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Editorial

Fractional Calculus—Theory and Applications

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In recent years, fractional calculus has witnessed tremendous progress in various areas of sciences and mathematics. On one hand, new definitions of fractional derivatives and integrals have appeared in recent years, extending the classical definitions in some sense or another. Moreover, the rigorous analysis of the functional properties of these new definitions has been an active area of research in mathematical analysis. Systems considering differential equations with fractional-order operators have been investigated rigorously from the analytical and numerical points of view, and potential applications have been proposed in the sciences and in technology. The purpose of this Special Issue is to serve as a specialized forum for the dissemination of recent progress in the theory of fractional calculus and its potential applications. We invite authors to submit high-quality reports on the analysis of fractional-order differential/integral equations, the analysis of new definitions of fractional derivatives, numerical methods for fractional-order equations, and applications to physical systems governed by fractional differential equations, among other interesting topics of research.

The present Special Issue includes 10 articles, which cover the following topics.

- Fractional-order differential/integral equations.
- Existence and regularity of solutions.
- Numerical methods for fractional equations.
- Analysis of convergence and stability.
- Applications to science and technology.

In one of the articles published in this Special Issue [1], the authors considered a fractional-order system of malaria pestilence. The stability of the model at equilibrium points was investigated by applying the Jacobian matrix technique. The contribution of the basic reproduction number, R_0, in the infection dynamics and stability analysis was elucidated. The results indicated that the given system is locally asymptotically stable at the disease-free steady-state solution when $R_0 < 1$. A similar result was obtained for the endemic equilibrium when $R_0 > 1$. The underlying system showed global stability at both steady states. The fractional-order system was then converted into a stochastic model. For a more realistic study of the disease dynamics, the non-parametric perturbation version of the stochastic epidemic model was developed and studied numerically. The general stochastic fractional Euler method, the Runge–Kutta method, and a proposed numerical method were applied to solve the model. The standard techniques failed to preserve the positivity property of the continuous system. Meanwhile, the proposed stochastic fractional nonstandard finite-difference method preserved the positivity. For the boundedness of the nonstandard finite-difference scheme, a result was established. All the analytical results were verified by numerical simulations.

The article [2] is devoted to studying GPU-based modeling for a parallel fractional-order derivative model of the spiral-plate heat exchanger. As pointed out by the authors, a spiral-plate heat exchanger with two fluids is a compact plant that only requires a small space and is excellent in high heat-transfer efficiency. However, the spiral-plate heat exchanger is a nonlinear plant with uncertainties, considering the difference between the

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heat fluid, the heated fluid, and other complex factors. The fractional-order derivation model is more accurate than the traditional integer-order model. In this paper, a parallel fractional order derivation model was proposed by considering the merit of the graphics processing unit (GPU). Then, the parallel fractional-order derivation model for the spiral-plate heat exchanger was constructed. Simulations show the relationships between the output temperature of heated fluid and the orders of fractional-order derivatives with two directional fluids impacted by complex factors, namely, the volume flow rate in hot fluid and the volume flow rate in cold fluid, respectively.

In turn, a forecasting of the economic growth of the Group of Seven (G7) via a fractional-order gradient descent approach was investigated in [3]. More concretely, this work established a model of economic growth for all G7 countries from 1973 to 2016, in which the gross domestic product (GDP) is related to land area, arable land, population, school attendance, gross capital formation, exports of goods and services, general government, final consumer spending and broad money. The fractional-order gradient descent and integer-order gradient descent were used to estimate the model parameters to fit the GDP and forecast GDP from 2017 to 2019. The results showed that the convergence rate of the fractional-order gradient descent is faster and has a better fitting accuracy and prediction effect.

In [4], the authors studied the approximate and analytic solutions of the time-fractional intermediate diffusion wave equation associated with the Fokker–Planck operator. More precisely, the time-fractional wave equation associated with the space-fractional Fokker–Planck operator and with the time-fractional-damped term were studied in this work. The concept of the Green function was implemented to drive the analytic solution of the three-term time-fractional equation. The explicit expressions for the green function of the three-term time-fractional wave equation with constant coefficients was also studied for two physical and biological models. The explicit analytic solutions for the two studied models were expressed in terms of the Weber, hypergeometric, exponential, and Mittag–Leffler functions. The relation to the diffusion equation was given therein. The asymptotic behaviors of the Mittag–Leffler function, the hypergeometric function, and the exponential functions were compared numerically. The Grünwald–Letnikov scheme was then used to derive the approximate difference schemes of the Caputo time-fractional operator and the Feller–Riesz space-fractional operator. The explicit difference scheme was numerically studied, and the simulations of the approximate solutions were plotted for different values of the fractional orders.

On the other hand, the authors of [5] reported on some new fractional estimates of inequalities for LR-p-convex interval-valued functions by means of pseudo order relation. Interval analysis provides tools to deal with data uncertainty. In general, interval analysis is typically used to deal with the models whose data are composed of inaccuracies that may occur from certain kinds of measurements. In this context, both the inclusion relation ($\subseteq$) and the pseudo-order relation ($\leq_p$) are two different concepts. By using the latter relation, the authors introduce the new class of nonconvex functions known as LR-p-convex interval-valued functions (LR-p-convex-IVFs). With the help of this relation, they establish a strong relationship between LR-p-convex-IVFs and Hermite–Hadamard-type inequalities (HH-type inequalities) via the Katugampola fractional integral operator. The results include a wide class of new and known inequalities for LR-p-convex-IVFs and their variant forms as special cases. Useful examples that demonstrate the applicability of the theory proposed in this study were given in that study.

Sequential Riemann–Liouville and Hadamard–Caputo fractional differential systems with nonlocal coupled fractional integral boundary conditions were studied in [6]. In that work, the authors investigated the existence of solutions for a fractional differential system that contains mixed Riemann–Liouville and Hadamard–Caputo fractional derivatives, complemented with nonlocal coupled fractional integral boundary conditions. They derived necessary conditions for the existence and uniqueness of solutions of those system by using standard fixed-point theorems, such as Banach contraction mapping principle and the

Leray–Schauder alternative. Numerical examples illustrating the theoretical results were also presented.

In [7], a numerical method for solving a fractional diffusion-wave and nonlinear Fredholm and Volterra integral equations with zero absolute error was presented. The method was based on Euler wavelet approximation and matrix inversion of $M \times M$ collocation points. The proposed equations were presented based on the Caputo fractional derivative, and the authors reduced the resulting system to a system of algebraic equations by implementing the Gaussian quadrature discretization. The reduced system was generated via the truncated Euler wavelet expansion. Several examples with known exact solutions were solved with zero absolute error. This method was also applied to the Fredholm and Volterra nonlinear integral equations and achieved the desired absolute error for all tested examples. The new numerical scheme is appealing in terms of its efficiency and accuracy in the field of numerical approximation.

On the other hand, some non-instantaneous impulsive boundary-value problems containing Caputo fractional derivatives of a function with respect to another function as well as Riemann–Stieltjes fractional integral boundary conditions were considered in [8]. In that work, the authors studied existence and uniqueness results for a new class of boundary-value problems consisting of non-instantaneous impulses and Caputo fractional derivative of a function with respect to another function, supplemented with Riemann–Stieltjes fractional integral boundary conditions. The existence of a unique solution was obtained via Banach's contraction mapping principle, while an existence result is established by using Leray–Schauder nonlinear alternative. Examples illustrating the main results were also constructed.

In article [9], the authors considered a retarded linear fractional differential system with distributed delays and Caputo-type derivatives of incommensurate orders. For this system, several a priori estimates for the solutions, applying the two traditional approaches (Gronwall's inequality and integral representations of the solutions) were obtained. As an application of the obtained estimates, different sufficient conditions that guarantee finite-time stability of the solutions were established. A comparison of the obtained different conditions was made with respect to the estimates and norms used.

Finally, a fractional coupled hybrid Sturm–Liouville differential equation with a multi-point boundary coupled hybrid condition was presented in [10]. It is worth recalling here that the Sturm–Liouville differential equation is an important tool for physics, applied mathematics, and other fields of engineering and science and has wide applications in quantum mechanics, classical mechanics, and wave phenomena. In this paper, the authors investigated the coupled hybrid version of the Sturm–Liouville differential equation. They studied the existence of solutions for the coupled hybrid Sturm–Liouville differential equation with multi-point boundary-coupled hybrid condition. Furthermore, they investigated the existence of solutions for the coupled hybrid Sturm–Liouville differential equation with an integral boundary coupled hybrid condition. To close that work, the authors gave an application and some examples to illustrate their results.

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