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A New Equilibrium Version of Ekeland's Variational Principle and Its Applications

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Abstract: In this note, a new equilibrium version of Ekeland's variational principle is presented. It is a modification and promotion of previous results. Subsequently, the principle is applied to discuss the equilibrium points for binary functions and the fixed points for nonlinear mappings.

Keywords: Ekeland-type variational principle; bi-function; equilibrium point; fixed point

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1. Introduction

Ekeland's variational principle (abbrev. **EVP**), which is considered to be the basis of modern calculus of variations, was presented in 1974 (see, for instance [1,2]). It is widely used in many fields, such as differential equations, optimization, fixed point theory, etc. It is precisely the wide application of this theorem that it has attracted the attention of a large number of scholars, and has been promoted from all directions. For example, Zhong [3] extended the form of EVP in metric space; we rewrite the result as follows.

Theorem 1 (EVP of Zhong-type [3])**.** *Let (X, d) be a complete metric space and $x_0 \in X$ fixed. The function $f : X \to R \cup \{+\infty\}$ is bounded from below, lower semi-continuous, and not identically $+\infty$.*

If $g : [0, +\infty) \to [0, +\infty)$ is a continuous non-decreasing function such satisfying

$$\int_0^{+\infty} \frac{1}{1+g(r)} dr = +\infty,$$

then, for any $\varepsilon > 0, y \in M$ such that

$$f(y) < \inf_{x \in X} f(x) + \varepsilon$$

and, for any $\lambda > 0$, there exists $z \in X$ satisfying

$$f(z) \leq f(y)$$

$$d(z, x_0) \leq \bar{r} + r_0$$

and

$$f(x) \geq f(z) - \frac{\varepsilon}{\lambda(1 + g(d(x_0, z)))} d(x, z) \qquad \forall x \in M$$

where $r_0 = d(x_0, y)$ *and* $\bar{r}$ *is such that*

$$\int_{r_0}^{\bar{r}+r_0} \frac{1}{1+g(r)}\, dr \geq \lambda.$$

Oettli and Théra [4] and Blum and Oettli [5] investigated the equilibrium versions of EVP. In [6], Bianchi et al. presented equilibrium versions of EVP as follows

Let X be an Euclidean space, $C \subseteq X$ be a closed set and $f : C \times C \to R$.

Theorem 2 ([6])**.** *Assume the following assumptions are satisfied:*

(i) $f(x, \cdot)$ *is lower bounded and lower semicontinuous, for every* $x \in C$;
(ii) $f(t, t) = 0$, *for every* $t \in C$;
(iii) $f(z, x) \leq f(z, t) + f(t, x)$ *for every* $x, t, z \in C$.

Then, for every $\varepsilon > 0$ *and for every* $x_0 \in C$, *there exists* $\overline{x} \in C$ *such that*

(a) $f(x_0, \overline{x}) + \varepsilon\|x_0 - \overline{x}\| \leq 0$;
(b) $f(\overline{x}, x) + \varepsilon\|\overline{x} - x\| > 0,\ \forall x \in C, x \neq \overline{x}$.

Farkas and Molnar [7] improved the conclusion in [6], and obtained a Zhong-type variational principle for bi-functions as follows:

Theorem 3 ([7])**.** *Let* (X, d) *be a complete metric space,* $C \subset X$ *be a closed set, and* $f : C \times C \to \mathbf{R}^+$ *be a mapping. Let* $g : [0, +\infty) \to (0, +\infty)$ *be a continuous nondecreasing function such that*

$$\int_0^{+\infty} \frac{1}{g(s)} ds = +\infty$$

Let $x_0 \in C$ *be fixed. Assume that the following assumptions be satisfied:*

(i) $f(x, \cdot)$ *is bounded from below and lower semicontinuous, for every* $x \in C$;
(ii) $f(z, z) = 0$, *for every* $z \in C$;
(iii) $f(z, x) \leq f(z, t) + f(t, x)$ *for every* $x, t, z \in C$;

Then, for every $\varepsilon > 0$ *and* $y \in C$ *for which we have*

$$\inf_{z \in C} f(y, z) > -\varepsilon \tag{1}$$

and for every $\lambda > 0$, *there exists* x_ε *such that*

(a) $d(x_0, x_\varepsilon) < r + \bar{r}$;
(b) $f(x_\varepsilon, x_0) + \frac{\varepsilon}{\lambda(1+g(d(x_0,x_\varepsilon)))} d(x_\varepsilon, x_0) \leq 0$;
(c) $f(x_\varepsilon, x) + \frac{\varepsilon}{\lambda(1+g(d(x_0,x_\varepsilon)))} d(x, x_\varepsilon) > 0, \forall x \in C, x \neq x_\varepsilon$;

where $r_0 = d(x_0, y)$ *and* $\bar{r}$ *are chosen such that*

$$\int_{r_0}^{r_0+\bar{r}} \frac{1}{1+g(r)} dr \geq \lambda.$$

However, when proving (a), there are some errors in [7].

In the process of proving $\{d(x_0, x_n) < r_0 + \bar{r}\}(14)$, they presented the following inequality,

$$\begin{aligned}\sum_{n=1}^{k-1} \frac{d(x_n, x_{n+1})}{1 + g(d(x_0, x_{n+1}))} &\geq \sum_{n=1}^{k-1} \frac{d(x_0, x_{n+1}) - d(x_0, x_n)}{1 + g(d(x_0, x_{n+1}))} \\ &\geq \sum_{n=1}^{k-1} \int_{d(x_0,x_n)}^{d(x_0,x_{n+1})} \frac{1}{1 + g(r)} dr \\ &= \int_{d(x_0,x_1)}^{d(x_0,x_k)} \frac{1}{1 + g(r)} dr\end{aligned}$$

But in fact, by the continuity and monotonicity of g and the definition of $W(x_n)$, we have $d(x_0, x_n) < d(x_0, x_{n+1})$, then for $d(x_0, x_n) \leq r \leq d(x_0, x_{n+1})$,

$$\frac{1}{1 + g(r)} \geq \frac{1}{1 + g(d(x_0, x_{n+1}))}$$

Hence,

$$\begin{aligned}\sum_{n=1}^{k-1} \int_{d(x_0,x_n)}^{d(x_0,x_{n+1})} \frac{1}{1 + g(r)} dr &\geq \sum_{n=1}^{k-1} \int_{d(x_0,x_n)}^{d(x_0,x_{n+1})} \frac{1}{1 + g(d(x_0, x_{n+1}))} dr \\ &= \sum_{n=1}^{k-1} \frac{d(x_0, x_{n+1}) - d(x_0, x_n)}{1 + g(d(x_0, x_{n+1}))},\end{aligned}$$

which contradicts their conclusion.

In this note, we aim at modifying the result of [7], and establish a new equilibrium form of the Ekeland's variational principle for bi-function. Then, the conclusions are used to discuss the equilibrium point problem and fixed point problem. Some recent advances in Ekeland's variational principles and applications can be seen in [8–19] and references therein.

This paper is organized as follows: In Section 2, we state a new version of Ekeland's variational principle for bi-functions. In Section 3, as applications of the main result, we discuss a equilibrium problem and a fixed point problem.

2. A New Equilibrium Version of EVP

In this section, we establish a new equilibrium version of EVP.

Theorem 4. *Let (X, d) be a complete metric space, $C \subset X$ be a closed set, $x_0 \in C$ fixed, and $g : [0, +\infty) \to (0, +\infty)$ be a continuous nondecreasing function such that*

$$\int_0^{+\infty} \frac{1}{g(s)} ds = m, \quad (0 < m \leq +\infty).$$

If $f : C \times C \to \mathbf{R}$ satisfies:

(i) *$f(x, \cdot)$ is bounded from below and lower semi-continuous, $\forall x \in C$;*
(ii) *$f(y, y) = 0$, $\forall y \in C$;*
(iii) *$f(x, z) \leq f(x, y) + f(y, z)$, $\forall x, y, z \in C$.*

Then, for any $\varepsilon > 0, 0 < \alpha < m$ fulfilling

$$\inf_{z \in C} f(x_0, z) > -\alpha\varepsilon \tag{2}$$

there is $x_\varepsilon \in C$ such that

(a) *$f(x_0, x_\varepsilon) + \frac{\varepsilon}{g(d(x_0,x_0))} d(x_0, x_\varepsilon) \leq 0$*
(b) *$f(x_\varepsilon, x) + \frac{\varepsilon}{g(d(x_0,x_\varepsilon))} d(x_\varepsilon, x) > 0, \forall x \in C, x \neq x_\varepsilon$;*

(c) $d(x_0, x_\varepsilon) \le l$,

where l satisfies

$$\int_0^l \frac{1}{g(s)} ds = \alpha.$$

Proof. Let

$$T(x) = \{y \in C \backslash B(x_0, d(x_0, x)) | f(x_0, x) + \frac{\varepsilon}{g(d(x_0, x))} d(x_0, x) \le 0\}.$$

In the same manner as the proof of Theorem 2.1 in [7], we can construct a sequence $\{x_n\}_{n=0}^{\infty} \subseteq C$ such that

(1) $x_{n+1} \in T(x_n), T(x_{n+1}) \subset T(x_n), n = 0, 1, 2, \cdots$;
(2) $\mathrm{diam} T(x_n) \to 0$.

Due to the completeness of X and the closeness of C, there is a unique $x_\varepsilon \in C$ such that

$$\lim_{n \to \infty} x_n = x_\varepsilon, \quad \bigcap_{n=0}^{\infty} T(x_n) = \{x_\varepsilon\}.$$

As $x_\varepsilon \in T(x_0)$, we have

$$f(x_0, x_\varepsilon) + \frac{\varepsilon}{g(d(x_0, x_0))} d(x_0, x_\varepsilon) \le 0.$$

This verifies assertion (a).

Due to $x_\varepsilon \in T(x_n), n = 0, 1, 2, \cdots$, we obtain $T(x_\varepsilon) \subset T(x_n), n = 0, 1, 2, \cdots$. Hence

$$T(x_\varepsilon) \in \bigcap_{n=0}^{\infty} T(x_n).$$

and $T(x_\varepsilon) = \{x_\varepsilon\}$.

Therefore, the assertion

$$(b) \;\; f(x_\varepsilon, x) + \frac{\varepsilon}{g(d(x_0, x_\varepsilon))} d(x_\varepsilon, x) > 0, \;\; \forall x \in C, x \ne x_\varepsilon,$$

holds.

In what follows, let us verify conclusion (c).

As $x_{n+1} \in T(x_n)$,

$$f(x_n, x_{n+1}) + \frac{\varepsilon}{g(d(x_0, x_n))} d(x_n, x_{n+1}) \le 0, \;\; (n = 0, 1, 2, \cdots)$$

Hence,

$$\sum_{j=0}^{n} f(x_j, x_{j+1}) + \sum_{j=0}^{n} \frac{\varepsilon}{g(d(x_0, x_j))} d(x_j, x_{j+1}) \le 0.$$

Noting that

$$\sum_{j=0}^{n} f(x_j, x_{j+1}) \ge f(x_0, x_{n+1}), \tag{3}$$

we obtain

$$\sum_{j=0}^{n} \frac{\varepsilon}{g(d(x_0, x_j))} d(x_j, x_{j+1}) \le -\sum_{j=0}^{n} f(x_j, x_{j+1}) \le -f(x_0, x_{n+1}) < \alpha\varepsilon,$$

which means

$$\sum_{j=0}^{\infty} \frac{1}{g(d(x_0, x_j))} d(x_j, x_{j+1}) < \alpha$$

We assert $d(x_0, x_\varepsilon) \leq l$. Contrarily, assume $d(x_0, x_\varepsilon) > l$.

Take $\{n_i\}$ as a subsequence of $\{n\}$ such that $\{d(x_0, x_{n_i})\}$ is monotone increasing, converges to $d(x_0, x_\varepsilon)$ and

$$d(x_0, x_k) \leq d(x_0, x_{n_{i-1}}) \ (k = n_{i-1} + 1, n_{i-1} + 2, \cdots, n_i - 1),$$

then

$$\begin{aligned} \sum_{k=n_{i-1}}^{n_i - 1} \frac{d(x_k, x_{k+1})}{g(d(x_0, x_k))} &\geq \sum_{k=n_{i-1}}^{n_i - 1} \frac{d(x_k, x_{k+1})}{g(d(x_0, x_{n_{i-1}}))} \geq \frac{d(x_{n_{i-1}}, x_{n_i})}{g(d(x_0, x_{n_{i-1}}))} \\ &\geq \frac{d(x_0, x_{n_i}) - d(x_0, x_{n_{i-1}})}{g(d(x_0, x_{n_{i-1}}))} \geq \int_{d(x_0, x_{n_{i-1}})}^{d(x_0, x_{n_i})} \frac{1}{g(s)} ds \end{aligned}$$

which implies

$$\alpha > \sum_{n=0}^{\infty} \frac{d(x_n, x_{n+1})}{g(d(x_0, x_n))} \geq \int_0^{d(x_0, x_\varepsilon)} \frac{1}{g(s)} ds > \int_0^l \frac{1}{g(s)} ds = \alpha,$$

a contradiction.

This completes the proof of conclusion (c). □

If there exists $\varphi : X \to \mathbf{R}^+$ such that $f(x, y) = \varphi(y) - \varphi(x)$, we have the following corollary.

Corollary 1. *Let (X, d) be a complete metric space, $C \subset X$ be a closed set, $x_0 \in C$ fixed and $\varphi : C \to \mathbf{R}^+$ be a bounded from below and lower semi-continuous mapping, $g : [0, +\infty) \to (0, +\infty)$ be a continuous nondecreasing function such that*

$$\int_0^{+\infty} \frac{1}{g(s)} ds = m, (0 < m \leq +\infty)$$

If and $\varepsilon > 0$, $0 < \alpha < m$ satisfy

$$\varphi(x_0) \leq \inf_{x \in C} \varphi + \alpha\varepsilon,$$

then there exists x_ε such that

(a) $\varphi(x_\varepsilon) \leq \varphi(x_0)$;

(b) $\varphi(x) > \varphi(x_\varepsilon) - \frac{\varepsilon}{g(d(x_0, x_\varepsilon))} d(x, x_\varepsilon) \quad \forall x \in C \ \text{with} \ x \neq x_\varepsilon$;

(c) $d(x_0, x_\varepsilon) \leq l$;

where l satisfies

$$\int_0^l \frac{1}{g(s)} ds = \alpha$$

Remark 1. *Corollary 1 can be seen as an extension of Theorem 2.1 in [8].*

3. Applications

As applications of Theorem 4, we first discuss the existence of equilibrium point for a bi-function.

By an equilibrium problem (abbrev. EP), we understand the problem of finding

$$\bar{x} \in X \ \ such \ \ that \ \ f(\bar{x}, x) \geq 0 \ \ , \forall x \in C.$$

where C is a given subset of a metric space X and $f : C \times C \to \mathbb{R}$ is a given bi-function.

Theorem 5. *Let (X, d) be a complete metric space, $C \subset X$ be a compact set. Assume $f : C \times C \to$ $\mathbf{R}$ satisfies*

(i) $f(x, \cdot)$ is bounded from below and lower semi-continuous, for every $x \in C$;
(ii) $f(z, z) = 0$, for every $z \in C$;
(iii) $f(z, x) \leq f(z, t) + f(t, x)$ for every $x, t, z \in C$;
(iv) $f(\cdot, y)$ is upper semi-continuous, for every $y \in C$.

Then, the equilibrium problem (EP) has a solution.

Proof. Let $g(s) \equiv 1$. It is a continuous nondecreasing function such and

$$\int_0^{+\infty} \frac{1}{g(s)} ds = +\infty$$

Let $x_0 \in C$ be fixed, for every $\varepsilon_n = \frac{1}{n}$ and $\alpha = n(b - 1)$, where $b = \inf_{z \in C} f(x_0, z)$. Then, by Theorem 4 (b), there exists $x_n \in C$ such that

$$f(x_n, x) + \frac{1}{n} d(x_n, x) \geq 0, \forall x \in C$$

Due to compactness of C, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is convergent, i.e., there exists $\bar{x} \in C$, such that

$$\lim_{k \to \infty} x_{n_k} = \bar{x}.$$

Hence, we have

$$f(\bar{x}, x) \geq \lim_{k \to \infty} \sup[f(x_{n_k}, x) + \frac{1}{n_k} d(x_{n_k}, x)] \geq 0, \forall x \in C$$

This implies that $\bar{x}$ is a solution to the equilibrium problem (EP). □

Then, we establish the following Caristi type fixed point theorem.

Theorem 6. *Let (X, d) be a complete metric space, $x_0 \in X$ fixed, and $\varphi : X \to \mathbf{R}^+$ be a bounded from below and lower semicontinuous mapping, $g : [0, +\infty) \to (0, +\infty)$ be a continuous nondecreasing function such that*

$$\int_0^{+\infty} \frac{1}{g(s)} \, ds = m$$

where $m \in \mathbf{R}^+ \bigcup \{+\infty\}$.

If a mapping $K : X \to X$ satisfies: for some $\varepsilon > 0$,

$$\frac{\varepsilon d(x, K(x))}{g(d(x_0, x))} \leq \varphi(x) - \varphi(K(x)) \ \ \forall x \in X, \tag{4}$$

then K has a fixed point in X.

Proof. Let $f(x, y) = \varphi(y) - \varphi(x), C = X$. By the proof of Theorem 4, for each $\varepsilon > 0$, there exists a sequence $\{x_n\}_{n \in N} \subset X$ and $x_\varepsilon \in X$, such that $x_n \to x_\varepsilon$ as $n \to \infty$ and

$$\varphi(x) > \varphi(x_\varepsilon) - \frac{\varepsilon}{g(d(x_0, x_\varepsilon))} d(x, x_\varepsilon) \qquad \forall x \in C, x \neq x_\varepsilon \tag{5}$$

In what follows, we will prove that x_ε is a fixed point of K.

Conversely, suppose that $x_\varepsilon \neq K(x_\varepsilon)$. Let $x = K(x_\varepsilon)$ and substitute it into (5), we find

$$\varphi(K(x_\varepsilon)) \geq \varphi(x_\varepsilon) - \frac{\varepsilon}{g(d(x_0, x_\varepsilon))} d(K(x_\varepsilon), x_\varepsilon). \tag{6}$$

Taking x_ε instead of x in (4), we have that

$$\frac{\varepsilon d(x_\varepsilon, K(x_\varepsilon))}{g(d(x_0, x_\varepsilon))} \leq \varphi(x_\varepsilon) - \varphi(K(x_\varepsilon)) \tag{7}$$

Combing the inequalities (6) with (7), we know

$$\frac{\varepsilon d(x_\varepsilon, K(x_\varepsilon))}{g(d(x_0, x_\varepsilon))} \leq \varphi(x_\varepsilon) - \varphi(K(x_\varepsilon)) < \frac{\varepsilon d(x_\varepsilon, K(x_\varepsilon))}{g(d(x_0, x_\varepsilon))}$$

which is a contradiction.

Thus $x_\varepsilon = K(x_\varepsilon)$, i.e., x_ε is a fixed point of K. □

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