



Article Fixed Point Results on Partial Modular Metric Space

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Abstract: In the present paper, we refine the notion of the partial modular metric defined by Hosseinzadeh and Parvaneh to eliminate the occurrence of discrepancies in the non-zero self-distance and triangular inequality. In support of this, we discuss non-trivial examples. Finally, we prove a common fixed-point theorem for four self-mappings in partial modular metric space and an application to our result; the existence of a solution for a system of Volterra integral equations is discussed.

Keywords: fixed point; partial metric space; modular space; partial modular space; weakly compatible mappings; C-class function; Volterra integral equation

MSC: 47H10; 54H25



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1. Introduction

In 1992, Matthews [1] initiated the idea of non-zero self-distance by introducing the notion of the partial metric as a part of the study of the denotational semantics of data flow programming languages in a topological model in computer sciences and also extended Banach's contraction principle [2] in such space. Subsequently, many authors have begun to report its topological properties and obtained many fixed-point theorems in this space (for more details and references, we refer to [3–8]). On the other hand, in 1950, Nakano [9] introduced the concept of the modular in connection with the theory of order spaces, which was later developed by Musielak and Orlicz [10], Khamsi [11] and Kozlowski [12] as modular function space.

In 2006, Chistyakov [13] introduced the notion of the metric modular on an arbitrary set and the corresponding modular space, which is more general than a metric space, and, based on this, he further studied Lipschitz continuity and a class of superposition (or Nemytskii) operators on modular metric space (see also [14,15]). Recently, Hosseinzadeh and Parvaneh [16] introduced the notion of partial modular metric spaces as a generalization partial metric space and gave some fixed-point results.

In this paper, we refine the concept of the partial modular metric to eliminate the occurrence of discrepancies in the non-zero self-distance and triangular inequality and prove a common fixed-point theorem for four self-mappings with a suitable example. As an application of our result, the existence of a solution for a system of Volterra integral equations is discussed.

2. Preliminaries

In this section, we recall some definitions and properties to use in our result.

Definition 1 ([1]). Let $X \neq \emptyset$. A function $p : X \times X \rightarrow [0, \infty)$ is called a partial metric on X if *it satisfies:*

 $(p_1): 0 \le p(x,y), \ \forall x, y \in X \ and \ p(x,y) = p(x,x) = p(y,y) \iff x = y;$ $(p_2): p(x,x) \le p(x,y), \ \forall x, y \in X;$ $(p_3): p(x,y) = p(y,x), \ \forall x, y \in X;$ $(p_4): p(x,y) \le p(x,z) + p(z,y) - p(z,z), \ \forall x, y, z \in X.$ Then, the pair (X, p) is called a partial metric space.

Obviously, if p(x, y) = 0, then, from (p_1) and (p_2) , we have x = y, but the converse may not be true. Moreover, if (X, p) is a partial metric space, then the function $d^p : X \times X \rightarrow [0, \infty)$ defined by

$$d^{p}(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on *X*.

Example 1 ([1]). Let $X \neq \emptyset$ and $c \ge 0$. Define p(x, y) = |x - y| + c; the *p* is a partial metric on *X* and the corresponding metric is $d_p(x, y) = 2|x - y|$, $\forall x, y \in X$.

Every partial metric p on X generates a T_0 topology τ_p on X with a base, which is defined by the family of open p-balls { $B_p(x, \epsilon) : x \in X, \epsilon > 0$ }, where $B_p(x, \epsilon) = \{u : p(x, u) < p(x, x) + \epsilon\}$, $\forall x \in X$ and $\epsilon > 0$.

Definition 2 ([13–15]). Let $X \neq \emptyset$. A function $\omega : (0, +\infty) \times X \times X \to [0, \infty)$, defined by $\omega(\lambda, x, y) = \omega_{\lambda}(x, y)$, is called a modular metric on X if it satisfies the following: $(\omega_1) : \omega_{\lambda}(x, y) = 0 \iff x = y, \forall \lambda > 0;$ $(\omega_2) : \omega_{\lambda}(x, y) = \omega_{\lambda}(y, x), \forall x, y \in X \text{ and } \forall \lambda > 0;$ $(\omega_3) : \omega_{\lambda+\mu}(x, y) \leq \omega_{\lambda}(x, z) + \omega_{\mu}(z, y) \forall x, y, z \in X \text{ and } \forall \lambda, \mu > 0.$

If in lieu of (ω_1) , we write

$$(\omega_{1'}): \omega_{\lambda}(x,x) = 0, \forall \lambda > 0,$$

and then ω is called the pseudomodular metric on *X*. Note that the function $\lambda \mapsto \omega_{\lambda} \in [0, \infty)$ is non-decreasing. Indeed, $\forall x, y \in X$ and $\forall \lambda, \mu >$ such that $0 < \mu < \lambda$; from (ω_1) and (ω_3) , we obtain

$$\omega_{\lambda}(x,y) \leq \omega_{\lambda-\mu}(x,x) + \omega_{\mu}(x,y) = \omega_{\mu}(x,y).$$

Moreover, we say that ω is convex if it satisfies the axioms (ω_1) , (ω_2) of Definition 2 and the following:

$$(\omega_4): \ \omega_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu}\omega_{\lambda}(x,z) + \frac{\mu}{\lambda+\mu}\omega_{\mu}(z,y), \ \forall x,y,z \in X \text{ and } \forall \lambda,\mu > 0.$$

Now, we define the following definition, a general form of convex modular metric on *X*.

Definition 3. A modular metric ω defined on a non-empty set X is said to be a weak convex modular if it satisfies the axioms (ω_1) , (ω_2) of Definition 2 such that there exists a function $\alpha : (0, \infty) \times (0, \infty) \to (0, 1)$ satisfying the following:

$$(\omega_{4'}): \ \omega_{\lambda+\mu}(x,y) \leq \alpha(\lambda,\mu)\omega_{\lambda}(x,z) + (1-\alpha(\lambda,\mu))\omega_{\mu}(z,y),$$

 $\forall x, y, z \in X \text{ and } \forall \lambda, \mu > 0.$

Obviously, every convex modular metric is a weak convex modular metric but the converse may not be true. Moreover, every (weak) convex modular metric is a modular

metric but the converse may not be true. In fact, by setting $\alpha = \frac{\lambda}{\lambda + \mu}$, then $0 < \alpha < 1$ and $0 < 1 - \alpha = \frac{\mu}{\lambda + \mu} < 1$, so (ω_4) and $(\omega_{4'})$ infer directly the axiom (ω_3) of Definition 2. Let $X \neq \emptyset$ be an arbitrary set. For given $x_0 \in X$, we define

$$X_{\omega}(x_0) = \{x \in X : \lim_{\lambda \to +\infty} \omega_{\lambda}(x_0, x) = 0\}$$

and

$$X^*_{\omega}(x_0) = \{ x \in X : \exists \lambda = \lambda(x) > 0, \ \omega_{\lambda}(x_0, x) < \infty \}.$$

Then, the two sets X_{ω} and X_{ω}^* are called modular spaces centered at x_0 . It is obvious that $X_{\omega} \subseteq X_{\omega}^*$. If $x_0 \in X$ is an arbitrary, then $X_{\omega}(x_0)$ and $X_{\omega}^*(x_0)$ are written as X_{ω} and X_{ω}^* . If ω is a modular metric on X, then the modular space X_{ω} is a metric space equipped with a non-trivial metric given by

$$d_{\omega}(x,y) = \inf\{\lambda : \omega_{\lambda}(x,y) \leq \lambda\}, \ \forall x,y \in X_{\omega}.$$

Further, if ω is a convex modular on X, then $X_{\omega} = X_{\omega}^*$, and this common space can be equipped with a metric d_{ω}^* defined by

$$d_{\omega}^*(x,y) = \inf\{\lambda : \omega_{\lambda}(x,y) \le 1\}, \ \forall x,y \in X_{\omega}.$$

If a modular metric ω on X is finite and $\omega_{\lambda}(x, y) = \omega_{\mu}(x, y)$, $\forall x, y \in X$ and $\forall \lambda, \mu > 0$, then $d(x, y) = \omega_{\lambda}(x, y)$ is a metric on X.

Example 2 ([17]). Let (X, d) be a metric space. Define $\omega_{\lambda}(x, y) = \frac{d(x,y)}{\lambda}$, $\forall x, y \in X$ and $\forall \lambda > 0$. Then, ω is a modular metric on X. Moreover, ω is convex and hence it is a weak convex modular metric on X.

Lemma 1 ([14]). Let ω be a modular metric on a set X, given a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_{ω} and $x \in X_{\omega}$. Then, $d_{\omega}(x_n, x) \to 0$ as $n \to \infty$ if and only if $\omega_{\lambda}(x_n, x) \to 0$ as $n \to \infty$, $\forall \lambda > 0$. A similar assertion holds for Cauchy sequences.

Example 3. Define $\omega_{\lambda}(x, y) = \frac{e^{-\lambda}|x-y|}{c}$, $c > 0 \ \forall x, y \in X$ and $\forall \lambda > 0$. Obviously, ω satisfies the axioms (ω_1) , (ω_2) and (ω_3) of Definition 2. Therefore, ω is a modular metric but not a convex modular metric on X.

In fact, $\forall \lambda, \mu > 0$, and we have

$$\begin{split} \omega_{\lambda+\mu}(x,y) &\leq \frac{e^{-(\lambda+\mu)}}{c}[|x-z|+|z-y|] \\ &= \frac{e^{\lambda}}{e^{\lambda+\mu}}\frac{e^{-\lambda}|x-z|}{c} + \frac{e^{\mu}}{e^{\lambda+\mu}}\frac{e^{-\mu}|z-y|}{c} \\ &= \frac{e^{\lambda}}{e^{\lambda+\mu}}\omega_{\lambda}(x,z) + \frac{e^{\mu}}{e^{\lambda+\mu}}\omega_{\mu}(z,y). \end{split}$$

Note that $0 < \frac{\lambda}{\lambda+\mu} < \frac{e^{\lambda}}{e^{\lambda+\mu}} < 1$ *and* $0 < \frac{\mu}{\lambda+\mu} < \frac{e^{\mu}}{e^{\lambda+\mu}} < 1$. *Thus,* ω *is not a convex modular metric on* X.

Definition 4 ([16]). Let $X \neq \emptyset$ and $\omega^p : (0, +\infty) \times X \times X \to [0, \infty)$ be a function defined by $\omega^p(\lambda, x, y) = \omega_\lambda^p(x, y)$, which is called a partial modular metric on X if it satisfies the following axioms:

 $\begin{aligned} & (\omega_1^p): \omega_{\lambda}^p(x,y) = \omega_{\lambda}^p(x,x) = \omega_{\lambda}^p(y,y) \Longleftrightarrow x = y, \ \forall \lambda > 0; \\ & (\omega_2^p): \omega_{\lambda}^p(x,x) \le \omega_{\lambda}^p(x,y), \ \forall \ x,y \in X \ and \ \forall \lambda > 0; \\ & (\omega_3^p): \omega_{\lambda}^p(x,y) = \omega_{\lambda}^p(y,x), \ \forall x,y \in X \ and \ \forall \lambda > 0; \end{aligned}$

$$(\omega_4^p): \omega_{\lambda+\mu}^p(x,y) \leq \omega_{\lambda}^p(x,z) + \omega_{\mu}^p(z,y) - \frac{\omega_{\lambda}^p(x,x) + \omega_{\lambda}^p(z,z) + \omega_{\mu}^p(z,z) + \omega_{\lambda}^p(y,y)}{2}, \ \forall x,y \in X \text{ and } \forall \lambda, \mu > 0.$$

As in Definition 1, the self-distance in Definition 4 of a partial modular metric need not be restricted to zero, i.e., $\omega_{\lambda}^{p}(x, x) = 0$. Note that if x = y = z, $\forall \lambda, \mu > 0$, then, from (ω_{4}^{p}) , it follows that $\omega_{\lambda}^{p}(x, x) = 0$. In order to avoid this limitation, we modify the axioms (ω_{1}^{p}) and (ω_{4}^{p}) in Definition 4 and restate them as follows.

Definition 5. Let $X \neq \emptyset$ and $\omega^p : (0, +\infty) \times X \times X \to [0, \infty)$ be a function defined by $\omega^p(\lambda, x, y) = \omega_\lambda^p(x, y)$, which is called a partial modular metric on X if it retains the axioms (ω_2^p) and (ω_3^p) of Definition 4 with the following: $(\omega_{1'}^p) : \omega_\lambda^p(x, x) = \omega_\mu^p(x, x)$ and $\omega_\lambda^p(x, x) = \omega_\lambda^p(x, y) = \omega_\lambda^p(y, y) \iff x = y, \forall \lambda, \mu > 0;$ $(\omega_{4'}^p) : \omega_{\lambda+\mu}^p(x, y) \leq \omega_\lambda^p(x, z) + \omega_\mu^p(z, y) - \omega_\lambda^p(z, z), \forall x, y \in X \text{ and } \forall \lambda, \mu > 0.$

Obviously, if $\omega_{\lambda}^{p}(x, y) = 0$, then, from $(\omega_{1'}^{p})$ and (ω_{2}^{p}) , we have x = y, but the converse may not be true. It is not difficult to see that a partial modular metric ω^{p} on X is a modular metric but the converse may not be true. If a partial modular metric ω^{p} on X possesses a finite value and is independent of the parameter $\lambda > 0$ that is $\omega_{\lambda}^{p}(x, y) = \omega_{\mu}^{p}(x, y)$, $\forall \lambda, \mu > 0$, then $p(x, y) = \omega_{\lambda}^{p}(x, y)$ is a partial metric on X.

Definition 6. A partial modular metric ω^p on X is said to be convex if, in addition to the axioms $(\omega_{1'}^p)$, (ω_2^p) and (ω_3^p) , it satisfies the following:

$$(\omega_5^p): \ \omega_{\lambda+\mu}^p(x,y) \leq \frac{\lambda}{\lambda+\mu} \omega_{\lambda}^p(x,y) + \frac{\mu}{\lambda+\mu} \omega_{\mu}^p(z,y) - \frac{\lambda}{\lambda+\mu} \omega_{\lambda}^p(z,z),$$

 $\forall x, y, z \in X \text{ and } \forall \lambda, \mu > 0.$

Definition 7. A partial modular metric ω^p on X is said to be weakly convex if it satisfies the axioms $(\omega_{1'}^p)$, (ω_2^p) , (ω_3^p) and the following:

$$(\omega_{5'}^p): \ \omega_{\lambda+\mu}^p(x,y) \leq \alpha(\lambda,\mu)\omega_{\lambda}^p(x,y) + (1-\alpha(\lambda,\mu))\omega_{\mu}^p(z,y) - \alpha(\lambda,\mu)\omega_{\lambda}^p(z,z),$$

 $\forall x, y, z \in X \text{ and } \forall \lambda, \mu > 0, \text{ where } \alpha : (0, \infty) \times (0, \infty) \rightarrow (0, 1) \text{ is a function.}$

Now, we define the following definitions as in the modular metric:

Definition 8. Let ω^p be a partial modular metric on a set X. For given $x_0 \in X$, we define

$$X_{\omega^p}(x_0) = \{x \in X : \lim_{\lambda \to +\infty} \omega_{\lambda}^p(x_0, x) = c\},\$$

for some $c \ge 0$ *and*

$$X^*_{\omega^p}(x_0) = \{ x \in X : \exists \lambda = \lambda(x) > 0, \ \omega^p_\lambda(x_0, x) < \infty \}.$$

Then, two sets X_{ω^p} and $X^*_{\omega^p}$ are called partial modular spaces centered at x_0 . It is obvious that $X_{\omega^p} \subset X^*_{\omega^p}$. We write $X_{\omega^p} \equiv X_{\omega^p}(x_0)$ and $X^* \equiv X^*_{\omega^p}(x_0)$, if $x_0 \in X$ is arbitrary.

Remark 1. For every $x, y \in X$, the function $\lambda \mapsto \omega_{\lambda}^{p} \in [0, \infty)$ is non-increasing. Indeed, $\forall x \in X \text{ and } 0 < \mu < \lambda$, from $(\omega_{1'}^{p})$ and $(\omega_{4'}^{p})$, and we obtain

$$\omega_{\lambda}^{p}(x,y) \leq \omega_{\lambda-\mu}^{p}(x,x) + \omega_{\mu}^{p}(x,y) - \omega_{\lambda-\mu}^{p}(x,x) = \omega_{\mu}^{p}(x,y).$$

Lemma 2. Let ω^p be a partial modular metric on a non-empty set X. Define

$$\omega_{\lambda}^{s}(x,y) = 2\omega_{\lambda}^{p}(x,y) - \omega_{\lambda}^{p}(x,x) - \omega_{\lambda}^{p}(y,y).$$

Then, ω^s is a modular metric on X.

Proof. Obviously, ω^s holds (ω_2) of Definition 2. For (ω_1) and (ω_3) , we have (ω_1) : If x = y, then $\omega_{\lambda}^s(x, y) = 0$, $\forall \lambda > 0$. Suppose $\omega_{\lambda}^s(x, y) = 0$, $\forall \lambda > 0$, then

$$2\omega_{\lambda}^{p}(x,y) = \omega_{\lambda}^{p}(x,x) + \omega_{\lambda}^{p}(y,y).$$

From (ω_2^p) of Definition 5, we obtain

$$2\omega_{\lambda}^{p}(x,x) \leq 2\omega_{\lambda}^{p}(x,y) = \omega_{\lambda}^{p}(x,x) + \omega_{\lambda}^{p}(y,y) \implies \omega_{\lambda}^{p}(x,x) \leq \omega_{\lambda}^{p}(y,y).$$

Similarly, we obtain

$$2\omega_{\lambda}^{p}(y,y) \leq 2\omega_{\lambda}^{p}(x,y) = \omega_{\lambda}^{p}(x,x) + \omega_{\lambda}^{p}(y,y) \implies \omega_{\lambda}^{p}(y,y) \leq \omega_{\lambda}^{p}(x,x).$$

Consequently, we obtain

$$\omega_{\lambda}^{p}(x,y) = \omega_{\lambda}^{p}(x,x) = \omega_{\lambda}^{p}(y,y).$$

Thus, by the second part of $(\omega_{1'}^p)$ of Definition 5, x = y. (ω_3) : From $(\omega_{1'}^p)$ of Definition 5, we obtain $\omega_{\lambda+\mu}^p(x,x) = \omega_{\lambda}^p(x,x)$ and $\omega_{\lambda+\mu}^p(y,y) = \omega_{\lambda}^p(y,y), \forall x, y \in X$ and $\forall \lambda, \mu > 0$. Now, by $(\omega_{4'}^p)$ of Definition 5, we have

$$\begin{split} \omega_{\lambda+\mu}^{s}(x,y) &= 2\omega_{\lambda+\mu}^{p}(x,y) - \omega_{\lambda+\mu}^{p}(x,x) - \omega_{\lambda+\mu}^{p}(y,y) \\ &= 2\omega_{\lambda+\mu}^{p}(x,y) - \omega_{\lambda}^{p}(x,x) - \omega_{\mu}^{p}(y,y) \\ &\leq 2\left(\omega_{\lambda}^{p}(x,z) + \omega_{\mu}^{p}(z,y) - \omega_{\lambda}^{p}(z,z))\right) - \omega_{\lambda}^{p}(x,x) - \omega_{\mu}^{p}(y,y) \\ &= \left(2\omega_{\lambda}^{p}(x,z) - \omega_{\lambda}^{p}(x,x) - \omega_{\lambda}^{p}(z,z)\right) + \left(2\omega_{\mu}^{p}(z,y) - \omega_{\lambda}^{p}(z,z)\right) - \omega_{\mu}^{p}(y,y) \Big) \\ &= \omega_{\lambda}^{s}(x,z) + \omega_{\mu}^{s}(z,y). \end{split}$$

Thus, ω^s satisfies the axioms (ω_1) , (ω_2) and (ω_3) of Definition 2 and hence ω^s is a modular metric. \Box

Remark 2. (i) Let ω^s be a modular metric induced by partial modular metric ω^p on a non-empty set X, and then X_{ω^s} shall denote the modular space with respect to modular metric ω^s . (ii) Let ω be a modular metric on X and $c \ge 0$; then,

$$\omega_{\lambda}^{p}(x,y) = \omega_{\lambda}(x,y) + c$$

defines a partial modular metric on X and the corresponding modular metric is $\omega_{\lambda}^{s}(x,y) = 2\omega_{\lambda}(x,y)$ or $\omega_{\lambda}^{s}(x,y) = 2(\omega_{\lambda}^{p}(x,y)-c)$. Moreover, ω^{p} is (weakly) convex if ω is a (weakly) convex modular metric with c = 0 on X.

Example 4. Let $X = \mathbb{R}$. Define a function $\omega_{\lambda}^{p} : (0, \infty) \times X \times X \to [0, \infty)$ by

$$\omega_{\lambda}^{p}(x,y) = e^{-\lambda}|x-y| + c,$$

where $c \ge 0$, $\lambda > 0$ and $\forall x, y \in X$. Then, ω^p is a partial modular metric on X.

Example 5. Let (X, d) be a metric space and a function ω^p be defined by

$$\omega_{\lambda}^{p}(x,y) = rac{d(x,y) + c}{\lambda}, \ \forall x,y \in X, \ and \ \forall \lambda > 0$$

where $c \ge 0$. We see that $\lim_{\lambda\to+\infty} \omega_{\lambda}^{p}(x,y) = 0$, $\forall x, y \in X$. However, ω^{p} is not a partial modular metric on X. Indeed, by the first part of $(\omega_{1'}^{p})$ of Definition 5, $\omega_{\lambda}^{p}(x,x) \neq \omega_{\mu}^{p}(x,x)$, $\forall x \in X$ and $\forall \lambda, \mu > 0, \lambda \neq \mu$.

Example 6. Let $X = \mathbb{R}$. Define

$$\omega_{\lambda}^{p}(x,y) = e^{-\lambda}|x-y| + |x| + |y|, \ \forall x,y \in X \text{ and } \forall \lambda > 0.$$

Then, ω^p is a partial modular metric on X. It is obvious that $(\omega_{1'}^p)$, (ω_2^p) and (ω_3^p) of Definition 5 hold. For $(\omega_{4'}^p)$, $\forall \lambda, \mu > 0$ and $\forall x, y, z \in X$, we have

$$\begin{split} \omega_{\lambda+\mu}^{p}(x,y) &= e^{-(\lambda+\mu)} |x-y| + |x| + |y| \\ &\leq e^{-(\lambda+\mu)} \left(|x-z| + |z-y| \right) + |x| + |y| \\ &= \left(e^{-(\lambda+\mu)} |x-z| + |x| \right) + \left(e^{-(\lambda+\mu)} |z-y| + |y| \right) \\ &\leq \left(e^{-\lambda} |x-z| + |x| + |z| \right) + \left(e^{-\mu} |z-y| + |z| + |y| \right) - 2|z| \\ &= \omega_{\lambda}^{p}(x,z) + \omega_{\mu}^{p}(z,y) - \omega_{\lambda}^{p}(z,z). \end{split}$$

Thus, ω^p *is a partial modular metric on X.*

Example 7. Let $X \neq \emptyset$ be a set. Define $\omega_{\lambda}^{p}(x,y) = \frac{|x-y|}{\lambda} + c, c > 0$, $\forall x, y \in X$ and $\forall \lambda > 0$. It is obvious that $(\omega_{1'}^{p})$, (ω_{2}^{p}) and (ω_{3}^{p}) of Definition 5 hold. Now, we show that ω^{p} is a partial modular metric and but not (weakly) convex on X. For $(\omega_{4'}^{p})$, $\forall x, y, z \in X$ and $\forall \lambda, \mu > 0$, we have

$$egin{aligned} &\omega_{\lambda+\mu}^p(x,y)=rac{|x-y|}{\lambda+\mu}+c\ &\leqrac{1}{\lambda+\mu}\Big(|x-z|+|z-y|\Big)+c\ &\leq \Big(rac{|x-z|}{\lambda}+c\Big)+\Big(rac{|z-y|}{\mu}+c\Big)-c\ &=\omega_\lambda^p(x,z)+\omega_\mu^p(z,y)-\omega_\mu^p(z,z). \end{aligned}$$

Then, ω^p is a partial modular metric on X. On the other hand, $\forall \lambda, \mu > 0$ and $\forall x.y, z \in X$, and we have

$$\begin{split} \omega_{\lambda+\mu}^{p}(x,y) &\leq \frac{1}{\lambda+\mu} \Big(|x-z|+|z-y| \Big) + c \\ &= \frac{\lambda}{\lambda+\mu} \Big(\frac{|x-z|}{\lambda} + c \Big) + \frac{\mu}{\lambda+\mu} \Big(\frac{|z-y|}{\mu} + c \Big) \\ &= \frac{\lambda}{\lambda+\mu} \omega_{\lambda}^{p}(x,z) + \frac{\mu}{\lambda+\mu} \omega_{\mu}^{p}(z,y). \end{split}$$

To show that ω^p is not convex on X, $\forall \lambda, \mu > 0$, taking x = 4, y = 1, z = 2, then

$$\begin{split} & \left(\frac{\lambda}{\lambda+\mu}\omega_{\lambda}^{p}(x,z) + \frac{\mu}{\lambda+\mu}\omega_{\mu}^{p}(z,y) - \frac{\lambda}{\lambda+\mu}\omega_{\mu}^{p}(z,z)\right) - \omega_{\lambda+\mu}^{p}(x,y) \\ &= \left(\frac{3}{\lambda+\mu} + c - \frac{\lambda}{\lambda+\mu}c\right) - \left(\frac{3}{\lambda+\mu} + c\right) \\ &= -\frac{\lambda}{\lambda+\mu}c < 0. \end{split}$$

This shows that ω^p is not convex and, hence, it is not a weakly convex partial modular metric on X.

Example 8. Let ω^p be a partial modular metric on a non-empty set X. Define $\omega_{\lambda}^p(x,y) = \frac{\omega_{\lambda}(x,y)}{\lambda}$, $\forall x, y \in X$ and $\forall \lambda > 0$. Then, ω^p is convex and hence it is a weakly convex partial modular metric on X.

Example 9. For any non-empty set X, define $\omega_{\lambda}^{p}(x, y) = e^{-\lambda}\omega_{\lambda}(x, y)$, $\forall x, y \in X$ and $\forall \lambda > 0$. Then, ω^{p} is weakly convex but is not a convex partial modular metric on X.

Definition 9. Let ω^p be a partial modular metric on a non-empty set X and $\{x_n\}$ be a sequence in a partial modular space X_{ω^p} ; then,

(*i*) $\{x_n\}$ is said to be convergent to a point $x \in X_{\omega^p}$, if and only if, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$|\omega_{\lambda}^{p}(x_{n},x)-\omega_{\lambda}^{p}(x,x)|\leq\epsilon,$$

 $\forall n \geq n_0 \text{ and } \forall \lambda > 0. \text{ We write } \lim_{n \to +\infty} \omega_{\lambda}^p(x_n, x) = \omega_{\lambda}^p(x, x), \ \forall \lambda > 0;$

- (ii) a sequence $\{x_n\}$ is a Cauchy in X_{ω^p} if $\lim_{n,m\to+\infty} \omega_{\lambda}^p(x_n, x_m) = c$, $\forall \lambda > 0$, for some $c \ge 0$. In this case, $\lim_{n\to+\infty} \omega_{\lambda}^p(x_n, x_n) = \lim_{m\to+\infty} \omega_{\lambda}^p(x_m, x_m) = c$. Thus, if $\{x_n\}$ is a Cauchy sequence in X_{ω^s} , then c = 0;
- (iii) a partial modular space X_{ω^p} is said to be complete if every Cauchy sequence converges to a point $x \in X_{\omega^p}$ such that

$$\lim_{n,m\to+\infty}\omega_{\lambda}^{p}(x_{n},x_{m})=\omega_{\lambda}^{p}(x,x), \ \forall \lambda>0.$$

Remark 3. (*i*) If $\{x_n\}$ is a Cauchy sequence in X_{ω^s} , i.e., $\lim_{n,m\to+\infty} \omega_{\lambda}^s(x_n, x_m) = 0$, then

$$\lim_{u,m\to+\infty}\omega_{\lambda}^{p}(x_{n},x_{m})=\lim_{n\to+\infty}\omega_{\lambda}^{p}(x_{n},x_{n})=\lim_{m\to+\infty}\omega_{\lambda}^{p}(x_{m},x_{m}).$$

(*ii*) If $\{x_n\}$ is a Cauchy sequence in X_{ω^s} that converges to some point $x \in X_{\omega^s}$, then

r

$$\lim_{n,m\to+\infty}\omega_{\lambda}^{p}(x_{n},x_{m})=\lim_{n\to+\infty}\omega_{\lambda}^{p}(x_{n},x_{n})=\omega_{\lambda}^{p}(x,x).$$

(iii) A sequence $\{x_n\}$ in X_{ω^p} is a Cauchy sequence if it is a Cauchy sequence in X_{ω^s} , i.e., $\lim_{n,m\to+\infty} \omega^s_{\lambda}(x_n, x_m) = 0.$

Lemma 3. Let ω^p be a partial modular on X and $\{x_n\}$ be a sequence in X_{ω^p} . Then,

- (*i*) $\{x_n\}$ is a Cauchy sequence in X_{ω^p} if it is a Cauchy sequence in the modular space X_{ω^s} induced by partial modular metric ω^p ;
- (ii) a partial modular space X_{ω^p} is complete if and only if the modular space X_{ω^s} induced by ω^p is complete. Furthermore,

$$\lim_{n \to +\infty} \omega_{\lambda}^{s}(x_{n}, x) = 0 \Longleftrightarrow \lim_{n \to \infty} [2\omega_{\lambda}^{p}(x_{n}, x) - \omega_{\lambda}^{p}(x_{n}, x_{n}) - \omega_{\lambda}^{p}(x.x)] = 0$$

or

$$\lim_{n\to\infty}\omega_{\lambda}^{s}(x_{n},x)=0 \Longleftrightarrow \lim_{n\to+\infty}\omega_{\lambda}^{p}(x_{n},x)=\lim_{n\to+\infty}\omega_{\lambda}^{p}(x_{n},x_{n})=\omega_{\lambda}^{p}(x,x), \ \forall \lambda>0.$$

Definition 10 ([18]). A continuous function $\mathcal{F} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a \mathcal{C} - class function if, for any $s, t \in \mathbb{R}$, the following conditions hold:

- (*i*) $\mathcal{F}(s,t) \leq s;$
- (*ii*) $\mathcal{F}(s,t) = s$ implies s = 0 or t = 0.

Example 10 ([18]). *The following are examples of the* C*-class function:*

- (*i*) $\mathcal{F}(s,t) = \alpha s, \ \alpha \in (0,1);$
- (ii) $\mathcal{F}(s,t) = \frac{s}{(1+t)^r}, \ r \in (0,\infty);$
- (iii) $\mathcal{F}(s,t) = \frac{\log(t+\alpha^s)}{(1+t)}, \ \alpha > 1.$

Definition 11 ([19]). A control function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance if the following conditions hold:

- (*i*) ψ is non-decreasing and continuous;
- (*ii*) $\psi(t) = 0$ *if and only if* t = 0.

We denote by Ψ *the set of all altering distance functions.*

Example 11 ([20]). *The following examples are the altering distance functions:*

- (*i*) $\psi(t) = e^{\alpha t} + \beta t 1;$
- (*ii*) $\psi(t) = \alpha t^2 + \ln(\beta t + 1)$, where $\alpha, \beta > 0$.

Definition 12 ([18]). A control function $\varphi : [0, \infty) \to [0, \infty)$ is called an ultra-altering distance *if the following conditions hold:*

- (*i*) φ *is continuous;*
- (*ii*) $\varphi(t) > 0$, t > 0 and $\varphi(0) \ge 0$.

 Φ denotes the set of all ultra-altering distance functions.

Definition 13 ([21]). A triplet $(\psi, \varphi, \mathcal{F})$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $\mathcal{F} \in C$ is monotonically increasing if

$$\forall x, y \in [0, \infty), \ x \leq y \Longrightarrow \mathcal{F}(\psi(x), \varphi(x)) \leq \mathcal{F}(\psi(y), \varphi(y)).$$

Further, we say that the triplet $(\psi, \varphi, \mathcal{F})$ is strictly monotonically increasing if

$$\forall x, y \in [0, \infty) \ x < y \Longrightarrow \mathcal{F}(\psi(x), \varphi(x)) < \mathcal{F}(\psi(y), \varphi(y)).$$

Example 12 ([21]). Consider a C-class function $\mathcal{F}(s,t) = s - t$. Define $\psi, \varphi : [0,\infty) \to [0,\infty)$ by $\varphi(x) = \sqrt{x}$ and

$$\psi(x) = egin{cases} \sqrt{x}, & 0 \leq x \leq 1; \ x^2, & x > 1. \end{cases}$$

Obviously, the triplet $(\psi, \varphi, \mathcal{F})$ *is monotonically increasing.*

Definition 14 ([22]). Let \mathcal{P} and \mathcal{Q} be two self-mappings on a non-empty set X; then, they are said to be weakly compatible if they commute at their coincidence points, i.e., $\mathcal{PQ}x = \mathcal{QP}x$, for some $x \in X$.

Definition 15 ([23]). Let $X \neq \emptyset$ and $\mathcal{P}, \mathcal{Q} : X \to X$ be two self-mappings. If $u = \mathcal{P}x = \mathcal{Q}x$, for some $x \in X$, then x is called a coincidence point of \mathcal{P} and \mathcal{Q} , and u is called a point of coincidence (briefly, poc) of \mathcal{P} and \mathcal{Q} .

Lemma 4 ([23]). *If* \mathcal{P} and \mathcal{Q} are weakly compatible self-mappings on a non-empty set X, and if \mathcal{P} and \mathcal{Q} have a unique point of coincidence $u = \mathcal{P}x = \mathcal{Q}x$, then u is the unique common fixed-point \mathcal{P} and \mathcal{Q} .

3. Main Results

Let ω^p be a partial modular metric on a non-empty set X and X_{ω^p} be a partial modular space. Suppose that $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S} : X_{\omega^p} \to X_{\omega^p}$ are four self-mappings such that

$$\mathcal{P}X_{\omega^p} \subseteq \mathcal{Q}X_{\omega^p} \text{ and } \mathcal{R}X_{\omega^p} \subseteq \mathcal{S}X_{\omega^p}.$$
 (1)

Let $x_0 \in X_{\omega^p}$ be any point. By virtue of (1), the two sequences $\{x_n\}$ and $\{y_n\}$ in X_{ω^p} are defined as follows:

$$y_{2n} = \mathcal{P}x_{2n} = \mathcal{Q}x_{2n+1} \text{ and } y_{2n+1} = \mathcal{R}x_{2n+1} = \mathcal{S}x_{2n+2}, \forall n \in \mathbb{N} \cup \{0\}.$$
 (2)

Inspired by Chandok et al. [4], we are ready to prove the following lemma, which plays a crucial role in the subsequent results.

Lemma 5. Let ω^p be a partial modular metric on a non-empty set X and X_{ω^p} be a partial modular space. Suppose that $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S} : X_{\omega^p} \to X_{\omega^p}$ are four self-mappings satisfying the condition (1). If there exist $\psi \in \Psi, \varphi \in \Phi$ and $\mathcal{F} \in C$ such that the triplet $(\psi, \varphi, \mathcal{F})$ is a monotonically increasing function satisfying the following:

$$\psi(\omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{R}y)) \leq \mathcal{F}(\psi(\mathcal{M}(x,y)),\varphi(\mathcal{M}(x,y))),$$
(3)

where

$$\mathcal{M}(x,y) = \max\{\omega_{\lambda}^{p}(\mathcal{S}x,\mathcal{Q}y),\omega_{\lambda}^{p}(\mathcal{S}x,\mathcal{P}x),\omega_{\lambda}^{p}(\mathcal{Q}y,\mathcal{R}y),\frac{1}{2}[\omega_{2\lambda}^{p}(\mathcal{Q}y,\mathcal{P}x)+\omega_{2\lambda}^{p}(\mathcal{S}x,\mathcal{R}y)]\},$$

 $\forall \lambda > 0$ and $\forall x, y \in X_{\omega^p}$. Then, the sequence $\{y_n\}$ defined by (2) is a Cauchy sequence in X_{ω^p} .

Proof. From (2), we recall that

$$y_{2n} = \mathcal{P}x_{2n} = \mathcal{Q}x_{2n+1} \text{ and } y_{2n+1} = \mathcal{R}x_{2n+1} = \mathcal{S}x_{2n+2}, \forall n \in \mathbb{N} \cup \{0\}$$

Using (3), we obtain

$$\psi(\omega_{\lambda}^{p}(y_{2n}, y_{2n+1})) = \psi(\omega_{\lambda}^{p}(\mathcal{P}x_{2n}, \mathcal{R}x_{2n+1}))$$

$$\leq \mathcal{F}(\psi(\mathcal{M}(x_{2n}, x_{2n+1})), \varphi(\mathcal{M}(x_{2n}, x_{2n+1}))),$$
(4)

where

$$\mathcal{M}(x_{2n}, x_{2n+1}) = \max\{\omega_{\lambda}^{p}(\mathcal{S}x_{2n}, \mathcal{Q}x_{2n+1}), \omega_{\lambda}^{p}(\mathcal{S}x_{2n}, \mathcal{P}x_{2n}), \omega_{\lambda}^{p}(\mathcal{Q}x_{2n+1}, \mathcal{R}x_{2n+1}), (5) \\ \frac{1}{2}[\omega_{2\lambda}^{p}(\mathcal{Q}x_{2n+1}, \mathcal{P}x_{2n}) + \omega_{2\lambda}^{p}(\mathcal{S}x_{2n}, \mathcal{R}x_{2n+1})]\} \\ = \max\{\omega_{\lambda}^{p}(y_{2n-1}, y_{2n}), \omega_{\lambda}^{p}(y_{2n-1}, y_{2n}), \omega_{\lambda}^{p}(y_{2n}, y_{2n+1}), \\ \frac{1}{2}[\omega_{2\lambda}^{p}(y_{2n}, y_{2n}) + \omega_{2\lambda}^{p}(y_{2n-1}, y_{2n+1})]\}.$$

and by $(\omega_{1'}^p)$ and $(\omega_{4'}^p)$, we have

$$\frac{1}{2} [\omega_{2\lambda}^{p}(y_{2n}, y_{2n}) + \omega_{2\lambda}^{p}(y_{2n-1}, y_{2n+1})]$$

$$\leq \frac{1}{2} [\omega_{\lambda}^{p}(y_{2n}, y_{2n}) + \omega_{\lambda}^{p}(y_{2n-1}, y_{2n}) + \omega_{\lambda}^{p}(y_{2n}, y_{2n+1}) - \omega_{\lambda}^{p}(y_{2n}, y_{2n})]$$

$$= \frac{1}{2} [\omega_{\lambda}^{p}(y_{2n-1}, y_{2n}) + \omega_{\lambda}^{p}(y_{2n}, y_{2n+1})].$$
(6)

Using (5), (6) and the monotonicity of the triplet $(\psi, \varphi, \mathcal{F})$, (4) becomes

$$\psi(\omega_{\lambda}^{p}(y_{2n}, y_{2n+1})) \leq \mathcal{F}(\psi(\max\{\omega_{\lambda}^{p}(y_{2n-1}, y_{2n}), \omega_{\lambda}^{p}(y_{2n}, y_{2n+1})\}), \qquad (7)$$
$$\varphi(\max\{\omega_{\lambda}^{p}(y_{2n-1}, y_{2n}), \omega_{\lambda}^{p}(y_{2n}), y_{2n+1})\})),$$

From the above inequality, the following cases arise: Case (*I*): Suppose $\omega_{\lambda}^{p}(y_{2n-1}, y_{2n}) < \omega_{\lambda}^{p}(y_{2n}, y_{2n+1})$; then, from (7) and by the strict monotonicity of $(\psi, \varphi, \mathcal{F})$, we obtain

$$\psi(\omega_{\lambda}^{p}(y_{2n}, y_{2n+1})) < \mathcal{F}(\psi(\omega_{\lambda}^{p}(y_{2n}, y_{2n+1}), \varphi(\omega_{\lambda}^{p}(y_{2n}, y_{2n+1})))$$
$$\leq \psi(\omega_{\lambda}^{p}(y_{2n}, y_{2n+1})).$$

Therefore, $\omega_{\lambda}^{p}(y_{2n}, y_{2n+1}) < \omega_{\lambda}^{p}(y_{2n}, y_{2n+1})$. This is a contradiction. Case (II): Suppose $\omega_{\lambda}^{p}(y_{2n}, y_{2n+1}) \leq \omega_{\lambda}^{p}(y_{2n-1}, y_{2n})$; then, from (7), we obtain

$$\psi(\omega_{\lambda}^{p}(y_{2n}, y_{2n+1})) \leq \mathcal{F}(\psi(\omega_{\lambda}^{p}(y_{2n-1}, y_{2n})), \varphi(\omega_{\lambda}^{p}(y_{2n-1}, y_{2n})).$$
(8)

Since ψ is a non-increasing function, then, from (8), we have

$$\psi(\omega_{\lambda}^{p}(y_{2n},y_{2n+1})) \leq \psi(\omega_{\lambda}^{p}(y_{2n-1},y_{2n})) \Longrightarrow \omega_{\lambda}^{p}(y_{2n},y_{2n+1})) \leq \omega_{\lambda}^{p}(y_{2n-1},y_{2n}).$$

This shows that $\{\omega_{\lambda}^{p}(y_{2n}, y_{2n+1})\}$ is a non-increasing sequence of non-negative real numbers. Thus, there exists $\epsilon \ge 0$ such that

$$\lim_{n\to+\infty}\omega_{\lambda}^{p}(y_{2n},y_{2n+1})=\epsilon, \ \forall \lambda>0.$$

Taking the limit as $n \to +\infty$ in (8), we obtain

$$\psi(\epsilon) \leq \mathcal{F}(\psi(\epsilon), \varphi(\epsilon)) \leq \psi(\epsilon) \Longrightarrow \mathcal{F}(\psi(\epsilon), \varphi(\epsilon)) = \psi(\epsilon)$$

so $\psi(\epsilon) = 0$ or $\varphi(\epsilon) = 0$ and hence $\epsilon = 0$, i.e.,

$$\lim_{n \to +\infty} \omega_{\lambda}^{p}(y_{2n}, y_{2n+1}) = 0, \ \forall \lambda > 0.$$
(9)

Now, we show that $\{y_n\}$ is a Cauchy sequence in X_{ω^p} . By Lemma 3, it is sufficient to prove that a subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in X_{ω^s} .

From (ω_2^p) of Definition 4, we have

$$0 \leq \omega_{\lambda}^{p}(y_{2n}, y_{2n}) \leq \omega_{\lambda}^{p}(y_{2n}, y_{2n+1}), \ \forall \lambda > 0,$$

so from (9), it follows that

$$\lim_{n \to +\infty} \omega_{\lambda}^{p}(y_{2n}, y_{2n}) = 0.$$
⁽¹⁰⁾

Similarly, $\lim_{n\to+\infty} \omega_{\lambda}^{p}(y_{2n+1}, y_{2n+1}) = 0, \forall \lambda > 0.$

If possible, let $\{y_{2n}\}$ be not a Cauchy in X_{ω^s} , and then there exists $\delta > 0$ such that, for each even +ve integer k, we can find subsequence $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ with $2n(k) > 2m(k) \ge k$ such that

$$\omega_{\lambda}^{s}(y_{2m(k)}, y_{2n(k)}) > \delta, \ \forall \ \lambda > 0.$$

$$(11)$$

Now, we choose 2n(k) corresponding to 2m(k) such that it is the smallest even integer with 2n(k) > 2m(k) and satisfies Inequality (11). Hence,

$$\omega_{\lambda}^{s}(y_{2m(k)}, y_{2n(k)-1}) \leq \delta, \ \forall \lambda > 0.$$
(12)

By triangular inequality (ω_3) and (12), we have

$$\omega_{\lambda}^{s}(y_{2m(k)}, y_{2n(k)}) \leq \omega_{\frac{\lambda}{2}}^{s}(y_{2m(k)}, y_{2n(k)-1}) + \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)-1}, y_{2n(k)})| \\ \leq \delta + \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)-1}, y_{2n(k)}).$$
(13)

On the other hand, by Lemma 2, $\forall \lambda > 0$, we have

$$\omega_{\lambda}^{s}(y_{2n(k)-1}, y_{2n(k)}) = 2\omega_{\lambda}^{p}(y_{2n(k)-1}, y_{2n(k)}) - \omega_{\lambda}^{p}(y_{2n(k)-1}, y_{2n(k)-1})$$

$$- \omega_{\lambda}^{p}(y_{2n(k)}, y_{2n(k)}).$$
(14)

Letting $k \to +\infty$ on (14), then from (9) and (10), $\forall \lambda > 0$, we have

$$\lim_{k \to +\infty} \omega_{\lambda}^{s}(y_{2n(k)-1}, y_{2n(k)}) = 0, \ \forall \lambda > 0.$$
(15)

From (13), using (11) and (15), we have

$$\delta < \lim_{k \to \infty} \omega_{\lambda}^{s}(y_{2m(k)}, y_{2n(k)}) \le \delta, \ \forall \lambda > 0.$$

This implies

$$\lim_{k \to +\infty} \omega_{\lambda}^{s}(y_{2m(k)}, y_{2n(k)}) = \delta.$$
(16)

Again, using the triangular inequality (ω_3) , we have

$$\omega_{\lambda}^{s}(y_{2n(k)}, y_{2m(k)}) \leq \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)}, y_{2n(k)-1}) + \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)-1}, y_{2m(k)}) \\ \leq \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)}, y_{2n(k)-1}) + \omega_{\frac{\lambda}{4}}^{s}(y_{2n(k)-1}, y_{2m(k)-1}) \\ + \omega_{\frac{\lambda}{2}}^{s}(y_{2m(k)-1}, y_{2m(k)}).$$
(17)

Furthermore, we have

$$\omega_{\lambda}^{s}(y_{2n(k)-1}, y_{2m(k)-1}) \leq \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)-1}, y_{2n(k)}) + \omega_{\frac{\lambda}{4}}^{s}(y_{2n(k)}, y_{2m(k)}) + \omega_{\frac{\lambda}{4}}^{s}(y_{2m(k)}, y_{2m(k)-1})$$
(18)

Letting the limit as $k \to +\infty$ in (17) and (18), using (15) and (16), we obtain

$$\lim_{k\to\infty}\omega_{\lambda}^{s}(y_{2n(k)-1},y_{2m(k)-1})=\delta.$$

Further, we have

$$\omega_{\lambda}^{s}(y_{2n(k)+1}, y_{2m(k)}) \leq \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)+1}, y_{2n(k)}) + \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)}, y_{2m(k)}).$$
(19)

However,

$$\omega_{2\lambda}^{s}(y_{2n(k)}, y_{2m(k)}) \le \omega_{\lambda}^{s}(y_{2n(k)}, y_{2n(k)+1}) + \omega_{\lambda}^{s}(y_{2n(k)+1}, y_{2m(k)}).$$
⁽²⁰⁾

Taking the limit on (19) and (20) as $k \to +\infty$ and using (15) and (16), we obtain

$$\lim_{k \to +\infty} \omega_{\lambda}^{s}(y_{2n(k)+1}, y_{2m(k)}) = \delta.$$
⁽²¹⁾

Since $\forall \lambda > 0$, we have

$$\omega_{\lambda}^{s}(y_{2n(k)}, y_{2m(k)}) = [2\omega_{\lambda}^{p}(y_{2n(k)}, y_{2m(k)}) - \omega_{\lambda}^{p}(y_{2n(k)}, y_{2n(k)}) - \omega_{\lambda}^{p}(y_{2m(k)}, y_{2m(k)})].$$

Taking the limit on the above equation as $k \to +\infty$, and then using (9) and (16), we obtain

$$\lim_{k \to +\infty} \omega_{\lambda}^{p}(y_{2n(k)}, y_{2m(k)}) = \frac{\delta}{2} = d \text{ (say)}$$
(22)

Similarly, we obtain

$$\lim_{k \to +\infty} \omega_{\lambda}^{p}(y_{2n(k)}, y_{2m(k)-1}) = \frac{\delta}{2} = d \text{ and } \lim_{k \to +\infty} \omega_{\lambda}^{p}(y_{2n(k)+1}, y_{2m(k)}) = \frac{\delta}{2} = d.$$
(23)

Now, from (3), we obtain

$$\psi(\omega_{\lambda}^{p}(y_{2m(k)}, y_{2n(k)+1})) = \psi(\omega_{\lambda}^{p}(\mathcal{P}x_{2m(k)}, \mathcal{R}x_{2n(k)+1}))$$

$$\leq \mathcal{F}(\psi(\mathcal{M}(x_{2m(k)}, x_{2n(k)+1})), \varphi(\mathcal{M}(x_{2m(k)}, x_{2n(k)+1}))),$$
(24)

where

$$\begin{aligned} \mathcal{M}(x_{2m(k)}, x_{2n(k)+1}) &= \max\{\omega_{\lambda}^{p}(Sx_{2m(k)}, Qx_{2n(k)+1}), \omega_{\lambda}^{p}(Sx_{2m(k)}, Px_{2m(k)}), \\ & \omega_{\lambda}^{p}(Qx_{2n(k)+1}, Rx_{2n(k)+1}), \\ & \frac{1}{2}[\omega_{2\lambda}^{p}(Qx_{2n(k)+1}, Px_{2m(k)}) + \omega_{2\lambda}^{p}(Sx_{2m(k)}, Rx_{2n(k)+1})]\} \\ &= \max\{\omega_{\lambda}^{p}(y_{2m(k)-1}, y_{2n(k)}), \omega_{\lambda}^{p}(y_{2m(k)-1}, y_{2m(k)}), \\ & \omega_{\lambda}^{p}(y_{2n(k)}, y_{2n(k)+1}), \\ & \frac{1}{2}[\omega_{2\lambda}^{p}(y_{2n(k)}, y_{2m(k)}) + \omega_{2\lambda}^{p}(y_{2m(k)-1}, y_{2n(k)+1})]\} \end{aligned}$$

and by $(\omega_{4'}^p)$, we have

$$\begin{aligned} &\frac{1}{2} [\omega_{2\lambda}^{p}(y_{2n(k)}, y_{2m(k)}) + \omega_{2\lambda}^{p}(y_{2m(k)-1}, y_{2n(k)+1})] \\ &= \frac{1}{2} \omega_{2\lambda}^{p}(y_{2n(k)}, y_{2m(k)}) + \frac{1}{2} [\omega_{\lambda}^{p}(y_{2m(k)-1}, y_{2m(k)}) + \omega_{\lambda}^{p}(y_{2m(k)}, y_{2n(k)+1}) \\ &- \omega_{\lambda}^{p}(y_{2m(k)}, y_{2m(k)})]. \end{aligned}$$

Taking the limit as $k \to +\infty$ on (24), and then using (9), (10), (20), (22) and (23), we obtain

$$\psi(d) \leq \mathcal{F}(\psi(d), \varphi(d)) \leq \psi(d) \Longrightarrow \mathcal{F}(\psi(d), \varphi(d)) = \psi(d),$$

which implies $\psi(d) = 0$ or $\varphi(d) = 0$; then, d = 0. This is a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in the modular space X_{ω^s} and hence the sequence $\{y_n\}$ is a Cauchy sequence in X_{ω^p} . \Box

Theorem 1. Suppose $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S} : X_{\omega^p} \to X_{\omega^p}$ to be four self-mappings defined on a complete partial modular space satisfying (1) and (3). Then, $poc(\mathcal{P}, \mathcal{S}) \neq \emptyset$ and $poc(\mathcal{Q}, \mathcal{R}) \neq \emptyset$. Further, if the pairs $(\mathcal{P}, \mathcal{S})$ and $(\mathcal{Q}, \mathcal{R})$ are weakly compatible in X_{ω^p} , then $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ and \mathcal{S} have a unique common fixed point in X_{ω^p} .

Proof. By Lemma 5, $\{y_n\}$ is a Cauchy sequence in the partial modular space X_{ω^p} . Since X_{ω^p} is complete, $\{y_n\}$ converges in X_{ω^p} . Then, there exists $z \in X_{\omega^p}$ such that

$$\lim_{n \to +\infty} \omega_{\lambda}^{p}(y_{n}, z) = \lim_{n \to +\infty} \omega_{\lambda}^{p}(y_{n}, x_{n}) = \omega_{\lambda}^{p}(z, z), \ \forall \lambda > 0.$$

By Lemma 3 and from (9), we obtain

$$\lim_{n \to +\infty} \omega_{\lambda}^{p}(y_{n}, z) = 0 \text{ and } \omega_{\lambda}^{p}(z, z) = 0, \ \forall \lambda > 0.$$
(25)

Since X_{ω^p} is complete, the subsequences $\{\mathcal{P}x_{2n}\}$, $\{\mathcal{Q}x_{2n+1}\}$, $\{\mathcal{R}x_{2n+1}\}$ and $\{\mathcal{S}x_{2n+2}\}$, $\forall n \in \mathbb{N} \cup \{0\}$ converge to $z \in X_{\omega^p}$. Now, we show that $poc(\mathcal{P}, \mathcal{S}) \neq \emptyset$ and $poc(\mathcal{Q}, \mathcal{R}) \neq \emptyset$. Since $\{\mathcal{S}x_{2n+2}\}$ converges to $z \in X_{\omega^p}$, there exists $u \in X_{\omega^p}$ such that $z = \mathcal{S}u$. We claim that $\mathcal{P}u = \mathcal{S}u$. Using (3), we obtain

$$\psi(\omega_{\lambda}^{p}(\mathcal{P}u, y_{2n+1})) = \psi(\omega_{\lambda}^{p}(\mathcal{P}u, \mathcal{R}x_{2n+1}))$$

$$\leq \mathcal{F}(\psi(\mathcal{M}(u, x_{2n+1})), \varphi(\mathcal{M}(u, x_{2n+1}))),$$
(26)

where

$$\mathcal{M}(u, x_{2n+1}) = \max\{\omega_{\lambda}^{p}(\mathcal{S}u, \mathcal{Q}x_{2n+1}), \omega_{\lambda}^{p}(\mathcal{S}u, \mathcal{P}u), \omega_{\lambda}^{p}(\mathcal{Q}x_{2n+1}, \mathcal{R}x_{2n+1}), \\ \frac{1}{2}[\omega_{2\lambda}^{p}(\mathcal{Q}x_{2n+1}, \mathcal{P}u) + \omega_{2\lambda}^{p}(\mathcal{S}u, \mathcal{R}x_{2n+1})]\} \\ = \max\{\omega_{\lambda}^{p}(z, y_{2n}), \omega_{\lambda}^{p}(z, \mathcal{P}u), \omega_{\lambda}^{p}(y_{2n}, y_{2n+1}), \\ \frac{1}{2}[\omega_{2\lambda}^{p}(y_{2n}, \mathcal{P}u) + \omega_{2\lambda}^{p}(z, y_{2n+1})]\}$$

and

$$\omega_{2\lambda}^p(y_{2n},\mathcal{P}u) \le \omega_{\lambda}^p(y_{2n},z) + \omega_{\lambda}^p(z,\mathcal{P}u) - \omega_{\lambda}^p(z,z).$$

Taking the limit as $n \to +\infty$ on (26), and then using (9), (25) and (27), and by the definition of $(\psi, \varphi, \mathcal{F})$, we obtain

$$\psi(\omega_{\lambda}^{p}(\mathcal{P}u,z)) \leq \lim_{n \to +\infty} \mathcal{F}(\psi(\mathcal{M}(u,x_{2n+1})),\varphi(\mathcal{M}(u,x_{2n+1}))),$$

where

$$\lim_{n \to +\infty} \mathcal{M}(u, x_{2n+1}) = \max\{\omega_{\lambda}^{p}(z, \mathcal{P}u), \frac{1}{2} \lim_{n \to +\infty} \omega_{2\lambda}^{p}(y_{2n}, \mathcal{P}u)\}$$

and

$$\lim_{n \to +\infty} \omega_{2\lambda}^p(y_{2n}, \mathcal{P}u) \leq \lim_{n \to +\infty} [\omega_{\lambda}^p(y_{2n}, z) + \omega_{\lambda}^p(z, \mathcal{P}u) - \omega_{\lambda}^p(z, z)]$$

$$\leq \omega_{\lambda}^p(z, \mathcal{P}u).$$
(27)

Therefore,

$$\begin{split} \psi(\omega_{\lambda}^{p}(\mathcal{P}u,z)) &\leq \lim_{n \to +\infty} \mathcal{F}(\psi(\mathcal{M}(u,x_{2n+1})),\varphi(\mathcal{M}(u,x_{2n+1}))) \\ &\leq \lim_{n \to +\infty} \mathcal{F}(\psi(\omega_{\lambda}^{p}(z,\mathcal{P}u)),\varphi(\omega_{\lambda}^{p}(z,\mathcal{P}u))) \\ &\leq \psi(\omega_{\lambda}^{p}(z,\mathcal{P}u)). \end{split}$$

It follows that

$$\mathcal{F}(\psi(\omega_{\lambda}^{p}(z,\mathcal{P}u)),\varphi(\omega_{\lambda}^{p}(z,\mathcal{P}u))) = \psi(\omega_{\lambda}^{p}(z,\mathcal{P}u)),$$

so $\psi(\omega_{\lambda}^{p}(z, \mathcal{P}u)) = 0$ or $\varphi(\omega_{\lambda}^{p}(z, \mathcal{P}u))$; then, $\omega_{\lambda}^{p}(z, \mathcal{P}u) = 0$ and hence $\mathcal{P}u = \mathcal{S}u = z$, i.e., $poc(\mathcal{P}, \mathcal{S}) \neq \emptyset$.

Since $\mathcal{P}X_{\omega^p} \subset \mathcal{Q}X_{\omega^p}$ and $u \in poc(\mathcal{P}, \mathcal{S})$, i.e., $\mathcal{P}u = \mathcal{S}u = z$, then there exists $v \in \mathcal{Q}X_{\omega^p}$ such that $\mathcal{P}u = \mathcal{Q}v = z$. Now, we show that $\mathcal{R}v = \mathcal{Q}v$. For this, from (3), we obtain

$$\psi(\omega_{\lambda}^{p}(z,\mathcal{R}v)) = \psi(\omega_{\lambda}^{p}(\mathcal{P}u,\mathcal{R}v)) \le \mathcal{F}(\psi(\mathcal{M}(u,v)),\varphi(\mathcal{M}(u,v))),$$
(28)

where

$$\mathcal{M}u, v) = \max\{\omega_{\lambda}^{p}(\mathcal{S}u, \mathcal{Q}v), \omega_{\lambda}^{p}(\mathcal{S}u, \mathcal{P}u), \omega_{\lambda}^{p}(\mathcal{Q}v, \mathcal{R}v), \\ \frac{1}{2}[\omega_{2\lambda}^{p}(\mathcal{Q}v, \mathcal{P}u) + \omega_{2\lambda}^{p}(\mathcal{S}u, \mathcal{R}v)]\} \\ = \max\{\omega_{\lambda}^{p}(z, z), \omega_{\lambda}^{p}(z, z), \omega_{\lambda}^{p}(z, \mathcal{R}v), \\ \frac{1}{2}[\omega_{2\lambda}^{p}(z, z) + \omega_{2\lambda}^{p}(z, \mathcal{R}v)]\}.$$

Then, (28) becomes

$$\psi(\omega_\lambda^p(z,\mathcal{R}v)) \leq \mathcal{F}(\psi(\omega_\lambda^p(z,\mathcal{R}v)), \varphi(\omega_\lambda^p(z,\mathcal{R}v))) \leq \psi(\omega_\lambda^p(z,\mathcal{R}v)).$$

Therefore,

$$\mathcal{F}(\psi(\omega_{\lambda}^{p}(z,\mathcal{R}v)),\varphi(\omega_{\lambda}^{p}(z,\mathcal{R}v)))=\psi(\omega_{\lambda}^{p}(z,\mathcal{R}v)),$$

yielding $\psi(\omega_{\lambda}^{p}(z, \mathcal{R}v)) = 0$ or $\varphi(\omega_{\lambda}^{p}(z, \mathcal{R}v)) = 0$; then, $\omega_{\lambda}^{p}(z, \mathcal{R}v) = 0$ and hence $\mathcal{R}v = z = \mathcal{Q}v$. Thus, $poc(\mathcal{R}, \mathcal{Q}) \neq \emptyset$.

Since $(\mathcal{P}, \mathcal{S})$ and $(\mathcal{R}, \mathcal{Q})$ are weakly compatible, then $\mathcal{P}z = \mathcal{P}Su = \mathcal{S}\mathcal{P}u = \mathcal{S}z$ and $\mathcal{R}z = \mathcal{R}\mathcal{Q}v = \mathcal{Q}\mathcal{R}v = \mathcal{Q}z$. Now, we claim that the pairs $(\mathcal{P}, \mathcal{S})$ and $(\mathcal{R}, \mathcal{Q})$ have a unique common point of coincidence. Suppose, if possible, that there exist $r, r^* \in X_{\omega^p}, r \neq r^*$ such that $\mathcal{P}z = \mathcal{S}z = r$ and $\mathcal{R}z = \mathcal{Q}z = r^*$.

From (3), we obtain

$$\psi(\omega_{\lambda}^{p}(r,r^{*})) = \psi(\omega_{\lambda}^{p}(\mathcal{P}z,\mathcal{R}z)) \leq \mathcal{F}(\psi(\mathcal{M}(z,z)),\varphi(\mathcal{M}(z,z)))$$

where

$$\begin{split} \mathcal{M}(z,z) &= \max\{\omega_{\lambda}^{p}(\mathcal{S}z,\mathcal{Q}z), \omega_{\lambda}^{p}(\mathcal{S}z,\mathcal{P}z), \omega_{\lambda}^{p}(\mathcal{Q}z,\mathcal{R}z), \\ &\qquad \frac{1}{2}[\omega_{2\lambda}^{p}(\mathcal{Q}z,\mathcal{P}z) + \omega_{2\lambda}^{p}(\mathcal{S}z,\mathcal{R}z)]\} \\ &= \max\{\omega_{\lambda}^{p}(r,r^{*}), \omega_{\lambda}^{p}(r,r), \omega_{\lambda}^{p}(r^{*},r^{*}), \\ &\qquad \frac{1}{2}[\omega_{2\lambda}^{p}(r^{*},r) + \omega_{2\lambda}^{p}(r,r^{*})]\}. \end{split}$$

From the above inequality, we obtain

$$\psi(\omega_{\lambda}^{p}(r,r^{*})) \leq \mathcal{F}(\psi(\omega_{\lambda}^{p}(r^{*},r)),\varphi(\omega_{\lambda}^{p}(r^{*},r))) \leq \psi(\omega_{\lambda}^{p}(r^{*},r))$$

It follows that

$$\mathcal{F}(\psi(\omega_{\lambda}^{p}(r^{*},r)),\varphi(\omega_{\lambda}^{p}(r^{*},r)))=\psi(\omega_{\lambda}^{p}(r^{*},r))$$

giving $\psi(\omega_{\lambda}^{p}(r^{*}, r)) = 0$ or $\varphi(\omega_{\lambda}^{p}(r^{*}, r)) = 0$. Then, $\omega_{\lambda}^{p}(r^{*}, r) = 0$ and hence $r = r^{*}$. This is a contradiction. Therefore, by Lemma 4, the pairs $(\mathcal{P}, \mathcal{S})$ and $(\mathcal{R}, \mathcal{Q})$ have a unique common fixed point in $X_{\omega^{p}}$. \Box

Example 13. Let $X = [0, \infty)$ and define $\omega_{\lambda}^{p}(x, y) = e^{-\lambda}|x - y| + |x| + |y|$; then, ω^{p} is a partial modular metric on X. Moreover, we can verify that $X_{\omega^{p}}$ is a complete partial modular space. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S} : X_{\omega^{p}} \to X_{\omega^{p}}$ be self-mappings defined by

$$\mathcal{P}x = \frac{1}{2}x, \ \mathcal{Q}x = \frac{1}{3}x, \ \mathcal{S}x = x \ and \ \mathcal{R}x = \frac{1}{6}x, \ \forall x \in X_{\omega^p}.$$

Clearly, $\mathcal{P}X_{\omega^p} \subset \mathcal{Q}X_{\omega^p}$ and $\mathcal{R}X_{\omega^p} \subset \mathcal{S}X_{\omega^p}$. Moreover, the pairs $(\mathcal{P}, \mathcal{S})$ and $(\mathcal{Q}, \mathcal{R})$ are weakly compatible. Setting $\psi(r) = \varphi(r) = kr$ and $\mathcal{F}(s, t) = k^2s$, where $k = \frac{1}{2}$. Then, the triplet $(\psi, \varphi, \mathcal{F})$ is monotonically increasing. Now, $\forall x, y \in X_{\omega^p}$ and $\forall \lambda > 0$, and we have

$$\begin{split} \omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{R}y) &= e^{-\lambda} |\frac{1}{2}x - \frac{1}{6}y| + |\frac{1}{2}x| + |\frac{1}{6}y| \\ &= \frac{1}{2} \Big(e^{-\lambda} |x - \frac{1}{3}y| + |x| + |\frac{1}{3}y| \Big) \\ &= \frac{1}{2} \Big(e^{-\lambda} |\mathcal{S}x - \mathcal{Q}y| + |\mathcal{S}x| + |\mathcal{Q}y| \Big) \\ &\leq \frac{1}{2} \mathcal{M}(x,y). \end{split}$$

Therefore,

$$\psi(\omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{R}y)) = k\omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{R}y) \leq k^{2}M(x,y)$$

$$\leq \mathcal{F}(\psi(\mathcal{M}(x,y)),\varphi(\mathcal{M}(x,y))), \ \forall x,y \in X_{\omega^{p}} \ and \ \forall \lambda > 0.$$

Thus, all the conditions of Theorem 1 are satisfied and 0 is the unique fixed point of $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ and \mathcal{S} in X_{ω^p} .

The following theorem is the direct consequence of Theorem 1, which is a counterpart of Banach's contraction in metric space.

Theorem 2. Let ω^p be a partial modular metric on a non-empty set X and X_{ω^p} be a complete partial modular metric space. Suppose $\mathcal{P} : X_{\omega^p} \to X_{\omega^p}$ to be a self-mapping satisfying

$$\omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{P}y) \leq k\omega_{\lambda}^{p}(x,y), \ \forall \ x,y \in X_{\omega^{p}} \ and \ \forall \lambda > 0,$$

where $0 \le k < 1$; then, \mathcal{P} has a unique fixed point in X_{ω^p} .

4. Application

In this section, inspired by Pant et al. [6], we establish the existence of a solution of a system of Volterra-type integral equations.

Consider a set of Volterra-type integral equations

$$x(t) = q(t) + \int_0^t \mathcal{K}_i(t, s, x(t)) ds,$$
(29)

where $t \in [0, k] = I \subset \mathbb{R}$, $\mathcal{K}_i : [0, k] \times [0, k] \times \mathbb{R} \to \mathbb{R}$, $i = \{1, 2, 3, 4\}$ and $q : [0, k] \to \mathbb{R}$ are continuous functions.

Let $X = C(I, \mathbb{R})$ be the space of real continuous functions defined on *I*. Define ω^p on *X* by

$$\omega_{\lambda}^{p}(x,y) = \max_{t \in [0,k]} \left[e^{-\lambda} |x(t) - y(t)| + |x(t)| + |y(t)| \right], \ \forall \lambda > 0.$$

Then, X_{ω^p} is a complete modular space. Suppose $\mathcal{H}_i : X_{\omega^p} \to X_{\omega^p}$ to be a self-mapping defined by

$$\mathcal{H}_i x(t) = q(t) + \int_0^t \mathcal{K}_i(t, s, x(t)) ds, \forall x \in X_{\omega^p} \text{ and } \forall t \in I, i = \{1, 2, 3, 4\}.$$

Clearly, x(t) is a solution of (29) if and only if it is a common fixed point of \mathcal{H}_i for $i = \{1, 2, 3, 4\}$.

Theorem 3. Under the above conditions, assume that the following hypotheses hold: (*h*₁): For any $x \in X_{\omega^p}$, there exist $u, v \in X_{\omega^p}$ such that

$$\mathcal{H}_1 x = \mathcal{H}_3 u, \ \mathcal{H}_2 x = \mathcal{H}_4 v;$$

 (h_2) : For any $t \in I$, there exist $u, v \in X_{\omega^p}$ such that

$$\mathcal{H}_1\mathcal{H}_4u(t) = \mathcal{H}_4\mathcal{H}_1u(t), \text{ if } \mathcal{H}_1u(t) = \mathcal{H}_4u(t)$$

and

$$\mathcal{H}_2\mathcal{H}_3v(t) = \mathcal{H}_3\mathcal{H}_2v(t), \text{ if } \mathcal{H}_2v(t) = \mathcal{H}_3v(t);$$

(*h*₃): There exists a continuous function $f : I \times I \rightarrow \mathbb{R}^+$ such that

$$\begin{aligned} |\mathcal{K}_1(t,s,x(s)) - \mathcal{K}_3(t,s,y(s))| &\leq f(t,s) \left[|\mathcal{H}_4 x(s) - \mathcal{H}_2 y(s)| + e^{\lambda} (|\mathcal{H}_4 x(s)| + |\mathcal{H}_2 y(s)|) - 2e^{\lambda} (|\mathcal{H}_1 x(s)| + |\mathcal{H}_3 y(s)|) \right] \end{aligned}$$

 $\forall \lambda > 0 \text{ and } \forall x, y \in X_{\omega^p}, \text{ where } t, s \in I;$

(*h*₄): $\max_{t \in [0,k]} \int_0^t f(t,s) ds \le \frac{1}{2}$.

Then, the system (29) of integral equations has a unique common solution in X_{ω^p} .

Proof. From (h_1) , $\mathcal{H}_1 X_{\omega^p} \subseteq \mathcal{H}_3 X_{\omega^p}$ and $\mathcal{H}_2 X_{\omega^p} \subseteq \mathcal{H}_4 X_{\omega^p}$.

From (h_2) , the pairs $(\mathcal{H}_1, \mathcal{H}_4)$ and $(\mathcal{H}_2, \mathcal{H}_3)$ are weakly compatible. Now, from (h_3) , we have

$$\begin{split} \omega_{\lambda}^{p}(\mathcal{H}_{1}x,\mathcal{H}_{3}y) &= \max_{t \in [0,k]} \left[e^{-\lambda} |\mathcal{H}_{1}x(t) - \mathcal{H}_{3}y(t)| + |\mathcal{H}_{1}x(t)| + |\mathcal{H}_{3}y(t)| \right] \\ &\leq \max_{t \in [0,k]} \left[e^{-\lambda} \int_{0}^{t} |\mathcal{K}_{1}(t,s,r(s)) - \mathcal{K}_{3}(t,s,r(s))| ds + |\mathcal{H}_{1}x(t)| + |\mathcal{H}_{3}y(t)| \right] \\ &\leq e^{-\lambda} \max_{t \in [0,k]} \int_{0}^{t} f(t,s) ds \Big[|\mathcal{H}_{4}x - \mathcal{H}_{2}y| + e^{\lambda} (|\mathcal{H}_{4}x| + |\mathcal{H}_{2}y|) \\ &- 2e^{\lambda} (|\mathcal{H}_{1}x| + |\mathcal{H}_{3}y|) \Big] + \Big(|\mathcal{H}_{1}x| + |\mathcal{H}_{3}y| \Big) \\ &\leq \frac{1}{2} \Big[e^{-\lambda} |\mathcal{H}_{4}x - \mathcal{H}_{2}y| + |\mathcal{H}_{4}x| + |\mathcal{H}_{2}y| \Big] \\ &= \frac{1}{2} \omega_{\lambda}^{p} (\mathcal{H}_{4}x, \mathcal{H}_{2}y) \\ &\leq \frac{1}{2} \mathcal{M}(x,y), \end{split}$$

where

$$\mathcal{M}(x,y) = \max\left\{\omega_{\lambda}^{p}(\mathcal{H}_{4}x,\mathcal{H}_{2}y),\omega_{\lambda}^{p}(\mathcal{H}_{4}x,\mathcal{H}_{1}x),\omega_{\lambda}^{p}(\mathcal{H}_{2}y,\mathcal{H}_{3}y),\frac{\omega_{2\lambda}^{p}(\mathcal{H}_{2}x,\mathcal{H}_{1}x)+\omega_{2\lambda}^{p}(\mathcal{H}_{4}x,\mathcal{H}_{3}y)}{2}\right\}.$$

Setting $\psi(t) = \varphi(t) = t$ and $\mathcal{F}(s, t) = \frac{1}{2}t$, then the triplet $(\psi, \varphi, \mathcal{F})$ is monotonically increasing. Therefore,

$$\omega_{\lambda}^{p}(\mathcal{H}_{1}x,\mathcal{H}_{3}y) \leq \mathcal{F}(\psi(\mathcal{M}(x,y)),\varphi(\mathcal{M}(x,y))), \ \forall x,y \in X_{\omega^{p}} \text{ and } \forall \lambda > 0.$$

Thus, all the conditions of Theorem 1 are satisfied, and hence the system (29) has a unique solution in X_{ω^p} .

5. Conclusions

We propose a refinement of the notion of the partial modular metric to eliminate the occurrence of discrepancies in the non-zero self-distance and triangular inequality. Using the altering distance functions, a common fixed-point theorem for four self-mappings via the C – class function is proven in such space. In addition, we apply our results to establish the existence of a solution for a system of Volterra integral equations as an application.

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