axioms

MDPI

Article

A New q-Hypergeometric Symbolic Calculus in the Spirit of Horn, Borngässer, Debiard and Gaveau

Thomas Ernst

Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden; thomas@math.uu.se

Abstract: The purpose of this article is to introduce a new complete multiple q-hypergeometric symbolic calculus, which leads to q-Euler integrals and a very similar canonical system of q-difference equations for multiple q-hypergeometric functions. q-analogues of recurrence formulas in Horns paper and Borngässers thesis lead to a more exact way to find these Frobenius solutions. To find the right formulas, the parameters in q-shifted factorials can be changed to negative integers, which give no extra q-factors. In proving these q-formulas, in the limit $q \to 1$ we obtain versions of the paper by Debiard and Gaveau for the solution of differential or q-difference equations. The paper is also a correction of some of the statements in the paper by Debiard and Gaveau, e.g., the Euler integrals and other solutions to differential equations for Appell functions, also without references to page numbers in the standard work of Appell and Kampé de Fériet. Sometimes the q-binomial theorem is used to simplify q-integral formulas. By the Horn method, we find another solution to the Appell Φ_1 function partial differential equation, which was not mentioned in the thesis by Le Vavasseur 1893.

Keywords: symbolic calculus; canonical system of q-difference equations; q-Euler integral

MSC: Primary 33D05; Secondary 33C65; 33D70; 33D60

check for updates

Citation: Ernst, T. A New q-Hypergeometric Symbolic Calculus in the Spirit of Horn, Borngässer, Debiard and Gaveau. *Axioms* **2022**, *11*, 64. https://doi.org/10.3390/axioms11020064

Academic Editors: Hans J. Haubold and Anna Maria Fino

Received: 28 December 2021
Accepted: 30 January 2022
Published: 4 February 2022

1. Introduction

We refer to our standard work [1] and to the paper on multiple q-hypergeometric functions [2]. The pathbreaking paper [3] by Debiard and Gaveau on a new umbral calculus led to the automatic solutions of differential equations for multiple hypergeometric functions according to Frobenius and Horn. In this paper, we generalize this method to the q-case and slightly change the notation for a better overview. As examples, the exponent in method of Frobenius is changed from α to λ and the Euler operator $x\frac{d}{dx}$ is changed to $\theta(_q)$ as in [1]. Our umbral calculus simply means that a $\theta_{q,1} \vee \theta_{q,2}$ before a double power series is replaced by the exponents of $x \vee y$. The same goes for additive arguments in the Γ_q function.

A proper notation is extremely important in papers on special functions, since long computations often occur and the origin of the variables is crucial for the understanding of the formulas. The notation and especially the computations in [3] are sometimes erroneous, one example is the notation on the top of page 789, where small a, α_i and α occur, together with a misprint. For operators, we mention the spaces of formal power series in their definitions. We also remember that in Horns paper ([4], p. 387) and in Borngässers thesis [5], recurrence formulas for the determination of the other solutions in the method of Frobenius were given, which was missed in [3].

The paper is organized as follows: In Section 1 we define all q-functions. In Section 2 we present Horns and Borngässers recurrence formulas for the coefficients in the method of Frobenius, which have a very similar form as before. In Section 3 we introduce the general symbolic calculus. In Section 4 we find bases for the spaces of solutions by the Frobenius method for the first q-Appell function. In Sections 5–8 we consider the q-Appell functions Φ_1, Φ_2, Φ_3, Φ_4.

Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi + \delta < \text{Im}\,(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The power function is defined by

$$q^a \equiv e^{a \log(q)}. \tag{1}$$

A q-analogue of a complex number is also a complex number.

Definition 1. *The q-analogue of a complex number a is defined as follows:*

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{0,1\}, \tag{2}$$

The q-shifted factorial is defined by

$$\langle a;q\rangle_n \equiv \prod_{m=0}^{n-1}(1-q^{a+m}), \tag{3}$$

The q-derivative is defined by

$$(D_q\varphi)(x) \equiv \begin{cases} \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{when } q \in \mathbb{C}\backslash\{1\},\ x \neq 0; \\ \frac{d\varphi}{dx}(x), & \text{when } q = 1; \\ \frac{d\varphi}{dx}(0), & \text{when } x = 0. \end{cases} \tag{4}$$

Definition 2. *The following operator will also be useful.*

$$\theta_{q,j} \equiv x_j D_{q,x_j}. \tag{5}$$

Definition 3. *[1]. The q-analogues of the Appell functions are*

$$\Phi_1(a;b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}, \quad max(|x_1|,|x_2|) < 1. \tag{6}$$

$$\Phi_2(a;b,b';c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \quad |x_1| \oplus_q |x_2| < 1. \tag{7}$$

$$\Phi_3(a,a';b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}, \quad max(|x_1|,|x_2|) < 1. \tag{8}$$

$$\Phi_4(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \quad |\sqrt{x_1}| \oplus_q |\sqrt{x_2}| < 1. \tag{9}$$

2. q-Analogues of Horns and Borngässers Recurrence Formulas

The purpose of this section is to introduce q-analogues of Horns and Borngässers recurrence formulas ([4], p. 387), ([5], p. 26 ff) for double series. We just state the formulas, the proofs are simple; in the process we slightly improve the notation. We start with the double q-hypergeometric series

$$z \equiv H(x,y) = \sum_{m,n=0}^{\infty} A_{mn} x^m y^n, \tag{10}$$

where the two quotients

$$f(m,n) \equiv \frac{A_{m+1,n}}{A_{mn}},\ g(m,n) \equiv \frac{A_{m,n+1}}{A_{mn}} \tag{11}$$

are rational functions in q-analogues of m, n. Now put

$$f(m,n) \equiv \frac{F(m,n)}{F'(m,n)},\ g(m,n) \equiv \frac{G(m,n)}{G'(m,n)}, \tag{12}$$

where $F(mn),\ G(mn),\ F'(mn),\ G'(mn)$ are entire products of q-analogues in m, n of maximal second order in m, n. We assume that $F'(mn)$ has the factor $\{1+m\}_q$, $G'(mn)$ has the factor $\{1+n\}_q$.

All these q-functions are q-analogues of the Appell, confluent Humbert, etc., and Horn functions.

We just state a q-analogue of a generalization of the Euler operator ([4], p. 387). Assume that $\alpha,\ \beta \in \mathbb{R}[\theta_{q,1}, \theta_{q,2}]$. $\alpha,\ \beta$ are linear functions of $\theta_{q,1}$ and $\theta_{q,2}$ with coefficients $\in \mathbb{Z}$, and z is defined by (10).

$$(1-q)^2\{\alpha\}_q\{\beta\}_q z = \sum_{m,n=0}^{\infty} \langle\alpha;q\rangle_1 \langle\beta;q\rangle_1 A_{mn} x^m y^n. \tag{13}$$

A q-analogue of an improved version of ([4], p.387), where we have skipped the sums $\sum_{\alpha,\beta}$. Assume that $\alpha,\ \beta,\ \gamma,\ \delta, \alpha',\ \beta',\ \gamma',\ \delta' \in \mathbb{R}[\theta_{q,1}, \theta_{q,2}]$ are linear functions of $\theta_{q,1}$ and $\theta_{q,2}$ with coefficients $\in \mathbb{Z}$. Furthermore, the function z in (10) satisfies the system of q-difference equations

$$\begin{aligned} (x\{\alpha\}_q\{\beta\}_q - \{\alpha'\}_q\{\beta'\}_q)z = 0, \\ (y\{\gamma\}_q\{\delta\}_q - \{\gamma'\}_q\{\delta'\}_q)z = 0, \end{aligned} \tag{14}$$

with convenient boundary values.

Case I. Assume instead that

$$z = \sum_{m,n=0}^{\infty} C_{mn} x^{m+\rho} y^{n+\sigma}, \tag{15}$$

where ρ and σ are unknown real constants. In the previous case, C_{mn} become A_{mn}. We now have the recurrence formulas

$$\begin{cases} F'(m+\rho, n+\sigma)C_{m+1,n} = F(m+\rho, n+\sigma)C_{m,n} \\ G'(m+\rho, n+\sigma)C_{m,n+1} = G(m+\rho, n+\sigma)C_{m,n}, \end{cases} \tag{16}$$

which follow from the previous recurrence formulas for A_{mn}.

By comparing the coefficients of

$$x^{m+\rho} y^{n+\sigma},\ m = -1; n \geq 0 \tag{17}$$

in the first recurrence, and the coefficients of

$$x^{m+\rho} y^{n+\sigma},\ m \geq 0,\ n = -1 \tag{18}$$

in the second recurrence, we obtain the equations ([5], p. 27), ([4], p. 388) for the determination of the exponents ρ and σ

$$\begin{cases} F'(\rho-1,\sigma)=0, \\ G'(\rho,\sigma-1)=0 \end{cases} \tag{19}$$

Case II. For the determination of the solutions in the vicinity of the point (∞,∞), we look at series of the form

$$z=\sum_{m,n=0}^{\infty} C_{mn}x^{\rho-m}y^{\sigma-n}, \tag{20}$$

where ρ and σ are unknown real constants. We now have the recurrence formulas

$$\begin{cases} F(\rho-m-1,\sigma-n)C_{m+1,n}=F'(\rho-m-1,\sigma-n)C_{m,n} \\ G(\rho-m,\sigma-n-1)C_{m,n+1}=G'(\rho-m,\sigma-n-1)C_{m,n}. \end{cases} \tag{21}$$

By comparing the coefficients of

$$x^{\rho-m}y^{\sigma-n}, \tag{22}$$

we obtain the equations ([5], p. 28), ([4], p. 388) for the determination of the exponents ρ and σ

$$\begin{cases} F(\rho,\sigma)=0, \\ G(\rho,\sigma)=0 \end{cases} \tag{23}$$

Case III. For the determination of the solutions in the vicinity of the point $(0,\infty)$, we look at series of the form

$$z=\sum_{m,n=0}^{\infty} C_{mn}x^{\rho+m}y^{\sigma-n}, \tag{24}$$

which leads to the recurrence formulas

$$\begin{cases} F'(\rho+m,\sigma-n)C_{m+1,n}=F(\rho+m,\sigma-n)C_{m,n} \\ G(\rho+m,\sigma-n-1)C_{m,n+1}=G'(\rho+m,\sigma-n-1)C_{m,n}. \end{cases} \tag{25}$$

By comparing the coefficients of

$$x^{\rho+m}y^{\sigma-n}, \tag{26}$$

we obtain the equations ([5], p. 29), ([4], p. 388) for the determination of the exponents ρ and σ

$$\begin{cases} F'(\rho-1,\sigma)=0, \\ G(\rho,\sigma)=0. \end{cases} \tag{27}$$

Case IV. Finally, for the determination of the solutions in the vicinity of the point $(\infty,0)$, we look at series of the form

$$z=\sum_{m,n=0}^{\infty} C_{mn}x^{\rho-m}y^{\sigma+n}, \tag{28}$$

which leads to the recurrence formulas

$$\begin{cases} F(\rho-m-1,\sigma+n)C_{m+1,n}=F'(\rho-m-1,\sigma+n)C_{m,n} \\ G'(\rho-m,\sigma+n)C_{m,n+1}=G(\rho-m,\sigma+n)C_{m,n}. \end{cases} \tag{29}$$

By comparing the coefficients of

$$x^{\rho-m}y^{\sigma+n}, \tag{30}$$

we obtain the equations ([5], p. 29), ([4], p. 388) for the determination of the exponents ρ and σ

$$\begin{cases} F(\rho,\sigma) = 0, \\ G'(\rho,\sigma-1) = 0 \end{cases} \tag{31}$$

3. General Symbolic Calculus

The purpose of this section is to introduce the general symbolic calculus for double series.

Definition 4. *We put*

$$H_q(a,b,c,x) \equiv \{\theta_{q,1}+c\}_q D_{q,x} - \{\theta_{q,1}+a\}_q\{\theta_{q,1}+b\}_q. \tag{32}$$

Then we have

$$H_q(a,b,c,x)\;{}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix}\middle| q;x\right] = 0. \tag{33}$$

We are always interested in solutions to the equation

$$H_q(a,b,c,x)(f(x)) = 0. \tag{34}$$

Around $x = 0$ another solution, apart from y_1 in (33) is (6.186 [1])

$$y_2 = x^{1-c}\;{}_2\phi_1\left[\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix}\middle| q;x\right]. \tag{35}$$

The purpose of the next definition is to keep the powers of the variables in the operator.

Definition 5. *Let A, B, C be three operators $\mathbb{R}[[x]] \to \mathbb{R}[[x]]$, which are linear in $\theta_{q,1}$, $\theta_{q,2}$. Then we define*

$$H_q(A,B,C,x) \equiv \{\theta_{q,1}+C\}_q D_{q,x} - \{\theta_{q,1}+A\}_q\{\theta_{q,1}+B\}_q. \tag{36}$$

Lemma 1. *Compare with ([3], p. 777). Let $\mathscr{F}(a,b,c,x)$ be a solution of*

$$H_q(a,b,c,x)\mathscr{F} = 0. \tag{37}$$

Then

$$\mathscr{F}(A,B,C,x) \tag{38}$$

is a solution of

$$H_q(A,B,C,x)\mathscr{F}(A,B,C,x) = 0. \tag{39}$$

Definition 6. *Compare with ([3], p. 777). Assume $x > 0$, C, $\lambda \in \mathbb{R}$, $\psi(y) \times y^{-\lambda} \in \mathbb{R}[[y]]$. Then, in the umbral sense,*

$$x^{C\theta_{q,2}}\psi(y) \doteqdot \psi(x^C y). \tag{40}$$

Assume that

$$\phi(\vec{a},\vec{b},y) \equiv y^\lambda \sum_{k=0}^{\infty} \frac{\langle \vec{a};q\rangle_k}{\langle 1,\vec{b};q\rangle_k} y^k. \tag{41}$$

Then, in the umbral sense,

$$\phi(\vec{a}+\vec{a}'\theta_{q,2},\vec{b}+\vec{b}'\theta_{q,2},y)\equiv y^{\lambda}\sum_{k=0}^{\infty}\frac{\langle\vec{a}+\vec{a}'(\lambda+k);q\rangle_k}{\langle 1,\vec{b}+\vec{b}'(\lambda+k);q\rangle_k}y^k. \tag{42}$$

Furthermore, for the Γ_q-function:

$$\Gamma_q(\alpha+\theta_{q,2})y^{\lambda}\doteq\Gamma_q(\alpha+\lambda)y^{\lambda}. \tag{43}$$

We can generalize this to many variables.

Definition 7. *Compare with (3.9 [3]). Let $H_q(x,y)$ be an operator $\mathbb{R}[[x]]\to\mathbb{R}[[x]]$:*

$$H_q(x,y)\equiv\{\theta_{q,1}+\gamma\theta_{q,2}+c\}_q D_{q,x}-\{\theta_{q,1}+\alpha\theta_{q,2}+a\}_q\{\theta_{q,1}+\beta\theta_{q,2}+b\}_q. \tag{44}$$

The parameters in $H_q(x,y)$ will always be the same.

Remark 1. *The function (44) generalizes the basic definition (32) and is a special case of the more general definition (36). The notation in ([3], 3.9) is slightly misleading.*

Theorem 1. *Compare with ([3], (3.12) p. 779). Let $\mathscr{F}_j(a,b,c,x),\ j=1,2$ of the form (41) be two independent solutions of*

$$H_q(a,b,c,x)\mathscr{F}_j=0. \tag{45}$$

Furthermore, let $\psi_j(y)\in\mathbb{R}[[y]],\ j=1,2$ be q-hypergeometric series, with suitable convergence radii. Then the general solution of the equation

$$H_q(x,y)f=0 \tag{46}$$

in the umbral form (42) is given by

$$f(x,y)=\sum_{j=1}^{2}\mathscr{F}_j(\alpha\theta_{q,2}+a,\beta\theta_{q,2}+b,\gamma\theta_{q,2}+c,x)\phi_j(y). \tag{47}$$

Proof. This follows from (39). □

Theorem 2. *Compare with ([3], (4.5) p. 780). Let $\psi(y)\in\mathbb{R}[[y]]$ be a q-hypergeometric series, with suitable convergence radius. Then the series*

$$F(x,y:q)\equiv{}_2\phi_1(\alpha\theta_{q,2}+a,\beta\theta_{q,2}+b,\gamma\theta_{q,2}+c|q;x)\psi(y) \tag{48}$$

is a double q-hypergeometric series, convergent in the vicinity of $(0,0)$.

Proof. Similar to ([3], p. 780). □

Definition 8. *Compare with (5.1 [3]). Introduce the two general operators $\mathbb{R}[[x,y]]\to\mathbb{R}[[x,y]]$:*

$$\begin{aligned}H_{1;q}(x,y)&\equiv\{\gamma_1\theta_{q,2}+c_1\}_q D_{q,x}-\{\alpha_1\theta_{q,2}+a_1\}_q\{\beta_1\theta_{q,2}+b_1\}_q,\\H_{2;q}(y,x)&\equiv\{\gamma_2\theta_{q,1}+c_2\}_q D_{q,y}-\{\alpha_2\theta_{q,1}+a_2\}_q\{\beta_2\theta_{q,1}+b_2\}_q.\end{aligned} \tag{49}$$

We wish to study the system of q-difference equations

$$\begin{cases}H_{1;q}(x,y)f(x,y)=0\\H_{2;q}(y,x)f(x,y)=0.\end{cases} \tag{50}$$

The system (50) is called q-compatible if it has common solutions $f(x,y)$.

Theorem 3. *Compare with ([3], p. 785). The system (50) is q-compatible if the following two products of q-analogues are equal,*

$$P_1(m,n;q) = P_2(m,n;q), \tag{51}$$

where

$$\begin{aligned} &P_1(m,n;q) \equiv \{m+\alpha_1 n + a_1\}_q\{m+\beta_1 n+b_1\}_q\{m+\gamma_1(n+1)+c_1\}_q \\ &\{n+\alpha_2(m+1)+a_2\}_q\{n+\beta_2(m+1)+b_2\}_q\{n+\gamma_2 m+c_2\}_q, \\ &P_2(m,n;q) \equiv \{m+\alpha_1(n+1)+a_1\}_q\{m+\beta_1(n+1)+b_1\}_q\{m+\gamma_1 n+c_1\}_q \\ &\{n+\alpha_2 m+a_2\}_q\{m+\beta_2 n+b_2\}_q\{n+\gamma_2(m+1)+c_2\}_q. \end{aligned} \tag{52}$$

For the following proof, compare with ([3], p. 786).

Proof. We put

$$f(x,y) = \sum_{m,n=0}^{\infty} A_{mn} x^m y^n. \tag{53}$$

We first calculate the following operator formulas.

$$\{\theta_{q,1} + \gamma_1\theta_{q,2} + c_1\}_q \mathrm{D}_{q,x} f = \sum_{m,n=0}^{\infty} a_{m+1,n} \frac{\{m+\gamma_1 n+c_1\}_q}{\langle 1;q\rangle_m \langle 1;q\rangle_n} x^m y^n \tag{54}$$

$$\begin{aligned} &\{\theta_{q,1} + \alpha_1\theta_{q,2} + a_1\}_q\{\theta_{q,1} + \beta_1\theta_{q,2} + b_1\}_q f \\ &= \sum_{m,n=0}^{\infty} a_{mn} \frac{\{m+\alpha_1 n+a_1\}_q\{m+\beta_1 n+b_1\}_q}{\langle 1;q\rangle_m \langle 1;q\rangle_n} x^m y^n, \end{aligned} \tag{55}$$

where

$$A_{mn} \equiv \frac{a_{mn}}{\langle 1;q\rangle_m \langle 1;q\rangle_n}. \tag{56}$$

The first Equation (50) is satisfied when

$$\frac{a_{m+1,n}}{a_{mn}} = \frac{\{m+\alpha_1 n+a_1\}_q\{m+\beta_1 n+b_1\}_q}{\{m+\gamma_1 n+c_1\}_q}. \tag{57}$$

The second Equation (50) is satisfied when

$$\frac{a_{m,n+1}}{a_{mn}} = \frac{\{n+\alpha_2 m+a_2\}_q\{n+\beta_2 m+b_2\}_q}{\{n+\gamma_2 m+c_2\}_q}. \tag{58}$$

Using Horn's notation, we have

$$f(m,n) \equiv \frac{A_{m+1,n}}{A_{mn}} = \frac{\{m+\alpha_1 n+a_1\}_q\{m+\beta_1 n+b_1\}_q}{\{m+\gamma_1 n+c_1\}_q\{m+1\}_q}, \tag{59}$$

$$g(m,n) \equiv \frac{A_{m,n+1}}{A_{mn}} = \frac{\{n+\alpha_2 m+a_2\}_q\{n+\beta_2 m+b_2\}_q}{\{n+\gamma_2 m+c_2\}_q\{n+1\}_q}. \tag{60}$$

Now (51) follows from the compatibility condition

$$f(m,n)g(m+1,n) = f(m,n+1)g(m,n). \tag{61}$$

□

Similarly, we find that the q-hypergeometric functions defined by (6)–(9), after rescaling, satisfy the systems (50).

4. Bases for the Spaces of Solutions by the Frobenius Method

Assuming that our system (50) is q-compatible, by using Lemma 1, we construct a basis for its solutions. Like in ([3], p. 788), the parameter $c_i \ni -\mathbb{Z}$. Many of these solutions are obtained simply by the same formula after q-deformation like in [1]. We note that other solutions are obtained by permutation of the variables.

Solutions in the Vicinity of $(0,0)$

Now we assume that

$$\psi(y) \equiv y^{\lambda} \sum_{k=0}^{\infty} a_n \frac{y^k}{\langle 1;q\rangle_k}. \tag{62}$$

Consider the basis $y_1(x),\ y_2(x)$ in (33) and (35). In all these assumptions, we put the first coefficient $a_0 = 1$. Note that this was not mentioned in [3]. In order to obtain solutions of our system of q-difference equations, define (49)

$$f(x,y;q) \equiv {}_2\phi_1(\alpha_1\theta_{q,2} + a_1, \beta_1\theta_{q,2} + b_1, \gamma_1\theta_{q,2} + c_1|q;x)\psi(y). \tag{63}$$

This implies

$$f(x,y;q) = \sum_{m,n=0}^{\infty} a_n \frac{\langle \alpha_1(n+\lambda) + a_1, \beta_1(n+\lambda) + b_1; q\rangle_m}{\langle 1, \gamma_1(n+\lambda) + c_1; q\rangle_m \langle 1;q\rangle_n} x^m y^{n+\lambda} \tag{64}$$

$$(\theta_{q,2} + \gamma_2\theta_{q,1} + c_2)\mathrm{D}_{q,y} f = y^{\lambda} \sum_{m,n=0}^{\infty} a_n x^m y^{n-1} \{n+\lambda\}_q \\ \frac{\langle \alpha_1(n+\lambda) + a_1, \beta_1(n+\lambda) + b_1; q\rangle_m \{n+\lambda-1+\gamma_2 m + c_2\}_q}{\langle 1, \gamma_1(n+\lambda) + c_1; q\rangle_m \langle 1;q\rangle_n} \tag{65}$$

$$(\theta_{q,2} + \alpha_2\theta_{q,1} + a_2)(\theta_{q,2} + \beta_2\theta_{q,1} + b_2) f \\ = y^{\lambda} \sum_{m,n=0}^{\infty} a_n x^m y^n \{n+\lambda+\alpha_2 m + a_2\}_q \\ \frac{\langle \alpha_1(n+\lambda) + a_1, \beta_1(n+\lambda) + b_1; q\rangle_m \{n+\lambda+\beta_2 m + b_2\}_q}{\langle 1, \gamma_1(n+\lambda) + c_1; q\rangle_m \langle 1;q\rangle_n} \tag{66}$$

By equating the last two formulas for $n = 0$, we obtain the indicial equation

$$\{\lambda\}_q\{\lambda - 1 + \gamma_2 m + c_2\}_q = 0,\ m \geq 0, \tag{67}$$

which implies

$$\lambda = 0,\ \forall\gamma_2, \\ \lambda = 0 \vee\ \lambda = 1 - c_2 \text{ if } \gamma_2 = 0. \tag{68}$$

5. First q-Appell Function

We now apply the general method from the previous section to the first q-Appell function. Put $\gamma_i = \alpha_i = 1,\ c_i = c,\ a_i = a$ in (49). Like before the system is denoted by $(H_i f(x,y;q))_{i=1}^2$.

$$\begin{cases} [\{\theta_{q,1} + \theta_{q,2} + c\}_q \mathrm{D}_{q,x} - \{\theta_{q,1} + \theta_{q,2} + a\}_q\{\theta_{q,1} + b_1\}_q] f(x,y;q) = 0 \\ [\{\theta_{q,1} + \theta_{q,2} + c\}_q \mathrm{D}_{q,y} - \{\theta_{q,1} + \theta_{q,2} + a\}_q\{\theta_{q,2} + b_2\}_q] f(x,y;q) = 0. \end{cases} \tag{69}$$

With (62), $a_0 = 1$ and y_1 in (33), we get the first q-Appell function. Next consider the function y_2 in (35).

We find that the following equation can be rewritten by (40) and (43) as

$$\begin{aligned} &f_D(x,y;q)\\ &\equiv x^{1-c-\theta_{q,2}}{}_2\phi_1(a+1-c,b_1+1-c-\theta_{q,2};2-c-\theta_{q,2}|q;x)\psi(y)\\ &=x^{1-c}\sum_{m,n=0}^{\infty}a_n\frac{\langle a+1-c,b_1+1-c-n-\lambda;q\rangle_m}{\langle 1,2-c-n-\lambda+c_1;q\rangle_m\langle 1;q\rangle_n}x^{m-n-\lambda}y^{n+\lambda}. \end{aligned} \tag{70}$$

Lemma 2.

$$\frac{\langle b_1+1-c-n;q\rangle_m\langle c-1;q\rangle_n\langle 2-c;q\rangle_{m-n}}{\langle 2-c-n;q\rangle_m\langle c-b_1;q\rangle_n\langle b_1+1-c;q\rangle_{m-n}}=q^{n(b_1-1)}. \tag{71}$$

This lemma is used in the proof (77). Similar to ([3], p. 793) we find that

$$\begin{aligned} &\left[\{\theta_{q,1}+\theta_{q,2}+c\}_q\mathrm{D}_{q,y}\right](x^{m-n-\lambda+1-c}y^{n+\lambda})\\ &=\{m\}_q\{n+\lambda\}_q x^{m-n-\lambda+1-c}y^{n+\lambda-1}. \end{aligned} \tag{72}$$

Again, $\lambda=0$, and we have

$$\begin{aligned} &H_2f_D(x,y;q)=x^{1-c}\Bigg[\sum_{m,n=1}^{\infty}a_n\frac{\langle a+1-c,b_1+1-c-n;q\rangle_m}{\langle 2-c-n;q\rangle_m\langle 1;q\rangle_{m-1}\langle 1;q\rangle_{n-1}}x^{m-n}y^{n-1}\\ &-\sum_{m,n=0}^{\infty}a_n\langle n+b_2,1+m+a-c;q\rangle_1\frac{\langle a+1-c,b_1+1-c-n;q\rangle_m}{\langle 1,2-c-n;q\rangle_m\langle 1;q\rangle_n}x^{m-n}y^n\Bigg]\\ &=x^{1-c}\sum_{m,n=0}^{\infty}\Bigg[a_{n+1}\frac{\langle a+1-c,b_1+1-c-n;q\rangle_{m+1}}{\langle 1-c-n;q\rangle_{m+1}\langle 1;q\rangle_m\langle 1;q\rangle_n}\\ &-a_n\langle n+b_2,1+m+a-c;q\rangle_1\frac{\langle a+1-c,b_1+1-c-n;q\rangle_m}{\langle 1,2-c-n;q\rangle_m\langle 1;q\rangle_n}\Bigg]x^{m-n}y^n. \end{aligned} \tag{73}$$

By the condition $H_2f=0$ we obtain

$$\frac{a_{n+1}}{a_n}=\frac{\langle n+b_2,1-c-n;q\rangle_1}{\langle b_1-c-n;q\rangle_1}. \tag{74}$$

This implies

$$a_n=\frac{\langle b_2,c-1;q\rangle_n}{\langle c-b_1;q\rangle_n}, \tag{75}$$

$$\psi(y)={}_2\phi_1(b_2,c-1;c-b_1|q;y). \tag{76}$$

Then, we can induce by (71)

$$\begin{aligned} &f_D(x,y;q)\\ &=x^{1-c}\sum_{m,n=0}^{\infty}\frac{\langle a+1-c,b_1+1-c-n;q\rangle_m\langle b_2,c-1;q\rangle_n}{\langle 1,2-c-n;q\rangle_m\langle 1,c-b_1;q\rangle_n}x^{m-n}y^nq^{n(1-b_1)}\\ &=x^{1-c}\sum_{m,n=0}^{\infty}\frac{\langle a+1-c;q\rangle_m\langle b_1+1-c;q\rangle_{m-n}\langle b_2;q\rangle_n}{\langle 2-c;q\rangle_{m-n}\langle 1;q\rangle_m\langle 1;q\rangle_n}x^{m-n}y^n. \end{aligned} \tag{77}$$

We can rewrite this in the form of q-Horn function, convenient for convergence aspects.

$$\begin{aligned} &f_D(x,y;q)=x^{1-c}\sum_{m,n=0}^{\infty}(-1)^{m-n}QE\left(-\binom{m-n}{2}+(m-n)(c-2)\right)\\ &\frac{\langle a+1-c;q\rangle_m\langle b_1+1-c;q\rangle_{m-n}\langle c-1;q\rangle_{n-m}\langle b_2;q\rangle_n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}x^{m-n}y^n. \end{aligned} \tag{78}$$

The operator form is

$$f_D(x,y;q) = x^{1-c}{}_2\phi_1(a+1-c, b_1+1-c-\theta_{q,2}; 2-c-\theta_{q,2}|q;x)\\ {}_2\phi_1\Big(b_2, c-1; c-b_1|q;\frac{y}{x}\Big). \tag{79}$$

The series (78) converges in a slightly larger region than

$$|x| < 1,\ |y/x| < 1. \tag{80}$$

5.1. First Horn Recurrence Solution

The Horn recurrence (16) for $\rho = 1-c,\ \sigma = 0$ gives

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle a+1-c+m+n, b_1+1-c+m;q\rangle_1}{\langle 1+m+n, 2-c+m;q\rangle_1} \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle a+1-c+m+n, b_2+n;q\rangle_1}{\langle 1+m+n, 1+n;q\rangle_1}. \end{cases} \tag{81}$$

The solution to this recurrence is

$$f_2(x,y;q) = x^{1-c}\sum_{m,n=0}^{\infty}\frac{\langle a+1-c;q\rangle_{m+n}\langle b_1+1-c\rangle_m\langle b_2\rangle_n}{\langle 2-c\rangle_m\langle 1;q\rangle_{m+n}\langle 1;q\rangle_n}x^m y^n. \tag{82}$$

This solution, not of usual q-hypergeometric type, was not given in the thesis by Le Vavasseur.

By symmetry, we get a third solution $f_3(x,y;q)$, the three functions $\{f_i(x,y;q)\}$ form a basis for the system Φ_1 around $(0,0)$.

5.2. Q-Integral Representations

We now turn to q-integral representations of solutions to the system for Φ_1. The operator form

$$\Phi_1(a;b_1,b_2;c|q;x,y)\\ = {}_2\phi_1(a+\theta_{q,2}, b_1; c+\theta_{q,2}|q;x)\ {}_2\phi_1(a,b_2;c|q;y) \tag{83}$$

together with the q-integral for ${}_2\phi_1$ (7.50 [1]) gives the q-Picard integral (10.104 [1]) for the first q-Appell function.

The operator form (79) together with (7.50 [1]) gives a q-analogue of (7.11 [3]).

Theorem 4.

$$f_D(x,y;q) \cong x^{1-c}\,\Gamma_q\left[\begin{matrix} 2-c \\ a+1-c, 1-a\end{matrix}\right]\int_0^1 t^{a-c}\frac{(qt;q)_{-a}}{(xt;q)_{b_1+1-c}}\\ {}_3\phi_2\left[\begin{matrix} a, b_2 \\ c-b_1\end{matrix}\Big|q; yq^{b_1-c+a}\Big|\Big|\begin{matrix}((xt)^{-1}q^{c-b_1};q)_k \\ (t^{-1}q^a;q)_k\end{matrix}\right] d_q(t). \tag{84}$$

Proof. We can apply (43) for following deduction.

$$f_D(x,y;q) \cong x^{1-c}\,\Gamma_q\left[\begin{matrix} 2-c-\theta_{q,2} \\ a+1-c, 1-a-\theta_{q,2}\end{matrix}\right]\int_0^1 t^{a-c}\frac{(qt;q)_{-a-\theta_{q,2}}}{(xt;q)_{b_1+1-c-\theta_{q,2}}}\\ {}_2\phi_1\left[\begin{matrix} c-1, b_2 \\ c-b_1\end{matrix}\Big|q;\frac{y}{x}\right] d_q(t)=\text{RHS}. \tag{85}$$

□

Similarly, we get an improved version of (7.12, [3]).

Theorem 5.

$$f_D(x,y) = x^{1-c}\,\Gamma\begin{bmatrix} 2-c \\ b_1+1-c, 1-b_1 \end{bmatrix} \int_0^1 t^{b_1-c}(1-t)^{-b_1}(1-xt)^{c-a-1} \left(1-\frac{y}{x}\right)^{-b_2} dt. \tag{86}$$

Proof. Permute the parameters in the proof. □

A q-analogue of (86)

$$f_D(x,y;q) \cong x^{1-c}\,\Gamma_q\begin{bmatrix} 2-c \\ b_1+1-c, 1-b_1 \end{bmatrix} \int_0^1 t^{b_1-c}\frac{(qt;q)_{-b_1}}{(xt;q)_{a+1-c}} \frac{1}{(\frac{y}{x}q^{b_1-1};q)_{b_2}}\,d_q(t). \tag{87}$$

5.3. Solutions Around $(0,\infty)$

From Ansatz III we obtain the equations

$$\begin{cases} \{\rho\}_q\{c+\sigma+\rho-1\}_q = 0, \\ \{b_2+\sigma\}_q\{a+\rho+\sigma)_q = 0. \end{cases} \tag{88}$$

This has the three solutions

$$\begin{aligned} &\rho = 0,\ \sigma = -b_2 \\ &\rho = 0,\ \sigma = -a \\ &\rho = b_2+1-c,\ \sigma = -b_2. \end{aligned} \tag{89}$$

Put

$$\psi(y) \equiv \sum_{n=0}^{\infty} a_n \frac{y^{-n-\lambda}}{\langle 1;q\rangle_n}. \tag{90}$$

According to (40) and (43), the condition $H_1 g_1(x,y;q) = 0$ gives,

$$\begin{aligned} &g_1(x,y;q) \\ &\equiv {}_2\phi_1(a+\theta_{q,2}, b_1; c+\theta_{q,2}|q;x)\psi(y) \\ &= \sum_{m,n=0}^{\infty} a_n \frac{\langle a-n-\lambda, b_1;q\rangle_m}{\langle 1, c-n-\lambda;q\rangle_m \langle 1;q\rangle_n} x^m y^{-n-\lambda}. \end{aligned} \tag{91}$$

Lemma 3.

$$\begin{aligned} &\frac{\langle a-b_2-n;q\rangle_m \langle b_2-c+1;q\rangle_n (-1)^{m-n}}{\langle c-b_2-n;q\rangle_m \langle b_2+1-a;q\rangle_n \langle a-b_2;q\rangle_{m-n} \langle b_2+1-c;q\rangle_{n-m}} \\ &= QE\left(-\binom{m}{2} - \binom{n}{2} - m(c-b_2-n) - n(2b_2+1-c)\right). \end{aligned} \tag{92}$$

This lemma is used in the following proof. Similar to ([3], p. 796) we find that

$$\begin{aligned} &\left[\{\theta_{q,1}+\theta_{q,2}+c\}_q \mathrm{D}_{q,y}\right](x^m y^{-n-\lambda}) \\ &= \{m-n-\lambda-1+c\}_q\{-n-\lambda\}_q x^m y^{-n-\lambda-1}. \end{aligned} \tag{93}$$

We have

$$H_2 f(x,y;q) = -\sum_{m,n=0}^{\infty} a_n \frac{\langle a-n-\lambda, b_1; q\rangle_m}{\langle 1, c-n-\lambda; q\rangle_m \langle 1; q\rangle_n} \Big[\{m-n-\lambda-1+c\}_q \{-n-\lambda\}_q x^m y^{-n-\lambda-1} + \{b_2-n-\lambda\}_q \{a+m-n-\lambda\}_q x^m y^{-n-\lambda}\Big]. \tag{94}$$

For $n=0$, the condition $H_2 f = 0$ implies $\lambda = b_2$, and we have

$$H_2 f(x,y;q) = y^{-b_2} \sum_{m,n=0}^{\infty} \left[-a_n \frac{\langle a-b_2-n, b_1; q\rangle_m \{b_2+n\}_q q^{-n-b_2}}{\langle c-b_2-n; q\rangle_{m-1} \langle 1; q\rangle_m \langle 1; q\rangle_n} x^{m-n} y^{n-1} + a_{n+1} \frac{\langle a-b_2-n-1; q\rangle_{m+1} \langle b_1; q\rangle_m}{\langle 1, c-b_2-n-1; q\rangle_m \langle 1; q\rangle_n}\right] x^m y^{-n-1} q^{-n}. \tag{95}$$

By the condition $H_2 f = 0$ we obtain

$$\frac{a_{n+1}}{a_n} = \frac{\langle n+b_2, c-b_2-n-1; q\rangle_1}{\langle a-b_2-n-1; q\rangle_1} q^{-b_2}. \tag{96}$$

This implies

$$a_n = \frac{\langle b_2, b_2+1-c; q\rangle_n}{\langle b_2+1-a; q\rangle_n} q^{n(c-a-b_2)}, \tag{97}$$

$$\psi(y) = {}_2\phi_1(b_2, b_2+1-c; b_2+1-a|q; yq^{c-a-b_2}). \tag{98}$$

According to (92), we should have

$$\begin{aligned} &g_1(x,y;q) \\ &= y^{-b_2} \sum_{m,n=0}^{\infty} \frac{\langle a-b_2-n, b_1; q\rangle_m \langle b_2+1-c, b_2; q\rangle_n}{\langle 1, c-b_2-n; q\rangle_m \langle 1, b_2+1-a; q\rangle_n} x^m y^{-n} q^{n(c-a-b_2)} \\ &= y^{-b_2} \sum_{m,n=0}^{\infty} \frac{\langle b_1; q\rangle_m \langle a-b_2; q\rangle_{m-n} \langle b_2; q\rangle_n \langle b_2+1-c; q\rangle_{n-m}}{\langle 1; q\rangle_m \langle 1; q\rangle_n} \\ &(-x)^m (-y)^{-n} QE\left(-\binom{m}{2} - \binom{n}{2} - m(c-b_2-n) + n(c-2b_2-1)\right). \end{aligned} \tag{99}$$

We can again rewrite this in the form of the q-Horn function. The operator form is

$$g_1(x,y;q) = {}_2\phi_1(a+\theta_{q,2}, b_1; c+\theta_{q,2}|q;x)\, {}_2\phi_1(b_2, b_2+1-c; b_2+1-a|q; yq^{c-a-b_2}). \tag{100}$$

Again put

$$\psi(y) \equiv y^{-\lambda} \sum_{n=0}^{\infty} a_n \frac{y^{-n}}{\langle 1; q\rangle_n}, \tag{101}$$

and use the other q-hypergeometric function solution around 0.

According to (40) and (43), we have

$$\begin{aligned} &g_2(x,y;q) \\ &\equiv x^{1-c-\theta_{q,2}} {}_2\phi_1(a+1-c, b_1+1-c-\theta_{q,2}; 2-c-\theta_{q,2}|q;x)\psi(y) \\ &= x^{1-c+\lambda} y^{-\lambda} \sum_{m,n=0}^{\infty} a_n \frac{\langle a+1-c, b_1+1-c+n+\lambda; q\rangle_m}{\langle 1, 2-c+n+\lambda; q\rangle_m \langle 1; q\rangle_n} x^{m+n} y^{-n}. \end{aligned} \tag{102}$$

Similar to ([3], p. 797) we find that

$$\begin{aligned}&\left[\{\theta_{q,1}+\theta_{q,2}+c\}_q \mathrm{D}_{q,y}\right](x^{m+n+\lambda+1-c}y^{-n-\lambda})\\&=\{m\}_q\{-n-\lambda\}_q x^{m+n+\lambda+1-c}y^{-n-\lambda-1}.\end{aligned} \tag{103}$$

Because of the factor $\{b_2-n-\lambda\}_q$, $\lambda=b_2$, and we have

$$\begin{aligned}&H_2g_2(x,y;q)(1-q)^2=x^{1-c+b_2}y^{-b_2}\\&\Bigg[\sum_{m=1,n=0}^{\infty}-a_n\frac{\langle a+1-c,b_1+1-c+n+b_2;q\rangle_m\langle n+b_2;q\rangle_1}{\langle 2-c+n+b_2;q\rangle_m\langle 1;q\rangle_{m-1}\langle 1;q\rangle_n}x^{m+n}y^{-n-1}q^{-n-b_2}\\&+\sum_{m=0,n=1}^{\infty}a_n\langle 1+m+a-c;q\rangle_1\frac{\langle a+1-c,b_1+1-c+n+b_2;q\rangle_m}{\langle 1,2-c+n+b_2;q\rangle_m\langle 1;q\rangle_{n-1}}x^{m+n}y^{-n}q^{-n}\Bigg]\\&=x^{2-c+b_2}y^{-b_2}\sum_{m,n=0}^{\infty}\Bigg[-a_n\frac{\langle a+1-c,b_1+1-c+n+b_2;q\rangle_{m+1}}{\langle 2-c+n+b_2;q\rangle_{m+1}\langle 1;q\rangle_m\langle 1;q\rangle_n}\langle n+b_2;q\rangle_1 q^{-b_2}\\&+a_{n+1}\langle 1+m+a-c;q\rangle_1\frac{\langle a+1-c,b_1+2-c+n+b_2;q\rangle_m}{q\langle 1,3-c+n+b_2;q\rangle_m\langle 1;q\rangle_n}\Bigg]x^{m+n}y^{-n-1}q^{-n}.\end{aligned} \tag{104}$$

By the condition $H_2g_2=0$ we obtain

$$a_n=\frac{\langle b_2,b_1+b_2+1-c;q\rangle_n}{\langle b_2+2-c;q\rangle_n}q^{n(1-b_2)}, \tag{105}$$

$$\psi(y)=y^{-b_2}{}_2\phi_1(b_2,b_1+b_2+1-c;b_2+2-c|q;y^{-1}q^{1-b_2}). \tag{106}$$

$$\begin{aligned}&g_2(x,y;q)=x^{b_2+1-c}y^{-b_2}\sum_{m,n=0}^{\infty}\\&\frac{\langle a+1-c,b_1+1-c-n;q\rangle_m\langle b_2,c-1;q\rangle_n}{\langle 1,2-c+n+b_2;q\rangle_m\langle 1,b_2+2-c;q\rangle_n}x^{m+n}y^{-n}q^{n(1-b_1)}\\&=x^{b_2+1-c}y^{-b_2}\Phi_1\left(b_1+b_2+1-c;a+1-c,b_2;b_2+2-c\middle|q;x,\frac{x}{y}q^{n(1-b_2)}\right).\end{aligned} \tag{107}$$

The operator form is

$$\begin{aligned}&g_2(x,y;q)=x^{1-c-\theta_{q,2}}{}_2\phi_1(a+1-c,b_1+1-c-\theta_{q,2};2-c-\theta_{q,2}|q;x)\\&y^{-b_2}{}_2\phi_1(b_2,b_1+b_2+1-c;b_2+2-c|q;y^{-1}q^{1-b_2}).\end{aligned} \tag{108}$$

Type B2. Use the same $\psi(x)$ and the function (6.187 [1]), according to (40) and (43)

$$\begin{aligned}&g_4(x,y;q)\\&\equiv y^{-a-\theta_{q,1}}{}_2\phi_1(a+\theta_{q,1},a+1-c;a+1-b_2+\theta_{q,1}|q;\frac{1}{y})\psi(x)\\&=y^{-a}\sum_{m,n=0}^{\infty}a_n\frac{\langle a+n+\lambda,a+1-c;q\rangle_m}{\langle 1,a+1-b_2+n+\lambda;q\rangle_m\langle 1;q\rangle_n}x^{n+\lambda}y^{-m-n-\lambda}.\end{aligned} \tag{109}$$

Similar to ([3], p. 798) we find that

$$\begin{aligned}&\left[\{\theta_{q,1}+\theta_{q,2}+c\}_q \mathrm{D}_{q,x}\right](x^{n+\lambda}y^{-m-n-a-\lambda})\\&=\{c-a-1-m\}_q\{n+\lambda\}_q x^{n+\lambda-1}y^{-m-n-a-\lambda}.\end{aligned} \tag{110}$$

Again, $\lambda = 0$, and we have

$$H_1 g_4(x,y;q) = y^{-a} \left[\sum_{m=0,n=1}^{\infty} a_n \frac{\langle c-a-1-m;q\rangle_1 \langle a+n,a+1-c;q\rangle_m}{\langle 1,a+1+n-b_2;q\rangle_m \langle 1;q\rangle_{n-1}} x^{n-1} y^{-n-m} - \sum_{m=1,n=0}^{\infty} a_n \langle b_1+n;q\rangle_1 \frac{\langle a+n,a+1-c;q\rangle_m}{\langle 1,2-c-n;q\rangle_m \langle 1;q\rangle_n} q^{-m} x^n y^{-m-n} \right] = y^{-a} \sum_{m,n=0}^{\infty} \left[a_{n+1} \frac{\langle c-a-1-m;q\rangle_1 \langle a+n+1,a+1-c;q\rangle_m}{\langle 1,a+2+n-b_2;q\rangle_m \langle 1;q\rangle_n} - a_n \langle n+b_1;q\rangle_1 \frac{\langle a+1-c,a+n;q\rangle_{m+1}}{\langle a+1+n-b_2;q\rangle_{m+1} \langle 1;q\rangle_m \langle 1;q\rangle_n} q^{-m} \right] x^n y^{-m-n}. \tag{111}$$

By the condition $H_1 g_4 = 0$ we obtain

$$\frac{a_{n+1}}{a_n} = -q^{a+1-c} \frac{\langle n+b_1,n+a;q\rangle_1}{\langle a+1+n-b_2;q\rangle_1}. \tag{112}$$

This implies

$$a_n = (-1)^n \frac{\langle a,b_1;q\rangle_n}{\langle a+1-b_2;q\rangle_n} q^{n(a+1-c)} \tag{113}$$

$$\psi(x) = {}_2\phi_1(a,b_1;a+1-b_2|q;-xq^{a+1-c}). \tag{114}$$

Finally, we obtain a q-analogue of the corrected version of Levavasseur.

$$g_4(x,y;q) = y^{-a-\theta_{q,1}} {}_2\phi_1\left(a+\theta_{q,1},a+1-c;a+1-b_2+\theta_{q,1}\middle|q;\frac{1}{y}\right) {}_2\phi_1(a,b_1;a+1-b_2|q;-xq^{a+1-c}) = y^{-a}\Phi_1\left(a;b_1,a+1-c;a+1-b_2\middle|q;-\frac{x}{y}q^{a+1-c},\frac{1}{y}\right). \tag{115}$$

Theorem 6. *The second solution is*

$$h_2(x,y;q) = y^{-a} \sum_{m,n=0}^{\infty} \frac{\langle a+1-c;q\rangle_{n-m} \langle a;q\rangle_n \langle b_1;q\rangle_m}{\langle a+1-b_2;q\rangle_n \langle 1;q\rangle_m \langle 1;q\rangle_{n-m}} x^m y^{-n} \, QE((c-a)(m-n)+(1-b_2)n). \tag{116}$$

This solution is a q-analogue of ([5], p. 31).

Proof. From the recurrences, using (6.14 [1]), we can find

$$h_2(x,y;q) = y^{-a} \sum_{m,n=0}^{\infty} \frac{\langle c-a+m-n;q\rangle_{n-m} \langle 1-a-n;q\rangle_n \langle b_1;q\rangle_m}{\langle -a-n+b_2;q\rangle_n \langle 1;q\rangle_m \langle m-n;q\rangle_{n-m}} x^m y^{-n} = \text{RHS}. \tag{117}$$

□

Theorem 7. *The third solution is*

$$h_3(x,y;q) = x^{b_2+1-c} y^{-b_2} \sum_{m,n=0}^{\infty} \frac{\langle a+1-c;q\rangle_{m-n} \langle b_2;q\rangle_n \langle b_1+b_2+1-c;q\rangle_m}{\langle 2+b_2-c;q\rangle_m \langle 1;q\rangle_n \langle 1;q\rangle_{m-n}} x^m y^{-n} \, QE(-nb_2+n). \tag{118}$$

This solution is a q-analogue of ([5], p. 31).

Proof. For the case $\rho = b_2 + 1 - c,\ \sigma = -b_2$ we obtain the recurrence

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle a+1-c+m-n, b_1+b_2+1-c+m; q\rangle_1}{\langle 1+m-n, 2+b_2-c+m; q\rangle_1} \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle 1+m-n, -b_2-n; q\rangle_1}{\langle a+m-n+1-c, -1-n; q\rangle_1}. \end{cases} \tag{119}$$

The solution to this recurrence, using (6.14 [1]), is

$$\begin{aligned} h_3(x,y;q) = x^{b_2+1-c} y^{-b_2} \sum_{m,n=0}^{\infty} \langle a+1-c; q\rangle_{m-n} x^m y^{-n} \\ \frac{\langle -b_2+1-n; q\rangle_n \langle b_1+b_2+1-c; q\rangle_m}{\langle 2+b_2-c; q\rangle_m \langle -n; q\rangle_n \langle 1; q\rangle_{m-n}} = \text{RHS}. \end{aligned} \tag{120}$$

□

6. Second q-Appell Function

Now we put $\gamma_i = 0,\ ,\ \alpha_i = \beta_i = 1,\ a_i = a$ in (49).

Theorem 8. *A q-analogue of ((1) [6]), ([7], p. 50). The q-difference equation for* Φ_2 *has the following four independent solutions in the vicinity of* $(0,0)$.

$$\begin{aligned} &f_1(x,y;q) \equiv \Phi_2(a; b_1, b_2; c_1, c_2|q; x, y), \\ &f_2(x,y;q) \equiv\ x^{1-c_1}\Phi_2(a-c_1+1; b_1-c_1+1, b_2; 2-c_1, c_2|q; x, y), \\ &f_3(x,y;q) \equiv\ y^{1-c_2}\Phi_2(a-c_2+1; b_1, b_2-c_2+1; c_1, 2-c_2|q; x, y), \\ &f_4(x,y;q) \equiv\ x^{1-c_1}y^{1-c_2} \\ &\Phi_2(a-c_1-c_2+2; b_1-c_1+1, b_2-c_2+1; 2-c_1, 2-c_2|q; x, y). \end{aligned} \tag{121}$$

Proof. According to (40) and (43), we find

$$\begin{aligned} &f_3(x,y;q) \\ &\equiv {}_2\phi_1(a+\theta_{q,2}, b_1; c_1|q; x) y^{1-c_2} {}_2\phi_1(a+1-c_2, b_2+1-c_2; 2-c_2|q; y) \\ &= \text{RHS}. \end{aligned} \tag{122}$$

$$\begin{aligned} &f_4(x,y;q) \\ &= x^{1-c_1} {}_2\phi_1(a+1-c_1+\theta_{q,2}, b+1-c_1; 2-c_1|q; x) \\ &y^{1-c_2} {}_2\phi_1(a+2-c_1-c_2, b+2-c_2+1; 2-c_2|q; y) \\ &= \text{RHS}. \end{aligned} \tag{123}$$

□

Remark 2. *The asymmetric expressions for* $f_4(x,y)$ *in ([3], p. 804 f) are in error.*

A q-analogue of ([3], p. 804).

Theorem 9. *A q-integral representation of* $\Phi_2(a; b_1, b_2; c_1, c_2|q; x, y)$

$$\Phi_2(a;b_1,b_2;c_1,c_2|q;x,y) \cong \Gamma_q\left[\begin{array}{c} c_1 \\ a, c_1-a \end{array}\right] \int_0^1 t^{a-1} \frac{(qt;q)_{c_1-a-1}}{(xt;q)_{b_1}}$$
$$\sum_{m=0}^{\infty} \frac{\langle b_2, a+1-c_1;q\rangle_m}{\langle 1,c_2;q\rangle_m} (-ty)^m (tq^{c_1-a};q)_{-m} \tag{124}$$
$$\mathrm{QE}\left(-\binom{m}{2}+m(c_1-a-1)\right) d_q(t).$$

Proof. According to (7.50 [1]) we have

$$\text{LHS} = {}_2\phi_1(a+\theta_{q,2},b_1;c_1|q;x){}_2\phi_1(a,b_2;c_2|q;y)$$
$$=\Gamma_q\left[\begin{array}{c} c_1 \\ a+\theta_{q,2}, c_1-a-\theta_{q,2} \end{array}\right] \int_0^1 t^{a+\theta_{q,2}-1} \frac{(qt;q)_{c_1-a-\theta_{q,2}-1}}{(xt;q)_{b_1}} \tag{125}$$
$${}_2\phi_1(a,b_2;c_2|q;y)\, d_q(t)=\text{RHS}.$$

□

Theorem 10. *The functions $f_3(x,y;q)$ and $f_4(x,y;q)$ have q-integral representations*

$$f_3(x,y;q) \cong x^{1-c_1}\,\Gamma_q\left[\begin{array}{c} c_2 \\ a, c_2-a \end{array}\right] \int_0^1 t^{a-1} \frac{(qt;q)_{c_2-a-1}}{(yt;q)_{b_2}}$$
$$\sum_{m=0}^{\infty} \frac{\langle a+1-c_1, a+1-c_2, b_1+1-c_1;q\rangle_m}{\langle 1,2-c_1,a;q\rangle_m} (-xt)^m (tq^{c_2-a};q)_{-m} \tag{126}$$
$$\mathrm{QE}\left(-\binom{m}{2}+m(c_2-a-1)\right) d_q(t).$$

$$f_4(x,y;q) \cong x^{1-c_1}y^{1-c_2}\,\Gamma_q\left[\begin{array}{c} 2-c_1 \\ a+1-c_1, 1-a \end{array}\right] \int_0^1 t^{a-c_1} \frac{(qt;q)_{-a}}{(xt;q)_{b_1+1-c_1}}$$
$$\sum_{m=0}^{\infty} \frac{\langle a+2-c_1-c_2, b_2-c_2+1, a;q\rangle_m}{\langle 1,2-c_2,a+1-c_1;q\rangle_m} (-yt)^m (tq^{1-a};q)_{-m} \tag{127}$$
$$\mathrm{QE}\left(-\binom{m}{2}-ma\right) d_q(t).$$

Proof. Use formulas (122) and (123). □

7. Third *q*-Appell Function

Let us put $\alpha_i=\beta_i=0,\ \gamma_i=1,\ c_i=c,\ a_i=a,\ b_i=b$ in (49).

Theorem 11. *A q-analogue of ([3], p. 805). The third q-Appell function has q-integral representation*

$$\Phi_3(a_1,a_2;b_1,b_2;c|q;x,y) \cong \Gamma_q\left[\begin{array}{c} c \\ a_1, c-a_1 \end{array}\right] \int_0^1 t^{a_1-1} \frac{(qt;q)_{c-a_1-1}}{(xt;q)_{b_1}}$$
$$\sum_{m=0}^{\infty} \frac{\langle a_2,b_2;q\rangle_m}{\langle 1,c-a_1;q\rangle_m} y^m (q^{c-a_1}t;q)_m\, d_q(t). \tag{128}$$

Proof. Using (7.50 [1]), we have

$$\text{LHS}=\Gamma_q\left[\begin{array}{c} c+\theta_{q,2} \\ a_1, c-a_1+\theta_{q,2} \end{array}\right] \int_0^1 t^{a_1-1} \frac{(qt;q)_{c-a_1-\theta_{q,2}-1}}{(xt;q)_{b_1}}$$
$${}_2\phi_1\left[\begin{array}{c} a_2,b_2 \\ c \end{array}\middle| q;y\right] \stackrel{\text{by}(43)}{=} \text{RHS}. \tag{129}$$

□

8. Fourth q-Appell Function

Finally, we put $\gamma_i = 0,\ \alpha_i = \beta_i = 1,\ a_i = a,\ b_i = b$ in (49). We have

$$\Phi_4(a,b;c_1,c_2|q;x,y) = {}_2\phi_1(a+\theta_{q,2}, b+\theta_{q,2};c_1|q;x){}_2\phi_1(a,b;c_2|q;y). \tag{130}$$

Theorem 12. *A q-analogue of ([3], p. 807). The fourth q-Appell function has q-integral representation*

$$\begin{aligned}&\Phi_4(a,b;c_1,c_2|q;x,y) \cong \Gamma_q\left[\begin{matrix} c_1 \\ a, c_1-a \end{matrix}\right]\int_0^1 t^{a-1}\frac{(qt;q)_{c_1-a-1}}{(xt;q)_b} \\ &\sum_{m=0}^{\infty}\frac{\langle a+1-c_1, b;q\rangle_m}{\langle 1,c_2;q\rangle_m}(-yt)^m\frac{(tq^{c_1-a};q)_{-m}}{(xtq^b t;q)_m} \\ &QE\left(-\binom{m}{2} - m(a+1-c_1)\right)d_q(t).\end{aligned} \tag{131}$$

Proof. Using (7.50 [1]) and (43), we have

$$\begin{aligned}&\text{LHS}{=}\Gamma_q\left[\begin{matrix} c_1+\theta_{q,2} \\ a+\theta_{q,2}, c_1-a-\theta_{q,2} \end{matrix}\right]\int_0^1 t^{a+\theta_{q,2}-1}\frac{(qt;q)_{c_1-a-\theta_{q,2}-1}}{(xt;q)_{b+\theta_{q,2}}} \\ &{}_2\phi_1\left[\begin{matrix} a,b \\ c_2 \end{matrix}\middle|q;y\right]{=}\text{RHS}.\end{aligned} \tag{132}$$

□

Theorem 13. *A q-analogue of ([3], pp. 807–808). The q-difference equation for Φ_4 has the following four independent solutions in the vicinity of $(0,0)$.*

$$\begin{aligned}&f_1(x,y;q) \equiv \Phi_4(a,b;c_1,c_2|q;x,y), \\ &f_2(x,y;q) \equiv\ y^{1-c_2}\Phi_4(a-c_2+1;b-c_2+1;2-c_2,c_1|q;x,y), \\ &f_3(x,y;q) \equiv\ x^{1-c_1}\Phi_4(a-c_1+1;b-c_1+1;2-c_1,c_2|q;x,y), \\ &f_4(x,y;q) \equiv\ x^{1-c_1}y^{1-c_2} \\ &\Phi_4(a-c_1-c_2+2, b_-c_1-c_2+2;2-c_1,2-c_2|q;x,y).\end{aligned} \tag{133}$$

Proof. According to (40) and (43), we find

$$\begin{aligned}&f_2(x,y;q) \\ &= {}_2\phi_1(a+\theta_{q,2}, b+\theta_{q,2};c_1|q;x)y^{1-c_2}{}_2\phi_1(a+1-c_2,b+1-c_2;2-c_2|q;y) \\ &{=}\text{RHS}.\end{aligned} \tag{134}$$

$$\begin{aligned}&f_4(x,y;q) \\ &= x^{1-c_1}{}_2\phi_1(a+1-c_1+\theta_{q,2}, b+1-c_1+\theta_{q,2};2-c_1|q;x) \\ &y^{1-c_2}{}_2\phi_1(a+2-c_1-c_2, b+2-c_1-c_2;2-c_2|q;y) \\ &{=}\text{RHS}.\end{aligned} \tag{135}$$

□

Theorem 14. *The functions $f_2(x,y;q)$ and $f_4(x,y;q)$ have q-integral representations*

$$f_2(x,y;q) \cong y^{1-c_2}\,\Gamma_q\left[\begin{array}{c} c_1 \\ a, c_1-a \end{array}\right] \int_0^1 t^{a-1}\frac{(qt;q)_{c_1-a-1}}{(xt;q)_b}$$
$$\sum_{m=0}^{\infty}\frac{\langle a+1-c_1, a+1-c_2, b+1-c_2;q\rangle_m}{\langle 1,a,2-c_2;q\rangle_m}(-yt)^m\frac{(tq^{c_1-a};q)_{-m}}{(xtq^b t;q)_m} \tag{136}$$
$$QE\left(-\binom{m}{2}-m(a+1-c_1)\right)d_q(t).$$

$$f_4(x,y;q) \cong x^{1-c_1}y^{1-c_2}\Gamma_q\left[\begin{array}{c} 2-c_1 \\ a+1-c_1, 1-a \end{array}\right] \int_0^1 t^{a-c_1}\frac{(qt;q)_{-a}}{(xt;q)_{b_1+1-c_1}}$$
$$\sum_{m=0}^{\infty}\frac{\langle a+2-c_1-c_2, b_2-c_2+1, a;q\rangle_m}{\langle 1,2-c_2,a+1-c_1;q\rangle_m}(-yt)^m\left(tq^{1-a};q\right)_{-m} \tag{137}$$
$$QE\left(-\binom{m}{2}-ma\right)d_q(t).$$

Proof. Use formulas (134) and (135). □

9. Conclusions

We have given the other solutions to the systems of q-difference equations in three forms

1. the factorized, umbral form
2. the series expansion, with convergence regions, q-analogues of [3]
3. possibly, a q-integral representation

These convergence regions are always larger than in the ordinary case, sometimes q-deformed cones arise. Our method leads to more direct computation of the other solutions of Appell differential and similar differential equations than the papers by Horn and Borngässer. We have illustrated the new symbolic calculus in the special case q-Appell functions, since more complex functions would lead to longer computations. These computations are similar to the solutions of differential equations by the Frobenius method. We started with the solutions in the vicinity of $(0,0)$ and obtained the usual indicial equation for the exponents. Then we found all solutions, which was treated by Borngässer [5]. With the help of a lemma, we found a recurrence for the unknown coefficients, and the unknown function was sometimes another q-Appell function and sometimes a q-Horn function.

Then, by the symbolic operator formulas, we found q-integral representations of the formulas in the basis. For the solutions around $(0,\infty)$ we found $\lambda = b_2$ and by using another lemma, we obtained another q-Horn function in the basis of solutions.

10. Discussion

The Frobenius method [8] for solutions of differential equations originates from papers by Thomae [9], who studied logarithmic solutions of the Euler equation and Thomé [10], who wrote about very general solutions of differential equations, convergent in disks around a point a.

Thanks to Debiard and Gaveau for their most interesting papers on multiple hypergeometric functions. We have retained their notation as much as possible. The Debiard–Gaveau umbral method was neither used in the thesis by Borngässer [5], nor in the papers by Horn [4]. However, the umbral q-difference equations for q-Appel functions in our book ([1], p. 436), in the spirit of Mellin [11] and Thomae [9], are equivalent to the q-difference equations in this paper. The paper is also interesting for the case $q = 1$, since Borngässer's thesis [5], in German, is almost unknown, and is now available, in part, in English. In a future paper, we will discuss the q-difference equations and q-integral representations of the corresponding q-Horn functions. Likewise, the confluent forms [12], as well as other multiple q-hypergeometric functions can be treated with this method.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Ernst, T. *A Comprehensive Treatment of q-Calculus*; Birkhäuser: Basel, Switzerland, 2012.
2. Ernst, T. On the q-analogues of Srivastava's triple hypergeometric functions. *Axioms* **2013**, *2*, 85–99. [CrossRef]
3. Debiard, A.; Gaveau, B. Hypergeometric symbolic calculus. I: Systems of two symbolic hypergeometric equations. *Bull. Sci. Math.* **2002**, *126*, 773–829. [CrossRef]
4. Horn, J. Hypergeometric functions of two variables. (Hypergeometrische Funktionen zweier Veränderlichen). *Math. Ann.* **1931**, *105*, 381–407. [CrossRef]
5. Borngässer, L. Über hypergeometrische Funktionen zweier Veränderlichen. Ph.D. Thesis, Pfeffer & Balzer, Darmstadt, Germany, 1933.
6. Olsson, P.O.M. On the integration of the differential equations of five-parametric double-hypergeometric functions of second order. *J. Math. Phys.* **1977**, *18*, 1285. [CrossRef]
7. Appell, P.; Kampé de Fériet, J. *Fonctions Hypergéométriques et Hypersphériques*; Polynomes D'Hermite: Paris, France, 1926.
8. Frobenius, G. On the integration of linear differential equations by means of series. (Über die Integration der linearen Differentialgleichungen durch Reihen.) *Crelle* **1873**, *76*, 214–235.
9. Thomae, J. On the higher hypergeometric series, especially the series ....(Über die höheren hypergeometrischen Reihen, insbesondere die Reihe ....) *Math. Ann.* **1870**, *2*, 427–444. (In Germany) [CrossRef]
10. Thomé, L.W. On the theory of linear differential equations. (Zur Theorie der linearen Differentialgleichungen). *Crelle* **1872**, *74*, 193–217. (In Germany)
11. Mellin, H.J. On the connection between the linear differential and difference equations, Über den Zusammenhang zwischen den linearen Differential- und Differenzengleichungen. *Acta Math.* **1902**, *25*, 139–164. (In Germany) [CrossRef]
12. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Ellis Horwood: New York, NY, USA, 1985.