

Bounded Sets in Topological Spaces

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Abstract: Let G be a monoid that acts on a topological space X by homeomorphisms such that there is a point $x_0 \in X$ with $GU = X$ for each neighbourhood U of x_0 . A subset A of X is said to be G -bounded if for each neighbourhood U of x_0 there is a finite subset F of G with $A \subseteq FU$. We prove that for a metrizable and separable G -space X , the bounded subsets of X are completely determined by the bounded subsets of any dense subspace. We also obtain sufficient conditions for a G -space X to be locally G -bounded, which apply to topological groups. Thereby, we extend some previous results accomplished for locally convex spaces and topological groups.

Keywords: bounded set; group action; G -space; barrelled space

MSC: 22A05; 22D35; 22B05; 54H11; 54A25



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1. Introduction and Basic Facts

The notion of a bounded subset is ubiquitous in many parts of mathematics, particularly in functional analysis and topological groups. Here, we approach this concept from a broader viewpoint. Namely, we consider the action of a monoid G on a topological space X and associate it with a canonical family of G -bounded subsets. This provides a very general notion of boundedness that includes both the bounded subsets considered in functional analysis and in topological groups. In this paper, we initiate the study of this new notion of G -bounded subset. Among other results, it is proved that for a metrizable and separable G -space X , the bounded subsets of X are completely determined by the bounded subsets of any dense subspace, extending results obtained by Grothendieck for metrizable separable locally convex spaces [1], generalized subsequently by Burke and Todorčević and, separately, Saxon and Sánchez-Ruiz for metrizable locally convex spaces [2,3] and by Chis, Ferrer, Hernández and Tsaban for metrizable groups [4,5]. We also obtain sufficient conditions for a G -space X to be locally G -bounded, which applies to topological groups. This also provides the frame for extending to this setting some results by Burke and Todorčević and, separately, Saxon and Sánchez-Ruiz (loc. cit.) for metrizable locally convex spaces. A different approach to the notion of the bounded set was given by Hejzman [6] and Hu [7], who studied this concept in the realm of uniform and even topological spaces. Vilenkin [8] applied this general approach in the realm of topological groups.

2. G -Spaces

Let X be a topological space and let G be a *monoid*, i.e, a semigroup with a neutral element, which will be denoted by e . A *left action* of G on X is a map $\pi: G \times X \rightarrow X$ satisfying that $ex = x$ and $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G$ and $x \in X$, where as usual, we write gx instead of $\pi(g, x) = \pi_g(x) = \pi_x(g)$. A topological space X is said to be a (left) G -space if all translations $\pi_g: X \rightarrow X$ are homeomorphisms. We sometimes denote the G -space X by the pair (G, X) . Let $G \times X \rightarrow X$ and $G \times Y \rightarrow Y$ be two actions. A map

$f: X \rightarrow Y$ between G -spaces is a G -map if $f(gx) = gf(x)$ for every $(g, x) \in G \times X$. Given $x \in X$, its orbit is the set $Gx = \{gx : g \in G\}$. Given $A \subseteq X$, we define $GA = \cup\{Gx : x \in A\}$.

A right G -space (X, G) can be defined analogously. If G^{op} is the opposite semigroup of G with the same topology then (X, G) can be treated as a left G -space (G^{op}, X) (and vice versa).

We say that a point $x \in X$ topologically generates a G -space X if for each neighborhood U of x we have $GU = X$. The set of generating points is denoted by X_{tg} . We say that X is point-generated when $X_{tg} \neq \emptyset$. We refer to [9] for unexplained topological definitions.

2.1. G -Boundedness

Let (G, X) be a point-generated G -space and let us fix a point $x_0 \in X_{tg}$. We say that a set $A \subseteq X$ is (G, x_0) -bounded (or G -bounded for short when there is no possible confusion) if for every neighborhood U of x_0 , there is a finite set $F \subseteq G$ such that $A \subseteq FU$. The set $\mathfrak{B}(G, X, x_0)$ (or $\mathfrak{B}(G, X)$ for short) of all G -bounded sets in X is called the canonical (G, x_0) -boundedness on X . The G -space (G, X) is said to be homogeneous if for every pair of points x, y in X , there is a homeomorphism $f_{xy}: X \rightarrow X$ such that $f_{xy}(x) = y$ and $f_{xy}(A)$ is G -bounded for every G -bounded subset $A \subseteq X$. The proof of the following proposition is straightforward.

Proposition 1. *Let (G, X) be a G -space with a generating point $x_0 \in X_{tg}$. The following assertions hold true:*

1. $A \subseteq X$ is (G, x_0) -bounded if and only if A is (G, x_1) -bounded for any other point $x_1 \in X_{tg}$.
2. Subsets of G -bounded sets are G -bounded.
3. If A and B are G -bounded so is $A \cup B$.
4. Finite sets are G -bounded.
5. If A is G -bounded so is gA for all $g \in G$.
6. Relatively compact subsets are G -bounded.
7. Every topological vector space E is an \mathbb{R}^* -space with the action $(r, v) \mapsto rv$, $r \in \mathbb{R}^*$ and $v \in E$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The usual family of bounded subsets of E coincides with the canonical \mathbb{R}^* -boundedness, with $0 \in E$ as the point that topologically generates E .
8. If H is a topological group, K is a closed subgroup and G is a dense submonoid of H then the coset space H/K defined by the quotient map $p: H \rightarrow H/K$ is canonically a G -space by the action $ghK := p(gh)$. We say that a set $A \subseteq H/K$ is G -bounded if for every neighborhood U of K (seen as an element of H/K) there is a finite set $F \subseteq G$ such that $A \subseteq FU$. This defines the canonical G -boundedness on H/K , where K is the point that topologically generates H/K . Here, the family of G -bounded subsets coincide with the family of all precompact subsets for the left uniformity on H/K .

Definition 1. *A point-generated G -space X is said to be locally G -bounded if for every point $x \in X$ there is a G -bounded open subset U containing it.*

The proof of the following proposition is straightforward.

Proposition 2. *Let X be a point-generated G -space. If there is a point $x \in X_{tg}$ and a neighborhood U of x that is G -bounded, then X is locally G -bounded.*

Remark 1. *From the above proposition, it follows that if a point-generated G -space X is not locally G -bounded then no neighborhood of a point $x \in X_{tg}$ can be G -bounded.*

2.2. Infinite Cardinals

In what follows, we shall use the notation ZFC for Zermelo-Fraenkel set theory including the axiom of choice, CH for the continuum hypothesis ($\mathcal{C} = \aleph_1$) and GCH for the generalized continuum hypothesis ($2^{\aleph_l} = \aleph_{l+1}$ for each cardinal \aleph_l). If CH is false, then there are cardinals strictly between \aleph_0 and \mathcal{C} .

Following [10], consider the set of functions $\mathbb{N}^{\mathbb{N}}$ from \mathbb{N} into \mathbb{N} endowed with the quasi-order \leq^* defined by

$$f \leq^* g \text{ if } \{n \in \mathbb{N} : f(n) > g(n)\} \text{ is finite.}$$

A subset C of $\mathbb{N}^{\mathbb{N}}$ is said to be *cofinal* if for each $f \in \mathbb{N}^{\mathbb{N}}$ there is some $g \in C$ with $f \leq^* g$. A subset of $\mathbb{N}^{\mathbb{N}}$ is said to be *unbounded* if it is unbounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. One defines

$$\mathfrak{b} = \min\{|B| : B \text{ is an unbounded subset of } \mathbb{N}^{\mathbb{N}}\}$$

and

$$\mathfrak{d} = \min\{|D| : D \text{ is a cofinal subset of } \mathbb{N}^{\mathbb{N}}\},$$

yielding $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

If instead of $f \leq^* g$ we consider $f \leq g$, that is $f(n) \leq g(n)$ for all $n \in \mathbb{N}$, the value of \mathfrak{b} would be \aleph_0 . As for \mathfrak{d} , it would not change its value. Indeed, let D be a \mathfrak{d} -sized cofinal subset of $\mathbb{N}^{\mathbb{N}}$. Thus, given any $f \in \mathbb{N}^{\mathbb{N}}$, there exists $g \in D$ with $f(n) \leq g(n)$ for almost all $n \in \mathbb{N}$. Now the set $\mathbb{D} = \{mg : m \in \mathbb{N} \text{ and } g \in D\}$ still has size $\aleph_0 \cdot \mathfrak{d} = \mathfrak{d}$.

3. Dense Subspaces

In [1], Grothendieck proved that, when E is a metrizable and separable locally convex space, the bounded subsets of E are completely determined by the bounded subsets of any dense subspace. This result has been extended by Burke and Todorčević [2] and, separately, Saxon and Sánchez-Ruiz [3] for some nonseparable spaces. Subsequently, Chis, Ferrer, Hernández and Tsaban [5] extended these results for metrizable groups. As we show next, the same assertion holds for point-generated G -spaces if G is a countable monoid. First, we need the following lemma, which is analogous to ([4], Lemma 2.2.10) (resp. [5], Th. 3.6). We include its proof here for the reader’s sake.

Lemma 1. *Let $G = \{g_i : i \in \mathbb{N}\}$ be a countable monoid and let X be a non locally G -bounded G -space with a generating point $x_0 \in X_{tg}$ that has a countable neighborhood basis. Then there are two order preserving maps*

$$\Phi_{\mathcal{V}} : \mathfrak{B}(G, X) \rightarrow \mathbb{N}^{\mathbb{N}} \quad \Psi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathfrak{B}(G, X)$$

such that $\Phi_{\mathcal{V}}(\mathfrak{B}(G, X))$ is cofinal in $\mathbb{N}^{\mathbb{N}}$ and $\Psi(\mathbb{N}^{\mathbb{N}})$ is cofinal in $\mathfrak{B}(G, X)$.

Proof. The map $\Phi_{\mathcal{V}}$ is defined in a similar way as in ([4], Section 2.2.4) (resp. [5], Def. 3.5). Indeed, let $\mathcal{U} = \{U_m\}_{m < \omega}$ be a countable neighborhood basis at x_0 . By Proposition 1, no neighborhood of x_0 is G -bounded. Therefore, there is $U_{m_0} \in \mathcal{U}$ such that $U_1 \not\subseteq \bigcup_{i \leq n} g_i U_{m_0}$, $\forall n < \omega$. Analogously there is $U_{m_1} \in \mathcal{U}$ such that $V_1 := U_1 \cap U_{m_0} \not\subseteq \bigcup_{i \leq n} g_i U_{m_1}$, $\forall n < \omega$. Repeating this procedure, we obtain a decreasing neighborhood base $\mathcal{V} = \{V_m\}_{m < \omega}$ at x_0 by $V_{m+1} := V_m \cap U_{n+1} \cap U_{m_n} \not\subseteq \bigcup_{i \leq n} g_i U_{m_{n+1}}$, $\forall n < \omega$. \square

Define

$$\Phi_{\mathcal{V}} : \mathfrak{B}(G, X) \rightarrow \mathbb{N}^{\mathbb{N}}$$

by the rule

$$\Phi_{\mathcal{V}}(K)(m) := \min \left\{ n : K \subseteq \bigcup_{i \leq n} g_i V_m \right\}.$$

Obviously,

$$\Phi_{\mathcal{V}}(K) := \{\Phi_{\mathcal{V}}(K)(m)\}_{m \leq \omega}.$$

This map is order preserving and relates the cofinality of $\mathfrak{B}(G, X)$ and $\mathbb{N}^{\mathbb{N}}$. Indeed, take $\alpha \in \mathbb{N}^{\mathbb{N}}$. Set $V_0 := U_1$ and take $x_m \in V_{m-1} \setminus \bigcup_{i=1}^{\alpha(m)} g_i V_m$. The sequence $K := \{x_m\}_{m < \omega}$ converges to x_0 . Thus $K \cup \{x_0\}$ is G -bounded and $\Phi_{\mathcal{V}}(K)(m) = \min\{n : K \subseteq \bigcup_{i \leq n} g_i V_m\}$. It follows that $\alpha \leq \Phi_{\mathcal{V}}(K)$.

As for the map Ψ , set

$$\Psi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathfrak{B}(G, X)$$

by

$$\Psi(\alpha)(n) := \bigcap_{m < \omega} \bigcup_{i=1}^{\alpha(n)} g_i V_m.$$

Obviously this map is order preserving. Moreover, $\Psi(\mathbb{N}^{\mathbb{N}})$ is cofinal in $\mathfrak{B}(G, X)$. To see this, take an arbitrary G -bounded subset K , then for every $n < \omega$ there is a finite subset $F_n \subseteq \mathbb{N}$ such that $K \subseteq \bigcup_{i \in F_n} g_i V_m$. Set $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha(m) := \max\{i : i \in F_m\}$ for every $m < \omega$. Then $K \subseteq \Psi(\alpha)$.

Theorem 1. Let $G = \{g_n : n \in \mathbb{N}\}$ be a countable monoid and let X be a first countable G -space with a generating point $x_0 \in X_{tg}$. If Y is a dense subset of X , then for each G -bounded $K \subseteq X$ whose density is less than \mathfrak{b} , there is a G -bounded $P \subseteq Y$ such that $\overline{P} \supseteq K$.

Proof. Suppose first that X is locally G -bounded and let U be a G -bounded neighborhood of x_0 . Let F be a finite subset of G such that $K \subseteq FU$. Since G acts on X by homeomorphisms and Y is dense in X , it follows that $\overline{F(U \cap Y)}^X \supseteq FU$. Therefore, it suffices to take $P = F(U \cap Y)$. \square

Assume without loss of generality that X is not locally G -bounded and set $D \subseteq K$ such that $|D| < \mathfrak{b}$ and $\overline{D}^K = K$. Since K is G -bounded, we take the map $\Phi_{\mathcal{V}}$ defined in Lemma 1 above, where $\mathcal{V} = \{V_m\}_{m < \omega}$ is a decreasing basis at x_0 . We have

$$K \subseteq \bigcup_{n=1}^{\Phi_{\mathcal{V}}(K)(m)} g_n V_m$$

for all $m < \omega$. On the other hand, since Y is dense in X , for all $d \in D \subseteq K$, there is a sequence $S_d \subseteq Y$ which converges to d . Therefore, since $S_d \cup \{d\}$ is compact, we have

$$\overline{S_d} = S_d \cup \{d\} \subseteq \bigcup_{n=1}^{\Phi_{\mathcal{V}}(\overline{S_d})(m)} g_n V_m$$

for all $m < \omega$. So, we have a family $\{\Phi_{\mathcal{V}}(\overline{S_d})\}_{d \in D} \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality less than \mathfrak{b} , then it is bounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. Therefore, there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\Phi_{\mathcal{V}}(\overline{S_d}) \leq^* \alpha \quad \forall d \in D$. That is, if $d \in D$, then there is $m_d < \omega$ with $\Phi_{\mathcal{V}}(\overline{S_d})(m) \leq \alpha(m) \quad \forall m \geq m_d$. We also assume that $\Phi_{\mathcal{V}}(K)(m) \leq \alpha(m) \quad \forall m < \omega$. Pick now a fixed element $d \in D$. If $m < m_d$, we have

$$K \subseteq \bigcup_{n=1}^{\Phi_{\mathcal{V}}(K)(m)} g_n V_m \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V_m.$$

Therefore,

$$K \subseteq \bigcap_{m=1}^{m_d-1} \left(\bigcup_{n=1}^{\alpha(m)} g_n V_m \right) = A_d$$

that is an open set. Since this open set contains the element $d \in D$ and the sequence S_d converges to d , there is $S'_d = S_d \setminus \{\text{a finite subset}\}$ such that $S'_d \subseteq A_d$. Consider now

$$P := \bigcup_{d \in D} S'_d \subseteq Y$$

and let us verify that P is G -bounded. Take an open set V of X such that $x_0 \in V$, then there is $V_m \in \mathcal{V}$ such that $V_m \subseteq V$. For each $d \in D$ we have one of the following two options:

- (1) $m < m_d$, which implies $S'_d \subseteq A_d \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V_m$.
- (2) $m \geq m_d$, then $S'_d \subseteq S_d \subseteq \bigcup_{n=1}^{\Phi_{\mathcal{V}}(\overline{S_d})(m)} g_n V_m \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V_m$.

In both cases, $S'_d \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V_m \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V$.

Therefore, $P = \bigcup_{d \in D} S'_d \subseteq \bigcup_{n=1}^{\alpha(m)} g_n V$, and since V is arbitrary this means that P is G -bounded.

It is readily seen that $\overline{P} \supseteq K$.

A consequence of this theorem is the following.

Corollary 1. *Let G be a countable monoid and let X be a point-generated, metrizable, G -space. If X contains a dense subset of cardinality less than \mathfrak{b} , and D is an arbitrary dense subset of X , then for each G -bounded $K \subseteq X$, there is a G -bounded $P \subseteq D$ such that $\overline{P} \supseteq K$.*

Proof. Since X is metrizable, it is first countable and the generating point x_0 has a countable neighborhood basis and K contains a dense subset of cardinality less than \mathfrak{b} . \square

The following result improves Corollary 2.3.3 in [4] (resp. Corollary 3.19 in [5]).

Corollary 2. *Let H be a topological group, K a closed subgroup of H such that H/K is metrizable and let L be a dense subgroup of H . If $P \subseteq H/K$ is precompact, then there is a precompact subset $Q \subseteq L/K$ such that $P \subseteq \overline{Q}$.*

Proof. Let $p: H \rightarrow H/K$ denote the canonical quotient map. Observe that P is separable because it is metrizable and precompact. Let D be a countable dense subset of P . For every $d \in D$, there is a sequence $S_d \subseteq L$ such that $p(S_d)$ converges to d . Consider the countable subset $E = D \cup \left(\bigcup_{d \in D} p(S_d) \right) = \bigcup_{d \in D} \overline{p(S_d)} = \{y_i\}_{i=1}^{\infty}$ and the set $H_E = \overline{\langle E \rangle}$ with the topology inherited from H/K . We have that $P \subseteq H_E$, and H_E is separable and metrizable. Let G be a countable subgroup of $p^{-1}(H_E)$ such that $p(G) = \langle \{y_i\}_{i=1}^{\infty} \rangle$, which is dense in H_E . Then H/K is a point generated G -space according to Proposition 1(viii), where the family of G -bounded subsets coincides with the family of precompact subsets of the left uniformity of H/K . On the other hand, $L \cap H_E$ is countable and dense in H_E and P is G -bounded. Accordingly, we apply Theorem 1 to deduce that there is $Q \subseteq L \cap H_E$, which is G -bounded (therefore, precompact) and $P \subseteq \overline{Q}^{H_E} \subseteq \overline{Q}$. It is readily seen that Q is precompact in L . \square

The metrizability condition in the previous theorem is essential even for the special case of topological groups ([4], Example 2.3.5) (resp. [5], [Remark 3.21]).

4. G-Barrelled Groups

In this section, we have a countable monoid $G = \{g_i : i \in \mathbb{N}\}$ and a metrizable G -space X . We assume WLOG that $g_1 = e_G$ is the neutral element of G .

Definition 2. Given a G -space X , we say that $A \subseteq X$ is G -absorbent (or simply A is absorbent for short) when $GA = X$. A G -space X is said to be barrelled when for every closed absorbent subset Q there is an index $i \in \mathbb{N}$ such that $g_i Q$ has a nonempty interior.

Theorem 2. Suppose that $G = \{g_i : i \in \mathbb{N}\}$ is a countable monoid and X is a homogeneous, barrelled G -space with a generating point $x_0 \in X_{tg}$ that has a countable neighborhood basis at x_0 . If X can be covered by less than \mathfrak{b} bounded subsets, then X is locally bounded.

Proof. Let $\mathcal{V} = \{V_m\}_{m < \omega}$ be a decreasing neighborhood base at x_0 defined as in Lemma 1 and let $\pi : G \times X \rightarrow X$ denote the action of G on X . For every $g_m \in G$ we define the map

$$p_m : X \rightarrow \mathbb{N} \quad \text{by} \quad p_m(x) = \min\{n : x \in \bigcup_{j \leq n} g_j V_m\}.$$

As a consequence, every element $x \in X$ defines a sequence $\{p_m(x)\}_{m < \omega}$ and, therefore, we have defined the map $p : X \rightarrow \mathbb{N}^{\mathbb{N}}$ as $p(x) = \{p_m(x)\}_{m < \omega}$ so that $p(x)[m] = p_m(x)$. Suppose there is a collection of G -bounded sets \mathfrak{B} such that $|\mathfrak{B}| < \mathfrak{b}$ and $X = \bigcup_{P \in \mathfrak{B}} P$. Every

$P \in \mathfrak{B}$ is associated with a map $\Phi_{\mathcal{V}}(P) \in \mathbb{N}^{\mathbb{N}}$ defined previously; that is

$$\Phi_{\mathcal{V}}(P)(m) = \min\{n : P \subseteq \bigcup_{j \leq n} g_j V_m\}.$$

Take $x \in X$. Then, there is $P \subseteq \mathfrak{B}$ such that $x \in P$. Therefore $p(x) \leq \Phi_{\mathcal{V}}(P)$. Since $|\mathfrak{B}| < \mathfrak{b}$ it follows that $\Phi_{\mathcal{V}}(\mathfrak{B}) = \{\Phi_{\mathcal{V}}(P) : P \in \mathfrak{B}\}$ is bounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. Thus, there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\Phi_{\mathcal{V}}(P) \leq^* \alpha$ and, since $p(x) \leq \Phi_{\mathcal{V}}(P)$, we have $p(x) \leq^* \alpha$ for all $x \in X$. So, for every $x \in X$, there is $m_x < \omega$ such that $p_m(x) \leq \alpha(m)$ for all $m \geq m_x$.

Define

$$Q_\alpha = \{x \in X : p_m(x) \leq \alpha(m) \quad \forall m < \omega\} = \bigcap_{m < \omega} \left(\bigcup_{j \leq \alpha(m)} g_j V_m \right).$$

Clearly, the set Q_α is bounded. Let us verify that Q_α is also absorbent. Take $x \in X$. Then, since $p_m(x) \leq \alpha(m) \quad \forall m \geq m_x$, we have

$$x \in \bigcap_{m \geq m_x} \left(\bigcup_{j \leq \alpha(m)} g_j V_m \right).$$

Thus,

$$x \in Q_\alpha \cup \left(\bigcap_{m < m_x} \left(\bigcup_{j \leq p_m(x)} g_j V_m \right) \right).$$

Set

$$F_x = \{i \in \mathbb{N} : i \leq p_m(x), m < m_x\}.$$

We claim that

$$x \in \bigcup_{i \in F_x} g_i Q_\alpha.$$

Indeed, since each map π_{g_i} is a bijection and g_1 is the neutral element of G , we have

$$\bigcup_{i \in F_x} g_i Q_\alpha = \bigcup_{i \in F_x} g_i \left(\bigcap_{m < \omega} \bigcup_{j \leq \alpha(m)} g_j V_m \right) = \bigcup_{i \in F_x} \left(\bigcap_{m < \omega} g_i \bigcup_{j \leq \alpha(m)} g_j V_m \right)$$

$$= \bigcup_{i \in F_x} \left(\bigcap_{m < \omega} \bigcup_{j \leq \alpha(m)} g_i g_j V_m \right) \supseteq \left(\bigcap_{m \geq m_x} \bigcup_{j \leq \alpha(m)} g_j V_m \right) \cap \left(\bigcap_{m < m_x} \bigcup_{i \in F_x} g_i V_m \right) \ni x.$$

This proves that Q_α is absorbent. Therefore $\overline{Q_\alpha}$ is absorbent too and, since X is G -barrelled, there is $g \in G$ such that $g\overline{Q_\alpha}$ has nonempty interior. Thus, $g\overline{Q_\alpha}$ is a G -bounded subset containing an open, G -bounded, subset U . Take any point $u \in U$. Since X is homogeneous, there is a homeomorphism $f_{ux_0}: X \rightarrow X$ such that $f_{ux_0}(u) = x_0$ and $f_{ux_0}(U)$ is an open, bounded subset containing x_0 . By Proposition 2, it follows that X is locally G -bounded. \square

As a consequence, we next obtain results that contain the previous results obtained by locally convex spaces [2] and topological groups [5].

Let G be a topological group, we say that a subset $A \subseteq G$ is *absorbent* when for every dense subgroup H of G it holds that $HA = G$. The group G is said to be *barrelled* when every closed absorbent subset Q has a nonempty interior. Remark that every separable Baire group is barrelled.

Corollary 3. *Let G be either a metrizable, barrelled, locally convex space or a separable, metrizable, barrelled group. If G is covered by less than \mathfrak{b} bounded (resp. precompact) subsets. Then G is normable (resp. locally precompact).*

Proof. In both cases, G is homogeneous and the homeomorphisms preserving bounded subsets are translations. If G is a metrizable, barrelled, locally convex space, applying Theorem 2, we obtain that G has a neighborhood basis of zero consisting of bounded subsets, which implies that G is normable. If G is a topological group, take any countable dense subgroup H of G and consider the canonical action of H on G that makes G an H -space. By Proposition 1, a subset A of G is H -bounded if and only if it is precompact. Again, it suffices now to apply Theorem 2. \square

5. Discussion

We have considered the action of a monoid G on a topological space X and associated it with a canonical family of G -bounded subsets. This provides a very general notion of boundedness that include both the bounded subsets considered in functional analysis and in topological groups. In this paper, we have initiated the study of this new notion of a G -bounded subset. Among other results, it is proved that for a metrizable and separable G -space X , the bounded subsets of X are completely determined by the bounded subsets of any dense subspace, extending results obtained by Grothendieck for metrizable separable locally convex spaces [1], generalized subsequently by Burke and Todorčević and, separately, Saxon and Sánchez-Ruiz for metrizable locally convex spaces [2,3] and by Chis, Ferrer, Hernández and Tsaban for metrizable groups [4,5]. We have also obtained sufficient conditions for a G -space X to be locally G -bounded, which applies to topological groups. This also provides the frame for extending to this setting some results by Burke and Todorčević and, separately, Saxon and Sánchez-Ruiz (loc. cit.) for metrizable locally convex spaces.

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