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A Straightforward Sufficiency Proof for a Nonparametric Problem of Bolza in the Calculus of Variations

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Abstract: We study a variable end-points calculus of variations problem of Bolza containing inequality and equality constraints. The proof of the principal theorem of the paper has a direct nature since it is independent of some classical sufficiency approaches invoking the Hamiltonian-Jacobi theory, Riccati equations, fields of extremals or the theory of conjugate points. In contrast, the algorithm employed to prove the principal theorem of the article is based on elementary tools of the real analysis.

Keywords: calculus of variations; nonparametric problems; variable end-points; inequality and equality restrictions; sufficiency; strong minima

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1. Introduction

In this paper, we study a nonparametric calculus of variations problem of Bolza having variable end-points, isoperimetric inequality and equality restrictions and mixed inequality and equality pointwise restraints. The fundamental sufficiency theorem presented in this article, assumes that a proposed optimal trajectory with an essentially bounded derivative is given, that the set of active indices of the mixed inequality restrictions is piecewise constant on the underlying interval of time, that the corresponding multipliers of the inequality restrictions are nonnegative at each point of the basic time interval and they are zero whenever the time-dependent index is inactive, that the matching Lagrange multipliers of the inequality isoperimetric constraints are nonnegative and they vanish whenever the corresponding index is inactive, that a sufficiency first order condition very related with the Euler–Lagrange equations holds, that a generalized transversality condition is verified, that an inequality hypothesis whose source comes from the proof of the main result of the paper is satisfied, that a very similar hypothesis of the Legendre necessary condition is satisfied, that the positivity of a quadratic integral over the cone of critical directions is fulfilled and, that three conditions involving the Weierstrass functions delimiting the calculus of variations problem are verified. Then the deviation between any admissible cost and the proposed optimal cost, can be estimated by a quadratic functional whose role is very similar with that of the square of the norm of the Banach space of the Lebesgue integrable functions. In particular, the result shows that if the proposed optimal trajectory satisfies the above sufficiency conditions, then it is a strict strong minimum of the problem in hand.

It is worthwhile mentioning that the proof of the main sufficiency theorem of the paper is self-contained in the sense that it is independent of some classical approaches such as the ones that invoke to the theory of Mayer fields by using independent path integrals, commonly called Hilbert integrals, Hamilton–Jacobi theory which frequently uses a fundamental inequality, symmetric solutions of some Riccati equations, generalizations of the conjugate point theory, local convexity processes or the insertion of the proposed optimal trajectories in some fields of extremals, see for instance [1–16]. On the other hand, it is important to point out that the calculus of variations has as its aim a generalization of the structure of classical analysis that will make possible the solution of some extremum problems having numerous applications in the qualitative analysis of various classes of differential equations and partial differential equations; see, e.g., the papers [17,18] for more details. The technique used in this article to obtain the main theorem of the paper corresponds to a generalization of a method originally introduced by Hestenes in [9]. This algorithm have been generalized in [19–21] for the case of a parametric problem of the calculus of variations, however, a direct sufficiency proof for the nonparametric problem of Bolza had not been provided. A crucial property of this direct sufficiency proof not only has the advantage that one does not need to invoke to a parametric problem as it is done in [19–21], but also the sufficiency result for the parametric problem, provides sufficient conditions to a strict strong minimum and not only for a strong minimum as it is the case of [20,21].

Some of the novelties of the main theorem of the paper as well as the technique employed to prove it can be described as follows: the problem has a wide range of applicability since the functions delimiting the problem only have to be continuous in their domain and they need to have first and second partial derivatives with respect to the state and the state-derivative independent variables. The smoothness of the first and second partial derivatives with respect to the previously mentioned variables is no longer imposed. The derivatives of the proposed optimal trajectories need not be continuous but only essentially bounded. This feature is a celebrated component since the derivatives of the admissible trajectories must only be essentially bounded. In fact, we have already provided concrete examples, in which our theory of sufficiency, indeed gives a response, meanwhile the classical sufficiency theories for optimality are not able to detect it, since they need the smoothness of the optimal trajectory in the basic time interval, see [21]. Finally, the technique used to prove the main theorem of the paper, allows us to avoid imposing some type of preliminary assumptions not appearing in the theorems, in contrast, with some classical necessary and sufficiency theories. To mention a few, in [12,22] it is indispensable that the gradients arising from the pointwise mixed constraints be linearly independent at each point of the underlying interval of time or see [22–24], where some preliminary assumptions of normality or regularity play a crucial role for obtaining the necessary optimality theory.

The paper is organized as follows. In Section 2, we pose the problem we are going to study, introduce some basic definitions and state the main result of the article. In Section 3, we illustrate the sufficient theorem of the paper by means of an example. In Section 4, we enunciate two auxiliary lemmas whose statements and proofs can be found in [21]. Finally, in Section 5, we develop the proof of Theorem 1.

2. The Problem and the Sufficiency Theorem

Suppose that an interval $T := [t_0, t_1]$ in $\mathbf{R}$ is given, that we have functions $l, l_\gamma \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ $(\gamma = 1, \ldots, K)$, $\Phi_i \colon \mathbf{R}^n \to \mathbf{R}^n$ $(i = 0, 1)$, $L(t, x, \dot{x}) \colon T \times \mathbf{R}^{2n} \to \mathbf{R}$, $L_\gamma(t, x, \dot{x}) \colon T \times \mathbf{R}^{2n} \to \mathbf{R}$ $(\gamma = 1, \ldots, K)$ and $\varphi(t, x, \dot{x}) \colon T \times \mathbf{R}^{2n} \to \mathbf{R}^s$. Let

$$\mathcal{A} := \{(t, x, \dot{x}) \in T \times \mathbf{R}^{2n} \mid \varphi_\alpha(t, x, \dot{x}) \leq 0 \ (\alpha \in R), \ \varphi_\beta(t, x, \dot{x}) = 0 \ (\beta \in S)\}$$

where $R := \{1, \ldots, r\}$ and $S := \{r + 1, \ldots, s\}$ $(r = 0, 1, \ldots, s)$. If $r = 0$ then $R = \emptyset$ and we disregard assertions concerning φ_α. Similarly, if $r = s$ then $S = \emptyset$ and we disregard assertions concerning φ_β.

Throughout the article we assume that L, L_γ $(\gamma = 1, \ldots, K)$ and φ have first and second derivatives with respect to x and $\dot{x}$. Furthermore, if we denote by $g(t, x, \dot{x})$ either $L(t, x, \dot{x})$, $L_\gamma(t, x, \dot{x})$ $(\gamma = 1, \ldots, K)$, $\varphi(t, x, \dot{x})$ or any of their partial derivatives of order less or equal than two with respect to x and $\dot{x}$, we are going to suppose that if $\mathcal{G}$ is any bounded subset of $T \times \mathbf{R}^{2n}$, then $|g(\mathcal{G})|$ is a bounded subset of $\mathbf{R}$. Additionally, we suppose that if $((\Lambda_q, \Gamma_q))$ is any sequence in $AC(T; \mathbf{R}^n) \times L^1(T; \mathbf{R}^n)$ such that for some $\Theta \subset T$ measurable and some $(\Lambda, \Gamma) \in AC(T; \mathbf{R}^n) \times L^\infty(T; \mathbf{R}^n)$, $(\Lambda_q(\cdot), \Gamma_q(\cdot)) \xrightarrow{L^\infty} (\Lambda(\cdot), \Gamma(\cdot))$ on Θ, then for all $q \in \mathbf{N}$, $g(\cdot, \Lambda_q(\cdot), \Gamma_q(\cdot))$ is measurable on Θ and

$$g(\cdot, \Lambda_q(\cdot), \Gamma_q(\cdot)) \xrightarrow{L^\infty} g(\cdot, \Lambda(\cdot), \Gamma(\cdot)) \text{ on } \Theta.$$

Note that all conditions given above are satisfied if the functions L, L_γ $(\gamma = 1, \ldots, K)$ and φ and their first and second derivatives with respect to x and $\dot{x}$ are continuous on $T \times \mathbf{R}^{2n}$. We shall also assume that the functions l, l_γ $(\gamma = 1, \ldots, K)$ are of class C^2 on $\mathbf{R}^n \times \mathbf{R}^n$ and Φ_i $(i = 0, 1)$ are of class C^2 on $\mathbf{R}^n$.

The calculus of variations problem we shall be concerned, labeled (P), is that of finding a minimum value to the functional

$$I(x) := l(x(t_0), x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt$$

over all absolutely continuous $x \colon T \to \mathbf{R}^n$ satisfying the constraints

$$\begin{cases} g(\cdot, x(\cdot), \dot{x}(\cdot)) \text{ is integrable on } T. \\ x(t_{-i}) = \Phi_{-i}(x(t_{i+1})) \text{ for } i = -1, 0. \\ I_i(x) := l_i(x(t_0), x(t_1)) + \int_{t_0}^{t_1} L_i(t, x(t), \dot{x}(t))dt \leq 0 \ (i = 1, \ldots, k). \\ I_j(x) := l_j(x(t_0), x(t_1)) + \int_{t_0}^{t_1} L_j(t, x(t), \dot{x}(t))dt = 0 \ (j = k+1, \ldots, K). \\ (t, x(t), \dot{x}(t)) \in \mathcal{A} \text{ (a.e. in } T). \end{cases}$$

Designate by $\mathcal{X}$ the space of absolutely continuous functions mapping T to $\mathbf{R}^n$ and by $\mathcal{U}_s$ the Banach space $L^\infty(T; \mathbf{R}^s)$ $(s \in \mathbf{N})$. Elements of $\mathcal{X}$ are named *arcs* or *trajectories* and an arc x is *admissible or feasible* if it satisfies the restrictions. A trajectory x *solves* (P) if it is feasible and $I(x) \leq I(y)$ for all feasible arcs y. An admissible arc x is called a *strong minimum* of (P) if it is a minimum of I relative to the norm

$$\|x\| := \sup_{t \in T} |x(t)|,$$

that is, if we have the existence of some $\epsilon > 0$ such that $I(x) \leq I(y)$ for all feasible trajectories y verifying $\|y - x\| < \epsilon$. It is a *strict strong minimum* when $I(x) = I(y)$ only if $x = y$.

The following definitions are going to be useful in the content of the paper. The notation $*$ means transpose.

- Given K real numbers λ_γ $(\gamma = 1, \ldots, K)$, take into consideration the functional $I_\lambda \colon \mathcal{X} \to \mathbf{R}$ defined by

$$I_\lambda(x) := I(x) + \sum_{\gamma=1}^{K} \lambda_\gamma I_\gamma(x) = l_\lambda(x(t_0), x(t_1)) + \int_{t_0}^{t_1} L_\lambda(t, x(t), \dot{x}(t))dt,$$

where $l_\lambda \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is given by

$$l_\lambda(a_1, a_2) := l(a_1, a_2) + \sum_{\gamma=1}^{K} \lambda_\gamma l_\gamma(a_1, a_2),$$

and $L_\lambda \colon T \times \mathbf{R}^{2n} \to \mathbf{R}$ is defined by

$$L_\lambda(t, x, \dot{x}) := L(t, x, \dot{x}) + \sum_{\gamma=1}^{K} \lambda_\gamma L_\gamma(t, x, \dot{x}).$$

- For all $(t, x, \dot{x}, \rho, \mu) \in T \times \mathbf{R}^{3n} \times \mathbf{R}^s$, set
$$H(t, x, \dot{x}, \rho, \mu) := \rho^* \dot{x} - L_\lambda(t, x, \dot{x}) - \mu^* \varphi(t, x, \dot{x}).$$
If $\rho \in \mathcal{X}$ and $\mu \in \mathcal{U}_s$ are given, set for all $(t, x, \dot{x}) \in T \times \mathbf{R}^{2n}$,
$$F_\lambda(t, x, \dot{x}) := -H(t, x, \dot{x}, \rho(t), \mu(t)) - \dot{\rho}^*(t)x,$$
and let
$$J_\lambda(x) := \rho^*(t_1)x(t_1) - \rho^*(t_0)x(t_0) + l_\lambda(x(t_0), x(t_1)) + \int_{t_0}^{t_1} F_\lambda(t, x(t), \dot{x}(t))dt.$$
- The *first variations* of J_λ and I_γ $(\gamma = 1, \ldots, K)$ along $x \in \mathcal{X}$ with $\dot{x} \in L^\infty(T; \mathbf{R}^n)$ in the direction $y \in \mathcal{X}$ are given, respectively, by

$$J'_\lambda(x, y) := l'_\lambda(x(t_0), x(t_1)) \begin{pmatrix} y(t_0) \\ y(t_1) \end{pmatrix} + \int_{t_0}^{t_1} \{F_{\lambda x}(t, x(t), \dot{x}(t))y(t) + F_{\lambda \dot{x}}(t, x(t), \dot{x}(t))\dot{y}(t)\}dt,$$
$$I'_\gamma(x, y) := l'_\gamma(x(t_0), x(t_1)) \begin{pmatrix} y(t_0) \\ y(t_1) \end{pmatrix} + \int_{t_0}^{t_1} \{L_{\gamma x}(t, x(t), \dot{x}(t))y(t) + L_{\gamma \dot{x}}(t, x(t), \dot{x}(t))\dot{y}(t)\}dt.$$

The *second variation of* J_λ along $x \in \mathcal{X}$ with $\dot{x} \in L^\infty(T; \mathbf{R}^n)$ in the direction $y \in \mathcal{X}$ with $\dot{y} \in L^2(T; \mathbf{R}^n)$ is given by

$$J''_\lambda(x, y) := (y^*(t_0), y^*(t_1)) l''_\lambda(x(t_0), x(t_1)) \begin{pmatrix} y(t_0) \\ y(t_1) \end{pmatrix} + \int_{t_0}^{t_1} 2\omega_\lambda(t, x(t), \dot{x}(t); t, y(t), \dot{y}(t))dt$$

where, for all $(t, y, \dot{y}) \in T \times \mathbf{R}^{2n}$,

$$2\omega_\lambda(t, x(t), \dot{x}(t); t, y, \dot{y}) := y^* F_{\lambda xx}(t, x(t), \dot{x}(t))y + 2y^* F_{\lambda x\dot{x}}(t, x(t), \dot{x}(t))\dot{y} + \dot{y}^* F_{\lambda \dot{x}\dot{x}}(t, x(t), \dot{x}(t))\dot{y}.$$

- Set
$$E_\lambda(t, x, \dot{x}, u) := F_\lambda(t, x, u) - F_\lambda(t, x, \dot{x}) - F_{\lambda \dot{x}}(t, x, \dot{x})(u - \dot{x}).$$
Similarly, for all $\gamma = 1, \ldots, K$, set
$$E_\gamma(t, x, \dot{x}, u) := L_\gamma(t, x, u) - L_\gamma(t, x, \dot{x}) - L_{\gamma \dot{x}}(t, x, \dot{x})(u - \dot{x}).$$
- For all $x \in \mathcal{X}$, set
$$D(x) := V(x(t_0)) + \int_{t_0}^{t_1} V(\dot{x}(t))dt$$
where for all $e \in \mathbf{R}^n$,
$$V(e) := (1 + |e|^2)^{1/2} - 1.$$
Finally, for all $(t, x, \dot{x}) \in T \times \mathbf{R}^{2n}$, designate by
$$\mathcal{I}_a(t, x, \dot{x}) := \{\alpha \in R \mid \varphi_\alpha(t, x, \dot{x}) = 0\},$$

the set of active indices of $(t, x, \dot{x})$ corresponding to the mixed inequality constraints. Given $x \in \mathcal{X}$, designate by
$$i_a(x) := \{i = 1, \ldots, k \mid I_i(x) = 0\},$$
the set of active indices of x corresponding to the isoperimetric inequality restrictions. For all $x \in \mathcal{X}$, let $\mathcal{Y}(x)$ be the set of all $y \in \mathcal{X}$ with $\dot{y} \in L^2(T; \mathbf{R}^n)$ verifying

$$\begin{cases} y(t_{-i}) = \Phi'_{-i}(x(t_{i+1}))y(t_{i+1}) \text{ for } i = -1, 0. \\ I'_i(x, y) \le 0 \ (i \in i_a(x)),\ I'_j(x, y) = 0 \ (j = k+1, \ldots, K). \\ \varphi_{\alpha x}(t, x(t), \dot{x}(t))y(t) + \varphi_{\alpha \dot{x}}(t, x(t), \dot{x}(t))\dot{y}(t) \le 0 \text{ (a.e. in } T,\ \alpha \in \mathcal{I}_a(t, x(t), \dot{x}(t))). \\ \varphi_{\beta x}(t, x(t), \dot{x}(t))y(t) + \varphi_{\beta \dot{x}}(t, x(t), \dot{x}(t))\dot{y}(t) = 0 \text{ (a.e. in } T,\ \beta \in S). \end{cases}$$

The cone $\mathcal{Y}(x)$ is commonly called the cone of *critical directions* along x.

Theorem 1. *Let x_0 be a feasible arc with $\dot{x}_0 \in L^\infty(T; \mathbf{R}^n)$. Assume that $\mathcal{I}_a(\cdot, x_0(\cdot), \dot{x}_0(\cdot))$ is piecewise constant on T, that there exist $\rho \in \mathcal{X}$, $\mu \in \mathcal{U}_s$ satisfying $\mu_\alpha(t) \geq 0$, $\mu_\alpha(t)$ $\varphi_\alpha(t, x_0(t), \dot{x}_0(t)) = 0$ ($\alpha \in R$, a.e. in T), $\delta, \epsilon > 0$, and multipliers λ_i ($i = 1, \ldots, K$) satisfying $\lambda_i \geq 0$, $\lambda_i I_i(x_0) = 0$ ($i = 1, \ldots, k$) such that*

$$\dot{\rho}(t) = -H_x^*(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) \text{ (a.e. in } T),$$

$$H_{\dot{x}}^*(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) = 0 \text{ (a.e. in } T),$$

and the following assumptions hold:

i. $l_\lambda'^*(x_0(t_0), x_0(t_1)) + \begin{pmatrix} \Phi_1'^*(x_0(t_0)) \\ 0_{n\times n} \end{pmatrix} \rho(t_1) - \begin{pmatrix} 0_{n\times n} \\ \Phi_0'^*(x_0(t_1)) \end{pmatrix} \rho(t_0) = 0.$

ii. $\sum_{i=-1}^{0} (-1)^{i+1} \rho^*(t_{-i}) \Phi''_{-i}(x_0(t_{i+1}); h) \geq 0$ *for all* $h \in \mathbf{R}^n$.

iii. $H_{\dot{x}\dot{x}}(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) \leq 0$ *(a.e. in T).*

iv. $J_\lambda''(x_0, y) > 0$ *for all* $y \neq 0$, $y \in \mathcal{Y}(x_0)$.

v. *For all x feasible satisfying $\|x - x_0\| < \epsilon$,*

(a) $E_\lambda(t, x(t), \dot{x}_0(t), \dot{x}(t)) \geq 0$ *(a.e. in T);*

(b) $\int_{t_0}^{t_1} E_\lambda(t, x(t), \dot{x}_0(t), \dot{x}(t))dt \geq \delta \int_{t_0}^{t_1} V(\dot{x}(t) - \dot{x}_0(t))dt;$

(c) $\int_{t_0}^{t_1} E_\lambda(t, x(t), \dot{x}_0(t), \dot{x}(t))dt \geq \delta \left| \int_{t_0}^{t_1} E_\gamma(t, x(t), \dot{x}_0(t), \dot{x}(t))dt \right|$ $(\gamma = 1, \ldots, K)$.

Then, there exist $\nu_1, \nu_2 > 0$ such that, if x is feasible with $\|x - x_0\| < \nu_1$, we have

$$I(x) \geq I(x_0) + \nu_2 D(x - x_0).$$

In particular, x_0 is a strict strong minimum of (P).

3. Example

In this section, we give an illustration of Theorem 1 by means of an example.

Let (P) be the problem of finding a minimum value to the functional

$$I(x) := x^2(-1) - 2x(-1) + \int_{-1}^{1} 2(x(t) + t)^2 dt$$

over all absolutely continuous $x \colon [-1, 1] \to \mathbf{R}$ verifying the constraints

$$\begin{cases} g(\cdot, x(\cdot), \dot{x}(\cdot)) \text{ is integrable on } [-1,1]. \\ x(-1) = -x(1). \\ I_1(x) := \int_{-1}^{1} \{(\dot{x}(t)+1)^2 + \dot{x}(t)(x(t)+t)^2\} dt \leq 0. \\ (t, x(t), \dot{x}(t)) \in \mathcal{A} \text{ (a.e. in } [-1,1]). \end{cases}$$

For this case, $T = [-1, 1]$, $n = 1$, $K = k = 1$, $r = s = 1$, $\Phi_0 = \Phi_1 = -\text{Id}$ where Id is the identity function, $l(a_1, a_2) = a_1^2 - 2a_1$, $l_1(a_1, a_2) = 0$, $L(t, x, \dot{x}) = 2(x + t)^2$, $L_1(t, x, \dot{x}) = (\dot{x} + 1)^2 + \dot{x}(x + t)^2$, $\varphi_1(t, x, \dot{x}) = -\dot{x} - 1$ and

$$\mathcal{A} = \{(t, x, \dot{x}) \in T \times \mathbf{R}^2 \mid \varphi_1(t, x, \dot{x}) \leq 0\}.$$

For all $(t, x, \dot{x}, \rho, \mu) \in T \times \mathbf{R}^4$, we have

$$H(t, x, \dot{x}, \rho, \mu) = \rho\dot{x} - 2(x + t)^2 - \lambda_1(\dot{x} + 1)^2 - \lambda_1\dot{x}(x + t)^2 + \mu_1[\dot{x} + 1],$$

$$H_x(t, x, \dot{x}, \rho, \mu) = -4(x + t) - 2\lambda_1\dot{x}(x + t),$$

$$H_{\dot{x}}(t, x, \dot{x}, \rho, \mu) = \rho - 2\lambda_1(\dot{x} + 1) - \lambda_1(x + t)^2 + \mu_1.$$

Let $x_0 \equiv -t$ on T and note that $x_0 \in \mathcal{X} = AC(T;\mathbf{R})$, $\dot{x}_0 \in L^\infty(T;\mathbf{R})$ and x_0 is admissible. Furthermore, note that $\mathcal{I}_a(\cdot, x_0(\cdot), \dot{x}_0(\cdot)) \equiv \{1\}$ on T, and hence it is constant on T. Set $\rho = \mu_1 \equiv 0$ on T and note that $\rho \in \mathcal{X}$ and $\mu = \mu_1 \in \mathcal{U}_1 = L^\infty(T;\mathbf{R})$. Moreover, observe that $\mu_1(t) \geq 0$ and $\mu_1(t)\varphi_1(t, x_0(t), \dot{x}_0(t)) = 0$ ($\alpha \in R = \{1\}$, a.e. in T). Additionally, let $\lambda_1 = 1$ and note that $\lambda_1 \geq 0$ and $\lambda_1 I_1(x_0) = 0$. With these concepts in mind, observe that

$$\dot{\rho}(t) = -H_x(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) \text{ (a.e. in } T),$$

$$H_{\dot{x}}(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) = 0 \text{ (a.e. in } T).$$

Now, note that $l_\lambda(a_1, a_2) = l(a_1, a_2) + \lambda_1 l_1(a_1, a_2) = a_1^2 - 2a_1$ and hence

$$l'_\lambda(a_1, a_2) = (2a_1 - 2, 0)$$

and $l'_\lambda(x_0(-1), x_0(1)) = (0, 0)$. As $\rho \equiv 0$ on T, then as one readily verifies, hypotheses (i) and (ii) of Theorem 1 are verified. Furthermore, observe that $H_{\dot{x}\dot{x}}(t, x, \dot{x}, \rho, \mu) = -2\lambda_1$ and so $H_{\dot{x}\dot{x}}(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) = -2$ (a.e. in T) and then, hypothesis (iii) of Theorem 1 is also verified. Now, note that since

$$H_{xx}(t, x, \dot{x}, \rho, \mu) = -4 - 2\lambda_1\dot{x} \quad \text{and} \quad H_{x\dot{x}}(t, x, \dot{x}, \rho, \mu) = -2\lambda_1(x + t),$$

then $H_{xx}(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) = -2$ and $H_{x\dot{x}}(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) = 0$ (a.e. in T).

Furthermore,

$$l''_\lambda(a_1, a_2) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

and so,

$$l''_\lambda(x_0(-1), x_0(1)) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, the second variation J''_λ is given by

$$J''_\lambda(x_0, y) = 2y^2(-1) + \int_{-1}^{1} 2\{y^2(t) + \dot{y}^2(t)\}dt$$

which is greater than zero for all $y \neq 0$, $y \in \mathcal{Y}(x_0)$ where $\mathcal{Y}(x_0)$ is given by all $y \in \mathcal{X}$ with $\dot{y} \in L^2(T;\mathbf{R})$ satisfying

$$\begin{cases} y(-1) = -y(1). \\ I'_1(x_0, y) \leq 0 \ (i \in i_a(x_0) = \{1\}). \\ -\dot{y}(t) \leq 0 \text{ (a.e. in } T). \end{cases}$$

Thus, hypothesis (iv) of Theorem 1 is satisfied. We also have that

$$F_\lambda(t, x, \dot{x}) = 2(x + t)^2 + (\dot{x} + 1)^2 + \dot{x}(x + t)^2.$$

Consequently, if x is admissible, then for almost all $t \in T$,

$$E_\lambda(t, x(t), \dot{x}_0(t), \dot{x}(t)) = (\dot{x}(t) + 1)^2 + \dot{x}(t)(x(t) + t)^2 + (x(t) + t)^2 \geq (\dot{x}(t) + 1)^2$$

and so, if x is admissible, then

(a) $E_\lambda(t, x(t), \dot{x}_0(t), \dot{x}(t)) \geq 0$ (a.e. in T);
(b) $\int_{-1}^{1} E_\lambda(t, x(t), \dot{x}_0(t), \dot{x}(t))dt \geq \int_{-1}^{1} (\dot{x}(t) + 1)^2 dt \geq \int_{-1}^{1} V(\dot{x}(t) - \dot{x}_0(t))dt$.
Moreover, as one readily verifies, if x is admissible, then for almost all $t \in T$,

$$E_1(t, x(t), \dot{x}_0(t), \dot{x}(t)) = (\dot{x}(t) + 1)^2 + \dot{x}(t)(x(t) + t)^2 + (x(t) + t)^2,$$

and hence, if x is admissible, then

(c) $\int_{-1}^{1} E_\lambda(t, x(t), \dot{x}_0(t), \dot{x}(t))dt = |\int_{-1}^{1} E_1(t, x(t), \dot{x}_0(t), \dot{x}(t))dt|$ implying that hypothesis (v) of Theorem 1 is verified with any $\epsilon > 0$ and $\delta = 1$. Then, there exist $\nu_1, \nu_2 > 0$ such that, if x is admissible with $\|x - x_0\| < \nu_1$, we have

$$I(x) \geq I(x_0) + \nu_2 D(x - x_0).$$

In particular, x_0 is a strict strong minimum of (P).

4. Auxiliary Lemmas

In this section, we are going to prove Theorem 1. First, we state two auxiliary lemmas whose statements and proofs are given in Lemmas 4.1 and 4.2 of [21].

In the following we suppose that we are given $x_0 \in \mathcal{X}$ and a subsequence (x_q) in $\mathcal{X}$ such that

$$\lim_{q\to\infty} D(x_q - x_0) = 0 \quad \text{and} \quad d_q := [2D(x_q - x_0)]^{1/2} > 0 \quad (q \in \mathbf{N}).$$

For all $q \in \mathbf{N}$, define

$$y_q := \frac{x_q - x_0}{d_q}.$$

We write $\dot{x}_q \xrightarrow{\text{au}} \dot{x}_0$ on T, if for any $\epsilon > 0$, there exists $\Theta_\epsilon \subset T$ measurable with $m(\Theta_\epsilon) < \epsilon$ such that $\dot{x}_q \xrightarrow{\text{u}} \dot{x}_0$ on $T \setminus \Theta_\epsilon$, that is, if $(\dot{x}_q)$ converges uniformly to $\dot{x}_0$ on $T \setminus \Theta_\epsilon$.

We shall not relabel the subsequences of a given sequence since this fact will not modify our results.

Lemma 1. *For some subsequence of (x_q), and some $y_0 \in \mathcal{X}$ with $\dot{y}_0 \in L^2(T; \mathbf{R}^n)$, $\dot{x}_q \xrightarrow{au} \dot{x}_0$ on T, $y_q \xrightarrow{u} y_0$ on T and $\dot{y}_q \xrightarrow{L^1} \dot{y}_0$ on T.*

Lemma 2. *Let $\Theta \subset T$ be measurable, $\mathcal{R}_\lambda \in L^\infty(\Theta; \mathbf{R}^{n\times n})$ and $(\mathcal{R}_q)$ a sequence in $L^\infty(\Theta; \mathbf{R}^{n\times n})$. If $\dot{x}_q \xrightarrow{u} \dot{x}_0$ on Θ, $\mathcal{R}_q \xrightarrow{u} \mathcal{R}_\lambda$ on Θ and $\mathcal{R}_\lambda(t) \geq 0$ $(t \in \Theta)$, then*

$$\liminf_{q\to\infty} \int_\Theta \dot{y}_q^*(t)\mathcal{R}_q(t)\dot{y}_q(t)dt \geq \int_\Theta \dot{y}_0^*(t)\mathcal{R}_\lambda(t)\dot{y}_0(t)dt.$$

5. Proof of Theorem 1

Proof. The proof of Theorem 1 will be made by contradiction, that is, we are going to assume that, for all $\nu_1, \nu_2 > 0$, there exists an admissible trajectory x such that

$$\|x - x_0\| < \nu_1 \quad \text{and} \quad I(x) < I(x_0) + \nu_2 D(x - x_0). \tag{1}$$

We recall also that $\mathcal{I}_a(\cdot, x_0(\cdot), \dot{x}_0(\cdot))$ is piecewise constant on T, (x_0, ρ, μ) satisfies the first order sufficiency conditions

$$\dot{\rho}(t) = -H_x^*(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) \text{ (a.e. in } T),$$

$$H_{\dot{x}}^*(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) = 0 \text{ (a.e. in } T),$$

and hypotheses (i), (ii), (iii) and (v) of Theorem 1. We are going to obtain the negation of hypothesis (iv) of Theorem 1.

First note that, as

$$\mu_\alpha(t) \geq 0 \ (\alpha \in R, \text{ a.e. in } T) \quad \text{and} \quad \lambda_i \geq 0 \ (i = 1, \ldots, k),$$

if x is feasible, then $I(x) \geq J_\lambda(x)$. Furthermore, as

$$\mu_\alpha(t)\varphi_\alpha(t, x_0(t), \dot{x}_0(t)) = 0 \ (\alpha \in R, \text{ a.e. in } T) \quad \text{and} \quad \lambda_i I_i(x_0) = 0 \ (i = 1, \ldots, k),$$

then $I(x_0) = J_\lambda(x_0)$. Consequently, (1) implies that, for all $\nu_1, \nu_2 > 0$, there exists x admissible with

$$\|x - x_0\| < \nu_1 \quad \text{and} \quad J_\lambda(x) < J_\lambda(x_0) + \nu_2 D(x - x_0). \tag{2}$$

Observe that by setting

$$[\theta] := (x_0(t_0) + \theta[x(t_0) - x_0(t_0)], x_0(t_1) + \theta[x(t_1) - x_0(t_1)]),$$

for all admissible trajectories x,

$$\begin{aligned} & J_\lambda(x) - \int_0^1 (1-\theta)(x^*(t_0) - x_0^*(t_0), x^*(t_1) - x_0^*(t_1)) l_\lambda''[\theta] \begin{pmatrix} x(t_0) - x_0(t_0) \\ x(t_1) - x_0(t_1) \end{pmatrix} d\theta \\ = \; & \rho^*(t_1)[x(t_1) - x_0(t_1)] - \rho^*(t_0)[x(t_0) - x_0(t_0)] + J_\lambda(x_0) + J_\lambda'(x_0, x - x_0) + \mathcal{K}_\lambda(x) + \mathcal{E}_\lambda(x) \end{aligned} \tag{3}$$

where

$$\mathcal{E}_\lambda(x) := \int_{t_0}^{t_1} E_\lambda(t, x(t), \dot{x}_0(t), \dot{x}(t)) dt,$$

$$\mathcal{K}_\lambda(x) := \int_{t_0}^{t_1} \{M_\lambda(t, x(t)) + [\dot{x}^*(t) - \dot{x}_0^*(t)] N_\lambda(t, x(t))\} dt,$$

and the functions M_λ and N_λ are given by

$$M_\lambda(t, x) := F_\lambda(t, x, \dot{x}_0(t)) - F_\lambda(t, x_0(t), \dot{x}_0(t)) - F_{\lambda x}(t, x_0(t), \dot{x}_0(t))(x - x_0(t)),$$

$$N_\lambda(t, x) := F_{\lambda \dot{x}}^*(t, x, \dot{x}_0(t)) - F_{\lambda \dot{x}}^*(t, x_0(t), \dot{x}_0(t)).$$

Note that

$$M_\lambda(t, x) = \tfrac{1}{2}[x^* - x_0^*(t)] P_\lambda(t, x)(x - x_0(t)), \quad N_\lambda(t, x) = Q_\lambda(t, x)(x - x_0(t)),$$

where

$$P_\lambda(t, x) := 2\int_0^1 (1-\theta) F_{\lambda xx}(t, x_0(t) + \theta[x - x_0(t)], \dot{x}_0(t)) d\theta,$$

$$Q_\lambda(t, x) := \int_0^1 F_{\lambda \dot{x} x}(t, x_0(t) + \theta[x - x_0(t)], \dot{x}_0(t)) d\theta.$$

Now, we claim that there exists $\eta > 0$ such that, for all x admissible with $\|x - x_0\| < 1$,

$$|\mathcal{K}_\lambda(x)| \leq \eta \|x - x_0\| [1 + D(x - x_0)]. \tag{4}$$

Indeed, observe that if x is admissible with $\|x - x_0\| < 1$, then for some α_i $(i = 1, 2)$ and almost all $t \in T$, we have that

$$\begin{aligned}
& |M_\lambda(t,x(t)) + [\dot{x}^*(t) - \dot{x}_0^*(t)]N_\lambda(t,x(t))| \\
= \; & |\tfrac{1}{2}[x^*(t) - x_0^*(t)]P_\lambda(t,x(t))(x(t)-x_0(t)) + [\dot{x}^*(t) - \dot{x}_0^*(t)]Q_\lambda(t,x(t))(x(t)-x_0(t))| \\
= \; & |\{\tfrac{1}{2}[x^*(t) - x_0^*(t)]P_\lambda(t,x(t)) + [\dot{x}^*(t) - \dot{x}_0^*(t)]Q_\lambda(t,x(t))\}(x(t)-x_0(t))| \\
\leq \; & |\tfrac{1}{2}[x^*(t) - x_0^*(t)]P_\lambda(t,x(t)) + [\dot{x}^*(t) - \dot{x}_0^*(t)]Q_\lambda(t,x(t))||x(t)-x_0(t)| \\
\leq \; & |x(t)-x_0(t)|(|\tfrac{1}{2}[x^*(t) - x_0^*(t)]P_\lambda(t,x(t))| + |[\dot{x}^*(t) - \dot{x}_0^*(t)]Q_\lambda(t,x(t))|) \\
\leq \; & |x(t)-x_0(t)|(\tfrac{1}{2}|x(t)-x_0(t)||P_\lambda(t,x(t))| + |\dot{x}(t)-\dot{x}_0(t)||Q_\lambda(t,x(t))|) \\
\leq \; & \alpha_1|x(t)-x_0(t)|(|x(t)-x_0(t)| + |\dot{x}(t)-\dot{x}_0(t)|) \\
\leq \; & \alpha_1|x(t)-x_0(t)|(1 + |\dot{x}(t)-\dot{x}_0(t)|) \\
\leq \; & \alpha_2|x(t)-x_0(t)|(1 + |\dot{x}(t)-\dot{x}_0(t)|^2)^{1/2}.
\end{aligned}$$

Setting $\eta := \max\{\alpha_2, (t_1 - t_0)\alpha_2\}$, x admissible with $\|x - x_0\| < 1$ implies that

$$\begin{aligned}
|\mathcal{K}_\lambda(x)| & \leq \alpha_2\|x - x_0\| \int_{t_0}^{t_1} (V(\dot{x}(t) - \dot{x}_0(t)) + 1)dt \\
& \leq \alpha_2\|x - x_0\|(D(x - x_0) + t_1 - t_0) \\
& = \alpha_2\|x - x_0\|D(x - x_0) + \alpha_2\|x - x_0\|(t_1 - t_0) \\
& \leq \eta\|x - x_0\|D(x - x_0) + \eta\|x - x_0\| \\
& = \eta\|x - x_0\|[1 + D(x - x_0)]
\end{aligned}$$

and then (4) is proved.

Now, by (2), for all $q \in \mathbf{N}$ there exists x_q admissible such that

$$\|x_q - x_0\| < \epsilon, \quad \|x_q - x_0\| < \frac{1}{q}, \quad J_\lambda(x_q) - J_\lambda(x_0) < \frac{1}{q}D(x_q - x_0). \tag{5}$$

The last inequality of (5) implies that for all $q \in \mathbf{N}$,

$$d_q := [2D(x_q - x_0)]^{1/2} > 0.$$

Since

$$\dot{\rho}(t) = -H_x^*(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) \text{ (a.e. in } T),$$
$$H_{\dot{x}}^*(t, x_0(t), \dot{x}_0(t), \rho(t), \mu(t)) = 0 \text{ (a.e. in } T),$$

we have that

$$J_\lambda'(x_0, y) = l_\lambda'(x_0(t_0), x_0(t_1)) \begin{pmatrix} y(t_0) \\ y(t_1) \end{pmatrix}$$

for all $y \in \mathcal{X}$. Having this in mind, by (3), (v)(b) of Theorem 1, (4) and (5),

$$\begin{aligned}
J_\lambda(x_q) - J_\lambda(x_0) = & \int_0^1 (1-\theta)(x_q^*(t_0) - x_0^*(t_0), x_q^*(t_1) - x_0^*(t_1)) l_\lambda''[\theta] \begin{pmatrix} x_q(t_0) - x_0(t_0) \\ x_q(t_1) - x_0(t_1) \end{pmatrix} d\theta \\
& + \rho^*(t_1)[x_q(t_1) - x_0(t_1)] - \rho^*(t_0)[x_q(t_0) - x_0(t_0)] + l_\lambda'(x_0(t_0), x_0(t_1)) \begin{pmatrix} x_q(t_0) - x_0(t_0) \\ x_q(t_1) - x_0(t_1) \end{pmatrix} \\
& + \mathcal{K}_\lambda(x_q) + \mathcal{E}_\lambda(x_q) \\
\geq & \int_0^1 (1-\theta)(x_q^*(t_0) - x_0^*(t_0), x_q^*(t_1) - x_0^*(t_1)) l_\lambda''[\theta] \begin{pmatrix} x_q(t_0) - x_0(t_0) \\ x_q(t_1) - x_0(t_1) \end{pmatrix} d\theta
\end{aligned}$$

$$
\begin{aligned}
&+\rho^*(t_1)[x_q(t_1) - x_0(t_1)] - \rho^*(t_0)[x_q(t_0) - x_0(t_0)] \\
&+l'_\lambda(x_0(t_0), x_0(t_1))\begin{pmatrix} x_q(t_0) - x_0(t_0) \\ x_q(t_1) - x_0(t_1) \end{pmatrix} - \eta\|x_q - x_0\| - \eta\|x_q - x_0\|D(x_q - x_0) \\
&\qquad\qquad\qquad +\delta\int_{t_0}^{t_1} V(\dot{x}_q(t) - \dot{x}_0(t))dt \\
= &\int_0^1 (1-\theta)(x_q^*(t_0) - x_0^*(t_0), x_q^*(t_1) - x_0^*(t_1)) l''_\lambda[\theta]\begin{pmatrix} x_q(t_0) - x_0(t_0) \\ x_q(t_1) - x_0(t_1) \end{pmatrix} d\theta \\
&+\rho^*(t_1)[x_q(t_1) - x_0(t_1)] - \rho^*(t_0)[x_q(t_0) - x_0(t_0)] \\
&+l'_\lambda(x_0(t_0), x_0(t_1))\begin{pmatrix} x_q(t_0) - x_0(t_0) \\ x_q(t_1) - x_0(t_1) \end{pmatrix} \\
&-\eta\|x_q - x_0\| - \eta\|x_q - x_0\|D(x_q - x_0) + \delta D(x_q - x_0) - \delta V(x_q(t_0) - x_0(t_0)).
\end{aligned}
$$

By (5), for all $q \in \mathbf{N}$,

$$
\begin{aligned}
&D(x_q - x_0)\left(\delta - \frac{\eta}{q} - \frac{1}{q}\right) < \frac{\eta}{q} \\
&+\delta V(x_q(t_0) - x_0(t_0)) - l'_\lambda(x_0(t_0), x_0(t_1))\begin{pmatrix} x_q(t_0) - x_0(t_0) \\ x_q(t_1) - x_0(t_1) \end{pmatrix} \\
&-\rho^*(t_1)[x_q(t_1) - x_0(t_1)] + \rho^*(t_0)[x_q(t_0) - x_0(t_0)] \\
&-\int_0^1 (1-\theta)(x_q^*(t_0) - x_0^*(t_0), x_q^*(t_1) - x_0^*(t_1)) l''_\lambda[\theta]\begin{pmatrix} x_q(t_0) - x_0(t_0) \\ x_q(t_1) - x_0(t_1) \end{pmatrix} d\theta.
\end{aligned}
$$

Consequently,

$$\lim_{q\to\infty} D(x_q - x_0) = 0.$$

For all $q \in \mathbf{N}$, define

$$y_q := \frac{x_q - x_0}{d_q}.$$

By Lemma 1, there exist $y_0 \in \mathcal{X}$ with $\dot{y}_0 \in L^2(T; \mathbf{R}^n)$ and some subsequence of (x_q) such that $\dot{y}_q \xrightarrow{L^1} \dot{y}_0$ on T. Once again, by Lemma 1, there exist some subsequence of (x_q) such that $y_q \xrightarrow{\text{u}} y_0$ on T.

We claim that

i. $J''_\lambda(x_0, y_0) \leq 0$, $y_0 \neq 0$.
ii. $y_0(t_{-i}) = \Phi'_{-i}(x_0(t_{i+1}))y_0(t_{i+1})$ for $i = -1, 0$.
iii. $I'_i(x_0, y_0) \leq 0$ $(i \in i_a(x_0))$, $I'_j(x_0, y_0) = 0$ $(j = k+1, \ldots, K)$.
iv. $\varphi_{\alpha x}(t, x_0(t), \dot{x}_0(t))y_0(t) + \varphi_{\alpha\dot{x}}(t, x_0(t), \dot{x}_0(t))\dot{y}_0(t) \leq 0$ (a.e. in T, $\alpha \in \mathcal{I}_a(t, x_0(t), \dot{x}_0(t))$).
v. $\varphi_{\beta x}(t, x_0(t), \dot{x}_0(t))y_0(t) + \varphi_{\beta\dot{x}}(t, x_0(t), \dot{x}_0(t))\dot{y}_0(t) = 0$ (a.e. in T, $\beta \in S$).

For all $q \in \mathbf{N}$,

$$\frac{\mathcal{K}_\lambda(x_q)}{d_q^2} = \int_{t_0}^{t_1}\left\{\frac{M_\lambda(t, x_q(t))}{d_q^2} + \dot{y}_q^*(t)\frac{N_\lambda(t, x_q(t))}{d_q}\right\}dt.$$

By Lemma 1,

$$\frac{M_\lambda(\cdot, x_q(\cdot))}{d_q^2} \xrightarrow{L^\infty} \tfrac{1}{2}y_0^*(\cdot)F_{\lambda xx}(\cdot, x_0(\cdot), \dot{x}_0(\cdot))y_0(\cdot),$$

$$\frac{N_\lambda(\cdot, x_q(\cdot))}{d_q} \xrightarrow{L^\infty} F_{\lambda\dot{x}x}(\cdot, x_0(\cdot), \dot{x}_0(\cdot))y_0(\cdot),$$

both on T and, as $\dot{y}_q \xrightarrow{L^1} \dot{y}_0$ on T,

$$\frac{1}{2}J''_\lambda(x_0,y_0) = \frac{1}{2}(y_0^*(t_0),y_0^*(t_1))l''_\lambda(x_0(t_0),x_0(t_1))\begin{pmatrix} y_0(t_0) \\ y_0(t_1) \end{pmatrix} + \lim_{q\to\infty}\frac{\mathcal{K}_\lambda(x_q)}{d_q^2} + \frac{1}{2}\int_{t_0}^{t_1}\dot{y}_0^*(t)F_{\lambda\dot{x}\dot{x}}(t,x_0(t),\dot{x}_0(t))\dot{y}_0(t)dt. \quad (6)$$

We have,

$$\liminf_{q\to\infty}\frac{\mathcal{E}_\lambda(x_q)}{d_q^2} \geq \frac{1}{2}\int_{t_0}^{t_1}\dot{y}_0^*(t)F_{\lambda\dot{x}\dot{x}}(t,x_0(t),\dot{x}_0(t))\dot{y}_0(t)dt. \quad (7)$$

Indeed, by Lemma 1, we can choose $\Theta \subset T$ measurable such that $\dot{x}_q \xrightarrow{\text{u}} \dot{x}_0$ on Θ. Additionally, for all $t \in \Theta$ and $q \in \mathbf{N}$,

$$\frac{1}{d_q^2}E_\lambda(t,x_q(t),\dot{x}_0(t),\dot{x}_q(t)) = \frac{1}{2}\dot{y}_q^*(t)\mathcal{R}_q(t)\dot{y}_q(t)$$

where

$$\mathcal{R}_q(t) := 2\int_0^1(1-\theta)F_{\lambda\dot{x}\dot{x}}(t,x_q(t),\dot{x}_0(t)+\theta[\dot{x}_q(t)-\dot{x}_0(t)])d\theta.$$

Clearly,

$$\mathcal{R}_q(\cdot) \xrightarrow{\text{u}} \mathcal{R}_\lambda(\cdot) := F_{\lambda\dot{x}\dot{x}}(\cdot,x_0(\cdot),\dot{x}_0(\cdot)) \text{ on } \Theta.$$

By hypothesis (iii) of Theorem 1, $\mathcal{R}_\lambda(t) \geq 0$ $(t \in \Theta)$. Moreover, by hypothesis (v)(a) of Theorem 1, and by Lemma 2,

$$\begin{aligned}\liminf_{q\to\infty}\frac{\mathcal{E}_\lambda(x_q)}{d_q^2} &= \liminf_{q\to\infty}\frac{1}{d_q^2}\int_{t_0}^{t_1}E_\lambda(t,x_q(t),\dot{x}_0(t),\dot{x}_q(t))dt \geq \liminf_{q\to\infty}\frac{1}{d_q^2}\int_\Theta E_\lambda(t,x_q(t),\dot{x}_0(t),\dot{x}_q(t))dt \\ &= \frac{1}{2}\liminf_{q\to\infty}\int_\Theta \dot{y}_q^*(t)\mathcal{R}_q(t)\dot{y}_q(t)dt \geq \frac{1}{2}\int_\Theta \dot{y}_0^*(t)\mathcal{R}_\lambda(t)\dot{y}_0(t)dt.\end{aligned}$$

As Θ can be selected to be different from T by a set of an arbitrarily small measure and the function $\dot{y}_0^*(\cdot)\mathcal{R}_\lambda(\cdot)\dot{y}_0(\cdot)$ is integrable on T, this inequality is verified when $\Theta = T$ and hence (7) is satisfied.

By, (3), (5), (6), (7) and hypotheses (i) and (ii) of Theorem 1, we have

$$\begin{aligned}&\frac{1}{2}J''_\lambda(x_0,y_0) \leq \frac{1}{2}(y_0^*(t_0),y_0^*(t_1))l''_\lambda(x_0(t_0),x_0(t_1))\begin{pmatrix} y_0(t_0) \\ y_0(t_1) \end{pmatrix} \\ &+ \lim_{q\to\infty}\frac{\mathcal{K}_\lambda(x_q)}{d_q^2} + \liminf_{q\to\infty}\frac{\mathcal{E}_\lambda(x_q)}{d_q^2} = \liminf_{q\to\infty}\frac{J_\lambda(x_q)-J_\lambda(x_0)}{d_q^2} \\ &- \lim_{q\to\infty}\frac{1}{d_q^2}\left\{\rho^*(t_1)[x_q(t_1)-x_0(t_1)] - \rho^*(t_0)[x_q(t_0)-x_0(t_0)] + l'_\lambda(x_0(t_0),x_0(t_1))\begin{pmatrix} x_q(t_0)-x_0(t_0) \\ x_q(t_1)-x_0(t_1) \end{pmatrix}\right\} \\ \leq\; & -\lim_{q\to\infty}\frac{1}{d_q^2}\{\rho^*(t_1)[\Phi_1(x_q(t_0)) - \Phi_1(x_0(t_0)) - \Phi'_1(x_0(t_0))(x_q(t_0)-x_0(t_0))] \\ &-\rho^*(t_0)[\Phi_0(x_q(t_1)) - \Phi_0(x_0(t_1)) - \Phi'_0(x_0(t_1))(x_q(t_1)-x_0(t_1))]\} \\ =\; & -\lim_{q\to\infty}\frac{1}{d_q^2}\left\{\rho^*(t_1)\int_0^1(1-\theta)\Phi''_1(x_0(t_0)+\theta[x_q(t_0)-x_0(t_0)];x_q(t_0)-x_0(t_0))d\theta\right. \\ &\left.-\rho^*(t_0)\int_0^1(1-\theta)\Phi''_0(x_0(t_1)+\theta[x_q(t_1)-x_0(t_1)];x_q(t_1)-x_0(t_1))d\theta\right\} \\ =\; & -\frac{1}{2}\sum_{i=-1}^{0}(-1)^{i+1}\rho^*(t_{-i})\Phi''_{-i}(x_0(t_{i+1});y_0(t_{i+1})) \leq 0.\end{aligned}$$

Now, if $y_0 = 0$, then

$$\lim_{q\to\infty} \frac{\mathcal{K}_\lambda(x_q)}{d_q^2} = 0,$$

and hence, by hypothesis (v)(b) of Theorem 1,

$$\begin{aligned} 0 \;\geq\; & \liminf_{q\to\infty} \frac{\mathcal{E}_\lambda(x_q)}{d_q^2} \geq \delta \liminf_{q\to\infty} \frac{1}{d_q^2}\int_{t_0}^{t_1} V(\dot{x}_q(t) - \dot{x}_0(t))dt = \delta \liminf_{q\to\infty}\left(\frac{D(x_q - x_0)}{d_q^2} - \frac{V(x_q(t_0) - x_0(t_0))}{d_q^2}\right) \\ = \; & \frac{\delta}{2} - \limsup_{q\to\infty} \frac{V(x_q(t_0) - x_0(t_0))}{d_q^2} \geq \frac{\delta}{2} - \frac{1}{2}\limsup_{q\to\infty} \frac{|x_q(t_0) - x_0(t_0)|^2}{d_q^2} = \frac{\delta}{2} - \frac{1}{2}|y_0(t_0)|^2 = \frac{\delta}{2} \end{aligned}$$

implying that δ cannot be positive, which is not the case and in this way we have obtained (i) of our claim.

Now, observe that since x_q is admissible, then for $i = -1, 0$ and all $q \in \mathbf{N}$, we have

$$y_q(t_{-i}) = \left(\int_0^1 \Phi'_{-i}(x_0(t_{i+1}) + \theta[x_q(t_{i+1}) - x_0(t_{i+1})])d\theta\right) y_q(t_{i+1}).$$

As $y_q \xrightarrow{\text{u}} y_0$ on T, then for $i = -1, 0$, we have

$$y_0(t_{-i}) = \Phi'_{-i}(x_0(t_{i+1}))y_0(t_{i+1})$$

and so (ii) of our claim is established.

Now, let us show that

$$I_i'(x_0, y_0) \leq 0 \quad (i \in i_a(x_0)). \tag{8}$$

Indeed, first observe that for all $\gamma = 1, \ldots, K$,

$$\begin{aligned} & I_\gamma(x) - \int_0^1 (1-\theta)(x^*(t_0) - x_0^*(t_0), x^*(t_1) - x_0^*(t_1)) l''_\gamma[\theta] \begin{pmatrix} x(t_0) - x_0(t_0) \\ x(t_1) - x_0(t_1) \end{pmatrix} d\theta \\ = \; & I_\gamma(x_0) + I'_\gamma(x_0, x - x_0) + \mathcal{K}_\gamma(x) + \mathcal{E}_\gamma(x) \end{aligned} \tag{9}$$

where

$$\mathcal{E}_\gamma(x) := \int_{t_0}^{t_1} E_\gamma(t, x(t), \dot{x}_0(t), \dot{x}(t))dt,$$

$$\mathcal{K}_\gamma(x) := \int_{t_0}^{t_1} \{M_\gamma(t, x(t)) + [\dot{x}^*(t) - \dot{x}_0^*(t)]N_\gamma(t, x(t))\}dt,$$

and the functions M_γ and N_γ are defined by

$$M_\gamma(t, x) := L_\gamma(t, x, \dot{x}_0(t)) - L_\gamma(t, x_0(t), \dot{x}_0(t)) - L_{\gamma x}(t, x_0(t), \dot{x}_0(t))(x - x_0(t)),$$

$$N_\gamma(t, x) := L^*_{\gamma\dot{x}}(t, x, \dot{x}_0(t)) - L^*_{\gamma\dot{x}}(t, x_0(t), \dot{x}_0(t)).$$

We have

$$M_\gamma(t, x) = [x^* - x_0^*(t)]P_\gamma(t, x)(x - x_0(t)), \quad N_\gamma(t, x) = Q_\gamma(t, x)(x - x_0(t)),$$

where

$$P_\gamma(t, x) := \int_0^1 (1-\theta)L_{\gamma xx}(t, x_0(t) + \theta(x - x_0(t)), \dot{x}_0(t))d\theta,$$

$$Q_\gamma(t, x) := \int_0^1 L_{\gamma\dot{x}x}(t, x_0(t) + \theta(x - x_0(t)), \dot{x}_0(t))d\theta.$$

It is clear that, for all $\gamma = 1, \dots, K$,

$$\frac{M_\gamma(\cdot, x_q(\cdot))}{d_q} = [x_q^*(\cdot) - x_0^*(\cdot)]P_\gamma(\cdot, x_q(\cdot))y_q(\cdot) \xrightarrow{L^\infty} 0,$$

$$N_\gamma(\cdot, x_q(\cdot)) = Q_\gamma(\cdot, x_q(\cdot))(x_q(\cdot) - x_0(\cdot)) \xrightarrow{L^\infty} 0,$$

all on T and, since $\dot{y}_q \stackrel{L^1}{\rightharpoonup} \dot{y}_0$ on T, then

$$\lim_{q\to\infty} \frac{\mathcal{K}_\lambda(x_q)}{d_q} = 0 \quad \text{and} \quad \lim_{q\to\infty} \frac{\mathcal{K}_\gamma(x_q)}{d_q} = 0 \quad (\gamma = 1, \dots, K). \tag{10}$$

By (5) and (10),

$$\begin{aligned} 0 \;&\geq\; \limsup_{q\to\infty} \frac{J_\lambda(x_q) - J_\lambda(x_0)}{d_q} \\ &= \lim_{q\to\infty} \frac{1}{d_q} \sum_{i=-1}^{0} (-1)^{i+1} \int_0^1 (1-\theta)\rho^*(t_{-i})\Phi''_{-i}(x_0(t_{i+1}) + \theta[x_q(t_{i+1}) - x_0(t_{i+1})]; x_q(t_{i+1}) - x_0(t_{i+1}))d\theta \\ &\quad + \limsup_{q\to\infty} \frac{\mathcal{E}_\lambda(x_q)}{d_q} = \limsup_{q\to\infty} \frac{\mathcal{E}_\lambda(x_q)}{d_q}. \end{aligned}$$

Since for all $q \in \mathbf{N}$, $\mathcal{E}_\lambda(x_q) \geq 0$, then

$$\lim_{q\to\infty} \frac{\mathcal{E}_\lambda(x_q)}{d_q} = 0.$$

Thus, by hypothesis (v)(c) of Theorem 1, for all $\gamma = 1, \dots, K$,

$$\lim_{q\to\infty} \frac{\mathcal{E}_\gamma(x_q)}{d_q} = 0. \tag{11}$$

Since for all $q \in \mathbf{N}$ and $i \in i_a(x_0)$,

$$\begin{aligned} 0 \;&\geq\; I_i(x_q) = I_i(x_q) - I_i(x_0) \\ &= \int_0^1 (1-\theta)(x_q^*(t_0) - x_0^*(t_0), x_q^*(t_1) - x_0^*(t_1)) l_i''[\theta] \begin{pmatrix} x_q(t_0) - x_0(t_0) \\ x_q(t_1) - x_0(t_1) \end{pmatrix} d\theta \\ &\quad + I_i'(x_0, x_q - x_0) + \mathcal{K}_i(x_q) + \mathcal{E}_i(x_q), \end{aligned}$$

then, by (10) and (11), for $i \in i_a(x_0)$,

$$0 \geq \lim_{q\to\infty} \frac{I_i'(x_0, x_q - x_0)}{d_q}.$$

As $y_q \xrightarrow{\text{u}} y_0$ and $\dot{y}_q \stackrel{L^1}{\rightharpoonup} \dot{y}_0$ both on T, then for $i \in i_a(x_0)$,

$$0 \geq \lim_{q\to\infty} \frac{I_i'(x_0, x_q - x_0)}{d_q} = I_i'(x_0, y_0)$$

establishing (8).

Let us prove that

$$I_j'(x_0, y_0) = 0 \quad (j = k+1, \dots, K). \tag{12}$$

Indeed, by (9), (10), (11) and the admissibility of x_q, for all $j = k+1, \ldots, K$,

$$0 = \lim_{q\to\infty} \frac{I_j'(x_0, x_q - x_0)}{d_q} = I_j'(x_0, y_0)$$

which is precisely (12), and hence we obtain (iii) of our claim.

Now, we claim that

$$\varphi_{\alpha x}(t, x_0(t), \dot{x}_0(t))y_0(t) + \varphi_{\alpha\dot{x}}(t, x_0(t), \dot{x}_0(t))\dot{y}_0(t) \leq 0 \quad \text{(a.e. in } T). \tag{13}$$

In fact, for all $\alpha \in R$, $q \in \mathbf{N}$, almost all $t \in T$ and $\theta \in [0,1]$, define

$$\Omega_q^\alpha(t;\theta) := \varphi_\alpha(t, x_0(t) + \theta[x_q(t) - x_0(t)], \dot{x}_0(t) + \theta[\dot{x}_q(t) - \dot{x}_0(t)]),$$

$$G_q^\alpha(t) := [-\varphi_\alpha(t, x_q(t), \dot{x}_q(t))]^{1/2},$$
$$O_\alpha(t) := -\varphi_{\alpha x}(t, x_0(t), \dot{x}_0(t))y_0(t) - \varphi_{\alpha\dot{x}}(t, x_0(t), \dot{x}_0(t))\dot{y}_0(t).$$

If $t \in [t_0, t_1)$ is a point of continuity of $\mathcal{I}_a(\cdot, x_0(\cdot), \dot{x}_0(\cdot))$ and $\alpha \in \mathcal{I}_a(t, x_0(t), \dot{x}_0(t))$, as $\mathcal{I}_a(\cdot, x_0(\cdot), \dot{x}_0(\cdot))$ is piecewise constant on T, we have the existence of an interval $[t, \bar{t}] \subset T$ satisfying $t < \bar{t}$ and such that $\varphi_\alpha(\sigma, x_0(\sigma), \dot{x}_0(\sigma)) = 0$ for almost all $\sigma \in [t, \bar{t}]$. Using the notation

$$\{\sigma\} := (\sigma, x_0(\sigma) + \theta[x_q(\sigma) - x_0(\sigma)], \dot{x}_0(\sigma) + \theta[\dot{x}_q(\sigma) - \dot{x}_0(\sigma)]),$$

we have

$$\begin{aligned}
0 &\leq \lim_{q\to\infty} \int_{[t,\bar{t}]\cap\Theta} \frac{(G_q^\alpha(\sigma))^2}{d_q} d\sigma \\
&= \lim_{q\to\infty} \frac{1}{d_q} \int_{[t,\bar{t}]\cap\Theta} \{-\varphi_\alpha(\sigma, x_q(\sigma), \dot{x}_q(\sigma)) + \varphi_\alpha(\sigma, x_0(\sigma), \dot{x}_0(\sigma))\} d\sigma \\
&= -\lim_{q\to\infty} \frac{1}{d_q} \int_{[t,\bar{t}]\cap\Theta} \{\Omega_q^\alpha(\sigma;1) - \Omega_q^\alpha(\sigma;0)\} d\sigma \\
&= -\lim_{q\to\infty} \frac{1}{d_q} \int_{[t,\bar{t}]\cap\Theta} \int_0^1 \frac{\partial}{\partial\theta} \Omega_q^\alpha(\sigma;\theta) d\theta d\sigma \\
&= -\lim_{q\to\infty} \frac{1}{d_q} \int_{[t,\bar{t}]\cap\Theta} \int_0^1 \{\varphi_{\alpha x}\{\sigma\}(x_q(\sigma) - x_0(\sigma)) + \varphi_{\alpha\dot{x}}\{\sigma\}(\dot{x}_q(\sigma) - \dot{x}_0(\sigma))\} d\theta d\sigma \\
&= -\lim_{q\to\infty} \int_{[t,\bar{t}]\cap\Theta} \int_0^1 \{\varphi_{\alpha x}\{\sigma\} y_q(\sigma) + \varphi_{\alpha\dot{x}}\{\sigma\}\dot{y}_q(\sigma)\} d\theta d\sigma \\
&= \int_{[t,\bar{t}]\cap\Theta} \{-\varphi_{\alpha x}(\sigma, x_0(\sigma), \dot{x}_0(\sigma))y_0(\sigma) - \varphi_{\alpha\dot{x}}(\sigma, x_0(\sigma), \dot{x}_0(\sigma))\dot{y}_0(\sigma)\} d\sigma \\
&= \int_{[t,\bar{t}]\cap\Theta} O_\alpha(\sigma) d\sigma.
\end{aligned}$$

As Θ can be chosen to be different from T by a set of an arbitrarily small measure, then

$$0 \leq \int_t^{\bar{t}} O_\alpha(\sigma) d\sigma.$$

If $O_\alpha < 0$ on a measurable set Σ such that $\Sigma \subset [t, \bar{t}]$ and $m(\Sigma) > 0$, then

$$0 > \int_{\Sigma\cap\Theta} O_\alpha(\sigma) d\sigma = \lim_{q\to\infty} \int_{\Sigma\cap\Theta} \frac{(G_q^\alpha(\sigma))^2}{d_q} d\sigma \geq 0$$

which is not the case. Consequently, $O_\alpha \geq 0$ almost everywhere on $[t, \bar{t}]$ with $t \in [t_0, t_1)$ an arbitrary point of continuity of $\mathcal{I}_a(\cdot, x_0(\cdot), \dot{x}_0(\cdot))$. Thus, $O_\alpha(t) \geq 0$ for almost all $t \in T$ showing that (13) is verified.

Now, let us prove that for all $\beta \in S$,

$$\varphi_{\beta x}(t, x_0(t), \dot{x}_0(t))y_0(t) + \varphi_{\beta \dot{x}}(t, x_0(t), \dot{x}_0(t))\dot{y}_0(t) = 0 \quad \text{(a.e. in } T). \tag{14}$$

Indeed, for all $\beta \in S$, $q \in \mathbf{N}$, almost all $t \in T$ and $\theta \in [0,1]$, set

$$\Upsilon_q^\beta(t;\theta) := \varphi_\beta(t, x_0(t) + \theta[x_q(t) - x_0(t)], \dot{x}_0(t) + \theta[\dot{x}_q(t) - \dot{x}_0(t)]).$$

For all $\beta \in S$, $q \in \mathbf{N}$ and almost all $t \in T$, we have

$$\begin{aligned} 0 &= \Upsilon_q^\beta(t;1) - \Upsilon_q^\beta(t;0) = \int_0^1 \frac{\partial}{\partial\theta}\Upsilon_q^\beta(t;\theta)d\theta \\ &= \int_0^1 [\varphi_{\beta x}\{t\}(x_q(t) - x_0(t)) + \varphi_{\beta \dot{x}}\{t\}(\dot{x}_q(t) - \dot{x}_0(t))]d\theta \end{aligned}$$

Then, for all $\beta \in S$, $q \in \mathbf{N}$ and almost all $t \in T$,

$$0 = \int_0^1 [\varphi_{\beta x}\{t\}y_q(t) + \varphi_{\beta \dot{x}}\{t\}\dot{y}_q(t)]d\theta. \tag{15}$$

By (15), for all $t \in T$ and $\beta \in S$,

$$0 = \int_{[t_0,t]\cap\Theta} \{\varphi_{\beta x}(\sigma, x_0(\sigma), \dot{x}_0(\sigma))y_0(\sigma) + \varphi_{\beta \dot{x}}(\sigma, x_0(\sigma), \dot{x}_0(\sigma))\dot{y}_0(\sigma)\}d\sigma.$$

Once again, since Θ can be chosen to be different from T by a set of an arbitrarily small measure, then for $t \in T$ and $\beta \in S$,

$$0 = \int_{t_0}^t \{\varphi_{\beta x}(\sigma, x_0(\sigma), \dot{x}_0(\sigma))y_0(\sigma) + \varphi_{\beta \dot{x}}(\sigma, x_0(\sigma), \dot{x}_0(\sigma))\dot{y}_0(\sigma)\}d\sigma$$

and hence (14) holds. Consequently, (iv) and (v) of our claim are satisfied. □

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