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Multiplicity Results of Solutions to Non-Local Magnetic Schrödinger–Kirchhoff Type Equations in $\mathbb{R}^N$

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Abstract: In this paper, we establish the existence of a nontrivial weak solution to Schrödinger-kirchhoff type equations with the fractional magnetic field without Ambrosetti and Rabinowitz condition using mountain pass theorem under a suitable assumption of the external force. Furthermore, we prove the existence of infinitely many large- or small-energy solutions to this problem with Ambrosetti and Rabinowitz condition. The strategy of the proof for these results is to approach the problem by applying the variational methods, that is, the fountain and the dual fountain theorem with Cerami condition.

Keywords: Schrödinger-kirchhoff equation; fractional magnetic operators; variational methods

MSC: 35A15; 35J60; 35R11; 47G20

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1. Introduction

The Schrödinger equation plays the role of Newton's laws and conservation of energy in classical mechanics. The linear Schrödinger equation represents one of the main results of quantum mechanics which is the evolution of a free non-relativistic quantum particle. The structure of the nonlinear Schrödinger equation is considerably complicated and requires more sophisticated analysis; see [1]. This equation has been studied extremely according to the pure or applied mathematical theory, because it stands out as a prototypical system that has shown to be crucial to model and understand the characteristics of numerous areas in nonlinear physics. In particular, the significant development of the Bose-Einstein condensates revived researches regarding the nonlinear waveforms for the nonlinear Schrödinger equations with external potentials and the related nonlinear partial differential equations. For further applications and more details we refer the reader to [2–8]. Indeed, the mathematical model for the remarkable Bose-Einstein condensate with attractive inter-particle interactions under a magnetic trap is a class of nonlinear Schrödinger equations with external potentials, which is sometimes called the Gross-Pitaevskii equation [9,10]. In this regard the present paper is motivated by some works (see [11–21]) concerning the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla + iA(x))^2\psi + W(x)\psi - f(x,|\psi|^2)\psi \quad \text{for} \quad x \in \mathbb{R}^N, \tag{1}$$

where $\hbar$ is Planck constant $A(x) = (A_1(x), A_2(x), \dots, A_N(x)) : \mathbb{R}^N \to \mathbb{R}^N$ is a real vector (magnetic) potential with magnetic field $B = \mathrm{curl}A$, and $W(x) : \mathbb{R}^N \to \mathbb{R}^N$ is a scalar electric potential. Particularly, we are interested in the existence of standing wave solutions, that is, solutions of type (1) when $\hbar$ is sufficiently small, where E is a real number and $u(x)$ is a complex-value function which satisfies

$$-(\nabla + iA(x))^2u(x) + \lambda V(x)u(x) = \lambda f(x,|u|^2)u, \quad x \in \mathbb{R}^N, \tag{2}$$

where $\lambda^{-1} = \frac{\hbar^2}{2m}$ and $V(x) = W(x) - E$. The transition from quantum mechanics to classical mechanics can be done formally with $\hbar$ approach 0. Thus the existence of solutions for $\hbar$ small, semi-classical solutions, has important physical interest. Very recently, authors in [22] established the Bourgain-Brezis-Mironescu type result which constructs a bridge between a fractional magnetic operator and the classical theory. Motivated by this paper, nonlocal fractional problems with magnetic fields has been extensively studied by many researchers; see [23–29] and the references therein. In this regard, the present paper is devoted to the existence of solutions for the following Kirchhoff type equation with the fractional magnetic field

$$K(|u|_{s,A}^{p})(-\Delta)_{p,A}^{s}u + V(x)|u|^{p-2}u = \lambda f(x,|u|)u \quad \text{in } \mathbb{R}^N, \tag{3}$$

where

$$|u|_{s,A}^{p} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{N+ps}}\,dxdy,$$

where $0 < s < 1 < p < +\infty$ and the fractional magnetic operator $(-\Delta)_A^s$ is defined as

$$(-\Delta)_{p,A}^{s}\phi(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|\phi(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}\phi(y)|^{p-2}(\phi(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}\phi(y))}{|x-y|^{N+ps}}dy, \quad x\in\mathbb{R}^N,$$

for all $\phi \in C_0^\infty(\mathbb{R}^N,\mathbb{C})$. Here, $B_\varepsilon(x)$ denotes a ball in $\mathbb{R}^N$ centered at $x\in\mathbb{R}^N$ and radius $\varepsilon > 0$ and $A:\mathbb{R}^N\to\mathbb{R}^N$ is the magnetic potential. Also, the nonlinear function $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ will be stated later (see Section 2). When $p = 2$, the fractional Laplacian $(-\Delta)_{p,A}^s$ is a fractional Laplacian contains the magnetic field. On the other hand, the standard fractional Laplacian $(-\Delta)^s$ has been a classical topic for a long time and it is applied in various research fields, such as social sciences, fractional quantum mechanics, materials science, continuum mechanics, phase transition phenomena, image process, game theory, and Lévy process, fractional Sobolev spaces and their corresponding nonlocal equations, see [30–32] and the references therein.

Kirchhoff in [33] first introduced a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|dx\Big)\frac{\partial^2 u}{\partial x^2} = 0,$$

which extends the classical D'Alembert's wave equation by taking into account the changes in the length of the strings during the vibrations. In this direction, the non-local problem of Kirchhoff type equations have been investigated in [34–37].

Now in order to confirm the existence of solutions to the nonlinear elliptic equations, the following Ambrosetti and Rabinowitz condition ((AR)-condition) given in [38] has been widely used;

(AR) There exists $\zeta > p$ such that

$$0 < \zeta F(x,\tau) \le f(x,\tau)\tau^2, \quad \text{for} \quad x\in\mathbb{R}^N \quad \text{and} \quad \tau > 0,$$

where $F(x,\tau) = \int_0^\tau f(x,s)sds$.

It is well known that (AR)-condition is essential to ensure the compactness condition of the Euler-Lagrange functional which plays a key role in applying the critical point theory. However, this condition is too restrictive and gets rid of many nonlinearities. Thus many researchers have tried to drop the (AR)-condition in the elliptic problem of nonlocal type (see e.g., [20,39–42]). In this respect, we are to prove the existence of a nontrivial solution for problem (3) without (AR)-condition using the mountain pass theorem with Cerami condition under a suitable assumption of the nonlinearity of f. Furthermore, we present the existence of infinitely many large- or small-energy solutions to our problem without (AR)-condition. Especially, following in ([43] Remark 1.8), there are many examples which are not fulfilling the condition on f in a elliptic problem. Thus, inspired by these examples, we investigate the existence and multiplicity of weak solutions to the fractional p-Laplacian Equation (3) with the external magnetic potential. The strategy of the proof for these results is to approach the problem by applying the variational methods, namely, the fountain and the dual fountain theorem with Cerami condition. As far as we are aware, none have reported such multiplicity results for our problem with the external magnetic field.

This present paper is organized as follows. in Section 2, we state some basic results to deal with this type equation with the fractional magnetic field and review well known facts for the fractional Sobolev space. And under certain assumptions of f, we establish the existence of a weak solution of problem (3) using mountain pass theorem.

2. Preliminaries

Let the potential function $V \in C(\mathbb{R}^N, \mathbb{R})$ be continuous and bounded from below. Assume that

(V) $V \in L^1_{loc}(\mathbb{R}^N)$, $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0$ and $\lim_{|x|\to\infty} V(x) = +\infty$.

Let $L^p_V(\mathbb{R}^N)$ denote the real valued Lebesgue space with $V(x)|u|^p \in L^1(\mathbb{R}^N)$, equipped with the norm

$$\|u\|^p_{p,V} = \int_{\mathbb{R}^N} V(x)|u|^p\,dx.$$

The fractional Sobolev space $\mathcal{H}^s_V(\mathbb{R}^N)$ is then defined as for $s \in (0,1)$ and $p \in (1,+\infty)$

$$\mathcal{H}^s_V(\mathbb{R}^N) = \left\{ u \in L^p_V(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy < +\infty \right\}.$$

The space $\mathcal{H}^s_V(\mathbb{R}^N)$ is endowed with the norm

$$\|u\|^p_{\mathcal{H}^s_V(\mathbb{R}^N)} := \left(\|u\|^p_{p,V} + [u]^p_s \right) \quad \text{with} \quad [u]^p_s := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy.$$

For further details on the fractional Sobolev spaces we refer the reader to [44] and the references therein. We recall the embedding theorem; see e.g., [45].

Lemma 1. *Let* (V) *hold and $s \in (0,1)$, $p \in (1,+\infty)$ and let p^*_s be the fractional critical Sobolev exponent, that is*

$$p^*_s := \begin{cases} \frac{Np}{N-sp} & \text{if} \quad sp < N, \\ +\infty & \text{if} \quad sp \geq N. \end{cases}$$

Then, the embedding $\mathcal{H}^s_V(\mathbb{R}^N) \to L^\gamma(\mathbb{R}^N)$ is continuous for any $\gamma \in [p, p^*_s]$ and moreover, the embedding $\mathcal{H}^s_V(\mathbb{R}^N) \hookrightarrow L^\gamma(\mathbb{R}^N)$ is compact for any $\gamma \in [p, p^*_s)$.

Let $L^p_V(\mathbb{R}^N, \mathbb{C})$ be the Lebesgue space of functions $u : \mathbb{R}^N \to \mathbb{C}$ with $V(x)|u|^p \in L^1(\mathbb{R}^N)$. Define $\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$ as the closure of $C^\infty_c(\mathbb{R}^N, \mathbb{C})$ with respect to the norm

$$\|u\|^p_{s,A} = (\|u\|^p_{p,V} + |u|^p_{s,A}),$$

where the magnetic Gagliardo seminorm is given by

$$|u|^p_{s,A} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{N+ps}} dxdy.$$

In fact, arguing as in ([46] Proposition 2.1), we can easily show that it is a reflexive and separable Banach space as the similar arguments in ([45,47] Appendix). The following Lemmas 2 and 3 can be shown by applying as a general exponent p instead of $p = 2$ the same argument in ([39] Lemmas 3.4 and 3.5).

Lemma 2. *If* (V) *holds and $r \in [p, p^*_s]$, then the embedding*

$$\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^r(\mathbb{R}^N, \mathbb{C})$$

is continuous. Furthermore, for any compact subset $\Gamma \subset \mathbb{R}^N$ and $r \in [1, p_s^*)$, then the embedding

$$\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow \mathcal{H}^s_V(\Gamma, \mathbb{C}) \hookrightarrow L^r(\Gamma, \mathbb{C})$$

is continuous and the latter is compact, where $\mathcal{H}^s_V(\Gamma, \mathbb{C})$ is endowed with the following norm:

$$\|u\|^p_{s,V} = \Big(\int_\Gamma V(x)|u|^p \, dx + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dxdy \Big).$$

Lemma 3. *Under the assumption* (V)*, for all bounded sequence $\{u_n\}$ in $\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$ the sequence $\{|u_n|\}$ admits a subsequence converging strongly to some u in $L^r(\mathbb{R}^N)$ for all $r \in [p, p_s^*)$.*

For our problem, we suppose that $K : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ satisfies the following conditions:

(K1) $K \in C(\mathbb{R}^+_0)$ satisfies $\inf_{\tau \in \mathbb{R}^+} K(\tau) \geq a > 0$, where $a > 0$ is a constant.

(K2) There is a positive constant $\theta \in [1, \frac{N}{N-ps})$ such that $\theta \mathcal{K}(\tau) = \theta \int_0^\tau K(\eta) d\eta \geq K(\tau)\tau$ for any $\tau \geq 0$.

A typical example for K is given by $K(\tau) = b_0 + b_1 \tau^m$ with $m > 0, b_0 > 0$, and $b_1 \geq 0$.

Now we assume that for $1 < p\theta < q < p_s^*$ and $p \in (1, +\infty)$,

(F1) $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ satisfies the Carathéodory condition.

(F2) $f \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$, and there exist constants $c_1, c_2 > 0$ such that

$$|f(x,\tau)| \leq c_1 \tau^{p-2} + c_2 \tau^{q-2}, \quad \text{for all } (x,\tau) \in \mathbb{R}^N \times \mathbb{R}^+, \quad q \in (p\theta, p_s^*).$$

(F3) $f(x,\tau) = o(\tau^{p-1})$ as $\tau \to 0$ for $x \in \mathbb{R}^N$ uniformly.

(F4) $\lim_{\tau \to \infty} \frac{F(x,\tau)}{\tau^{p\theta}} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$, where the number θ is given in (K2), and $F(x,\tau) = \int_0^\tau f(x,\eta)\eta \, d\eta$.

(F5) There exist $\mu > p$ and $r > 0$ such that

$$f(x,\tau)\tau^2 - \mu F(x,\tau) \geq -\varrho \tau^p - \beta(x) \quad \text{for all} \quad x \in \mathbb{R}^N \quad \text{and} \quad \tau \geq r,$$

where $\varrho \geq 0$ and $\beta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\beta(x) \geq 0$.

The Euler functional corresponding to the problem (3) is $\mathcal{J}_\lambda : \mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ defined as follows

$$\mathcal{J}_\lambda(u) = \frac{1}{p}(\mathcal{K}(|u|^p_{s,A}) + \|u\|^p_{p,V}) - \lambda \int_{\mathbb{R}^N} F(x,|u|) \, dx.$$

The functional $\mathcal{J}_\lambda$ is Fréchet differentiable on $\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$,

$$\langle \mathcal{J}'_\lambda(z), v \rangle = \Re \Bigg(K(|u|^p_{s,A}) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - E(x,y)u(y)|^{p-2}(u(x) - E(x,y)u(y)) \cdot \overline{[v(x) - E(x,y)v(y)]}}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u|^{p-2} u\bar{v} \, dx - \lambda \int_{\mathbb{R}^N} f(x,|u|)u\bar{v} \, dx \Bigg)$$

for any $u, v \in \mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$, where $E(x,y) := e^{i(x-y)\cdot A(\frac{x+y}{2})}$ and $\bar{v}$ denotes complex conjugation of $v \in \mathbb{C}$. Hereafter, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C}))'$ and $\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$. Following in [39], we observe that the critical points of $\mathcal{J}_\lambda$ are exactly the weak solutions of (1.1) and the functional $\mathcal{J}_\lambda$ is weakly lower semi-continuous in $\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$.

The following result is to show that the energy functional $\mathcal{J}_\lambda$ fulfills the geometric conditions.

Lemma 4. *Let $s \in (0,1)$, $p \in (1,+\infty)$ and $N > ps$. Assume that* (V), (K1), (K2) *and* (F1)–(F4) *hold. Then the geometric conditions in the mountain pass theorem hold, i.e.,*

(1) $u = 0$ is a strict local minimum for $\mathcal{J}_\lambda$.
(2) $\mathcal{J}_\lambda$ is unbounded from below on $\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$.

Proof. According to (F2) and (F3), for any $\varepsilon > 0$, we can choose a positive constant denoted $C(\varepsilon)$ such that

$$|f(x,\tau)\tau| \le \varepsilon \tau^{p-1} + C(\varepsilon)\tau^{q-1}, \quad \text{for all } (x,\tau) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{4}$$

Assume that $\|u\|_{s,A} < 1$. Owing to (K1), (K2) and (4), one has

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \frac{1}{p}(\mathcal{K}(|u|^p_{s,A}) + \|u\|^p_{p,V}) - \lambda \int_{\mathbb{R}^N} F(x,|u|)dx \\
&\ge \frac{\min\{1, a\theta^{-1}\}}{p}\|u\|^p_{s,A} - \frac{\lambda\epsilon}{p}\|u\|^p_{L^p(\mathbb{R}^N)} - \frac{\lambda C(\varepsilon)}{q}\|u\|^q_{L^q(\mathbb{R}^N)} \\
&\ge \frac{\min\{1, a\theta^{-1}\}}{p}\|u\|^p_{s,A} - \frac{\lambda\epsilon C}{p}\|u\|^p_{s,A} - \frac{\lambda C C(\varepsilon)}{q}\|u\|^q_{s,A}
\end{aligned}$$

for some constant C. Choose $\epsilon > 0$ so small that $0 < \lambda\epsilon C < \frac{\min\{1,a\theta^{-1}\}}{2p}$. Then

$$\mathcal{J}_\lambda(u) \ge \frac{\min\{1, a\theta^{-1}\}}{2p}\|u\|^p_{s,A} - C(\lambda,\epsilon)C\|u\|^q_{s,A}.$$

Since $q > p$, there is $R > 0$ small sufficiently and $\delta > 0$ such that $\mathcal{J}_\lambda(u) \ge \delta > 0$ when $\|u\|_{s,A} = R$. Therefore $u = 0$ is a strict local minimum for $\mathcal{J}_\lambda$.

Next we prove the condition (2). By the condition (F4), for any $\tilde{C} > 0$, we can choose a constant $\delta > 0$ such that

$$F(x,\tau) \ge \widetilde{C}\tau^{p\theta} \tag{5}$$

for $\tau > \delta$ and for almost all $x \in \mathbb{R}^N$. Under the assumption (K2), we note that for all $\xi \ge 1$,

$$\mathcal{K}(\xi) \le \mathcal{K}(1)(1 + \xi^\theta). \tag{6}$$

Relations (5) and (6) with Lemma 3 imply that for $v \in \mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$

$$\begin{aligned}
\mathcal{J}_\lambda(tv) &= \frac{1}{p}(\mathcal{K}(|tv|^p_{s,A}) + \|tv\|^p_{p,V}) - \lambda \int_{\mathbb{R}^N} F(x,|tv|)dx \\
&\le \frac{1}{p}\Big(\mathcal{K}(1)(1 + |tv|^{p\theta}_{s,A}) + \|tv\|^p_{p,V}\Big) - \lambda\widetilde{C}\int_{\{|tv|>\delta\}} |tv|^{p\theta}\,dx \\
&\le \frac{1}{p}\Big(2\mathcal{K}(1)t^{p\theta}|v|^{p\theta}_{s,A}) + t^{p\theta}\|v\|^p_{p,V}\Big) - \lambda t^{p\theta}\widetilde{C}\int_{\{|tv|>\delta\}} |v|^{p\theta}\,dx \\
&= t^{p\theta}\Big(\frac{1}{p}\Big(2\mathcal{K}(1)\|v\|^{p\theta}_{s,A} + \|v\|^p_{p,V}\Big) - \lambda\widetilde{C}\int_{\{|tv|>\delta\}} |v|^{p\theta}\,dx\Big)
\end{aligned}$$

for $t > 0$. If $\tilde{C}$ is large sufficiently, then we deduce that $\mathcal{J}_\lambda(tv) \to -\infty$ as $t \to \infty$. Hence the functional $\mathcal{J}_\lambda$ is unbounded from below. The proof is completed. □

First of all, we introduce the Cerami condition, which was initially provided by Cerami [48].

Definition 1. *Let the functional Ψ be C^1 and $c \in \mathbb{R}$. If any sequence $\{u_n\}$ satisfying*

$$\Psi(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\|\Psi'(u_n)\| \to 0$$

possesses a convergent subsequence, we say that Ψ fulfils Cerami condition ($(C)_c$-condition in short) at the level c.

Definition 2. *A function $u \in \mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ is called weak solution of problem (3) if u satisfies*

$$\Re\Big(K(|u|^p_{s,A})\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-E(x,y)u(y)|^{p-2}(u(x)-E(x,y)u(y))\cdot\overline{[\phi(x)-E(x,y)\phi(y)]}}{|x-y|^{N+ps}}dxdy$$
$$+\int_{\mathbb{R}^N}V(x)|u|^{p-2}u\bar{\phi}\,dx\Big)=\Re\Big(\lambda\int_{\Omega}f(x,|u(x)|)u\bar{\phi}\,dx\Big)$$

for all $\phi \in \mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$.

The following lemma plays a crucial role in establishing the existence of a nontrivial weak solution to the given problem.

Lemma 5. *Let $s \in (0,1)$, $p \in (1,+\infty)$ and $N > ps$. Assume that* (V), (K1), (K2), (F1)–(F2), *and* (F4)–(F5) *hold. Then the functional $\mathcal{J}_\lambda$ satisfies the $(C)_c$-condition for any $\lambda > 0$.*

Proof. For $c \in \mathbb{R}$, let $\{u_n\}$ be a $(C)_c$-sequence in $\mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, that is,

$$\mathcal{J}_\lambda(u_n)\to c \quad \text{and} \quad \|\mathcal{J}'_\lambda(u_n)\|_{s,A'}(1+\|u_n\|_{s,A})\to 0 \qquad \text{as} \quad n\to\infty,$$

which means

$$c=\mathcal{J}_\lambda(u_n)+o(1) \quad \text{and} \quad \langle\mathcal{J}'_\lambda(u_n),u_n\rangle=o(1), \tag{7}$$

where $o(1) \to 0$ as $n \to \infty$. If $\{u_n\}$ is bounded in $\mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, it follows from the analogous argument as in the proof of Lemma 4.2 in [39] that sequence $\{u_n\}$ converges strongly to u in $\mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. Hence, it suffices to ensure that the sequence $\{u_n\}$ is bounded in $\mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. We argue by contradiction. Assume that the sequence $\{u_n\}$ is unbounded in $\mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. So then we may assume that

$$\|u_n\|_{s,A}\to\infty, \quad \text{as} \quad n\to\infty.$$

Due to the condition (7), we have that

$$c=\mathcal{J}_\lambda(u_n)+o(1)=\frac{1}{p}(\mathcal{K}(|u_n|^p_{s,A})+\|u_n\|^p_{p,V})-\lambda\int_{\mathbb{R}^N}F(x,|u_n|)\,dx+o(1). \tag{8}$$

Since $\|u_n\|_{s,A} \to \infty$ as $n \to \infty$, we assert by (8) that

$$\begin{aligned}\int_{\mathbb{R}^N}F(x,u_n)\,dx &\geq \frac{1}{p\lambda}(\mathcal{K}(|u_n|^p_{s,A})+\|u_n\|^p_{p,V})-\frac{c}{\lambda}+\frac{o(1)}{\lambda}\\ &\geq \frac{1}{p\lambda}\min\{1,a\theta^{-1}\}\|u_n\|^p_{s,A}-\frac{c}{\lambda}+\frac{o(1)}{\lambda}\to\infty \quad \text{as} \quad n\to\infty.\end{aligned} \tag{9}$$

Define a sequence $\{\omega_n\}$ by $\omega_n = u_n/\|u_n\|_{s,A}$. Then it is immediate that $\{\omega_n\} \subset \mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $\|\omega_n\|_{s,A} = 1$. Hence, up to a subsequence, still denoted by $\{\omega_n\}$, we obtain $\omega_n \rightharpoonup \omega$ in $\mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ as $n \to \infty$, we have

$$\omega_n(x)\to\omega(x) \text{ for a.e. } x\in\mathbb{R}^N \quad \text{and} \quad |\omega_n|\to|\omega| \text{ in } L^r(\mathbb{R}^N) \quad \text{as} \quad n\to\infty \tag{10}$$

for $p \le r < p^*_s$. Set $\Sigma = \{x \in \mathbb{R}^N : \omega(x) \neq 0\}$. By the convergence (10), we know that

$$|u_n|=|w_n|\|u_n\|_{s,A}\to\infty \quad \text{as} \quad n\to\infty$$

for all $x \in \Sigma$. Then it follows from (K2) and (F3) that for all $x \in \Sigma$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{F(x,|u_n|)}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p} &\geq \lim_{n\to\infty} \frac{F(x,|u_n|)}{\mathcal{K}(1)(1+|u_n|_{s,A}^{p\theta}) + \|u_n\|_{p,V}^p} \\
&\geq \lim_{n\to\infty} \frac{F(x,|u_n|)}{2\mathcal{K}(1)\|u_n\|_{s,A}^{p\theta} + \|u_n\|_{p,V}^{p\theta}} \\
&\geq \lim_{n\to\infty} \frac{F(x,|u_n|)}{(2\mathcal{K}(1)+1)\|u_n\|_{s,A}^{p\theta}} \\
&\geq \lim_{n\to\infty} \frac{F(x,|u_n|)}{(2\mathcal{K}(1)+1)|u_n|^{p\theta}}|w_n|^{p\theta} \\
&= \infty,
\end{aligned} \tag{11}$$

where the inequality $\mathcal{K}(\eta) \leq \mathcal{K}(1)(1+\eta^\theta)$ is used for all $\eta \in \mathbb{R}_0^+$ because if $0 \leq \eta < 1$, then $\mathcal{K}(\eta) = \int_0^\eta K(s)\,ds \leq \mathcal{K}(1)$, and if $\eta > 1$, then $\mathcal{K}(\eta) \leq \mathcal{K}(1)\eta^\theta$. Thus we obtain that $|\Sigma| = 0$, where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^N$. Indeed, assume that $|\Sigma| \neq 0$. Taking account into (F4) we can choose $\tau_0 > 1$ such that $F(x,\tau) > \tau^{p\theta}$ for all $x \in \mathbb{R}^N$ and $\tau_0 < \tau$. By means of (F1) and (F2), we derive that there is $M > 0$ such that $|F(x,\tau)| \leq M$ for all $(x,\tau) \in \mathbb{R}^N \times (0,\tau_0]$. Hence there is a $M_0 \in \mathbb{R}$ such that $F(x,\tau) \geq M_0$ for all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}^+$, and thus

$$\frac{F(x,|u_n|) - M_0}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p} \geq 0, \tag{12}$$

for all $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. In accordance with (9), (11), (12) and the Fatou lemma, we infer that

$$\begin{aligned}
\frac{1}{\lambda} &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,|u_n|)\,dx}{\lambda \int_{\mathbb{R}^N} F(x,|u_n|)\,dx + c - o(1)} \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{2F(x,|u_n|)}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p}\,dx \\
&= \liminf_{n\to\infty} \int_\Sigma \frac{2F(x,|u_n|)}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p}\,dx - \limsup_{n\to\infty} \int_\Sigma \frac{2M_0}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p}\,dx \\
&= \liminf_{n\to\infty} \int_\Sigma \frac{2(F(x,|u_n|) - M_0)}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p}\,dx \\
&\geq \int_\Sigma \liminf_{n\to\infty} \frac{2(F(x,|u_n|) - M_0)}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p}\,dx \\
&= \int_\Sigma \liminf_{n\to\infty} \frac{2F(x,|u_n|)}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p}\,dx - \int_\Sigma \limsup_{n\to\infty} \frac{2M_0}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p}\,dx = \infty,
\end{aligned}$$

which is a contradiction. This means $\omega(x) = 0$ for almost all $x \in \mathbb{R}^N$.

Notice that $V(x) \to +\infty$ as $|x| \to \infty$, then

$$\begin{aligned}
&\left(\frac{1}{p\theta} - \frac{1}{\mu}\right)\|u_n\|_{p,V}^p - C_8 \int_{|u_n|\leq r} (|u_n|^p + |u_n|^q)\,dx \\
&\geq \frac{1}{2}\left(\frac{1}{p\theta} - \frac{1}{\mu}\right)\|u_n\|_{p,V}^p - \mathcal{M}_0,
\end{aligned}$$

where $\mathcal{M}_0$ is a positive constant. Combining this with (F2) and (F5), one has

$$
\begin{aligned}
c+1 &\geq \mathcal{J}_\lambda(u_n) - \frac{1}{\mu}\langle \mathcal{J}_\lambda'(u_n), u_n\rangle \\
&\geq \frac{1}{p}(\mathcal{K}(|u_n|_{s,A}^p) - \frac{1}{\mu}(K(|u_n|_{s,A}^p)|u_n|_{s,A}^p + \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|_{p,V}^p \\
&+ \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu} f(x,|u_n|)|u_n|^2 - F(x,|u_n|)\right) dx \\
&\geq \frac{1}{p}(\mathcal{K}(|u_n|_{s,A}^p) - \frac{1}{\mu}(K(|u_n|_{s,A}^p)|u_n|_{s,A}^p + \left(\frac{1}{p\theta} - \frac{1}{\mu}\right)\|u_n\|_{p,V}^p \\
&+ \lambda \int_{|u_n|>r} \left(\frac{1}{\mu} f(x,|u_n|)|u_n|^2 - F(x,|u_n|)\right) dx - C_8 \int_{|u_n|\leq r} (|u_n|^p + |u_n|^q)\, dx \\
&\geq \frac{1}{p\theta}(K(|u_n|_{s,A}^p)|u_n|_{s,A}^p - \frac{1}{\mu}(K(|u_n|_{s,A}^p)|u_n|_{s,A}^p + \frac{1}{2}\left(\frac{1}{p\theta} - \frac{1}{\mu}\right)\|u_n\|_{p,V}^p \\
&- \frac{\lambda}{\mu}\int_{\mathbb{R}^N} (\varrho|u_n|^p + \beta(x))\, dx - \mathcal{M}_0 \\
&\geq \left(\frac{1}{p\theta} - \frac{1}{\mu}\right)\min\left\{a, \frac{1}{2}\right\}\|u_n\|_{s,A}^p - \frac{\lambda\varrho}{\mu}\|u_n\|_{L^p(\mathbb{R}^N)}^p - \frac{\lambda}{\mu}\|\beta\|_{L^1(\mathbb{R}^N)} - \mathcal{M}_0,
\end{aligned}
$$

which implies

$$
\begin{aligned}
1 &\leq \frac{\lambda\varrho}{\mu\left(\frac{1}{p\theta} - \frac{1}{\mu}\right)\min\left\{a, \frac{1}{2}\right\}} \limsup_{n\to\infty} \|\omega_n\|_{L^p(\mathbb{R}^N)}^p \\
&= \frac{\lambda\varrho}{\mu\left(\frac{1}{p\theta} - \frac{1}{\mu}\right)\min\left\{a, \frac{1}{2}\right\}} \|\omega\|_{L^p(\mathbb{R}^N)}^p.
\end{aligned} \tag{13}
$$

Hence, it follows from (13) that $\omega \neq 0$. Thus, we can conclude a contradiction. Therefore, $\{u_n\}$ is bounded in $\mathcal{H}_{A,V}^s(\mathbb{R}^N, \mathbb{C})$. This complete the proof. □

Using Lemma 5, we prove the existence of a nontrivial weak solution to our problem.

Theorem 1. *Under the same assumptions of Lemma 5, the problem (3) has a nontrivial weak solution for all $\lambda > 0$.*

Proof. Note that $\mathcal{J}_\lambda(0) = 0$. By Lemma 4, the mountain pass geometric conditions are satisfied. From Lemma 5, $\mathcal{J}_\lambda$ fulfils the $(C)_c$-condition for any $\lambda > 0$. Subsequently, problem (3) admits a nontrivial weak solution for any $\lambda > 0$ by Lemmas 4 and 5. □

Next, applying the fountain theorem in ([49] Theorem 3.6), we indicate infinitely many weak solutions for problem (3). To do this, we refer to the following lemma.

Lemma 6 ([49])**.** *Let E be a reflexive and separable Banach space. Then there exist $\{e_n\} \subseteq E$ and $\{f_n^*\} \subseteq E^*$ such that*

$$
E = \overline{span\{e_n : n = 1, 2, \cdots\}}, \quad E^* = \overline{span\{f_n^* : n = 1, 2, \cdots\}},
$$

and

$$
\langle f_i^*, e_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}
$$

Let us denote $\mathcal{E}_n = \text{span}\{e_n\}$, $\mathcal{Y}_k = \bigoplus_{n=1}^k \mathcal{E}_n$, and $\mathcal{Z}_k = \overline{\bigoplus_{n=k}^\infty \mathcal{E}_n}$. In order to obtain the existence result, we use the following Fountain theorem.

Lemma 7 ([49,50])**.** *Let E be a real Banach space, $\mathcal{I} \in C^1(E, \mathbb{R})$ satisfies the $(C)_c$-condition for any $c > 0$ and $\mathcal{I}$ is even. If for each sufficiently large $k \in \mathbb{N}$, there exist $\varrho_k > \sigma_k > 0$ such that the following conditions hold:*

(1) $\beta_k := \inf\{\mathcal{I}(u) : z \in \mathcal{Z}_k, \|u\|_E = \sigma_k\} \to \infty$ *as* $k \to \infty$;

(2) $\alpha_k := \max\{\mathcal{I}(u) : u \in \mathcal{Y}_k, \|u\|_E = \varrho_k\} \le 0$.

Then the functional $\mathcal{I}$ has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_n\} \subset E$ such that $\mathcal{I}'(u_n) = 0$ and $\mathcal{I}(u_n) \to +\infty$ as $n \to +\infty$.

Theorem 2. *Let $s \in (0,1)$, $p \in (1, +\infty)$ and $N > ps$. Assume that* (V), (K1), (K2) *and* (F1)–(F4) *hold. Then for any $\lambda > 0$, problem* (3) *has an unbounded sequence of nontrivial weak solutions $\{u_n\}$ in $\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$ such that $\mathcal{J}_\lambda(u_n) \to \infty$ as $n \to \infty$.*

Proof. The proof follows the lines of that of Lemma 3.2 in [51]. To apply Lemma 7, let us denote $E := \mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$ and $\mathcal{I} := \mathcal{J}_\lambda$. Plainly, $\mathcal{J}_\lambda$ is an even functional and ensures the $(C)_c$-condition. It suffices to show that there exist $\varrho_k > \sigma_k > 0$ with the conditions (1) and (2) in Lemma 7. Let us denote

$$\varsigma_k = \sup_{\|u\|_{s,A}=1, z \in \mathcal{Z}_k} \|z\|_{L^q(\mathbb{R}^N)}.$$

Then, it is obvious to verify that $\varsigma_k \to 0$ as $k \to \infty$. For any $z \in \mathcal{Z}_k$, assume that $\|u\|_{s,A} > 1$. Choose $\epsilon > 0$ so small that $0 < \lambda \epsilon C < \frac{\min\{1, a\theta^{-1}\}}{2p}$. Then it follows from (4) that

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \frac{1}{p}(\mathcal{K}([u]^p_{s,A}) + \|u\|^p_{p,V}) - \lambda \int_{\mathbb{R}^N} F(x, |u|)dx \\
&\ge \frac{\min\{1, a\theta^{-1}\}}{p} \|u\|^p_{s,A} - \lambda \int_{\mathbb{R}^N} F(x, |u|)dx \\
&\ge \frac{\min\{1, a\theta^{-1}\}}{p} \|u\|^p_{s,A} - \frac{\lambda \epsilon}{p} \|u\|^p_{L^p(\mathbb{R}^N)} - \frac{\lambda C(\epsilon)}{q} \|u\|^q_{L^q(\mathbb{R}^N)} \\
&\ge \frac{\min\{1, a\theta^{-1}\}}{2p} \|u\|^p_{s,A} - \lambda C(\epsilon) \varsigma_k^q \|u\|^q_{s,A} \\
&= \Big(\frac{\min\{1, a\theta^{-1}\}}{2p} - \lambda C(\epsilon) \varsigma_k^q \|u\|^{q-p}_{s,A} \Big) \|u\|^p_{s,A}
\end{aligned} \tag{14}$$

Choose $\sigma_k = \left[\frac{2p\lambda C(\epsilon)}{\min\{1, a\theta^{-1}\}} \varsigma_k^q\right]^{\frac{1}{p-q}}$. Since $p < q$, $p \in (1, +\infty)$ and $\varsigma_k \to 0$ as $k \to \infty$, we infer $\sigma_k \to \infty$ as $k \to \infty$. Hence, if $u \in Z_k$ and $\|u\|_{s,A} = \sigma_k$, then we deduce that

$$\mathcal{J}_\lambda(u) \ge \frac{\min\{1, a\theta^{-1}\}}{2p} \sigma_k^p \to \infty \quad \text{as} \quad k \to \infty,$$

which implies (1).

Now we prove condition (2). To do this, we claim that $\mathcal{J}_\lambda(u) \to -\infty$ as $\|u\|_{s,A} \to \infty$ for all $u \in \mathcal{Y}_k$. Let us assume that this is false for some k. Then we can choose a sequence $\{u_n\}$ in $\mathcal{H}^s_{A,V}(\mathbb{R}^N, \mathbb{C})$ such that

$$\|u_n\|_{s,A} \to \infty \text{ as } n \to \infty \quad \text{and} \quad \mathcal{J}_\lambda(u_n) \ge -M.$$

Let $\omega_n = u_n / \|u_n\|_{s,A}$. Then it is obvious that $\|\omega_n\|_{s,A} = 1$. Since $\dim \mathcal{Y}_k < \infty$, there is $\omega \in \mathcal{Y}_k \setminus \{0\}$ such that up to a subsequence,

$$\|\omega_n - \omega\|_{s,A} \to 0 \quad \text{and} \quad \omega_n(x) \to \omega(x)$$

for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. Thus we have (14) that,

$$\begin{aligned}
\frac{1}{p} + \frac{M}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p} &\geq \frac{1}{p} - \frac{\mathcal{J}_\lambda(u_n)}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p} \\
&= \lambda \int_{\mathbb{R}^N} \frac{F(x,|u_n|)}{\mathcal{K}(|u_n|_{s,A}^p) + \|u_n\|_{p,V}^p} \, dx \\
&\geq \lambda \int_{\{\omega_n(x) \neq 0\}} \frac{F(x,|u_n|)}{(2\mathcal{K}(1)+1)\|u_n\|_{s,A}^{p\theta}} \, dx. \qquad (15)
\end{aligned}$$

If we follow the analogous argument as in the proof of Lemma 5, we derive by (12), (15), (F4) and Fatou's lemma that

$$\begin{aligned}
\frac{1}{p\lambda} &\geq \liminf_{n\to\infty} \int_{\{\omega_n(x)\neq 0\}} \frac{F(x,|u_n|)}{(2\mathcal{K}(1)+1)\|u_n\|_{s,A}^{p\theta}} \, dx - \limsup_{n\to\infty} \int_{\{\omega_n(x)\neq 0\}} \frac{M_0}{(2\mathcal{K}(1)+1)\|u_n\|_{s,A}^{p\theta}} \, dx \\
&= \liminf_{n\to\infty} \int_{\{\omega_n(x)\neq 0\}} \frac{F(x,|u_n|) - M_0}{(2\mathcal{K}(1)+1)\|u_n\|_{s,A}^{p\theta}} \, dx \geq \int_{\{\omega_n(x)\neq 0\}} \liminf_{n\to\infty} \frac{F(x,|u_n|) - M_0}{(2\mathcal{K}(1)+1)\|u_n\|_{s,A}^{p\theta}} \, dx \\
&= \int_{\{\omega_n(x)\neq 0\}} \liminf_{n\to\infty} \frac{F(x,|u_n|)}{(2\mathcal{K}(1)+1)\|u_n\|_{s,A}^{p\theta}} \, dx - \int_{\{\omega_n(x)\neq 0\}} \limsup_{n\to\infty} \frac{M_0}{(2\mathcal{K}(1)+1)\|u_n\|_{s,A}^{p\theta}} \, dx \\
&\geq \frac{1}{2\mathcal{K}(1)+1} \int_{\{\omega_n(x)\neq 0\}} \liminf_{n\to\infty} \left(\frac{F(x,|u_n|)}{|u_n|^{p\theta}} |\omega_n|^{p\theta} \right) dx = \infty,
\end{aligned}$$

where M_0 was given in the proof of Lemma 5. This is impossible. Thus, $\mathcal{J}_\lambda(u) \to -\infty$ as $\|u\|_{s,\mathcal{A}} \to \infty$ for all $u \in \mathcal{Y}_k$. Choose $\varrho_k > \sigma_k > 0$ large sufficiently and let $\|u\|_{s,A} = \varrho_k$, we finally obtain

$$a_k = \max\{\mathcal{J}_\lambda(u) : u \in \mathcal{Y}_k, \|u\|_{s,\mathcal{A}} = \varrho_k\} \leq 0.$$

This completes the proof. □

Definition 3. Let E be a real separable and reflexive Banach space. We say that $\mathcal{I}$ satisfies the $(\mathrm{C})_c^*$-condition (with respect to $\mathcal{Y}_n$) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ for which $u_n \in \mathcal{Y}_n$, for any $n \in \mathbb{N}$,

$$\mathcal{I}(u_n) \to c \quad \text{and} \quad \|(\mathcal{I}|_{\mathcal{Y}_n})'(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0 \text{ as } n \to \infty,$$

contains a subsequence converging to a critical point of E.

Lemma 8 (Dual Fountain Theorem ([52] Theorem 3.11))**.** *Assume that E is a real Banach space, $\mathcal{I} \in C^1(E,\mathbb{R})$ is an even functional. If there is $k_0 > 0$ so that, for each $k \geq k_0$, there are $\varrho_k > \sigma_k > 0$ such that*

(A1) $\inf\{\mathcal{I}(u) : u \in \mathcal{Z}_k, \|u\|_E = \varrho_k\} \geq 0$.
(A2) $\beta_k := \max\{\mathcal{I}(u) : u \in \mathcal{Y}_k, \|u\|_E = \sigma_k\} < 0$.
(A3) $\gamma_k := \inf\{\mathcal{I}(u) : u \in \mathcal{Z}_k, \|u\|_E \leq \varrho_k\} \to 0$ *as* $k \to \infty$.
(A4) $\mathcal{I}$ *satisfies the* $(\mathrm{C})_c^*$*-condition for every* $c \in [d_{k_0}, 0)$.

Then $\mathcal{I}$ has a sequence of negative critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to \infty$.

Lemma 9. *Let $s \in (0,1)$, $p \in (1,+\infty)$ and $N > ps$. Assume that* (V), (K1), (K2) *and* (F1)–(F5) *hold. Then the functional $\mathcal{J}_\lambda$ satisfies the $(\mathrm{C})_c^*$-condition.*

Proof. The proof is carried out by the analogous argument as in [51]. □

With the help of Lemmas 8 and 9 we are ready to demonstrate our second assertion.

Theorem 3. *Let $s \in (0,1)$, $p \in (1,+\infty)$ and $N > ps$. Assume that* (V), (K1), (K2) *and* (F1)–(F5) *hold. Then the problem* (3) *has a sequence of nontrivial weak solutions $\{u_n\}$ in $\mathcal{H}^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ such that $\mathcal{J}_\lambda(u_n) \to 0$ as $n \to \infty$ for any $\lambda > 0$.*

Proof. Invoking Lemma 9, we get that $\mathcal{J}_\lambda$ is even and satisfies the $(C)^*_c$-condition for all $c \in \mathbb{R}$. Now it remains to show that conditions (A1), (A2) and (A3) of Lemma 8 are satisfied.

(A1): Let us denote

$$\theta_{1,k} = \sup_{\|u\|_{s,A}=1, u\in\mathcal{Z}_k} \|u\|_{L^p(\mathbb{R}^N)}, \quad \theta_{2,k} = \sup_{\|u\|_{s,A}=1, u\in\mathcal{Z}_k} \|u\|_{L^q(\mathbb{R}^N)}.$$

Then, it is immediate to verify that $\theta_{1,k} \to 0$ and $\theta_{2,k} \to 0$ as $k \to \infty$. Set $\vartheta_k = \max\{\theta_{1,k}, \theta_{2,k}\}$. Then it follows that

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \frac{1}{p}(\mathcal{K}(|u|^p_{s,A}) + \|u\|^p_{p,V}) - \lambda \int_{\mathbb{R}^N} F(x,|u|)dx \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{p}\|u\|^p_{s,A} - \frac{\lambda c_1}{p}\|u\|^p_{L^p(\mathbb{R}^N)} - \frac{\lambda c_2}{q}\|u\|^q_{L^q(\mathbb{R}^N)} \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{p}\|u\|^p_{s,A} - \frac{\lambda c_1}{p}\theta^p_{1,k}\|u\|^p_{s,A} - \frac{\lambda c_2}{q}\theta^q_{2,k}\|u\|^q_{s,A} \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{p}\|u\|^p_{s,A} - \lambda\left(\frac{c_1}{p} + \frac{c_2}{q}\right)\vartheta^p_k\|u\|^q_{s,A}
\end{aligned}$$

for sufficiently large k and $\|u\|_{s,A} \geq 1$. Choose

$$\varrho_k = \left[\frac{2p\lambda}{\min\{1, a\theta^{-1}\}}\left(\frac{c_1}{p} + \frac{c_2}{q}\right)\vartheta^p_k\right]^{\frac{1}{p-2q}}.$$

Let $u \in \mathcal{Z}_k$ with $\|u\|_{s,A} = \varrho_k > 1$ for k large enough. Then, there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\mathcal{J}_\lambda(u) &\geq \frac{\min\{1, a\theta^{-1}\}}{p}\|u\|^p_{s,A} - \lambda\left(\frac{c_1}{p} + \frac{c_2}{q}\right)\vartheta^p_k\|u\|^{2q}_{s,A} \\
&= \frac{\min\{1, a\theta^{-1}\}}{2p}\varrho^p_k \geq 0
\end{aligned}$$

for all $k \in \mathbb{N}$ with $k \geq k_0$, because

$$\lim_{k\to\infty} \frac{\min\{1, a\theta^{-1}\}}{2p}\varrho^p_k = \infty.$$

Therefore,

$$\inf\{\mathcal{J}_\lambda(u) : u \in \mathcal{Z}_k, \|u\|_{s,A} = \varrho_k\} \geq 0.$$

(A2): Observe that $\|\cdot\|_{L^p(\mathbb{R}^N)}$, $\|\cdot\|_{L^{p\theta}(\mathbb{R}^N)}$ and $\|\cdot\|_{s,A}$ are equivalent on $\mathcal{Y}_k$. Then there exist positive constants $\varsigma_{1,k}$ and $\varsigma_{2,k}$ such that

$$\|u\|_{L^p(\mathbb{R}^N)} \leq \varsigma_{1,k}\|u\|_{s,A} \text{ and } \|u\|_{s,A} \leq \varsigma_{2,k}\|u\|_{L^{p\theta}(\mathbb{R}^N)}$$

for any $u \in \mathcal{Y}_k$. From (F2)–(F4), for any $\mathcal{M} > 0$ there are positive constants $C_7(\mathcal{M})$ such that

$$F(x,\tau) \geq \mathcal{M}\varsigma^{p\theta}_{2,k}\tau^{p\theta} - C_7(\mathcal{M})\tau^p$$

for almost all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}^+$. Since $\mathcal{K}(\eta) \le \mathcal{K}(1)(1+\eta^\theta)$ for all $\eta \in \mathbb{R}_0^+$, it follows that

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \frac{1}{p}(\mathcal{K}(|u|_{s,A}^p) + \|u\|_{p,V}^p) - \lambda \int_{\mathbb{R}^N} F(x,|u|)dx \\
&\le \frac{1}{p}\Big(\mathcal{K}(1)(1+|u|_{s,A}^{p\theta}) + \|u\|_{p,V}^p\Big) - \lambda \mathcal{M} \varsigma_{2,k}^{p\theta} \int_{\mathbb{R}^N} u^{p\theta} dx + \lambda C_7(\mathcal{M}) \int_{\mathbb{R}^N} |u|^p dx \\
&\le \frac{1}{p}\Big(2\mathcal{K}(1)\|u\|_{s,A}^{p\theta} + \|u\|_{s,A}^{p\theta}\Big) - \lambda \mathcal{M} \varsigma_{2,k}^{p\theta} \int_{\mathbb{R}^N} u^{p\theta} dx + \lambda C_7(\mathcal{M}) \int_{\mathbb{R}^N} |u|^p dx \\
&\le \frac{1}{p}(2\mathcal{K}(1)+1)\|u\|_{s,A}^{p\theta} - \lambda \mathcal{M}\|u\|_{s,A}^{p\theta} + \lambda C_7(\mathcal{M}) \varsigma_{1,k}^p \|u\|_{s,A}^p
\end{aligned}$$

for any $u \in \mathcal{Y}_k$ with $\|u\|_{s,A} \ge 1$. Let $f(\tau) = \frac{1}{p}(2\mathcal{K}(1)+1)\tau^{p\theta} - \lambda \mathcal{M}\tau^{p\theta} + \lambda C_7(\mathcal{M})\varsigma_{1,k}^p \tau^p$. If $\mathcal{M}$ is large thoroughly, then $\lim_{\tau\to\infty} f(\tau) = -\infty$, and thus there is $\tau_0 \in (1,\infty)$ such that $f(\tau) < 0$ for all $\tau \in [\tau_0,\infty)$. Hence $\mathcal{J}_\lambda(u) < 0$ for all $u \in \mathcal{Y}_k$ with $\|u\|_{s,A} = \tau_0$. Choosing $\sigma_k = \tau_0$ for all $k \in \mathbb{N}$, one has

$$\beta_k := \max\{\mathcal{J}_\lambda(u) : u \in \mathcal{Y}_k, \|u\|_{s,A} = \sigma_k\} < 0.$$

If necessary, we can change k_0 to a large value, so that $\varrho_k > \sigma_k > 0$ for all $k \ge k_0$.

(A3): Because $\mathcal{Y}_k \cap \mathcal{Z}_k \ne \phi$ and $0 < \sigma_k < \varrho_k$, we have $\gamma_k \le \beta_k < 0$ for all $k \ge k_0$. For any $u \in \mathcal{Z}_k$ with $\|u\|_{s,A} = 1$ and $0 < \tau < \varrho_k$, one has

$$\begin{aligned}
\mathcal{J}_\lambda(\tau u) &\ge \frac{\min\{1, a\theta^{-1}\}}{p}\|\tau u\|_{s,A}^p - \frac{\lambda c_1}{p}\|\tau u\|_{L^p(\mathbb{R}^N)}^p - \frac{\lambda c_2}{q}\|\tau u\|_{L^q(\mathbb{R}^N)}^q \\
&\ge -\frac{\lambda c_1}{p}\tau^p \|u\|_{L^p(\mathbb{R}^N)}^p - \frac{\lambda c_2}{q}\tau^q\|u\|_{L^q(\mathbb{R}^N)}^q \\
&\ge -\frac{\lambda c_1}{p}\varrho_k^p \vartheta_k^p - \frac{\lambda c_2}{q}\varrho_k^q\vartheta_k^q
\end{aligned}$$

for large enough k. Hence, it follows from the definition of ϱ_k that

$$\begin{aligned}
\gamma_k &\ge -\frac{\lambda c_1}{p}\varrho_k^p\vartheta_k^p - \frac{\lambda c_2}{q}\varrho_k^q\vartheta_k^q \\
&= -\frac{\lambda c_1}{p}\left[\frac{2p\lambda}{\min\{1,a\theta^{-1}\}}\left(\frac{c_1}{p}+\frac{c_2}{q}\right)\right]^{\frac{p}{p-2q}} \vartheta_k^{\frac{(p-q)p}{p-2q}} - \frac{\lambda c_2}{q}\left[\frac{2p\lambda}{\min\{1,a\theta^{-1}\}}\left(\frac{c_1}{p}+\frac{c_2}{q}\right)\right]^{\frac{q}{p-2q}} \vartheta_k^{\frac{(p-q)q}{p-2q}}.
\end{aligned}$$

Because $p < q$ and $\vartheta_k \to 0$ as $k \to \infty$, we derive that $\lim_{k\to\infty}\gamma_k = 0$.

Hence all conditions of Lemma 8 are required. Consequently, we assert that problem (3) has a sequence of nontrivial weak solutions $\{u_n\}$ in $\mathcal{H}_{A,p}^s(\mathbb{R}^N,\mathbb{C})$ such that $\mathcal{J}_\lambda(u_n) \to 0$ as $n \to \infty$ for any $\lambda > 0$. □

3. Conclusions

In this paper, we investigate the existence and multiplicity of weak solutions to the fractional p-Laplacian Equation (3) with the external magnetic potential. The strategy of the proof for these results is to approach the problem variationally by applying the variational methods, namely, the fountain and the dual fountain theorem with Cerami condition. As far as we are aware, the present paper is the first attempt to study the multiplicity of nontrivial weak solutions to Schrödinger-Kirchhoff-type problems with the external magnetic potential in these circumstances. We point out that with a similar analysis, our main consequences continue to hold when $(-\Delta)_{p,A}^s v$ in (3) is changed into any non-local integro-differential operator $\mathcal{L}_M$, defined as follows;

$$\mathcal{L}_M v(x) = 2\int_{\mathbb{R}^N} |v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)|^{p-2}(v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y))M(x-y)dy \quad \text{for all } x \in \mathbb{R}^N, \tag{16}$$

where $M : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a kernel function satisfying properties that
(M1) $mM \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^p, 1\}$;
(M2) there exists $\theta > 0$, such that $M(x) \geq \theta |x|^{-(N+ps)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;
(K3) $M(x) = M(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

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