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Nonlinear Eigenvalue Problems for the Dirichlet $(p, 2)$-Laplacian

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Abstract: We consider a nonlinear eigenvalue problem driven by the Dirichlet $(p, 2)$-Laplacian. The parametric reaction is a Carathéodory function which exhibits $(p - 1)$-sublinear growth as $x \to +\infty$ and as $x \to 0^+$. Using variational tools and truncation and comparison techniques, we prove a bifurcation-type theorem describing the "spectrum" as $\lambda > 0$ varies. We also prove the existence of a smallest positive eigenfunction for every eigenvalue. Finally, we indicate how the result can be extended to (p, q)-equations ($q \neq 2$).

Keywords: $(p, 2)$ and (p, q)-Laplacians; nonlinear regularity; positive solutions; strong comparison principle; sublinear reaction; bifurcation-type results

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear eigenvalue problem for the Dirichlet $(p, 2)$-Laplacian

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) - \Delta u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u \geqslant 0, \ \lambda > 0, \ 2 < p. \end{cases}$$

For every $r \in (1, \infty)$ by Δ_r we denote the r-Laplacian differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2} Du) \quad \forall u \in W_0^{1,p}(\Omega)$$

(Du stands for the gradient of u). When $r = 2$, we have the usual Laplacian denoted by Δ.

In the reaction, $\lambda > 0$ is a parameter and $f(z, x)$ is a Carathéodory function. Such a function is jointly measurable. We assume that for almost all $z \in \Omega$, $f(z, \cdot)$ is $(p - 1)$-sublinear as $x \to +\infty$. We are looking for positive solutions as the parameter $\lambda > 0$ varies. Our work complements those by Gasiński and Papageorgiou [1] and Papageorgiou, Rădulescu and Repovš [2] where the reaction is $(p - 1)$-superlinear in $x \in \mathbb{R}$. Moreover, in the aforementioned works, the equation is driven by the p-Laplacian differential operator which is homogeneous, a property used by the authors in the proof of their results. In contrast, here, the $(p, 2)$-Laplace differential operator is not homogeneous.

We mention that equations driven by the sum of two differential operators of different structures (such as $(p, 2)$-equations) arise in the mathematical models of many physical processes. We refer to the survey papers of Marano and Mosconi [3], Rădulescu [4] and the references therein.

2. Mathematical Background—Hypotheses

The main spaces in the analysis of problem (P_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

By $\|\cdot\|$, we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\Omega)$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\Omega) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int}C_+ = \{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega, \ \tfrac{\partial u}{\partial n}|_{\partial\Omega} < 0\},$$

with n being the outward unit normal on $\partial\Omega$ and $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$.

We know that if $r \in (1, +\infty)$, then $W_0^{1,r}(\Omega)^* = W^{-1,r'}(\Omega)$ $(\frac{1}{r} + \frac{1}{r'} = 1)$. Let $A_r \colon W_0^{1,r}(\Omega) \longrightarrow W^{-1,r'}(\Omega)$ by the operator defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

The next proposition gathers the main properties of this operator (see Gasiński and Papageorgiou [5]).

Proposition 1. *The operator $A_r \colon W_0^{1,r}(\Omega) \longrightarrow W^{-1,r'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$, that is, A_r has the following property:*
if $u_n \longrightarrow u$ weakly in $W_0^{1,r}(\Omega)$ and $\limsup_{n\to\infty}\langle A_r(u_n), u_n - u\rangle \leqslant 0$, then $u_n \longrightarrow u$ in $W_0^{1,r}(\Omega)$.

If $r = 2$, then we write $A_2 = A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

The Dirichlet r-Laplace differential operator has a principal eigenvalue denoted by $\widehat{\lambda}_1(r)$. Therefore, if we consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_r u(z) = \widehat{\lambda}|u(z)|^{r-2}u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

then this problem has a smallest eigenvalue $\widehat{\lambda}_1(r) > 0$ which is isolated and simple. It has the following variational characterization:

$$\widehat{\lambda}_1(r) = \inf_{u \in W_0^{1,r}(\Omega), u \neq 0} \frac{\|Du\|_r^r}{\|u\|_r^r}. \tag{1}$$

For $x \in \mathbb{R}$, we define $x^\pm = \max\{\pm x, 0\}$. Then, for $u \in W_0^{1,p}(\Omega)$, we set $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ = u^-, \quad |u| = u^+ + u^-.$$

A set $S \subseteq W_0^{1,p}(\Omega)$ is said to be "downward directed", if given $u_1, u_2 \in S$, we can find $u \in S$ such that $u \leqslant u_1$, $u \leqslant u_2$.

If $u, v \colon \Omega \longrightarrow \mathbb{R}$ are measurable functions, then we write $u \prec v$ if and only if for all compact sets $K \subseteq \Omega$, we have

$$0 < c_K \leqslant v(z) - u(z) \quad \text{for a.a. } z \in K.$$

Evidently if $u, v \in C(\overline{\Omega})$ and $u(z) < v(z)$ for all $z \in \Omega$, then $u \prec v$.

Now, we introduce the hypotheses on the reaction $f(z, x)$.

H: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x) > 0$ for all $x > 0$ and

(i) For every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)$ such that

$$f(z,x) \leqslant a_\varrho(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \varrho;$$

(ii) $\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(iii) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(iv) for every $\varrho > 0$, there exists s$\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$, the function $x \longmapsto f(z,x) + \widehat{\xi}_\varrho x^{p-1}$ is nondecreasing on $[0,\varrho]$.

Remark 1. *Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we may assume that*

$$f(z,x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \leqslant 0. \tag{2}$$

Hypothesis $H(ii)$ implies that $f(z,\cdot)$ is $(p-1)$-sublinear as $x \to +\infty$ while hypothesis $H(iii)$ says that $f(z,\cdot)$ is sublinear near 0^+. Hypothesis $H(iv)$ is essentially a one-sided local Lipschitz condition.

3. Positive Solutions

We introduce the following two sets:

$$\begin{aligned} \mathcal{L} &= \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ admits a positive solution}\}; \\ S_\lambda &= \text{the set of positive solutions for problem } (P_\lambda). \end{aligned}$$

We also set

$$\lambda_* = \inf \mathcal{L}.$$

First, we establish the existence of admissible parameters (eigenvalues) and determine the regularity properties of the corresponding solutions (eigenfunctions).

Proposition 2. *If hypotheses H hold, then $\mathcal{L} \neq \emptyset$ and $S_\lambda \subseteq \operatorname{int} C_+$ for all $\lambda > 0$.*

Proof. For every $\lambda > 0$, let $\varphi_\lambda\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u^+)\,dz \quad \forall u \in W_0^{1,p}(\Omega),$$

with $F(z,x) = \int_0^x f(z,s)\,ds$. From hypotheses $H(i),(ii)$, we see that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$0 \leqslant F(z,x) \leqslant \frac{\varepsilon}{p}x^p + c_\varepsilon \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{3}$$

For $u \in W_0^{1,p}(\Omega)$, using (3) we have

$$\varphi_\lambda(u) \geqslant \frac{1}{p}\left(\|Du\|_p^p - \lambda\varepsilon\|u\|_p^p\right) + \frac{1}{2}\|Du\|_p^p - \lambda c_\varepsilon|\Omega|_N,$$

with $|\cdot|_N$ being the Lebesgue measure on $\mathbb{R}^N$. Using (1) with $r = p$, we obtain

$$\varphi_\lambda(u) \geqslant \frac{1}{p}\left(1 - \frac{\lambda\varepsilon}{\widehat{\lambda}_p(p)}\right)\|Du\|_p^p - \lambda c_\varepsilon|\Omega|_N.$$

Choosing $\varepsilon \in \left(0, \frac{\widehat{\lambda}_1(p)}{\lambda}\right)$, we infer that

$$\varphi_\lambda(u) \geqslant c_1\|u\|^p - \lambda c_\varepsilon |\Omega|_N,$$

for some $c_1 > 0$ and thus φ_λ is coercive.

Additionally, using the Sobolev imbedding theorem, we see that φ_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_0) = \min_{u \in W_0^{1,p}(\Omega)} \varphi_\lambda(u). \tag{4}$$

On account of the strict positivity of $f(z,\cdot)$, if $\overline{u} \in \mathrm{int}C_+$, then

$$\int_\Omega F(z,\overline{u})\,dz > 0. \tag{5}$$

Then, we have

$$\begin{aligned} \varphi_\lambda(\overline{u}) &= \frac{1}{p}\|D\overline{u}\|_p^p + \frac{1}{2}\|D\overline{u}\|_2^2 - \lambda\int_\Omega F(z,\overline{u})\,dz \\ &= c_2 - \lambda\int_\Omega F(z,\overline{u})\,dz, \end{aligned}$$

with $c_2 = c_2(\overline{u}) > 0$. From (5) and by choosing $\lambda > 0$ big, we have

$$\varphi_\lambda(\overline{u}) < 0,$$

so

$$\varphi_\lambda(u_0) < 0 = \varphi_\lambda(0)$$

(see (4)) and thus

$$u_0 \neq 0.$$

From (4), we have

$$\varphi'_\lambda(u_0) = 0,$$

so

$$\langle A_p(u_0), h\rangle + \langle A(u_0), h\rangle = \lambda\int_\Omega f(z,u_0^+)h\,dz \quad \forall h \in W_0^{1,p}(\Omega). \tag{6}$$

In (6), we choose $h = -u_0^- \in W_0^{1,p}(\Omega)$. We obtain

$$\|Du_0^-\|_p \leqslant 0,$$

thus $u_0 \geqslant 0$ and $u_0 \neq 0$.

Then, from (6), we have

$$\begin{cases} -\Delta_p u_0(z) - \Delta u_0(z) = \lambda f(z, u_0(z)) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0, \end{cases} \tag{7}$$

for $\lambda > 0$ big and so $\mathcal{L} \neq \emptyset$.

From Theorem 7.1 of Ladyzhenskaya and Ural'tseva [6], we have that $u_0 \in L^\infty(\Omega)$. Then, the nonlinear regularity theory of Lieberman [7] implies that $u_0 \in C_+ \setminus \{0\}$. Let $\varrho = \|u_0\|_\infty$ and let $\widehat{\xi}_\varrho > 0$ be as postulated by hypothesis $H(iv)$. From (7), we have

$$-\Delta_p u_0(z) - \Delta u_0(z) + \lambda\widehat{\xi}_\varrho u_0(z)^{p-1} \geqslant 0 \quad \text{in } \Omega,$$

so

$$\Delta_p u_0(z) + \Delta u_0(z) \leqslant \lambda \widehat{\xi}_\varrho u_0(z)^{p-1} \quad \text{in } \Omega,$$

and thus $u_0 \in \text{int}C_+$ (see Pucci and Serrin [8] (pp. 111, 120)). Therefore, we conclude that $S_\lambda \subseteq \text{int}C_+$ for all $\lambda > 0$. □

Next, we show that $\mathcal{L}$ is connected (more precisely, an upper half-line).

Proposition 3. *If hypotheses H hold, $\lambda \in \mathcal{L}$ and $\vartheta > \lambda$, then $\vartheta \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S_\lambda \in \text{int}C_+$ (see Proposition 2). We introduce the Carathéodory function $k(z,x)$ defined by

$$k(z,x) = \begin{cases} f(z, u_\lambda(z)) & \text{if} \quad x \leqslant u_\lambda(z) \\ f(z,x) & \text{if} \quad u_\lambda(z) < x. \end{cases} \tag{8}$$

We set

$$K(z,x) = \int_0^x k(z,s)\, ds$$

and consider the C^1-functional $\psi_\vartheta \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\psi_\vartheta(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \vartheta K(z,u)\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Note that (8) and hypotheses $H(i),(ii)$ imply that, given $\varepsilon > 0$, we can find $\widehat{c}_\varepsilon > 0$ such that

$$K(z,x) \leqslant \frac{\varepsilon}{p} x^p + \widehat{c}_\varepsilon \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{9}$$

Using (9) and choosing $\varepsilon > 0$ small, as in the proof of Proposition 2, we show that ψ_ϑ is coercive. In addition, it is sequentially weakly lower semicontinuous. Therefore, we can find $u_\vartheta \in W_0^{1,p}(\Omega)$ such that

$$\psi_\vartheta(u_\vartheta) = \min_{u \in W_0^{1,p}(\Omega)} \psi_\vartheta(u),$$

so $\psi'_\vartheta(u_\vartheta) = 0$ and thus

$$\langle A_p(u_\vartheta), h\rangle + \langle A(u_\vartheta), h\rangle = \int_\Omega \vartheta k(z, u_\vartheta) h\, dz \quad \forall h \in W_0^{1,p}(\Omega). \tag{10}$$

In (10), we choose $h = (u_\lambda - u_\vartheta)^+ \in W_0^{1,p}(\Omega)$. Then, using (8), we have

$$\begin{aligned} & \langle A_p(u_\vartheta), (u_\lambda - u_\vartheta)^+\rangle + \langle A(u_\vartheta), (u_\lambda - u_\vartheta)^+\rangle \\ = & \int_\Omega \vartheta f(z, u_\lambda)(u_\lambda - u_\vartheta)^+\, dz \\ \geqslant & \int_\Omega \lambda f(z, u_\lambda)(u_\lambda - u_\vartheta)^+\, dz \\ = & \langle A_p(u_\lambda), (u_\lambda - u_\vartheta)^+\rangle + \langle A(u_\lambda), (u_\lambda - u_\vartheta)^+\rangle \end{aligned}$$

since $f \geqslant 0$ and $u_\lambda \in S_\lambda$. Thus,

$$u_\lambda \leqslant u_\vartheta \tag{11}$$

(see Proposition 1).

From (8), (10) and (11), we infer that

$$\begin{cases} -\Delta_p u_\vartheta(z) - \Delta u_\vartheta(z) = \vartheta f(z, u_\vartheta(z)) & \text{in } \Omega, \\ u_\vartheta|_{\partial\Omega} = 0, \end{cases}$$

so $u_\vartheta \in S_\vartheta \subseteq C_+$ and thus $\vartheta \in \mathcal{L}$. □

A byproduct of the above proof is the following corollary.

Corollary 1. *If hypotheses H hold, $\lambda \in \mathcal{L}$ and $u_\lambda \in S_\lambda \subseteq \mathrm{int}C_+$ and $\vartheta > \lambda$, then $\vartheta \in \mathcal{L}$ and we can find $u_\vartheta \in S_\vartheta \subseteq \mathrm{int}C_+$ such that $u_\lambda \leqslant u_\vartheta$.*

We can improve this corollary using the strong comparison principle of Gasiński and Papageorgiou [1] (Proposition 3.2).

Proposition 4. *If hypotheses H hold, $\lambda \in \mathcal{L}$ and $u_\lambda \in S_\lambda \subseteq \mathrm{int}C_+$ and $\vartheta > \lambda$, then $\vartheta \in \mathcal{L}$ and we can find $u_\vartheta \in S_\vartheta \subseteq \mathrm{int}C_+$ such that $u_\vartheta - u_\lambda \in \mathrm{int}C_+$.*

Proof. From Corollary 1, we already know that $\vartheta \in \mathcal{L}$ and there exists $u_\vartheta \in S_\vartheta \subseteq \mathrm{int}C_+$ such that

$$u_\lambda \leqslant u_\vartheta, \quad u_\lambda \neq u_\vartheta. \tag{12}$$

Consider the function $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ defined by

$$a(y) = |y|^{p-2}y + y \quad \forall y \in \mathbb{R}^N.$$

Evidently, $a \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ (recall that $2 < p$) and we have

$$\nabla a(y) = |y|^{p-2}\left(\mathrm{id} + (p-2)\frac{y \otimes y}{|y|^2}\right) + \mathrm{id} \quad \forall y \neq 0,$$

so

$$(\nabla a(y), \xi, \xi)_{\mathbb{R}^N} \geqslant |\xi|^2 \quad \forall y, \xi \in \mathbb{R}^N.$$

Then, the tangency principle of Pucci and Serrin [8] (Theorem 2.5.2, p. 35) implies that

$$u_\lambda(z) < u_\vartheta(z) \quad \forall z \in \Omega \tag{13}$$

(see (12)). Let $\varrho = \|u_\vartheta\|_\infty$ and let $\widehat{\xi}_\varrho > 0$ be as postulated by hypothesis $H(iv)$. We pick $\widetilde{\xi}_\varrho > \widehat{\xi}_\varrho$ and using (12), hypothesis $H(iv)$ and the facts that $f \geqslant 0$ and $u_\lambda \leqslant u_\vartheta$, we have

$$\begin{aligned}
& -\Delta_p u_\vartheta - \Delta u_\vartheta + \vartheta\widetilde{\xi}_\varrho u_\vartheta^{p-1} \\
= \; & \vartheta(f(z,u_\vartheta) + \widehat{\xi}_\varrho u_\vartheta^{p-1}) + \vartheta(\widetilde{\xi}_\varrho - \widehat{\xi}_\varrho)u_\vartheta^{p-1} \\
\geqslant \; & \vartheta(f(z,u_\lambda) + \widehat{\xi}_\varrho u_\lambda^{p-1}) + \vartheta(\widetilde{\xi}_\varrho - \widehat{\xi}_\varrho)u_\vartheta^{p-1} \\
\geqslant \; & \lambda f(z,u_\lambda) + \vartheta\widetilde{\xi}_\varrho u_\lambda^{p-1} \\
= \; & -\Delta_p u_\lambda - \Delta u_\lambda + \vartheta\widetilde{\xi}_\varrho u_\lambda^{p-1} \quad \text{in } \Omega.
\end{aligned} \tag{14}$$

Note that on account of (13), we have

$$0 \prec \vartheta(\widetilde{\xi}_\varrho - \widehat{\xi}_\varrho)(u_\vartheta^{p-1} - u_\lambda^{p-1}). \tag{15}$$

Then, (14), (15) and Proposition 3.2 of Gasiński and Papageorgiou [1] imply that $u_\vartheta - u_\lambda \in \mathrm{int}C_+$. □

Proposition 5. *If hypotheses H hold, then $\lambda_* > 0$.*

Proof. We argue by contradiction. Suppose that $\lambda_* = 0$. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ be such that $\lambda_n \to 0^+$ and consider $u_n = u_{\lambda_n} \subseteq \mathrm{int}C_+$ for all $n \in \mathbb{N}$. We have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = \int_\Omega \lambda_n f(z,u_n)h\,dz \quad \forall h \in W_0^{1,p}(\Omega),\ n \in \mathbb{N}. \tag{16}$$

On account of hypotheses $H(i), (ii)$, given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$0 \leqslant f(z, u_n(z)) \leqslant \varepsilon u_n(z)^{p-1} + c_\varepsilon \quad \text{for a.a. } z \in \Omega,\ n \in \mathbb{N}. \tag{17}$$

In (16), first, we choose $h = u_n \in W_0^{1,p}(\Omega)$ and then on the right hand side we use (17). We obtain

$$\|Du_n\|_p^p \leqslant \varepsilon \|u_n\|_p^p + c_3\|u_n\| \quad \forall n \in \mathbb{N},$$

for some $c_3 = c_3(\varepsilon) > 0$, so

$$\left(1 - \frac{\varepsilon}{\widehat{\lambda}_1(p)}\right)\|u_n\|^{p-1} \leqslant c_3 \quad \forall n \in \mathbb{N}$$

(see (1) with $r = p$). Choosing $\varepsilon \in (0, \widehat{\lambda}_1(p))$, we see that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. We may assume that

$$u_n \longrightarrow u_* \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u_* \quad \text{in } L^p(\Omega). \tag{18}$$

In (16), we choose $h = u_n - u_* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (18). We obtain

$$\lim_{n\to+\infty} \left(\langle A_p(u_n), u_n - u_*\rangle + \langle A(u_n), u_n - u_*\rangle\right) = 0,$$

so, using the monotonicity of A, we obtain

$$\limsup_{n\to+\infty} \left(\langle A_p(u_n), u_n - u_*\rangle + \langle A(u), u_n - u_*\rangle\right) = 0,$$

thus

$$\limsup_{n\to+\infty} \left(\langle A_p(u_n), u_n - u_*\rangle\right) \leqslant 0$$

and hence

$$u_n \longrightarrow u_* \quad \text{in } W_0^{1,p}(\Omega) \tag{19}$$

(see Proposition 1). Hypotheses $H(i), (ii), (iii)$ imply that given $\varepsilon > 0$, we can find $c_4 = c_4(\varepsilon) > 0$ such that

$$0 \leqslant f(z, x) \leqslant \varepsilon x + c_4 x^{p-1} \quad \text{for a.a. } z \in \Omega,\ x \geqslant 0, \tag{20}$$

so

$$0 \leqslant f(z, u_n(z)) \leqslant \varepsilon u_n(z) + c_4 u_n(z)^{p-1} \quad \text{for a.a. } z \in \Omega,\ n \in \mathbb{N},$$

thus the sequence $\{f(\cdot, u_n(\cdot)) \subseteq L^{p'}(\Omega)$ is bounded (see (19) and recall that $p' < 2 < p$). Therefore, if in (16) we pass to the limit as $n \to +\infty$, we obtain

$$\langle A_p(u_*), h\rangle + \langle A(u_*), h\rangle = 0 \quad \forall h \in W_0^{1,p}(\Omega).$$

Choosing $h = u_* \in W_0^{1,p}(\Omega)$, we obtain

$$\|Du_*\|_p \leqslant 0,$$

so

$$u_* = 0. \tag{21}$$

From (19) and the nonlinear regularity theory of Lieberman [7], we know that there exist $\alpha \in (0,1)$ and $c_5 > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_5 \quad \forall n \in \mathbb{N}. \tag{22}$$

Since the embedding $C_0^{1,\alpha}(\overline{\Omega}) \subseteq C_0^1(\overline{\Omega})$ is compact, from (19), (21) and (22), we infer that

$$u_n \longrightarrow 0 \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty. \tag{23}$$

Let $y_n = \frac{u_n}{\|u_n\|_{1,2}}$, for $n \in \mathbb{N}$, with $\|\cdot\|_{1,2}$ denoting the norm of $H_0^1(\Omega)$. We have

$$\|y_n\|_{1,2} = 0, \quad y_n \geqslant 0 \quad \forall n \in \mathbb{N}.$$

We may assume that

$$y_n \longrightarrow y \quad \text{weakly in } H_0^1(\Omega), \quad y_n \longrightarrow y \quad \text{in } L^2(\Omega), \quad y \geqslant 0. \tag{24}$$

From (16), we have

$$\|u_n\|_{1,2}^{p-2}\langle A_p(y_n), h\rangle + \langle A(y_n), h\rangle = \lambda_n \int_\Omega \frac{f(z,u_n)}{\|u_n\|_{1,2}} h\, dz \quad \forall h \in W_0^{1,p}(\Omega). \tag{25}$$

On account of (20), we have

$$0 \leqslant \frac{f(z,u_n(z))}{\|u_n\|_{1,2}} \leqslant \varepsilon y_n(z) + u_n(z)^{p-2} y_n(z) \leqslant c_6 y_n(z) \quad \text{for a.a. } z \in \Omega,\ n \in \mathbb{N},$$

for some $c_6 > 0$ and thus

$$\text{the sequence } \left\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|}\right\}_{n\in\mathbb{N}} \subseteq L^p(\Omega) \text{ is bounded} \tag{26}$$

(recall that, if $2 < p$, then $p' < 2$). Therefore, if in (25) we pass to the limit as $n \to +\infty$ and use (23), (24) and (26), we obtain

$$\langle A(y), h\rangle \leqslant 0 \quad \forall h \in W_0^{1,p}(\Omega),$$

so $y = 0$ and hence $\|Dy_n\|_2 \longrightarrow 0$ and $n \to +\infty$ (see (25)), a contradiction since $\|y_n\|_{1,2} = 1$ for all $n \in \mathbb{N}$. Therefore, we conclude that $\lambda_* > 0$. □

Next, we prove a multiplicity result when $\lambda > \lambda_*$.

Proposition 6. *If hypotheses H hold and $\lambda > \lambda_*$, then problem (P_λ) has at least two positive solutions*

$$u_0, \widehat{u} \in \mathrm{int}C_+, \quad u_0 \neq \widehat{u}.$$

Proof. Let $\mu \in (\lambda_*, \lambda)$. We have $\mu, \lambda \in \mathcal{L}$ and then, according to Proposition 4, we can find $u_0 \in S_\lambda \subseteq \mathrm{int}C_+$ and $u_\mu \in S_\mu \subseteq \mathrm{int}C_+$ such that

$$u_0 - u_\mu \in \mathrm{int}C_+. \tag{27}$$

We truncate $f(z,\cdot)$ from below at $u_\mu(z)$ and introduce the Carathéodory function $e(z,x)$ defined by

$$e(z,x) = \begin{cases} f(z,u_\mu(z)) & \text{if} \quad x \leqslant u_\mu(z), \\ f(z,x) & \text{if} \quad u_\mu(z) < x. \end{cases} \tag{28}$$

We set

$$E(z,x) = \int_0^x e(z,s)\, ds$$

and consider the C^1-functional $\widehat{\varphi}_\lambda : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \lambda E(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Let

$$[u_\mu) = \{u \in W_0^{1,p}(\Omega) : \ u_\mu(z) \leqslant u(z) \text{ for a.a. } z \in \Omega\}.$$

Then, from (28), we see that

$$\widehat{\varphi}_\lambda|_{[u_\mu)} = \varphi_\lambda|_{[u_\mu)} + \xi, \tag{29}$$

with $\xi \in \mathbb{R}$. From the proof of Proposition 2, we know that φ_λ is coercive. Hence φ_λ is coercive. Additionally, φ_λ is sequentially weakly lower semicontinuous. Therefore, we can find $\widehat{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\lambda(\widehat{u}_0) = \min_{u \in W_0^{1,p}(\Omega)} \widehat{\varphi}_\lambda(u), \tag{30}$$

so

$$\widehat{\varphi}'_\lambda(\widehat{u}_0) = 0,$$

and hence

$$\langle A_p(\widehat{u}_0), h\rangle + \langle A(\widehat{u}_0), h\rangle = \int_\Omega \lambda e(z,\widehat{u}_0)h\,dz \quad \forall h \in W_0^{1,p}(\Omega). \tag{31}$$

Choose $h \in (u_\mu - \widehat{u}_0)^+ \in W_0^{1,p}(\Omega)$. Using (28), we have

$$\begin{aligned} & \langle A_p(\widehat{u}_0), (u_\mu - \widehat{u}_0)^+\rangle + \langle A(\widehat{u}_0), (u_\mu - \widehat{u}_0)^+\rangle \\ = & \int_\Omega \lambda f(z,u_\mu)(u_\mu - \widehat{u}_0)^+\,dz \\ \geqslant & \int_\Omega \mu f(z,u_\mu)(u_\mu - \widehat{u}_0)^+\,dz \\ = & \langle A_p(u_\mu), (u_\mu - \widehat{u}_0)^+\rangle + \langle A(u_\mu), (u_\mu - \widehat{u}_0)^+\rangle \end{aligned}$$

(since $f \geqslant 0$, $\mu < \lambda$ and $u_\mu \in S_\mu$), so

$$u_\mu \leqslant \widehat{u}_0$$

(see Proposition 1).

Then, from (28) and (31), we infer that $\widehat{u}_0 \in S_\lambda \subseteq \text{int}C_+$.

If $\widehat{u}_0 \neq u_0$, then this is the second positive solution of (P_λ). Therefore, we assume that

$$\widehat{u}_0 = u_0.$$

From (27), (29) and (30), it follows that

$$u_0 \in \text{int}C_+ \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda$$

and so

$$u_0 \in \text{int}C_+ \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \tag{32}$$

(see Gasiński and Papageorgiou [9]).

Hypothesis $H(iii)$ implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\varepsilon}{2}x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leqslant \delta \tag{33}$$

(see (2)). Let $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta$. We have

$$\varphi_\lambda(u) \geqslant \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{\lambda\varepsilon}{2}\|u\|_2^2$$

$$\geqslant \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\left(1 - \frac{\lambda\varepsilon}{\widehat{\lambda}_1(2)}\right)\|Du\|_2^2$$

(see (1) with $r = 2$). Choosing $\varepsilon \in (0, \frac{\widehat{\lambda}_1(2)}{\lambda})$, we obtain

$$\varphi_\lambda(u) \geqslant \frac{1}{p}\|u\|^p \quad \forall u \in C_0^1(\overline{\Omega}),\ \|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta,$$

so

$$u = 0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda$$

and thus

$$u = 0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \tag{34}$$

(see Gasiński and Papageorgiou [9]).

We assume that $\varphi_\lambda(0) = 0 \leqslant \varphi_\lambda(u_0)$. The reasoning is similar if the opposite inequality holds, using (34) instead of (32).

We also assume that

$$K_{\varphi_\lambda} = \{u \in W_0^{1,p}(\Omega) :\ \varphi'_\lambda(u) = 0\}$$

(the critical set of φ_λ) is finite. Otherwise, we already have an infinity of distinct positive solutions of (P_λ). On account of (32) and using Theorem 5.7.6 of Papageorgiou, Rădulescu and Repovš [2] (p. 449), we can find $\varrho \in (0,1)$ small such that

$$\varphi_\lambda(0) = 0 \leqslant \varphi_\lambda(u_0) < \inf_{\|u-u_0\|=\varrho} \varphi_\lambda(u) = m_\lambda,\ 0 < \varphi < \|u_0\|. \tag{35}$$

Recall that φ_λ is coercive (see the proof of Proposition 2). Therefore, from Proposition 5.1.15 of Papageorgiou, Rădulescu and Repovš [2] (p. 449), we have that

$$\varphi_\lambda \text{ satisfies the PS-condition.} \tag{36}$$

Then, (35) and (36) permit the use of the mountain pass theorem. Therefore, we can find
$\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\varphi'_\lambda(\widehat{u}) = 0 \quad \text{and} \quad m_\lambda \leqslant \varphi_\lambda(\widehat{u}). \tag{37}$$

From (35) and (37), we conclude that

$$\widehat{u} \in S_\lambda \subseteq \text{int}C_+ \quad \text{and} \quad \widehat{u} \neq u_0.$$

□

It remains to be decided what we can say for the critical parameter value λ_*. We show that $\lambda_* > 0$ is admissible too.

Proposition 7. *If hypotheses H hold, then $\lambda_* \in \mathcal{L}$.*

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ be such that $\lambda_n \longrightarrow \lambda_*^+$. We can find $u_n \in S_{\lambda_n} \subseteq \text{int}C_+$ such that

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = \lambda_n \int_\Omega f(z, u_n) h\, dz \quad \forall h \in W_0^{1,p}(\Omega),\ n \in \mathbb{N}. \tag{38}$$

In (38), we use $h = u_n \in W_0^{1,p}(\Omega)$. Then,

$$\|u_n\|^p \leqslant \lambda_1 \int_\Omega f(z, u_n) u_n\, dz \quad \forall n \in \mathbb{N}. \tag{39}$$

On account of hypotheses $H(i), (ii)$, given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$0 \leqslant f(z,x)x \leqslant \varepsilon x^p + c_\varepsilon \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{40}$$

We use (40) in (39) and have

$$\|u_n\|^p \leqslant \lambda_1 \frac{\varepsilon}{\widehat{\lambda}_1(p)} \|u_n\|^p + c_\varepsilon |\Omega|_N$$

(see (1) with $r = p$ and recall that $|\cdot|_N$ is the Lebesgue measure on $\mathbb{R}^N$), so

$$\left(1 - \frac{\lambda_1}{\widehat{\lambda}_1(p)} \varepsilon\right) \|u_n\|^p \leqslant c_\varepsilon |\Omega|_N \quad \forall n \in \mathbb{N}.$$

We choose $\varepsilon \in (0, \frac{\widehat{\lambda}_1(p)}{\lambda_1})$ and infer that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. Therefore, we may assume that

$$u_n \longrightarrow u_* \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u_* \quad \text{in } L^p(\Omega).$$

Then, reasoning as in the proof of Proposition 5 (see the part of the proof after (18)), we show that

$$u_n \longrightarrow u_* \quad \text{in } W_0^{1,p}(\Omega),\ u_* \neq 0.$$

Therefore, if in (38) we pass to the limit as $n \to +\infty$, then

$$\langle A_p(u_*), h\rangle + \langle A(u_*), h\rangle = \lambda_* \int_\Omega f(f, u_*) h \, dz \quad \forall h \in W_0^{1,p}(\Omega),$$

so $u_* \in S_{\lambda_*} \subseteq \text{int} C_+$ and so $\lambda_* \in \mathcal{L}$. □

We have proved that

$$\mathcal{L} = [\lambda_*, \infty).$$

Next, we show that for every $\lambda \in \mathcal{L}$, problem (P_λ) admits a smallest positive solution (minimal positive solution).

Proposition 8. *If hypotheses H hold and $\lambda \in \mathcal{L}$, then problem (P_λ) admits a smallest solution $u_\lambda^* \in S_\lambda \subseteq \text{int} C_+$ (that is, $u_\lambda^* \leqslant u$ for all $u \in S_\lambda$).*

Proof. From Proposition 7 of Papageorgiou, Rădulescu and Repovš [10], we know that S_λ is downward directed. Using Lemma 3.10 of Hu and Papageorgiou [11] (p. 178), we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda$ such that

$$\inf_{n\in\mathbb{N}} u_n = \inf S_\lambda.$$

We have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = \int_\Omega \lambda f(z, u_n) h \, dz \quad \forall h \in W_0^{1,p}(\Omega),\ n \in \mathbb{N} \tag{41}$$

and

$$0 \leqslant u_n \leqslant u_1 \quad \forall n \in \mathbb{N}. \tag{42}$$

In (41), we choose $h = u_n \in W_0^{1,p}(\Omega)$ and then use (42) and hypothesis $H(i)$ to establish that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. Therefore, we may assume that

$$u_n \longrightarrow u_\lambda^* \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u_\lambda^* \quad \text{in } L^p(\Omega). \tag{43}$$

Then, as before (see the proof of Proposition 5 after (18)), using (43) we obtain

$$u_n \longrightarrow u_\lambda^* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_\lambda^* \neq 0. \tag{44}$$

If in (41) we pass to the limit as $n \to +\infty$ and use (44), then

$$\langle A_p(u_\lambda^*), h\rangle + \langle A(u_\lambda^*), h\rangle = \int_\Omega \lambda f(z, u_\lambda^*) h \, dz \quad \forall h \in W_0^{1,p}(\Omega),$$

so $u_\lambda^* \in S_\lambda \subseteq \mathrm{int}C_+$, $u_\lambda^* = \inf S_\lambda$. $\square$

The theorem that follows summarizes our findings concerning the changes in the set of positive solutions of (P_λ) as $\lambda > 0$ moves.

Theorem 1. *If hypotheses H hold, then there exists $\lambda_* > 0$ such that*
(a) *for all $\lambda > \lambda_*$ problem (P_λ) has at least two positive solutions $u_0, \widehat{u} \in \mathrm{int}C_+$, $u_0 \neq \widehat{u}$;*
(b) *for $\lambda = \lambda_*$, problem (P_λ) has at least one positive solution $u_* \in \mathrm{int}C_+$;*
(c) *for every $\lambda \in (0, \lambda_*)$ problem (P_λ) has no positive solution;*
(d) *for every $\lambda \in \mathcal{L} = [\lambda_*, \infty)$, problem (P_λ) has a smallest positive solution $u_\lambda^* \in \mathrm{int}C_+$.*

Remark 2. From Proposition 4, we know that the minimal solution map $\widehat{k}\colon \mathcal{L} \longrightarrow C_0^1(\overline{\Omega})$ defined by $\widehat{k}(\lambda) = u_\lambda^*$ is strictly increasing in the sense that

$$\text{if } \lambda_* \leqslant \mu \leqslant \lambda, \text{ then } u_\lambda^* - u_\mu^* \in \mathrm{int}C_+.$$

It is worth mentioning that when the reaction $f(z,\cdot)$ is $(p-1)$-superlinear, then we have the "bifurcation" in $\lambda > 0$, for small values of the parameter (see [1], [2]). Here, $f(z,\cdot)$ is $(p-1)$-sublinear, and the "bifurcation" in $\lambda > 0$ occurs for large values of the parameter.

4. (p,q)-Equations

In this section, we briefly mention the situation for the more general (p,q)-equations, $q \neq 2$. We now deal with the following nonlinear Dirichlet eigenvalue problem:

$$(P_\lambda)' \qquad \begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u \geqslant 0, \ \lambda > 0, 1 < q < p. \end{cases}$$

If we strengthen the conditions on $f(z,\cdot)$, we can have a similar "bifurcation-type" result for problem $(P_\lambda)'$.

The new conditions on $f(z,x)$ are the following:

<u>H′:</u> $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, $f(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses $H'(i), (ii), (iii)$ are the same as the corresponding hypotheses $H(i), (ii), (iii)$ and (iv) for a.a. $z \in \Omega$, $f(z,\cdot)$ is strictly increasing on $\mathbb{R}^+$.

Remark 3. According to hypothesis $H'(iv)$, we have

$$0 < f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x > 0.$$

The function $f(z,x) = a(z)x^{\tau-1}$ for a.a. $z \in \Omega$, all $x \geqslant 0$ with $a \in L^\infty(\Omega)$ and $1 < \tau < q < p$ satisfies hypotheses H'.

For the (p,q)-equation ($q \neq 2$), we cannot use the tangency principle of Pucci and Serrin [8] (p. 35) (see the proof of Proposition 4). Instead, on account of the stronger condition $H'(iv)$, we can use Proposition 3.4 of Gasiński and Papageorgiou [1] (strong comparison principle) and have that $u_\vartheta - u_\lambda \in \mathrm{int}C_+$. Then, all the other results remain valid and so we can have the following bifurcation-type result for problem $(P_\lambda)'$.

Theorem 2. *If hypotheses H' hold, then there exists $\lambda'_* > 0$ such that*
(a) *for all $\lambda > \lambda'_*$, problem $(P_\lambda)'$ has at least two positive solutions $u_0, \widehat{u} \in \mathrm{int}C_+$, $u_0 \neq \widehat{u}$;*
(b) *for $\lambda = \lambda'_*$, problem $(P_\lambda)'$ has at least one positive solution $u_* \in \mathrm{int}C_+$;*
(c) *for every $\lambda \in (0, \lambda_*)'$, problem $(P_\lambda)'$ has no positive solution;*
(d) *for every $\lambda \in \mathcal{L}' = [\lambda'_*, \infty)$, problem $(P_\lambda)'$ has a smallest positive solution $u^*_\lambda \in \mathrm{int}C_+$.*

Remark 4. *The function $f(z,x)$ defined by*

$$f(z,x) = \begin{cases} a(z)\big((x^+)^{r-1} + (x^+)^{\eta-1}\big) & \text{if} \quad |x| \leqslant 1, \\ a(z)\ln(x^+) & \text{if} \quad 1 < |x|, \end{cases}$$

with $a \in L^\infty(\Omega)$, $p < r < \eta$ satisfies hypotheses H but not hypotheses H'.

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