



# *Article* **Fixed Point Results on Partial Modular Metric Space**

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**Abstract:** In the present paper, we refine the notion of the partial modular metric defined by Hosseinzadeh and Parvaneh to eliminate the occurrence of discrepancies in the non-zero self-distance and triangular inequality. In support of this, we discuss non-trivial examples. Finally, we prove a common fixed-point theorem for four self-mappings in partial modular metric space and an application to our result; the existence of a solution for a system of Volterra integral equations is discussed.

**Keywords:** fixed point; partial metric space; modular space; partial modular space; weakly compatible mappings; C-class function; Volterra integral equation

**MSC:** 47H10; 54H25



**Citation:** Das, D.; Narzary, S.; Singh, Y.M.; Khan, M.S.; Sessa, S. Fixed Point Results on Partial Modular Metric Space. *Axioms* **2022**, *11*, 62. [https://doi.org/10.3390/](https://doi.org/10.3390/axioms11020062) [axioms11020062](https://doi.org/10.3390/axioms11020062)

Academic Editor: Yurii Kharkevych

Received: 9 November 2021 Accepted: 28 January 2022 Published: 1 February 2022

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# **1. Introduction**

In 1992, Matthews [\[1\]](#page-16-0) initiated the idea of non-zero self-distance by introducing the notion of the partial metric as a part of the study of the denotational semantics of data flow programming languages in a topological model in computer sciences and also extended Banach's contraction principle [\[2\]](#page-16-1) in such space. Subsequently, many authors have begun to report its topological properties and obtained many fixed-point theorems in this space (for more details and references, we refer to  $[3-8]$  $[3-8]$ ). On the other hand, in 1950, Nakano [\[9\]](#page-16-4) introduced the concept of the modular in connection with the theory of order spaces, which was later developed by Musielak and Orlicz [\[10\]](#page-16-5), Khamsi [\[11\]](#page-16-6) and Kozlowski [\[12\]](#page-16-7) as modular function space.

In 2006, Chistyakov [\[13\]](#page-16-8) introduced the notion of the metric modular on an arbitrary set and the corresponding modular space, which is more general than a metric space, and, based on this, he further studied Lipschitz continuity and a class of superposition (or Nemytskii) operators on modular metric space (see also [\[14](#page-16-9)[,15\]](#page-16-10)). Recently, Hosseinzadeh and Parvaneh [\[16\]](#page-17-0) introduced the notion of partial modular metric spaces as a generalization partial metric space and gave some fixed-point results.

In this paper, we refine the concept of the partial modular metric to eliminate the occurrence of discrepancies in the non-zero self-distance and triangular inequality and prove a common fixed-point theorem for four self-mappings with a suitable example. As an application of our result, the existence of a solution for a system of Volterra integral equations is discussed.

# **2. Preliminaries**

In this section, we recall some definitions and properties to use in our result.

<span id="page-1-1"></span>**Definition 1** ([\[1\]](#page-16-0)). Let  $X \neq \emptyset$ . A function  $p : X \times X \rightarrow [0, \infty)$  is called a partial metric on X if *it satisfies:*

 $(p_1) : 0 \leq p(x, y), \forall x, y \in X \text{ and } p(x, y) = p(x, x) = p(y, y) \Longleftrightarrow x = y;$  $(p_2) : p(x, x) \leq p(x, y), \ \forall x, y \in X;$  $(p_3)$ :  $p(x, y) = p(y, x)$ ,  $\forall x, y \in X$ ;  $(p_4): p(x, y) \leq p(x, z) + p(z, y) - p(z, z), \ \forall x, y, z \in X.$ *Then, the pair* (*X*, *p*) *is called a partial metric space.*

Obviously, if  $p(x, y) = 0$ , then, from  $(p_1)$  and  $(p_2)$ , we have  $x = y$ , but the converse may not be true. Moreover, if  $(X, p)$  is a partial metric space, then the function  $d^p: X \times X \rightarrow$  $[0, \infty)$  defined by

$$
d^{p}(x, y) = 2p(x, y) - p(x, x) - p(y, y)
$$

is a metric on *X*.

**Example 1** ([\[1\]](#page-16-0)). Let  $X \neq \emptyset$  and  $c \geq 0$ . Define  $p(x, y) = |x - y| + c$ ; the p is a partial metric on *X* and the corresponding metric is  $d_p(x, y) = 2|x - y|$ ,  $\forall x, y \in X$ .

Every partial metric *p* on *X* generates a  $T_0$  topology  $τ_p$  on *X* with a base, which is defined by the family of open *p*− balls { $B_p(x, \epsilon) : x \in X$ ,  $\epsilon > 0$ }, where  $B_p(x, \epsilon) = \{u :$  $p(x, u) < p(x, x) + \epsilon$ ,  $\forall x \in X$  and  $\epsilon > 0$ .

<span id="page-1-0"></span>**Definition 2** ([\[13](#page-16-8)[–15\]](#page-16-10)). Let  $X \neq \emptyset$ . A function  $\omega : (0, +\infty) \times X \times X \rightarrow [0, \infty)$ , defined by  $\omega(\lambda, x, y) = \omega_{\lambda}(x, y)$ , is called a modular metric on X if it satisfies the following:  $(\omega_1) : \omega_\lambda(x, y) = 0 \Longleftrightarrow x = y, \ \forall \lambda > 0;$  $(\omega_2) : \omega_\lambda(x, y) = \omega_\lambda(y, x), \ \forall x, y \in X \text{ and } \forall \lambda > 0;$  $(\omega_3): \omega_{\lambda+\mu}(x, y) \leq \omega_{\lambda}(x, z) + \omega_{\mu}(z, y) \,\forall x, y, z \in X$  and  $\forall \lambda, \mu > 0$ .

If in lieu of  $(\omega_1)$ , we write

$$
(\omega_{1'}) : \omega_{\lambda}(x, x) = 0, \ \forall \lambda > 0,
$$

and then  $\omega$  is called the pseudomodular metric on *X*. Note that the function  $\lambda \mapsto \omega_{\lambda} \in$ [0, ∞) is non-decreasing. Indeed, ∀*x*, *y* ∈ *X* and ∀*λ*, *µ* > such that 0 < *µ* < *λ*; from (*ω*1) and  $(\omega_3)$ , we obtain

$$
\omega_{\lambda}(x,y) \leq \omega_{\lambda-\mu}(x,x) + \omega_{\mu}(x,y) = \omega_{\mu}(x,y).
$$

Moreover, we say that  $\omega$  is convex if it satisfies the axioms  $(\omega_1)$ ,  $(\omega_2)$  $(\omega_2)$  $(\omega_2)$  of Definition 2 and the following:

$$
(\omega_4): \ \omega_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu} \omega_{\lambda}(x,z) + \frac{\mu}{\lambda+\mu} \omega_{\mu}(z,y), \ \forall x,y,z \in X \text{ and } \forall \lambda,\mu > 0.
$$

Now, we define the following definition, a general form of convex modular metric on *X*.

**Definition 3.** *A modular metric ω defined on a non-empty set X is said to be a weak convex modular if it satisfies the axioms*  $(\omega_1)$ ,  $(\omega_2)$  $(\omega_2)$  $(\omega_2)$  *of Definition* 2 *such that there exists a function*  $\alpha$  :  $(0, \infty) \times (0, \infty) \rightarrow (0, 1)$  *satisfying the following:* 

$$
(\omega_{4'}) : \omega_{\lambda+\mu}(x,y) \leq \alpha(\lambda,\mu)\omega_{\lambda}(x,z) + (1-\alpha(\lambda,\mu))\omega_{\mu}(z,y),
$$

 $∀x, y, z ∈ X$  and  $∀λ, μ > 0$ .

Obviously, every convex modular metric is a weak convex modular metric but the converse may not be true. Moreover, every (weak) convex modular metric is a modular metric but the converse may not be true. In fact, by setting  $\alpha = \frac{\lambda}{\lambda + \mu}$ , then  $0 < \alpha < 1$  and  $0 < 1 - \alpha = \frac{\mu}{\lambda + \mu} < 1$ , so  $(\omega_4)$  and  $(\omega_{4'})$  infer directly the axiom  $(\omega_3)$  of Definition [2.](#page-1-0) Let *X*  $\neq$  *Ø* be an arbitrary set. For given  $x_0 \in X$ , we define

$$
X_{\omega}(x_0) = \{x \in X : \lim_{\lambda \to +\infty} \omega_{\lambda}(x_0, x) = 0\}
$$

and

$$
X^*_{\omega}(x_0)=\{x\in X:\exists \lambda=\lambda(x)>0,\ \omega_{\lambda}(x_0,x)<\infty\}.
$$

Then, the two sets  $X_\omega$  and  $X^*_\omega$  are called modular spaces centered at  $x_0$ . It is obvious that  $X_\omega \subseteq X_\omega^*$ . If  $x_0 \in X$  is an arbitrary, then  $X_\omega(x_0)$  and  $X_\omega^*(x_0)$  are written as  $X_\omega$  and  $X_\omega^*$ . If  $\omega$  is a modular metric on *X*, then the modular space  $X_{\omega}$  is a metric space equipped with a non-trivial metric given by

$$
d_{\omega}(x,y)=\inf\{\lambda:\omega_{\lambda}(x,y)\leq\lambda\},\ \forall x,y\in X_{\omega}.
$$

Further, if  $\omega$  is a convex modular on *X*, then  $X_{\omega} = X_{\omega}^*$ , and this common space can be equipped with a metric *d* ∗ *<sup>ω</sup>* defined by

$$
d^*_\omega(x,y)=\inf\{\lambda:\omega_\lambda(x,y)\leq 1\},\ \forall x,y\in X_\omega.
$$

If a modular metric  $\omega$  on *X* is finite and  $\omega_{\lambda}(x, y) = \omega_{\mu}(x, y)$ ,  $\forall x, y \in X$  and  $\forall \lambda, \mu > 0$ , then  $d(x, y) = \omega_{\lambda}(x, y)$  is a metric on *X*.

**Example 2** ([\[17\]](#page-17-1)). Let  $(X, d)$  be a metric space. Define  $\omega_{\lambda}(x, y) = \frac{d(x, y)}{\lambda}$ ,  $\forall x, y \in X$  and  $\forall \lambda > 0$ . *Then, ω is a modular metric on X. Moreover, ω is convex and hence it is a weak convex modular metric on X.*

**Lemma 1** ([\[14\]](#page-16-9)). Let  $\omega$  be a modular metric on a set *X*, given a sequence  $\{x_n\}_{n\in\mathbb{N}}$  in  $X_\omega$  and  $x \in X_\omega$ . Then,  $d_\omega(x_n, x) \to 0$  as  $n \to \infty$  if and only if  $\omega_\lambda(x_n, x) \to 0$  as  $n \to \infty$ ,  $\forall \lambda > 0$ . A *similar assertion holds for Cauchy sequences.*

**Example 3.** *Define*  $\omega_{\lambda}(x, y) = \frac{e^{-\lambda}|x - y|}{c}$  $\frac{f(x-y)}{c}$ ,  $c > 0 \ \forall x, y \in X$  and  $\forall \lambda > 0$ . Obviously,  $\omega$  satisfies *the axioms*  $(\omega_1)$ ,  $(\omega_2)$  *and*  $(\omega_3)$  *of Definition* [2.](#page-1-0) Therefore,  $\omega$  *is a modular metric but not a convex modular metric on X.*

*In fact,* ∀*λ*, *µ* > 0*, and we have*

$$
\omega_{\lambda+\mu}(x,y) \leq \frac{e^{-(\lambda+\mu)}}{c} [|x-z|+|z-y|]
$$
  
= 
$$
\frac{e^{\lambda}}{e^{\lambda+\mu}} \frac{e^{-\lambda}|x-z|}{c} + \frac{e^{\mu}}{e^{\lambda+\mu}} \frac{e^{-\mu}|z-y|}{c}
$$
  
= 
$$
\frac{e^{\lambda}}{e^{\lambda+\mu}} \omega_{\lambda}(x,z) + \frac{e^{\mu}}{e^{\lambda+\mu}} \omega_{\mu}(z,y).
$$

*Note that*  $0 < \frac{\lambda}{\lambda + \mu} < \frac{e^{\lambda}}{e^{\lambda + \mu}}$  $\frac{e^{\lambda}}{e^{\lambda+\mu}} < 1$  and  $0 < \frac{\mu}{\lambda+\mu} < \frac{e^{\mu}}{e^{\lambda+\mu}}$  $\frac{e^{\mu}}{e^{\lambda+\mu}} < 1$ *. Thus,*  $\omega$  *is not a convex modular metric on X.*

<span id="page-2-0"></span>**Definition 4** ([\[16\]](#page-17-0)). Let  $X \neq \emptyset$  and  $\omega^p : (0, +\infty) \times X \times X \rightarrow [0, \infty)$  be a function defined by  $\omega^p(\lambda, x, y) = \omega_\lambda^p$ *λ* (*x*, *y*)*, which is called a partial modular metric on X if it satisfies the following axioms:*

 $(\omega_1^p)$  $\binom{p}{1}$ :  $\omega_{\lambda}^p$  $\omega_{\lambda}^{p}(x, y) = \omega_{\lambda}^{p}$  $\omega_{\lambda}^{p}(x,x) = \omega_{\lambda}^{p}$  $\chi^p(y,y) \Longleftrightarrow x = y, \,\forall \lambda > 0;$  $(\omega_2^{\bar{p}})$  $\binom{p}{2}$  :  $\omega_{\lambda}^{p}$  $\frac{p}{\lambda}(x,x) \leq \omega_{\lambda}^p$  $\frac{d^p}{dt^p}(x,y)$ ,  $\forall$   $x,y \in X$  and  $\forall \lambda > 0$ ;  $(\omega_3^{\bar{p}})$  $\left(\frac{\overline{p}}{3}\right):\omega_{\lambda}^{j\overline{p}}$  $\frac{\widetilde{p}}{\lambda}(x,y) = \omega_{\lambda}^{\widetilde{p}}$  $\chi^p_{\lambda}(y, x)$ ,  $\forall x, y \in X$  and  $\forall \lambda > 0$ ;

$$
(\omega_4^p) : \omega_{\lambda+\mu}^p(x,y) \leq \omega_{\lambda}^p(x,z) + \omega_{\mu}^p(z,y) - \frac{\omega_{\lambda}^p(x,x) + \omega_{\lambda}^p(z,z) + \omega_{\mu}^p(z,z) + \omega_{\lambda}^p(y,y)}{2}, \forall x,y \in X \text{ and }
$$
  

$$
\forall \lambda, \mu > 0.
$$

As in Definition [1,](#page-1-1) the self-distance in Definition [4](#page-2-0) of a partial modular metric need not be restricted to zero, i.e.,  $\omega_\lambda^p$  $\chi^p(\mathbf{x}, \mathbf{x}) = 0$ . Note that if  $\mathbf{x} = \mathbf{y} = \mathbf{z}$ ,  $\forall \lambda, \mu > 0$ , then, from  $(\omega_4^p)$  $\binom{p}{4}$ , it follows that  $\omega_{\lambda}^{p}$  $\chi^p(x, x) = 0$ . In order to avoid this limitation, we modify the axioms  $(\omega_1^{\bar{p}})$  $\binom{\tilde{p}}{1}$  and  $(\omega_4^p)$  $\binom{p}{4}$  $\binom{p}{4}$  $\binom{p}{4}$  in Definition 4 and restate them as follows.

<span id="page-3-0"></span>**Definition 5.** Let  $X \neq \emptyset$  and  $\omega^p$  :  $(0, +\infty) \times X \times X \rightarrow [0, \infty)$  be a function defined by  $\omega^p(\lambda, x, y) = \omega_\lambda^p$  $_{\lambda}^{p}(x,y)$ , which is called a partial modular metric on X if it retains the axioms  $(\omega_{2}^{p})$  $\binom{p}{2}$ and  $(\omega_3^p)$  of Definition [4](#page-2-0) with the following:  $(\omega_1^p)$  :  $\omega_2^p$  $\begin{pmatrix} p \\ 1' \end{pmatrix}$  :  $\omega_{\lambda}^p$  $\omega_{\mu}^{p}(x, x) = \omega_{\mu}^{p}(x, x)$  and  $\omega_{\lambda}^{p}$  $\omega_{\lambda}^{p}(x,x) = \omega_{\lambda}^{p}$  $\omega_{\lambda}^{p}(x,y) = \omega_{\lambda}^{p}$  $\chi^p(\mathcal{Y}, \mathcal{Y}) \Longleftrightarrow x = \mathcal{Y}, \ \forall \lambda, \mu > 0;$  $(\omega_4^{\bar{p}})$  $\binom{\tilde{p}}{4'}$  :  $\omega_{\lambda}^{i\tilde{p}}$  $\frac{\ddot{p}}{\lambda+\mu}(x,y) \leq \omega_{\lambda}^{p}$  $\omega_{\lambda}^{\hat{p}}(x, z) + \omega_{\mu}^{\hat{p}}(z, y) - \omega_{\lambda}^{\hat{p}}$  $\bigvee^p_\lambda(z,z)$ ,  $\forall x,y \in X$  and  $\forall \lambda, \mu > 0$ .

Obviously, if  $\omega_{\lambda}^{p}$  $\chi^p(x,y) = 0$ , then, from  $(\omega_1^p)$  $\binom{p}{1'}$  and  $\left(\omega_2^p\right)$  $\binom{p}{2}$ , we have  $x = y$ , but the converse may not be true. It is not difficult to see that a partial modular metric *ω<sup>p</sup>* on *X* is a modular metric but the converse may not be true. If a partial modular metric  $\omega^p$  on *X* possesses a finite value and is independent of the parameter  $\lambda > 0$  that is  $\omega_{\lambda}^p$  $\omega_{\mu}^{p}(x,y) = \omega_{\mu}^{p}(x,y),$  $\forall \lambda, \mu > 0$ , then  $p(x, y) = \omega_{\lambda}^{p}$  $\chi^p_{\lambda}(x,y)$  is a partial metric on *X*.

**Definition 6.** *A partial modular metric*  $\omega^p$  *on X is said to be convex if, in addition to the axioms*  $(\omega_1^p)$  $\binom{p}{1'}$ ,  $\left(\omega_2^p\right)$  $\binom{p}{2}$  and  $\left(\omega_3^p\right)$ 3 )*, it satisfies the following:*

$$
(\omega_5^p): \ \omega_{\lambda+\mu}^p(x,y) \leq \frac{\lambda}{\lambda+\mu} \omega_{\lambda}^p(x,y) + \frac{\mu}{\lambda+\mu} \omega_{\mu}^p(z,y) - \frac{\lambda}{\lambda+\mu} \omega_{\lambda}^p(z,z),
$$

 $∀x, y, z ∈ X$  and  $∀λ, μ > 0$ .

**Definition 7.** *A partial modular metric*  $ω^p$  *on X is said to be weakly convex if it satisfies the* axioms  $(\omega_1^p)$  $\binom{p}{1'}$ ,  $\left(\omega_2^p\right)$  $\binom{p}{2}$ ,  $(\omega_3^p)$ 3 ) *and the following:*

$$
(\omega_{5'}^p): \ \omega_{\lambda+\mu}^p(x,y) \leq \alpha(\lambda,\mu)\omega_{\lambda}^p(x,y) + (1-\alpha(\lambda,\mu))\omega_{\mu}^p(z,y) - \alpha(\lambda,\mu)\omega_{\lambda}^p(z,z),
$$

 $\forall x, y, z \in X$  and  $\forall \lambda, \mu > 0$ , where  $\alpha : (0, \infty) \times (0, \infty) \rightarrow (0, 1)$  *is a function.* 

Now, we define the following definitions as in the modular metric:

**Definition 8.** Let  $\omega^p$  be a partial modular metric on a set X. For given  $x_0 \in X$ , we define

$$
X_{\omega^p}(x_0) = \{x \in X : \lim_{\lambda \to +\infty} \omega_\lambda^p(x_0, x) = c\},\
$$

*for some*  $c \geq 0$  *and* 

$$
X^*_{\omega^p}(x_0)=\{x\in X:\exists \lambda=\lambda(x)>0,\ \omega^p_{\lambda}(x_0,x)<\infty\}.
$$

*Then, two sets*  $X_{\omega^p}$  *and*  $X_{\omega^p}^*$  *are called partial modular spaces centered at*  $x_0$ *. It is obvious* that  $X_{\omega^p} \subset X_{\omega^p}^*$ . We write  $X_{\omega^p} \equiv X_{\omega^p}(x_0)$  and  $X^* \equiv X_{\omega^p}^*(x_0)$ , if  $x_0 \in X$  is arbitrary.

**Remark 1.** *For every*  $x, y \in X$ , the function  $\lambda \mapsto \omega_{\lambda}^{p}$  $\lambda_{\lambda}^{p} \in [0, \infty)$  *is non-increasing.* Indeed,  $\forall x \in X$  and  $0 < \mu < \lambda$ , from  $(\omega_1^p)$  $\binom{p}{1'}$  and  $\left(\omega_4^p\right)$ 4 <sup>0</sup>)*, and we obtain*

$$
\omega_{\lambda}^p(x,y) \leq \omega_{\lambda-\mu}^p(x,x) + \omega_{\mu}^p(x,y) - \omega_{\lambda-\mu}^p(x,x) = \omega_{\mu}^p(x,y).
$$

<span id="page-4-0"></span>**Lemma 2.** *Let ω<sup>p</sup> be a partial modular metric on a non-empty set X. Define*

$$
\omega_{\lambda}^{s}(x,y) = 2\omega_{\lambda}^{p}(x,y) - \omega_{\lambda}^{p}(x,x) - \omega_{\lambda}^{p}(y,y).
$$

*Then, ω<sup>s</sup> is a modular metric on X.*

**Proof.** Obviously,  $\omega^s$  holds  $(\omega_2)$  of Definition [2.](#page-1-0) For  $(\omega_1)$  and  $(\omega_3)$ , we have  $(\omega_1)$ : If  $x = y$ , then  $\omega_\lambda^s(x, y) = 0$ ,  $\forall \lambda > 0$ . Suppose  $\omega_\lambda^s(x, y) = 0$ ,  $\forall \lambda > 0$ , then

$$
2\omega_{\lambda}^p(x,y)=\omega_{\lambda}^p(x,x)+\omega_{\lambda}^p(y,y).
$$

From  $(\omega_2^p)$  $2^{p \choose 2}$  of Definition [5,](#page-3-0) we obtain

$$
2\omega_\lambda^p(x,x) \leq 2\omega_\lambda^p(x,y) = \omega_\lambda^p(x,x) + \omega_\lambda^p(y,y) \implies \omega_\lambda^p(x,x) \leq \omega_\lambda^p(y,y).
$$

Similarly, we obtain

$$
2\omega_{\lambda}^p(y,y) \leq 2\omega_{\lambda}^p(x,y) = \omega_{\lambda}^p(x,x) + \omega_{\lambda}^p(y,y) \implies \omega_{\lambda}^p(y,y) \leq \omega_{\lambda}^p(x,x).
$$

Consequently, we obtain

$$
\omega_\lambda^p(x,y) = \omega_\lambda^p(x,x) = \omega_\lambda^p(y,y).
$$

Thus, by the second part of  $(\omega_1^p)$  $\binom{p}{1'}$  of Definition [5,](#page-3-0)  $x = y$ .  $(\omega_3)$ : From  $(\omega_1^p)$  of Definition [5,](#page-3-0) we obtain 1 *ω p*  $\omega_{\lambda+\mu}^p(x,x) = \omega_{\lambda}^p$  $\frac{p}{\lambda}(x, x)$  and  $\omega_{\lambda}^{p}$  $\mu_{\lambda+\mu}^p(y,y) = \omega_{\lambda}^p$ *λ* (*y*, *y*), ∀*x*, *y* ∈ *X* and ∀*λ*, *µ* > 0. Now, by  $(\omega_4^p)$  $\binom{p}{4}$  of Definition [5,](#page-3-0) we have

$$
\omega_{\lambda+\mu}^{s}(x,y) = 2\omega_{\lambda+\mu}^{p}(x,y) - \omega_{\lambda+\mu}^{p}(x,x) - \omega_{\lambda+\mu}^{p}(y,y)
$$
  
\n
$$
= 2\omega_{\lambda+\mu}^{p}(x,y) - \omega_{\lambda}^{p}(x,x) - \omega_{\mu}^{p}(y,y)
$$
  
\n
$$
\leq 2(\omega_{\lambda}^{p}(x,z) + \omega_{\mu}^{p}(z,y) - \omega_{\lambda}^{p}(z,z)) - \omega_{\lambda}^{p}(x,x) - \omega_{\mu}^{p}(y,y)
$$
  
\n
$$
= (2\omega_{\lambda}^{p}(x,z) - \omega_{\lambda}^{p}(x,x) - \omega_{\lambda}^{p}(z,z)) + (2\omega_{\mu}^{p}(z,y) - \omega_{\lambda}^{p}(z,z)) - \omega_{\mu}^{p}(y,y))
$$
  
\n
$$
= \omega_{\lambda}^{s}(x,z) + \omega_{\mu}^{s}(z,y).
$$

Thus,  $\omega^s$  satisfies the axioms  $(\omega_1)$ ,  $(\omega_2)$  $(\omega_2)$  $(\omega_2)$  and  $(\omega_3)$  of Definition 2 and hence  $\omega^s$  is a modular metric.

**Remark 2.** *(i) Let ω<sup>s</sup> be a modular metric induced by partial modular metric ω<sup>p</sup> on a non-empty*  $s$ et  $X$ , and then  $X_{\omega^s}$  shall denote the modular space with respect to modular metric  $\omega^s.$ *(ii)* Let  $\omega$  be a modular metric on X and  $c \geq 0$ ; then,

$$
\omega_\lambda^p(x,y)=\omega_\lambda(x,y)+c
$$

*defines a partial modular metric on X and the corresponding modular metric is*  $\omega_{\lambda}^{s}(x, y)$  =  $2\omega_{\lambda}(x, y)$  or  $\omega_{\lambda}^{s}(x, y) = 2(\omega_{\lambda}^{p})$  $\int_{\lambda}^{p}(x,y)-c$ ). Moreover,  $\omega^{p}$  *is (weakly) convex if*  $\omega$  *is a (weakly) convex modular metric with*  $c = 0$  *on X.* 

**Example 4.** Let  $X = \mathbb{R}$ . Define a function  $\omega_{\lambda}^{p}$  $\lambda_{\lambda}^{p}$  :  $(0, \infty) \times X \times X \rightarrow [0, \infty)$  by

$$
\omega_\lambda^p(x,y) = e^{-\lambda}|x-y| + c,
$$

 $\omega$  *where c*  $\geq$  0,  $\lambda$   $>$  0 and  $\forall$ x,  $y \in X$ . Then,  $\omega^p$  is a partial modular metric on X.

**Example 5.** Let  $(X, d)$  be a metric space and a function  $\omega^p$  be defined by

$$
\omega_{\lambda}^{p}(x,y)=\frac{d(x,y)+c}{\lambda}, \ \forall x,y\in X, \ and \ \forall \lambda>0
$$

*where*  $c \geq 0$ *. We see that*  $\lim_{\lambda \to +\infty} \omega_{\lambda}^p$  $\chi^p(x,y) = 0$ ,  $\forall x, y \in X$ . However,  $\omega^p$  is not a partial *modular metric on X. Indeed, by the first part of*  $(\omega_1^p)$  $_{1'}^p$ ) of Definition [5,](#page-3-0)  $\omega_\lambda^p$  $\omega_{\mu}^{p}(x, x) \neq \omega_{\mu}^{p}(x, x), \ \forall x \in$ *X* and  $\forall \lambda, \mu > 0, \lambda \neq \mu$ .

**Example 6.** *Let*  $X = \mathbb{R}$ *. Define* 

$$
\omega_{\lambda}^{p}(x,y)=e^{-\lambda}|x-y|+|x|+|y|, \ \forall x,y\in X \ and \ \forall \lambda>0.
$$

*Then,*  $\omega^p$  is a partial modular metric on X. It is obvious that  $(\omega_1^p)$  $\binom{p}{1'}$ ,  $\left(\omega_2^p\right)$  $\binom{p}{2}$  and  $\left(\omega_3^p\right)$  $\binom{p}{3}$  of *Definition [5](#page-3-0) hold. For*  $(\omega_4^p)$ 4 <sup>0</sup>)*,* ∀*λ*, *µ* > 0 *and* ∀*x*, *y*, *z* ∈ *X, we have*

$$
\omega_{\lambda+\mu}^{p}(x,y) = e^{-(\lambda+\mu)}|x-y| + |x| + |y|
$$
  
\n
$$
\leq e^{-(\lambda+\mu)}\Big(|x-z| + |z-y|\Big) + |x| + |y|
$$
  
\n
$$
= \Big(e^{-(\lambda+\mu)}|x-z| + |x|\Big) + \Big(e^{-(\lambda+\mu)}|z-y| + |y|\Big)
$$
  
\n
$$
\leq \Big(e^{-\lambda}|x-z| + |x| + |z|\Big) + \Big(e^{-\mu}|z-y| + |z| + |y|\Big) - 2|z|
$$
  
\n
$$
= \omega_{\lambda}^{p}(x,z) + \omega_{\mu}^{p}(z,y) - \omega_{\lambda}^{p}(z,z).
$$

*Thus, ω<sup>p</sup> is a partial modular metric on X.*

**Example 7.** Let  $X \neq \emptyset$  be a set. Define  $\omega_{\lambda}^{p}$  $\int_{\lambda}^{p}(x,y) = \frac{|x-y|}{\lambda} + c, c > 0, \ \forall x, y \in X$  and  $\forall \lambda > 0$ . *It is obvious that*  $(\omega_1^p)$  $\binom{p}{1'}$ ,  $\left(\omega_2^p\right)$  $\binom{p}{2}$  and  $\left(\omega_3^p\right)$  $\binom{p}{3}$  of Definition [5](#page-3-0) hold. Now, we show that  $\omega^p$  is a partial *modular metric and but not (weakly) convex on X. For*  $(\omega_4^p)$ 4 <sup>0</sup>)*,* ∀*x*, *y*, *z* ∈ *X and* ∀*λ*, *µ* > 0*, we have*

$$
\omega_{\lambda+\mu}^p(x,y) = \frac{|x-y|}{\lambda+\mu} + c
$$
  
\n
$$
\leq \frac{1}{\lambda+\mu} (|x-z| + |z-y|) + c
$$
  
\n
$$
\leq \left(\frac{|x-z|}{\lambda} + c\right) + \left(\frac{|z-y|}{\mu} + c\right) - c
$$
  
\n
$$
= \omega_{\lambda}^p(x,z) + \omega_{\mu}^p(z,y) - \omega_{\mu}^p(z,z).
$$

*Then,*  $\omega^p$  *is a partial modular metric on X.* On the other hand,  $\forall \lambda, \mu > 0$  and  $\forall x, y, z \in X$ , *and we have*

$$
\omega_{\lambda+\mu}^p(x,y) \le \frac{1}{\lambda+\mu} \left( |x-z| + |z-y| \right) + c
$$
  
= 
$$
\frac{\lambda}{\lambda+\mu} \left( \frac{|x-z|}{\lambda} + c \right) + \frac{\mu}{\lambda+\mu} \left( \frac{|z-y|}{\mu} + c \right)
$$
  
= 
$$
\frac{\lambda}{\lambda+\mu} \omega_{\lambda}^p(x,z) + \frac{\mu}{\lambda+\mu} \omega_{\mu}^p(z,y).
$$

*To show that*  $\omega^p$  *is not convex on*  $X$ *,*  $\forall \lambda$ *,*  $\mu > 0$ *, taking*  $x = 4$ *,*  $y = 1$ *,*  $z = 2$ *, then* 

$$
\left(\frac{\lambda}{\lambda+\mu}\omega_{\lambda}^{p}(x,z)+\frac{\mu}{\lambda+\mu}\omega_{\mu}^{p}(z,y)-\frac{\lambda}{\lambda+\mu}\omega_{\mu}^{p}(z,z)\right)-\omega_{\lambda+\mu}^{p}(x,y)
$$
\n
$$
=\left(\frac{3}{\lambda+\mu}+c-\frac{\lambda}{\lambda+\mu}c\right)-\left(\frac{3}{\lambda+\mu}+c\right)
$$
\n
$$
=-\frac{\lambda}{\lambda+\mu}c<0.
$$

*This shows that*  $ω<sup>p</sup>$  *is not convex and, hence, it is not a weakly convex partial modular metric on X.*

**Example 8.** Let  $\omega^p$  be a partial modular metric on a non-empty set X. Define  $\omega^p_\lambda$  $\frac{f}{\lambda}(x,y) =$  $\frac{ω_λ(x,y)}{λ}$ ,  $∀x, y ∈ X$  *and* ∀ $λ > 0$ . Then,  $ω^p$  is convex and hence it is a weakly convex partial *modular metric on X.*

**Example 9.** For any non-empty set *X*, define  $\omega_{\lambda}^{p}$  $\int_{\lambda}^{p}(x,y) = e^{-\lambda} \omega_{\lambda}(x,y)$ ,  $\forall x, y \in X$  and  $\forall \lambda > 0$ . *Then, ω<sup>p</sup> is weakly convex but is not a convex partial modular metric on X.*

**Definition 9.** Let  $\omega^p$  be a partial modular metric on a non-empty set *X* and  $\{x_n\}$  be a sequence in *a partial modular space Xω<sup>p</sup> ; then,*

(*i*)  $\{x_n\}$  *is said to be convergent to a point*  $x \in X_{\omega^p}$ *, if and only if, for every*  $\epsilon > 0$ *, there exists*  $n_0 \in \mathbb{N} \cup \{0\}$  *such that* 

$$
|\omega_{\lambda}^p(x_n,x)-\omega_{\lambda}^p(x,x)|\leq \epsilon,
$$

 $\forall n \ge n_0$  and  $\forall \lambda > 0$ . We write  $\lim_{n \to +\infty} \omega_{\lambda}^p(x_n, x) = \omega_{\lambda}^p(x, x)$ ,  $\forall \lambda > 0$ ;

- (*ii*) *a* sequence  $\{x_n\}$  *is a Cauchy in*  $X_{\omega^p}$  *if*  $\lim_{n,m\to+\infty} \omega^p_{\lambda}(x_n, x_n)$  $\chi^p(\mathbf{x}_n, \mathbf{x}_m) = c$ ,  $\forall \lambda > 0$ , for some  $c \geq 0$ . *In this case,*  $\lim_{n\to+\infty}\omega_\lambda^p$  $\int_{\lambda}^{p} (x_n, x_n) = \lim_{m \to +\infty} \omega_{\lambda}^{p}$  $\chi^p(\mathbf{x}_m, \mathbf{x}_m) = c$ . Thus, if  $\{\mathbf{x}_n\}$  is a Cauchy *sequence in*  $X_{\omega^s}$ *, then*  $c = 0$ *;*
- (*iii*) *a partial modular space Xω<sup>p</sup> is said to be complete if every Cauchy sequence converges to a point*  $x \in X_{\omega^p}$  *such that*

$$
\lim_{n,m\to+\infty}\omega_\lambda^p(x_n,x_m)=\omega_\lambda^p(x,x),\,\,\forall\lambda>0.
$$

**Remark 3.** (*i*) *If*  $\{x_n\}$  *is a Cauchy sequence in*  $X_{\omega^s}$ *, i.e.*,  $\lim_{n,m\to+\infty}\omega^s_\lambda(x_n,x_m)=0$ *, then* 

$$
\lim_{n,m\to+\infty}\omega_\lambda^p(x_n,x_m)=\lim_{n\to+\infty}\omega_\lambda^p(x_n,x_n)=\lim_{m\to+\infty}\omega_\lambda^p(x_m,x_m).
$$

(*ii*) *If*  $\{x_n\}$  *is a Cauchy sequence in*  $X_{\omega^s}$  *that converges to some point*  $x \in X_{\omega^s}$ *, then* 

$$
\lim_{n,m\to+\infty}\omega_\lambda^p(x_n,x_m)=\lim_{n\to+\infty}\omega_\lambda^p(x_n,x_n)=\omega_\lambda^p(x,x).
$$

(*iii*) *A sequence*  $\{x_n\}$  *in*  $X_{\omega^p}$  *is a Cauchy sequence if it is a Cauchy sequence in*  $X_{\omega^s}$ *, i.e.,*  $\lim_{n,m\to+\infty}\omega_{\lambda}^{s}(x_{n},x_{m})=0.$ 

<span id="page-6-0"></span>**Lemma 3.** Let  $\omega^p$  be a partial modular on X and  $\{x_n\}$  be a sequence in  $X_{\omega^p}$ . Then,

- $(i)$   $\{x_n\}$  *is a Cauchy sequence in*  $X_{\omega^p}$  *if it is a Cauchy sequence in the modular space*  $X_{\omega^s}$  *induced by partial modular metric ω<sup>p</sup> ;*
- (*ii*) *a partial modular space*  $X_{\omega^p}$  *is complete if and only if the modular space*  $X_{\omega^s}$  *induced by*  $\omega^p$ *is complete. Furthermore,*

$$
\lim_{n\to+\infty}\omega_\lambda^s(x_n,x)=0\Longleftrightarrow\lim_{n\to\infty}[2\omega_\lambda^p(x_n,x)-\omega_\lambda^p(x_n,x_n)-\omega_\lambda^p(x.x)]=0
$$

*or*

$$
\lim_{n\to\infty}\omega_{\lambda}^{s}(x_{n}, x)=0 \iff \lim_{n\to+\infty}\omega_{\lambda}^{p}(x_{n}, x)=\lim_{n\to+\infty}\omega_{\lambda}^{p}(x_{n}, x_{n})=\omega_{\lambda}^{p}(x, x), \ \forall \lambda>0.
$$

**Definition 10** ([\[18\]](#page-17-2)). *A continuous function*  $\mathcal{F} : [0, \infty) \times [0, \infty) \to \mathbb{R}$  *is called a*  $C-$  *class function if, for any s,*  $t \in \mathbb{R}$ *, the following conditions hold:* 

- $(f_i)$   $\mathcal{F}(s,t) \leq s$ ;
- (*ii*)  $\mathcal{F}(s,t) = s$  *implies*  $s = 0$  *or*  $t = 0$ *.*

**Example 10** ([\[18\]](#page-17-2))**.** *The following are examples of the* C−*class function:*

- (*i*)  $\mathcal{F}(s,t) = \alpha s, \ \alpha \in (0,1);$
- $(i)$   $\mathcal{F}(s,t) = \frac{s}{(1+t)^r}, r \in (0,\infty);$
- $(iii)$   $\mathcal{F}(s,t) = \frac{\log(t+\alpha^s)}{(1+t)}$  $\frac{g(t+\alpha^{2})}{(1+t)}$ ,  $\alpha > 1$ .

**Definition 11** ([\[19\]](#page-17-3)). *A control function*  $\psi$  :  $[0, \infty) \to [0, \infty)$  *is called an altering distance if the following conditions hold:*

- (*i*) *ψ is non-decreasing and continuous;*
- $(iii) \quad \psi(t) = 0$  *if and only if*  $t = 0$ .

*We denote by* Ψ *the set of all altering distance functions.*

**Example 11** ([\[20\]](#page-17-4))**.** *The following examples are the altering distance functions:*

- $(i)$   $\psi(t) = e^{\alpha t} + \beta t 1;$
- (*ii*)  $\psi(t) = \alpha t^2 + \ln(\beta t + 1)$ , *where*  $\alpha, \beta > 0$ .

**Definition 12** ([\[18\]](#page-17-2)). *A control function*  $\varphi$  :  $[0,\infty) \to [0,\infty)$  *is called an ultra-altering distance if the following conditions hold:*

- (*i*) *ϕ is continuous;*
- (*ii*)  $\varphi(t) > 0$ ,  $t > 0$  and  $\varphi(0) \ge 0$ .

Φ *denotes the set of all ultra-altering distance functions.*

**Definition 13** ([\[21\]](#page-17-5)). *A triplet*  $(\psi, \varphi, \mathcal{F})$ *, where*  $\psi \in \Psi$ *,*  $\varphi \in \Phi$  *and*  $\mathcal{F} \in \mathcal{C}$  *is monotonically increasing if*

$$
\forall x, y \in [0, \infty), \ x \leq y \Longrightarrow \mathcal{F}(\psi(x), \varphi(x)) \leq \mathcal{F}(\psi(y), \varphi(y)).
$$

*Further, we say that the triplet* (*ψ*, *ϕ*, F) *is strictly monotonically increasing if*

$$
\forall x, y \in [0, \infty) \ x < y \Longrightarrow \mathcal{F}(\psi(x), \varphi(x)) < \mathcal{F}(\psi(y), \varphi(y)).
$$

**Example 12** ([\[21\]](#page-17-5)). *Consider a*  $C -$  *class function*  $\mathcal{F}(s,t) = s - t$ . Define  $\psi$ ,  $\varphi : [0,\infty) \to [0,\infty)$ *by*  $\varphi(x) = \sqrt{x}$  and

$$
\psi(x) = \begin{cases} \sqrt{x}, & 0 \le x \le 1; \\ x^2, & x > 1. \end{cases}
$$

*Obviously, the triplet*  $(\psi, \varphi, \mathcal{F})$  *is monotonically increasing.* 

**Definition 14** ([\[22\]](#page-17-6))**.** *Let* P *and* Q *be two self-mappings on a non-empty set X; then, they are said to be weakly compatible if they commute at their coincidence points, i.e.,* PQ*x* = QP*x, for some x* ∈ *X.*

**Definition 15** ([\[23\]](#page-17-7)). Let  $X \neq \emptyset$  and  $P$ ,  $Q : X \rightarrow X$  be two self-mappings. If  $u = Px = Qx$ , *for some*  $x \in X$ *, then*  $x$  *is called a coincidence point of*  $P$  *and*  $Q$ *, and*  $u$  *is called a point of coincidence (briefly, poc) of*  $P$  *and*  $Q$ *.* 

<span id="page-8-6"></span>**Lemma 4** ([\[23\]](#page-17-7))**.** *If* P *and* Q *are weakly compatible self-mappings on a non-empty set X, and if* P *and*  $Q$  *have a unique point of coincidence*  $u = Px = Qx$ , then *u is the unique common fixed-point* P *and* Q*.*

### **3. Main Results**

Let  $\omega^p$  be a partial modular metric on a non-empty set *X* and  $X_{\omega^p}$  be a partial modular space. Suppose that  $\mathcal{P}$ ,  $\mathcal{Q}$ ,  $\mathcal{R}$ ,  $\mathcal{S}: X_{\omega^p} \to X_{\omega^p}$  are four self-mappings such that

<span id="page-8-1"></span><span id="page-8-0"></span>
$$
\mathcal{P}X_{\omega^p} \subseteq \mathcal{Q}X_{\omega^p} \text{ and } \mathcal{R}X_{\omega^p} \subseteq \mathcal{S}X_{\omega^p}.
$$
 (1)

Let  $x_0 \in X_{\omega^p}$  be any point. By virtue of [\(1\)](#page-8-0), the two sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X_{\omega^p}$ are defined as follows:

$$
y_{2n} = \mathcal{P}x_{2n} = \mathcal{Q}x_{2n+1} \text{ and } y_{2n+1} = \mathcal{R}x_{2n+1} = \mathcal{S}x_{2n+2}, \forall n \in \mathbb{N} \cup \{0\}. \tag{2}
$$

Inspired by Chandok et al. [\[4\]](#page-16-11), we are ready to prove the following lemma, which plays a crucial role in the subsequent results.

<span id="page-8-5"></span>**Lemma 5.** *Let ω<sup>p</sup> be a partial modular metric on a non-empty set X and Xω<sup>p</sup> be a partial modular space. Suppose that*  $P$ ,  $Q$ ,  $R$ ,  $S$  :  $X_{\omega^p} \to X_{\omega^p}$  *are four self-mappings satisfying the condition [\(1\)](#page-8-0). If there exist*  $\psi \in \Psi$ ,  $\varphi \in \Phi$  *and*  $\mathcal{F} \in \mathcal{C}$  *such that the triplet*  $(\psi, \varphi, \mathcal{F})$  *is a monotonically increasing function satisfying the following:*

<span id="page-8-2"></span>
$$
\psi(\omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{R}y)) \leq \mathcal{F}(\psi(\mathcal{M}(x,y)),\varphi(\mathcal{M}(x,y))),\tag{3}
$$

*where*

$$
\mathcal{M}(x,y) = \max \{ \omega_{\lambda}^{p}(\mathcal{S}x,\mathcal{Q}y), \omega_{\lambda}^{p}(\mathcal{S}x,\mathcal{P}x), \omega_{\lambda}^{p}(\mathcal{Q}y,\mathcal{R}y), \\ \frac{1}{2} [\omega_{2\lambda}^{p}(\mathcal{Q}y,\mathcal{P}x) + \omega_{2\lambda}^{p}(\mathcal{S}x,\mathcal{R}y)] \},
$$

 $∀λ > 0$  and  $∀x, y ∈ X<sub>ω</sub><sup>*p*</sup>$ . Then, the sequence  ${y<sub>n</sub>}$  defined by [\(2\)](#page-8-1) is a Cauchy sequence in  $X<sub>ω</sub><sup>*p*</sup>$ .

**Proof.** From [\(2\)](#page-8-1), we recall that

$$
y_{2n} = \mathcal{P}x_{2n} = \mathcal{Q}x_{2n+1}
$$
 and  $y_{2n+1} = \mathcal{R}x_{2n+1} = \mathcal{S}x_{2n+2}, \forall n \in \mathbb{N} \cup \{0\}.$ 

Using [\(3\)](#page-8-2), we obtain

<span id="page-8-4"></span><span id="page-8-3"></span>
$$
\psi(\omega_{\lambda}^p(y_{2n}, y_{2n+1})) = \psi(\omega_{\lambda}^p(\mathcal{P}x_{2n}, \mathcal{R}x_{2n+1}))
$$
  
\n
$$
\leq \mathcal{F}(\psi(\mathcal{M}(x_{2n}, x_{2n+1})), \varphi(\mathcal{M}(x_{2n}, x_{2n+1}))),
$$
\n(4)

where

$$
\mathcal{M}(x_{2n}, x_{2n+1}) = \max \{ \omega_{\lambda}^{p}(\mathcal{S}x_{2n}, \mathcal{Q}x_{2n+1}), \omega_{\lambda}^{p}(\mathcal{S}x_{2n}, \mathcal{P}x_{2n}), \omega_{\lambda}^{p}(\mathcal{Q}x_{2n+1}, \mathcal{R}x_{2n+1}), \quad (5) \}
$$
  
\n
$$
= \frac{1}{2} [\omega_{2\lambda}^{p}(\mathcal{Q}x_{2n+1}, \mathcal{P}x_{2n}) + \omega_{2\lambda}^{p}(\mathcal{S}x_{2n}, \mathcal{R}x_{2n+1})] \}
$$
  
\n
$$
= \max \{ \omega_{\lambda}^{p}(y_{2n-1}, y_{2n}), \omega_{\lambda}^{p}(y_{2n-1}, y_{2n}), \omega_{\lambda}^{p}(y_{2n}, y_{2n+1}), \quad \frac{1}{2} [\omega_{2\lambda}^{p}(y_{2n}, y_{2n}) + \omega_{2\lambda}^{p}(y_{2n-1}, y_{2n+1})] \}.
$$

and by  $(\omega_1^p)$  $\binom{p}{1'}$  and  $\left(\omega_4^p\right)$  $\binom{p}{4}$ , we have

<span id="page-9-0"></span>
$$
\frac{1}{2} [\omega_{2\lambda}^p (y_{2n}, y_{2n}) + \omega_{2\lambda}^p (y_{2n-1}, y_{2n+1})]
$$
\n
$$
\leq \frac{1}{2} [\omega_{\lambda}^p (y_{2n}, y_{2n}) + \omega_{\lambda}^p (y_{2n-1}, y_{2n}) + \omega_{\lambda}^p (y_{2n}, y_{2n+1}) - \omega_{\lambda}^p (y_{2n}, y_{2n})]
$$
\n
$$
= \frac{1}{2} [\omega_{\lambda}^p (y_{2n-1}, y_{2n}) + \omega_{\lambda}^p (y_{2n}, y_{2n+1})].
$$
\n(6)

Using [\(5\)](#page-8-3), [\(6\)](#page-9-0) and the monotonicity of the triplet  $(\psi, \varphi, \mathcal{F})$ , [\(4\)](#page-8-4) becomes

<span id="page-9-1"></span>
$$
\psi(\omega_{\lambda}^{p}(y_{2n}, y_{2n+1})) \leq \mathcal{F}(\psi(\max{\{\omega_{\lambda}^{p}(y_{2n-1}, y_{2n}), \omega_{\lambda}^{p}(y_{2n}, y_{2n+1})\}}),
$$
\n
$$
\varphi(\max{\{\omega_{\lambda}^{p}(y_{2n-1}, y_{2n}), \omega_{\lambda}^{p}(y_{2n}), y_{2n+1})\}})),
$$
\n(7)

From the above inequality, the following cases arise:

Case (*I*): Suppose  $\omega_{\lambda}^{p}$  $\frac{p}{\lambda}(y_{2n-1}, y_{2n}) < \omega_{\lambda}^{\tilde{p}}$  $\chi_l^p(y_{2n}, y_{2n+1})$ ; then, from [\(7\)](#page-9-1) and by the strict monotonicity of (*ψ*, *ϕ*, F), we obtain

$$
\psi(\omega_\lambda^p(y_{2n},y_{2n+1})) < \mathcal{F}(\psi(\omega_\lambda^p(y_{2n},y_{2n+1}),\varphi(\omega_\lambda^p(y_{2n},y_{2n+1})))\leq \psi(\omega_\lambda^p(y_{2n},y_{2n+1})).
$$

Therefore,  $\omega_\lambda^p$  $\frac{p}{\lambda}(y_{2n}, y_{2n+1}) < \omega_{\lambda}^{p}$  $\chi^p(y_{2n}, y_{2n+1})$ . This is a contradiction. Case (*II*): Suppose  $\omega_{\lambda}^{p}$  $\omega_{\lambda}^{p}(y_{2n}, y_{2n+1}) \leq \omega_{\lambda}^{p}$ *λ* (*y*2*n*−1, *y*2*n*); then, from [\(7\)](#page-9-1), we obtain

$$
\psi(\omega_{\lambda}^p(y_{2n},y_{2n+1})) \leq \mathcal{F}(\psi(\omega_{\lambda}^p(y_{2n-1},y_{2n})) , \varphi(\omega_{\lambda}^p(y_{2n-1},y_{2n})). \tag{8}
$$

Since  $\psi$  is a non-increasing function, then, from  $(8)$ , we have

$$
\psi(\omega_{\lambda}^p(y_{2n},y_{2n+1})) \leq \psi(\omega_{\lambda}^p(y_{2n-1},y_{2n})) \Longrightarrow \omega_{\lambda}^p(y_{2n},y_{2n+1})) \leq \omega_{\lambda}^p(y_{2n-1},y_{2n}).
$$

This shows that  $\{\omega_\lambda^p\}$  $\chi^p(\mathcal{Y}_{2n}, \mathcal{Y}_{2n+1})$ } is a non-increasing sequence of non-negative real numbers. Thus, there exists  $\epsilon \geq 0$  such that

<span id="page-9-2"></span>
$$
\lim_{n\to+\infty}\omega_{\lambda}^p(y_{2n},y_{2n+1})=\epsilon,\,\,\forall\lambda>0.
$$

Taking the limit as  $n \rightarrow +\infty$  in [\(8\)](#page-9-2), we obtain

$$
\psi(\epsilon) \leq \mathcal{F}(\psi(\epsilon), \varphi(\epsilon)) \leq \psi(\epsilon) \Longrightarrow \mathcal{F}(\psi(\epsilon), \varphi(\epsilon)) = \psi(\epsilon),
$$

so  $\psi(\epsilon) = 0$  or  $\varphi(\epsilon) = 0$  and hence  $\epsilon = 0$ , i.e.,

<span id="page-9-3"></span>
$$
\lim_{n \to +\infty} \omega_{\lambda}^p(y_{2n}, y_{2n+1}) = 0, \ \forall \lambda > 0. \tag{9}
$$

Now, we show that  $\{y_n\}$  is a Cauchy sequence in  $X_{\omega^p}$ . By Lemma [3,](#page-6-0) it is sufficient to prove that a subsequence  $\{y_{2n}\}$  of  $\{y_n\}$  is a Cauchy sequence in  $X_{\omega^s}$ .

From  $(\omega_2^p)$  $2^{p\choose 2}$  of Definition [4,](#page-2-0) we have

$$
0\leq \omega_\lambda^p(y_{2n},y_{2n})\leq \omega_\lambda^p(y_{2n},y_{2n+1}),\,\,\forall \lambda>0,
$$

so from [\(9\)](#page-9-3), it follows that

<span id="page-9-4"></span>
$$
\lim_{n \to +\infty} \omega_{\lambda}^{p}(y_{2n}, y_{2n}) = 0. \tag{10}
$$

Similarly,  $\lim_{n\to+\infty}\omega_\lambda^p$ *λ* (*y*2*n*+1, *y*2*n*+1) = 0, ∀*λ* > 0.

If possible, let  $\{y_{2n}\}$  be not a Cauchy in  $X_{\omega^s}$ , and then there exists  $\delta > 0$  such that, for each even  $+ve$  integer *k*, we can find subsequence  $\{y_{2m(k)}\}$  and  $\{y_{2n(k)}\}$  of  $\{y_{2n}\}$  with  $2n(k) > 2m(k) \geq k$  such that

<span id="page-10-0"></span>
$$
\omega_{\lambda}^{s}(y_{2m(k)}, y_{2n(k)}) > \delta, \ \forall \ \lambda > 0. \tag{11}
$$

Now, we choose  $2n(k)$  corresponding to  $2m(k)$  such that it is the smallest even integer with  $2n(k) > 2m(k)$  and satisfies Inequality [\(11\)](#page-10-0). Hence,

<span id="page-10-3"></span><span id="page-10-2"></span><span id="page-10-1"></span>
$$
\omega_{\lambda}^{s}(y_{2m(k)}, y_{2n(k)-1}) \leq \delta, \ \forall \lambda > 0. \tag{12}
$$

By triangular inequality  $(\omega_3)$  and  $(12)$ , we have

$$
\omega_{\lambda}^{s}(y_{2m(k)}, y_{2n(k)}) \leq \omega_{\frac{\lambda}{2}}^{s}(y_{2m(k)}, y_{2n(k)-1}) + \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)-1}, y_{2n(k)})
$$
\n
$$
\leq \delta + \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)-1}, y_{2n(k)}).
$$
\n(13)

On the other hand, by Lemma [2,](#page-4-0)  $\forall \lambda > 0$ , we have

$$
\omega_{\lambda}^{s}(y_{2n(k)-1}, y_{2n(k)}) = 2\omega_{\lambda}^{p}(y_{2n(k)-1}, y_{2n(k)}) - \omega_{\lambda}^{p}(y_{2n(k)-1}, y_{2n(k)-1}) - \omega_{\lambda}^{p}(y_{2n(k)}, y_{2n(k)}).
$$
\n(14)

Letting  $k \to +\infty$  on [\(14\)](#page-10-2), then from [\(9\)](#page-9-3) and [\(10\)](#page-9-4),  $\forall \lambda > 0$ , we have

<span id="page-10-4"></span>
$$
\lim_{k \to +\infty} \omega_{\lambda}^{s} (y_{2n(k)-1}, y_{2n(k)}) = 0, \ \forall \lambda > 0.
$$
 (15)

From  $(13)$ , using  $(11)$  and  $(15)$ , we have

$$
\delta < \lim_{k \to \infty} \omega_{\lambda}^s (y_{2m(k)}, y_{2n(k)}) \leq \delta, \ \forall \lambda > 0.
$$

This implies

<span id="page-10-7"></span><span id="page-10-5"></span>
$$
\lim_{k \to +\infty} \omega^s_\lambda(y_{2m(k)}, y_{2n(k)}) = \delta. \tag{16}
$$

Again, using the triangular inequality  $(\omega_3)$ , we have

$$
\omega_{\lambda}^{s}(y_{2n(k)}, y_{2m(k)}) \leq \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)}, y_{2n(k)-1}) + \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)-1}, y_{2m(k)})
$$
\n
$$
\leq \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)}, y_{2n(k)-1}) + \omega_{\frac{\lambda}{4}}^{s}(y_{2n(k)-1}, y_{2m(k)-1})
$$
\n
$$
+ \omega_{\frac{\lambda}{4}}^{s}(y_{2m(k)-1}, y_{2m(k)}).
$$
\n(17)

Furthermore, we have

$$
\omega_{\lambda}^{s}(y_{2n(k)-1}, y_{2m(k)-1}) \leq \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)-1}, y_{2n(k)}) + \omega_{\frac{\lambda}{4}}^{s}(y_{2n(k)}, y_{2m(k)}) + \omega_{\frac{\lambda}{4}}^{s}(y_{2m(k)}, y_{2m(k)-1})
$$
\n(18)

Letting the limit as  $k \to +\infty$  in [\(17\)](#page-10-5) and [\(18\)](#page-10-6), using [\(15\)](#page-10-4) and [\(16\)](#page-10-7), we obtain

<span id="page-10-8"></span><span id="page-10-6"></span>
$$
\lim_{k\to\infty}\omega_\lambda^s(y_{2n(k)-1},y_{2m(k)-1})=\delta.
$$

Further, we have

$$
\omega_{\lambda}^{s}(y_{2n(k)+1}, y_{2m(k)}) \leq \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)+1}, y_{2n(k)}) + \omega_{\frac{\lambda}{2}}^{s}(y_{2n(k)}, y_{2m(k)}).
$$
 (19)

However,

$$
\omega_{2\lambda}^{s}(y_{2n(k)}, y_{2m(k)}) \leq \omega_{\lambda}^{s}(y_{2n(k)}, y_{2n(k)+1}) + \omega_{\lambda}^{s}(y_{2n(k)+1}, y_{2m(k)}).
$$
 (20)

Taking the limit on [\(19\)](#page-10-8) and [\(20\)](#page-11-0) as  $k \rightarrow +\infty$  and using [\(15\)](#page-10-4) and [\(16\)](#page-10-7), we obtain

<span id="page-11-0"></span>
$$
\lim_{k \to +\infty} \omega_{\lambda}^{s}(y_{2n(k)+1}, y_{2m(k)}) = \delta. \tag{21}
$$

Since  $\forall \lambda > 0$ , we have

$$
\omega_{\lambda}^{s}(y_{2n(k)}, y_{2m(k)}) = [2\omega_{\lambda}^{p}(y_{2n(k)}, y_{2m(k)}) - \omega_{\lambda}^{p}(y_{2n(k)}, y_{2n(k)}) - \omega_{\lambda}^{p}(y_{2m(k)}, y_{2m(k)}].
$$

Taking the limit on the above equation as  $k \to +\infty$ , and then using [\(9\)](#page-9-3) and [\(16\)](#page-10-7), we obtain

<span id="page-11-3"></span><span id="page-11-2"></span><span id="page-11-1"></span>
$$
\lim_{k \to +\infty} \omega_{\lambda}^{p}(y_{2n(k)}, y_{2m(k)}) = \frac{\delta}{2} = d \text{ (say)}
$$
 (22)

Similarly, we obtain

$$
\lim_{k \to +\infty} \omega_{\lambda}^p(y_{2n(k)}, y_{2m(k)-1}) = \frac{\delta}{2} = d \text{ and } \lim_{k \to +\infty} \omega_{\lambda}^p(y_{2n(k)+1}, y_{2m(k)}) = \frac{\delta}{2} = d. \tag{23}
$$

Now, from [\(3\)](#page-8-2), we obtain

$$
\psi(\omega_{\lambda}^{p}(y_{2m(k)}, y_{2n(k)+1})) = \psi(\omega_{\lambda}^{p}(\mathcal{P}x_{2m(k)}, \mathcal{R}x_{2n(k)+1}))
$$
\n
$$
\leq \mathcal{F}(\psi(\mathcal{M}(x_{2m(k)}, x_{2n(k)+1})), \varphi(\mathcal{M}(x_{2m(k)}, x_{2n(k)+1}))),
$$
\n(24)

where

$$
\mathcal{M}(x_{2m(k)}, x_{2n(k)+1}) = \max \{ \omega_{\lambda}^{p}(Sx_{2m(k)}, Qx_{2n(k)+1}), \omega_{\lambda}^{p}(Sx_{2m(k)}, Px_{2m(k)}),
$$
  
\n
$$
\omega_{\lambda}^{p}(Qx_{2n(k)+1}, Rx_{2n(k)+1}),
$$
  
\n
$$
\frac{1}{2} [\omega_{2\lambda}^{p}(Qx_{2n(k)+1}, Px_{2m(k)}) + \omega_{2\lambda}^{p}(Sx_{2m(k)}, Rx_{2n(k)+1})] \}
$$
  
\n
$$
= \max \{ \omega_{\lambda}^{p}(y_{2m(k)-1}, y_{2n(k)}), \omega_{\lambda}^{p}(y_{2m(k)-1}, y_{2m(k)}),
$$
  
\n
$$
\omega_{\lambda}^{p}(y_{2n(k)}, y_{2n(k)+1}),
$$
  
\n
$$
\frac{1}{2} [\omega_{2\lambda}^{p}(y_{2n(k)}, y_{2m(k)}) + \omega_{2\lambda}^{p}(y_{2m(k)-1}, y_{2n(k)+1})] \}
$$

and by  $(\omega_4^p)$  $\binom{p}{4}$ , we have

$$
\frac{1}{2} [\omega_{2\lambda}^p(y_{2n(k)}, y_{2m(k)}) + \omega_{2\lambda}^p(y_{2m(k)-1}, y_{2n(k)+1})]
$$
\n
$$
= \frac{1}{2} \omega_{2\lambda}^p(y_{2n(k)}, y_{2m(k)}) + \frac{1}{2} [\omega_{\lambda}^p(y_{2m(k)-1}, y_{2m(k)}) + \omega_{\lambda}^p(y_{2m(k)}, y_{2n(k)+1})
$$
\n
$$
- \omega_{\lambda}^p(y_{2m(k)}, y_{2m(k)})].
$$

Taking the limit as *k* → +∞ on [\(24\)](#page-11-1), and then using [\(9\)](#page-9-3), [\(10\)](#page-9-4), [\(20\)](#page-11-0), [\(22\)](#page-11-2) and [\(23\)](#page-11-3), we obtain

$$
\psi(d) \le \mathcal{F}(\psi(d), \varphi(d)) \le \psi(d) \Longrightarrow \mathcal{F}(\psi(d), \varphi(d)) = \psi(d),
$$

which implies  $\psi(d) = 0$  or  $\varphi(d) = 0$ ; then,  $d = 0$ . This is a contradiction. Therefore,  $\{y_n\}$ is a Cauchy sequence in the modular space  $X_{\omega^s}$  and hence the sequence  $\{y_n\}$  is a Cauchy sequence in  $X_{\omega^p}$ .  $\square$ 

<span id="page-12-3"></span>**Theorem 1.** *Suppose*  $P$ ,  $Q$ ,  $R$ ,  $S$  :  $X_{\omega^p} \to X_{\omega^p}$  to be four self-mappings defined on a complete *partial modular space satisfying [\(1\)](#page-8-0) and [\(3\)](#page-8-2). Then,*  $poc(\mathcal{P}, \mathcal{S}) \neq \emptyset$  *and*  $poc(\mathcal{Q}, \mathcal{R}) \neq \emptyset$ *. Further, if the pairs*  $(P, S)$  *and*  $(Q, R)$  *are weakly compatible in*  $X_{\omega^p}$ , *then*  $P$ ,  $Q$ ,  $R$  *and*  $S$  *have a unique common fixed point in Xω<sup>p</sup> .*

**Proof.** By Lemma [5,](#page-8-5)  $\{y_n\}$  is a Cauchy sequence in the partial modular space  $X_{\omega^p}$ . Since *X*<sup>*w*</sup> is complete,  $\{y_n\}$  converges in *X*<sup>*w*</sup>. Then, there exists *z* ∈ *X*<sup>*w*</sup> such that

$$
\lim_{n\to+\infty}\omega_\lambda^p(y_n,z)=\lim_{n\to+\infty}\omega_\lambda^p(y_n,x_n)=\omega_\lambda^p(z,z),\,\,\forall\lambda>0.
$$

By Lemma [3](#page-6-0) and from [\(9\)](#page-9-3), we obtain

<span id="page-12-1"></span><span id="page-12-0"></span>
$$
\lim_{n \to +\infty} \omega_{\lambda}^{p}(y_{n}, z) = 0 \text{ and } \omega_{\lambda}^{p}(z, z) = 0, \ \forall \lambda > 0. \tag{25}
$$

Since  $X_{\omega^p}$  is complete, the subsequences  $\{\mathcal{P}x_{2n}\}, \{\mathcal{Q}x_{2n+1}\}, \{\mathcal{R}x_{2n+1}\}$  and  $\{\mathcal{S}x_{2n+2}\},$ *∀n* ∈ N ∪ {0} converge to *z* ∈ *X*<sub>ω</sub>*p*. Now, we show that  $poc(P, S) \neq \emptyset$  and  $poc(Q, R) \neq \emptyset$ . Since  $\{Sx_{2n+2}\}$  converges to  $z \in X_{\omega^p}$ , there exists  $u \in X_{\omega^p}$  such that  $z = S u$ . We claim that  $Pu = Su$ . Using [\(3\)](#page-8-2), we obtain

$$
\psi(\omega_{\lambda}^p(\mathcal{P}u, y_{2n+1})) = \psi(\omega_{\lambda}^p(\mathcal{P}u, \mathcal{R}x_{2n+1}))
$$
\n
$$
\leq \mathcal{F}(\psi(\mathcal{M}(u, x_{2n+1})), \varphi(\mathcal{M}(u, x_{2n+1}))),
$$
\n(26)

where

$$
\mathcal{M}(u, x_{2n+1}) = \max \{ \omega_{\lambda}^{p}(\mathcal{S}u, \mathcal{Q}x_{2n+1}), \omega_{\lambda}^{p}(\mathcal{S}u, \mathcal{P}u), \omega_{\lambda}^{p}(\mathcal{Q}x_{2n+1}, \mathcal{R}x_{2n+1}), \n= \frac{1}{2} [\omega_{2\lambda}^{p}(\mathcal{Q}x_{2n+1}, \mathcal{P}u) + \omega_{2\lambda}^{p}(\mathcal{S}u, \mathcal{R}x_{2n+1})] \}\n= \max \{\omega_{\lambda}^{p}(z, y_{2n}), \omega_{\lambda}^{p}(z, \mathcal{P}u), \omega_{\lambda}^{p}(y_{2n}, y_{2n+1}), \n= \frac{1}{2} [\omega_{2\lambda}^{p}(y_{2n}, \mathcal{P}u) + \omega_{2\lambda}^{p}(z, y_{2n+1})] \}
$$

and

$$
\omega_{2\lambda}^p(y_{2n}, \mathcal{P}u) \leq \omega_{\lambda}^p(y_{2n}, z) + \omega_{\lambda}^p(z, \mathcal{P}u) - \omega_{\lambda}^p(z, z).
$$

Taking the limit as  $n \to +\infty$  on [\(26\)](#page-12-0), and then using [\(9\)](#page-9-3), [\(25\)](#page-12-1) and [\(27\)](#page-12-2), and by the definition of  $(\psi, \varphi, \mathcal{F})$ , we obtain

$$
\psi(\omega_{\lambda}^p(\mathcal{P}u,z)) \leq \lim_{n \to +\infty} \mathcal{F}(\psi(\mathcal{M}(u,x_{2n+1})),\varphi(\mathcal{M}(u,x_{2n+1}))),
$$

where

$$
\lim_{n\to+\infty} \mathcal{M}(u,x_{2n+1}) = \max\{\omega_\lambda^p(z,\mathcal{P}u),\frac{1}{2}\lim_{n\to+\infty}\omega_{2\lambda}^p(y_{2n},\mathcal{P}u)\}\
$$

and

<span id="page-12-2"></span>
$$
\lim_{n \to +\infty} \omega_{2\lambda}^p(y_{2n}, \mathcal{P}u) \leq \lim_{n \to +\infty} [\omega_{\lambda}^p(y_{2n}, z) + \omega_{\lambda}^p(z, \mathcal{P}u) - \omega_{\lambda}^p(z, z)] \tag{27}
$$
\n
$$
\leq \omega_{\lambda}^p(z, \mathcal{P}u).
$$

Therefore,

$$
\psi(\omega_{\lambda}^{p}(\mathcal{P}u,z)) \leq \lim_{n \to +\infty} \mathcal{F}(\psi(\mathcal{M}(u,x_{2n+1})), \varphi(\mathcal{M}(u,x_{2n+1})))
$$
  

$$
\leq \lim_{n \to +\infty} \mathcal{F}(\psi(\omega_{\lambda}^{p}(z,\mathcal{P}u)), \varphi(\omega_{\lambda}^{p}(z,\mathcal{P}u)))
$$
  

$$
\leq \psi(\omega_{\lambda}^{p}(z,\mathcal{P}u)).
$$

It follows that

<span id="page-13-0"></span>
$$
\mathcal{F}(\psi(\omega_\lambda^p(z,\mathcal{P}u)),\varphi(\omega_\lambda^p(z,\mathcal{P}u)))=\psi(\omega_\lambda^p(z,\mathcal{P}u)),
$$

so  $ψ(ω<sub>λ</sub><sup>p</sup>)$  $\phi_{\lambda}^{p}(z,\mathcal{P}u)) = 0$  or  $\phi(\omega_{\lambda}^{p})$  $\frac{p}{\lambda}(z,\mathcal{P}u)$ ); then,  $\omega_{\lambda}^{p}$  $\mathcal{P}^p_\lambda(z, \mathcal{P}u) = 0$  and hence  $\mathcal{P}u = \mathcal{S}u = z$ , i.e.,  $poc(\mathcal{P}, \mathcal{S}) \neq \emptyset$ .

Since  $\mathcal{P}X_{\omega^p} \subset \mathcal{Q}X_{\omega^p}$  and  $u \in \textit{poc}(\mathcal{P}, \mathcal{S})$ , i.e.,  $\mathcal{P}u = \mathcal{S}u = z$ , then there exists  $v \in \mathcal{Q}X_{\omega^p}$  such that  $\mathcal{P}u = \mathcal{Q}v = z$ . Now, we show that  $\mathcal{R}v = \mathcal{Q}v$ . For this, from [\(3\)](#page-8-2), we obtain

$$
\psi(\omega_{\lambda}^p(z, \mathcal{R}v)) = \psi(\omega_{\lambda}^p(\mathcal{P}u, \mathcal{R}v)) \le \mathcal{F}(\psi(\mathcal{M}(u,v)), \varphi(\mathcal{M}(u,v))), \tag{28}
$$

where

$$
\mathcal{M}u, v) = \max \{ \omega_{\lambda}^p(\mathcal{S}u, \mathcal{Q}v), \omega_{\lambda}^p(\mathcal{S}u, \mathcal{P}u), \omega_{\lambda}^p(\mathcal{Q}v, \mathcal{R}v), \frac{1}{2} [\omega_{2\lambda}^p(\mathcal{Q}v, \mathcal{P}u) + \omega_{2\lambda}^p(\mathcal{S}u, \mathcal{R}v)] \} \n= \max \{ \omega_{\lambda}^p(z, z), \omega_{\lambda}^p(z, z), \omega_{\lambda}^p(z, \mathcal{R}v), \frac{1}{2} [\omega_{2\lambda}^p(z, z) + \omega_{2\lambda}^p(z, \mathcal{R}v)] \}.
$$

Then, [\(28\)](#page-13-0) becomes

$$
\psi(\omega_\lambda^p(z,\mathcal{R}v)) \leq \mathcal{F}(\psi(\omega_\lambda^p(z,\mathcal{R}v)),\varphi(\omega_\lambda^p(z,\mathcal{R}v))) \leq \psi(\omega_\lambda^p(z,\mathcal{R}v)).
$$

Therefore,

$$
\mathcal{F}(\psi(\omega_\lambda^p(z,\mathcal{R}v)),\varphi(\omega_\lambda^p(z,\mathcal{R}v)))=\psi(\omega_\lambda^p(z,\mathcal{R}v)),
$$

yielding  $\psi(\omega^p_{\lambda})$  $\phi_{\lambda}^{p}(z, \mathcal{R}v)) = 0$  or  $\phi(\omega_{\lambda}^{p})$  $\chi^p_\lambda(z,\mathcal{R}v))=0$ ; then,  $\omega_\lambda^p$  $\chi^p(z,\mathcal{R}v)=0$  and hence  $\mathcal{R}v=z=0$  $Qv$ . Thus,  $poc(R, Q) \neq \emptyset$ .

Since  $(\mathcal{P}, \mathcal{S})$  and  $(\mathcal{R}, \mathcal{Q})$  are weakly compatible, then  $\mathcal{P}z = \mathcal{PS}u = \mathcal{SP}u = \mathcal{S}z$  and  $Rz = RQv = QRv = Qz$ . Now, we claim that the pairs  $(P, S)$  and  $(R, Q)$  have a unique common point of coincidence. Suppose, if possible, that there exist  $r, r^* \in X_{\omega^p}$ ,  $r \neq r^*$  such that  $Pz = Sz = r$  and  $Rz = Qz = r^*$ .

From [\(3\)](#page-8-2), we obtain

$$
\psi(\omega_{\lambda}^p(r,r^*)) = \psi(\omega_{\lambda}^p(\mathcal{P}z,\mathcal{R}z)) \leq \mathcal{F}(\psi(\mathcal{M}(z,z)),\varphi(\mathcal{M}(z,z)))
$$

where

$$
\mathcal{M}(z,z) = \max \{ \omega_{\lambda}^{p}(\mathcal{S}z, \mathcal{Q}z), \omega_{\lambda}^{p}(\mathcal{S}z, \mathcal{P}z), \omega_{\lambda}^{p}(\mathcal{Q}z, \mathcal{R}z),
$$
  
\n
$$
\frac{1}{2} [\omega_{2\lambda}^{p}(\mathcal{Q}z, \mathcal{P}z) + \omega_{2\lambda}^{p}(\mathcal{S}z, \mathcal{R}z)] \}
$$
  
\n
$$
= \max \{ \omega_{\lambda}^{p}(r, r^{*}), \omega_{\lambda}^{p}(r, r), \omega_{\lambda}^{p}(r^{*}, r^{*}),
$$
  
\n
$$
\frac{1}{2} [\omega_{2\lambda}^{p}(r^{*}, r) + \omega_{2\lambda}^{p}(r, r^{*})] \}.
$$

From the above inequality, we obtain

$$
\psi(\omega_\lambda^p(r,r^*)) \leq \mathcal{F}(\psi(\omega_\lambda^p(r^*,r)),\varphi(\omega_\lambda^p(r^*,r))) \leq \psi(\omega_\lambda^p(r^*,r))
$$

It follows that

$$
\mathcal{F}(\psi(\omega_{\lambda}^p(r^*,r)),\varphi(\omega_{\lambda}^p(r^*,r)))=\psi(\omega_{\lambda}^p(r^*,r))
$$

giving  $\psi(\omega_\lambda^p)$  $\phi_{\lambda}^{p}(r^{*}, r)$ ) = 0 or  $\phi(\omega_{\lambda}^{p})$  $\phi_{\lambda}^{p}(r^*, r)$ ) = 0. Then,  $\omega_{\lambda}^{p}$  $\frac{p}{\lambda}(r^*, r) = 0$  and hence  $r = r^*$ . This is a contradiction. Therefore, by Lemma [4,](#page-8-6) the pairs  $(\mathcal{P}, \mathcal{S})$  and  $(\mathcal{R}, \mathcal{Q})$  have a unique common fixed point in  $X_{\omega^p}$ .  $\square$ 

**Example 13.** Let  $X = [0, \infty)$  and define  $\omega_{\lambda}^{p}$  $\int_{\lambda}^{p}(x,y) = e^{-\lambda}|x-y| + |x| + |y|$ ; then,  $\omega^{p}$  is a partial *modular metric on X. Moreover, we can verify that Xω<sup>p</sup> is a complete partial modular space. Let*  $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}: X_{\omega^p} \to X_{\omega^p}$  be self-mappings defined by

$$
\mathcal{P}x = \frac{1}{2}x, \ \mathcal{Q}x = \frac{1}{3}x, \ \mathcal{S}x = x \ \text{and} \ \mathcal{R}x = \frac{1}{6}x, \ \forall x \in X_{\omega^p}.
$$

*Clearly,*  $\mathcal{P}X_{\omega^p} \subset \mathcal{Q}X_{\omega^p}$  and  $\mathcal{R}X_{\omega^p} \subset \mathcal{S}X_{\omega^p}$ . Moreover, the pairs  $(\mathcal{P}, \mathcal{S})$  and  $(\mathcal{Q}, \mathcal{R})$  are *weakly compatible. Setting*  $\psi(r) = \varphi(r) = kr$  *and*  $\mathcal{F}(s,t) = k^2 s$ *, where*  $k = \frac{1}{2}$ *. Then, the triplet*  $(\psi, \varphi, \mathcal{F})$  *is monotonically increasing.* Now,  $\forall x, y \in X_{\omega^p}$  *and*  $\forall \lambda > 0$ *, and we have* 

$$
\omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{R}y) = e^{-\lambda} |\frac{1}{2}x - \frac{1}{6}y| + |\frac{1}{2}x| + |\frac{1}{6}y|
$$
  
=  $\frac{1}{2} (e^{-\lambda} |x - \frac{1}{3}y| + |x| + |\frac{1}{3}y|)$   
=  $\frac{1}{2} (e^{-\lambda} |\mathcal{S}x - \mathcal{Q}y| + |\mathcal{S}x| + |\mathcal{Q}y|)$   
 $\leq \frac{1}{2} \mathcal{M}(x,y).$ 

*Therefore,*

$$
\psi(\omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{R}y)) = k\omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{R}y) \le k^{2}M(x,y)
$$
  
 
$$
\le \mathcal{F}(\psi(\mathcal{M}(x,y)), \varphi(\mathcal{M}(x,y))), \ \forall x, y \in X_{\omega^{p}} \ and \ \forall \lambda > 0.
$$

*Thus, all the conditions of Theorem [1](#page-12-3) are satisfied and* 0 *is the unique fixed point of* P, Q, R *and*  $S$  *in*  $X_{\omega}$ *p*.

The following theorem is the direct consequence of Theorem [1,](#page-12-3) which is a counterpart of Banach's contraction in metric space.

**Theorem 2.** Let  $\omega^p$  be a partial modular metric on a non-empty set *X* and  $X_{\omega^p}$  be a complete *partial modular metric space. Suppose*  $P: X_{\omega^p} \to X_{\omega^p}$  to be a self-mapping satisfying

$$
\omega_{\lambda}^{p}(\mathcal{P}x,\mathcal{P}y)\leq k\omega_{\lambda}^{p}(x,y),\ \forall\ x,y\in X_{\omega^{p}}\ and\ \forall \lambda>0,
$$

*where*  $0 \leq k < 1$ *; then,*  $P$  *has a unique fixed point in*  $X_{\omega^p}$ *.* 

#### **4. Application**

In this section, inspired by Pant et al. [\[6\]](#page-16-12), we establish the existence of a solution of a system of Volterra-type integral equations.

Consider a set of Volterra-type integral equations

<span id="page-14-0"></span>
$$
x(t) = q(t) + \int_0^t \mathcal{K}_i(t, s, x(t))ds,
$$
\n(29)

where  $t \in [0,k] = I \subset \mathbb{R}$ ,  $\mathcal{K}_i : [0,k] \times [0,k] \times \mathbb{R} \to \mathbb{R}$ ,  $i = \{1,2,3,4\}$  and  $q : [0,k] \to \mathbb{R}$  are continuous functions.

Let  $X = C(I, \mathbb{R})$  be the space of real continuous functions defined on *I*. Define  $\omega^p$  on *X* by

$$
\omega_{\lambda}^{p}(x,y)=\max_{t\in[0,k]}\left[e^{-\lambda}|x(t)-y(t)|+|x(t)|+|y(t)|\right],\ \forall \lambda>0.
$$

Then,  $X_{\omega^p}$  is a complete modular space. Suppose  $\mathcal{H}_i:X_{\omega^p}\to X_{\omega^p}$  to be a self-mapping defined by

$$
\mathcal{H}_i x(t) = q(t) + \int_0^t \mathcal{K}_i(t, s, x(t)) ds, \forall x \in X_{\omega^p} \text{ and } \forall t \in I, i = \{1, 2, 3, 4\}.
$$

Clearly,  $x(t)$  is a solution of [\(29\)](#page-14-0) if and only if it is a common fixed point of  $\mathcal{H}_i$  for  $i = \{1, 2, 3, 4\}.$ 

**Theorem 3.** *Under the above conditions, assume that the following hypotheses hold:*  $(h_1)$ *: For any*  $x \in X_{\omega^p}$ *, there exist*  $u, v \in X_{\omega^p}$  *such that* 

$$
\mathcal{H}_1x=\mathcal{H}_3u,\ \mathcal{H}_2x=\mathcal{H}_4v;
$$

 $(h_2)$ *: For any t*  $\in$  *I, there exist u, v*  $\in$  *X<sub>ω</sub>p such that* 

$$
\mathcal{H}_1\mathcal{H}_4u(t) = \mathcal{H}_4\mathcal{H}_1u(t), \text{ if } \mathcal{H}_1u(t) = \mathcal{H}_4u(t)
$$

*and*

$$
\mathcal{H}_2\mathcal{H}_3v(t)=\mathcal{H}_3\mathcal{H}_2v(t), \text{ if } \mathcal{H}_2v(t)=\mathcal{H}_3v(t);
$$

(*h*<sub>3</sub>)*: There exists a continuous function*  $f : I \times I \rightarrow \mathbb{R}^+$  *such that* 

$$
|\mathcal{K}_1(t,s,x(s)) - \mathcal{K}_3(t,s,y(s))| \leq f(t,s) \left[ |\mathcal{H}_4 x(s) - \mathcal{H}_2 y(s)| + e^{\lambda} (|\mathcal{H}_4 x(s)| + |\mathcal{H}_2 y(s)|) - 2e^{\lambda} (|\mathcal{H}_1 x(s)| + |\mathcal{H}_3 y(s)|) \right]
$$

 $∀ \lambda > 0$  *and*  $∀x, y ∈ X<sub>ω</sub><sup>*p*</sup>$ *, where t*, *s* ∈ *I*;

 $(h_4)$ :  $\max_{t \in [0,k]} \int_0^t f(t,s) ds \leq \frac{1}{2}$ .

*Then, the system [\(29\)](#page-14-0) of integral equations has a unique common solution in*  $X_{\omega^p}$ *.* 

**Proof.** From  $(h_1)$ ,  $\mathcal{H}_1 X_{\omega^p} \subseteq \mathcal{H}_3 X_{\omega^p}$  and  $\mathcal{H}_2 X_{\omega^p} \subseteq \mathcal{H}_4 X_{\omega^p}$ .

From  $(h_2)$ , the pairs  $(\mathcal{H}_1, \mathcal{H}_4)$  and  $(\mathcal{H}_2, \mathcal{H}_3)$  are weakly compatible. Now, from  $(h_3)$ , we have

$$
\omega_{\lambda}^{p}(\mathcal{H}_{1}x,\mathcal{H}_{3}y) = \max_{t\in[0,k]} \left[e^{-\lambda}|\mathcal{H}_{1}x(t) - \mathcal{H}_{3}y(t)| + |\mathcal{H}_{1}x(t)| + |\mathcal{H}_{3}y(t)|\right]
$$
  
\n
$$
\leq \max_{t\in[0,k]} \left[e^{-\lambda} \int_{0}^{t} |\mathcal{K}_{1}(t,s,r(s)) - \mathcal{K}_{3}(t,s,r(s))|ds + |\mathcal{H}_{1}x(t)| + |\mathcal{H}_{3}y(t)|\right]
$$
  
\n
$$
\leq e^{-\lambda} \max_{t\in[0,k]} \int_{0}^{t} f(t,s)ds \left[|\mathcal{H}_{4}x - \mathcal{H}_{2}y| + e^{\lambda}(|\mathcal{H}_{4}x| + |\mathcal{H}_{2}y|)\right]
$$
  
\n
$$
-2e^{\lambda}(|\mathcal{H}_{1}x| + |\mathcal{H}_{3}y|)\right] + (|\mathcal{H}_{1}x| + |\mathcal{H}_{3}y|)
$$
  
\n
$$
\leq \frac{1}{2} \left[e^{-\lambda}|\mathcal{H}_{4}x - \mathcal{H}_{2}y| + |\mathcal{H}_{4}x| + |\mathcal{H}_{2}y|\right]
$$
  
\n
$$
= \frac{1}{2}\omega_{\lambda}^{p}(\mathcal{H}_{4}x,\mathcal{H}_{2}y)
$$
  
\n
$$
\leq \frac{1}{2}\mathcal{M}(x,y),
$$

where

$$
\mathcal{M}(x,y) = \max \Big\{ \omega_{\lambda}^{p}(\mathcal{H}_{4}x, \mathcal{H}_{2}y), \omega_{\lambda}^{p}(\mathcal{H}_{4}x, \mathcal{H}_{1}x), \omega_{\lambda}^{p}(\mathcal{H}_{2}y, \mathcal{H}_{3}y), \frac{\omega_{2\lambda}^{p}(\mathcal{H}_{2}x, \mathcal{H}_{1}x) + \omega_{2\lambda}^{p}(\mathcal{H}_{4}x, \mathcal{H}_{3}y)}{2} \Big\}.
$$

Setting  $\psi(t) = \varphi(t) = t$  and  $\mathcal{F}(s, t) = \frac{1}{2}t$ , then the triplet  $(\psi, \varphi, \mathcal{F})$  is monotonically increasing. Therefore,

$$
\omega_\lambda^p(\mathcal{H}_1x,\mathcal{H}_3y)\leq\mathcal{F}(\psi(\mathcal{M}(x,y)),\varphi(\mathcal{M}(x,y))),\ \forall x,y\in X_{\omega^p}\ \text{and}\ \forall \lambda>0.
$$

Thus, all the conditions of Theorem [1](#page-12-3) are satisfied, and hence the system [\(29\)](#page-14-0) has a unique solution in  $X_{\omega^p}$ .  $\square$ 

#### **5. Conclusions**

We propose a refinement of the notion of the partial modular metric to eliminate the occurrence of discrepancies in the non-zero self-distance and triangular inequality. Using the altering distance functions, a common fixed-point theorem for four self-mappings via the C− class function is proven in such space. In addition, we apply our results to establish the existence of a solution for a system of Volterra integral equations as an application.

**Author Contributions:** D.D., S.N. and Y.M.S. contributed to the methodology and the original draft preparation. Y.M.S., M.S.K. and S.S. reviewed and edited the manuscript. S.S. designed the research and supported funding acquisition. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors thank the anonymous referees for their valuable constructive comments and suggestions, which improved the quality of this paper in the present form.

**Conflicts of Interest:** The authors declare no conflicts of interest.

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