 3, we are now able to prove Theorem 1.

Proof of Theorem 1. It was established in Theorem 2 that the solution set of problem (1) is uniformly bounded by a constant $C = C(N, p, \Omega, a_0, c_1, c_2, c_3, d_1, d_2, \|\sigma\|_{L^1(\Omega)})$, where a_0 is a lower bound of the function G. Since the truncated function G_R has the lower bound a_0 too for all $R > 0$ (see (22)) and the reaction term $F(x, t, \xi)$ is unchanged in problems (1) and (31) and is subject to the same hypotheses (H1)-(H2), Theorem 2 applies to the truncated problem (31) and provides the same bound C for its solution set whenever $R > 0$. In particular, the solution $u_R \in W_0^{1,p}(\Omega)$ of problem (31) provided by Theorem 3 satisfies the estimate $\|u_R\|_{L^\infty(\Omega)} \leq C$.

Owing to the crucial information that C is independent of $R > 0$, we can choose $R \geq C$. Hence, the estimate $\|u_R\|_{L^\infty(\Omega)} \leq C$ and (22) render that the functions G_R and G coincide along the values $u_R(x)$ for all $x \in \Omega$. According to Theorem 3, $u_R \in W_0^{1,p}(\Omega)$ solves problem (31), and thus it becomes a bounded weak solution of the original problem (1). The conclusion of Theorem 1 is achieved. □

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