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g-Expectation for Conformable Backward Stochastic Differential Equations

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Abstract: In this paper, we study the applications of conformable backward stochastic differential equations driven by Brownian motion and compensated random measure in nonlinear expectation. From the comparison theorem, we introduce the concept of *g*-expectation and give related properties of *g*-expectation. In addition, we find that the properties of conformable backward stochastic differential equations can be deduced from the properties of the generator *g*. Finally, we extend the nonlinear Doob–Meyer decomposition theorem to more general cases.

1. Introduction

The initial research motivation of nonlinear expectations came from risk measurement and option pricing in financial applications. The Allais paradox, Ellsberg paradox and Simon's "bounded rationality" theory, and so forth, all show that decision-making in reality is contrary to the hypothesis of expected utility theory. Economists have found that the linearity of classical mathematical expectation (that is, the additivity of probability measures) is the main reason for this kind of problem so researchers wanted to find a new tool which can not only retain some properties of classical mathematical expectations, but also solve financial problems with highly dynamic and complex characteristics.

In the 1950s, Choquet [1] extended the Lebesgue integral to non-additive measure and obtained the Choquet expectation. However, this nonlinear expectation does not have dynamic compatibility and is not suitable for solving practical financial problems. In 1997, Peng [2] introduced a new nonlinear expectation, namely the *g*-expectation, based on the backward stochastic differential equation driven by Brownian motion. The *g*-expectation retains all the basic properties of the classical expectation except linearity [3], and it can be applied to the dynamic risk measurement of actuarial and financial valuation. Subsequently, Royer [4] studied the backward stochastic differential equation driven by Brownian motion and Poisson random measure, and introduced the corresponding *g*-expectation and a large number of studies show that this *g*-expectation can be applied to financial problems (see [5–9]). Recently, Long et al. [10] proposed a multi-step scheme on time-space grids for solving backward stochastic differential equations, and Chen and Ye [11] investigated solutions of backward stochastic differential equations in the framework of Riemannian manifold. From the paper [12], we could get the averaging principle for backward stochastic differential equations and the solutions can be approximated by the solutions to averaged stochastic systems in the sense of mean square under some appropriate assumptions.

In addition, coupled forward backward stochastic differential equations driven by the G-Brownian motion were studied in [13], while [14] investigated the solvability of fully coupled forward–backward stochastic differential equations with irregular coefficients.

The above papers concern research on integer order derivative, while the works of conformable type derivative are very few ([15–20]). The conformable derivative not only has some properties of fractional derivative, but also some properties of integer order derivatives. We discussed the necessity of studying conformable backward stochastic differential equations in [21]. In the present paper, we study g-expectation for conformable backward stochastic differential equations.

This paper is mainly divided into four parts. In the second section, we give some definitions and theorems. In the third section, we study the relationship between g-expectation and the filtered consensus expectation, and we give some properties of g-expectation. We find that the g-expectation can be considered as a nonlinear extension of the Girsanov transformation. In the final section, we prove the Doob–Meyer decomposition theorem under mild assumptions.

2. Preliminaries

Let $B(\cdot)$ be a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual hypotheses of completeness and right continuity. $\mathfrak{B}(\mathbb{R})$ denotes the Borel sets of $\mathbb{R}$ and $\mathbb{E}$ denotes the expected value. A stochastic process $V(\omega, t)$ is a real function defined on $\Omega \times [0, T]$ such that $\omega \to V(\omega, t)$ is $\mathcal{F}$-measurable for any $t \in [0, T]$. A stochastic process V is called $\mathcal{F}_t$-adapted if $\omega \to V(\omega, t)$ is $\mathcal{F}_t$-measurable for any $t \in [0, T]$. The natural filtration is completed with sets of measure zero. By $\mathcal{P}$ we denote the σ-field. A process $V : \Omega \times [0, T] \to \mathbb{R}$ is called $\mathcal{F}$-predictable if it is $\mathcal{F}$-adapted and $\mathcal{P}$-measurable. A process is called càdlàg if its trajectories are right-continuous and have left limits. The term *a.s.* means *almost surely* with respect to the probability measure. Inspired by [22], we define some spaces that we will use:

$$L_Q^2(\mathbb{R}) = \{measurable\ function\ \varphi : \mathbb{R} \to \mathbb{R}; \ \int_{\mathbb{R}} |\varphi(s)|^2 Q(ds) < \infty; \ Q\ is\ a\ \sigma - finite\ measure\},$$

$$\mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}) = \{\mathcal{F}_T - measurable\ random\ variable\ \xi : \Omega \to \mathcal{F}_T; \ \mathbb{E}[|\zeta|^2] < \infty\},$$

$$\mathbb{H}^2(\mathbb{R}) = \{predictable\ process\ Y : \Omega \times [a, T] \to \mathbb{R}; \ \mathbb{E}\left[\int_a^T |Y(t)|^2 dt\right] < \infty\},$$

$$\mathbb{H}_N^2(\mathbb{R}) = \{predictable\ process\ Z : \Omega \times [a, T] \times \mathbb{R} \to \mathbb{R}; \ \mathbb{E}\left[\int_a^T \int_{\mathbb{R}} |Z(t, s)|^2 Q(t, ds)\eta(t)dt\right] < \infty\},$$

$$\mathbb{S}^2(\mathbb{R}) = \{adapted,\ càdlàg\ process\ X : \Omega \times [a, T] \to \mathbb{R}; \ \mathbb{E}[\sup_{t \in [a,T]} |X(t)|^2] < \infty\},$$

$$\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}) = \{\mathcal{F}_t - measurable\ random\ variable\ \xi : \Omega \to \mathcal{F}_t; \ \mathbb{E}[|\xi|^2] < \infty]\}.$$

Furthermore, for any constant σ, we introduce the norms of spaces $\mathbb{H}^2$, $\mathbb{H}_N^2$ and $\mathbb{S}^2$ as:

$$\|Y\|_{\mathbb{H}^2}^2 = \mathbb{E}\left[\int_a^T e^{\sigma t} |Y(t)|^2 dt\right],$$

$$\|Z\|_{\mathbb{H}_N^2}^2 = \mathbb{E}\left[\int_a^T \int_{\mathbb{R}} e^{\sigma t} |Z(t, s)|^2 Q(t, ds)\eta(t)dt\right],$$

$$\|X\|_{\mathbb{S}^2}^2 = \mathbb{E}[\sup_{t \in [a,T]} e^{\sigma t} |X(t)|^2].$$

Definition 1. *(see [2] (Definition 3.1)) A functional $\mathcal{E} : \mathbb{L}^2(\Omega, \mathcal{F}_T, P) \to \mathbb{R}$ is called a nonlinear expectation if it satisfies the following properties:*
(i) Strict monotonicity: if $X_1 \geq X_2$ a.s., $\mathcal{E}[X_1] \geq \mathcal{E}[X_2]$, and if $X_1 \geq X_2$ a.s., $\mathcal{E}[X_1] = \mathcal{E}[X_2] \Leftrightarrow X_1 = X_2$ a.s.
(ii) preserving of constants: $\mathcal{E}[c] = c$, for any constant c.

Definition 2. *(see [2] (Definition 3.2)) A nonlinear expectation $\mathcal{E}$ is a filtration consistent expectation ($\mathcal{F}$-consistent expectation) if for any $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and $a \leq t \leq T$, there exists a random variable $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ such that $\mathcal{E}[\zeta 1_A] = \mathcal{E}[\xi 1_A]$, $\forall A \in \mathcal{F}_t$, where ξ is uniquely defined. We denote $\xi = \mathcal{E}[\zeta|\mathcal{F}_t]$, which is called the conditional expectation of ζ with respect to $\mathcal{F}_t$. Therefore, we can write it as $\mathcal{E}[\zeta 1_A] = \mathcal{E}[\mathcal{E}[\zeta|\mathcal{F}_t]1_A]$, $\forall A \in \mathcal{F}_t$.*

Lemma 1. *(see [4](Lemma A.1)) Let $A(\cdot)$ be an increasing predictable process. We consider its decomposition as a sum of a continuous and a purely discontinuous process: $A(t) = A^1(t) + A^2(t)$. We also consider a càdlàg martingale $W(\cdot)$, bounded in $\mathbb{L}^2$.*
(i) For any stopping time τ such that $a \leq \tau \leq T$,

$$\mathbb{E}\left[\int_a^\tau \triangle W(s) dA^1(s) \right] = 0.$$

(ii) For any predictable stopping time τ such that $a \leq \tau \leq T$,

$$\mathbb{E}\left[\int_a^\tau \triangle W(s) dA^2(s) \right] = \mathbb{E}\left[\sum_{a \leq s \leq \tau} \triangle W(s) \triangle A^2(s) \right].$$

Lemma 2. *(see [21] (Theorem 3.5)) Suppose $U(\cdot) = U(X(\cdot), \cdot) \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$. Then, for any $a \leq t \leq T$, we have*

$$\begin{aligned}
D_\rho^a U(t) d \frac{(t-a)^\rho}{\rho} &= \left(\frac{\partial u}{\partial t} - (t-a)^{\rho-1} g(t, X(t), Y(t), Z(t, \cdot)) \frac{\partial u}{\partial x} \right. \\
&\quad + \left. \frac{1}{2}(t-a)^{2(\rho-1)} Y^2(t) \frac{\partial^2 u}{\partial x^2} \right) dt + (t-a)^{\rho-1} Y(t) \frac{\partial u}{\partial x} dB(t) \\
&\quad + (t-a)^{\rho-1} \frac{\partial u}{\partial x} \int_{\mathbb{R}} Z(t,s) \tilde{N}(dt, ds) \\
&\quad + \frac{1}{2}(t-a)^{2(\rho-1)} \frac{\partial^2 u}{\partial x^2} \int_{\mathbb{R}} Z^2(t,s) N(dt, ds), \ 0 < \rho \leq 1.
\end{aligned}$$

Lemma 3. *(see [22] (Theorem 2.5.1)) Let B be a $(\mathbb{P}, \mathcal{F})$-Brownian motion, N be a $(\mathbb{P}, \mathcal{F})$-random measure with compensator $\vartheta(d\tau, ds) = Q(\tau, ds)\eta(\tau)d\tau$. Assume an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ with a positive $\mathcal{F}$-martingale:*

$$\frac{dM(t)}{M(t)} = \phi(t)dB(t) + \int_{\mathbb{R}} \kappa(t,s)\tilde{N}(dt, ds), \ M(a) = 1,$$

where $\phi(\cdot)$ and $\kappa(\cdot, \cdot)$ are the $\mathcal{F}$-predictable processes satisfying

$$\int_a^T |\phi(t)|^2 dt < \infty, \ \int_a^T \int_{\mathbb{R}} |\kappa(t,s)|^2 Q(t, ds)\eta(t)dt < \infty,$$
$$\kappa(t,s) > -1, \ a \leq t \leq T, \ s \in \mathbb{R}.$$

Then,

$$B^{\mathbb{Q}}(t) = B(t) - \int_a^t \phi(\tau)d\tau, \ a \leq t \leq T,$$
$$\tilde{N}^{\mathbb{Q}}(t, A) = N(t, A) - \int_a^t \int_{\mathbb{R}} (1 + \kappa(\tau, s)) Q(\tau, ds)\eta(\tau)d\tau, \ a \leq t \leq T, \ A \in \mathfrak{B}(\mathbb{R}),$$

are $(\mathbb{Q}, \mathcal{F})$-Brownian motion and a $(\mathbb{Q}, \mathcal{F})$-random measure.

Lemma 4. *(see [22] (p. 42)) Let $\gamma > 0$ and $x_1, x_2 \in \mathbb{R}$. Then,*

$$2|x_1 x_2| \leq \frac{1}{\gamma}|x_1|^2 + \gamma|x_2|^2. \tag{1}$$

Lemma 5. *Consider the following family of conformable backward stochastic differential equation parameterized by $n = 1, 2, \cdots$*

$$
\begin{aligned}
X^n(t) &= \zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau)) d\tau + n \int_t^T (\tau - a)^{\rho-1}(X(\tau) - X^n(\tau)) \\
&- \int_t^T (\tau - a)^{\rho-1} Y^n(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z^n(\tau, s) \tilde{N}(d\tau, ds), \ t \in [a, T], \ 0 < \rho \leq 1,
\end{aligned}
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, X is an adapted process, Y and Z are given control processes, $g : \Omega \times [a, T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R}) \rightarrow \mathbb{R}$ is predictable, $B(\cdot)$ is a given Brownian motion and $\tilde{N}$ is a compensated random measure. For any $n = 1, 2, \cdots$, we have $X(t) \geq X^n(t)$.

Proof. Following [23] (Lemma 3.4), we assume that $X(t) < X^n(t)$. Then there exists $\wp > 0$ such that the measure of $\{(\omega, t) : X^n(t) - X(t) - \wp \geq 0\} \subset \Omega \times [a, T]$ non-zero. Define the following two stopping times:

$$\bar{\sigma} = \min[T, \inf\{t : X^n(t) \geq X(t) + \wp\}],$$
$$\bar{\tau} = \inf\{t \geq \sigma : X^n(t) \leq X(t)\}.$$

Then we get $\bar{\sigma} \leq \bar{\tau} \leq T$. Since $X(t) - X^n(t)$ is right continuous, we have:

$$X^n(\bar{\sigma}) \geq X(\bar{\sigma}) + \wp,$$

$$X^n(\bar{\tau}) \leq X(\bar{\tau}). \tag{2}$$

Suppose $\bar{X}(t)$ is the solution with the terminal value $\bar{X}(\bar{\tau}) = X^n(\bar{\tau})$ on $[a, \bar{\tau}]$. From (2) and the comparison theorem, we get $X^n(\bar{\sigma}) \leq X(\bar{\sigma})$. This is a contradiction. Thus, $X(t) \geq X^n(t)$. $\square$

3. The Main Results of g-Expectations

Consider the following conformable backward stochastic differential equation

$$
\begin{aligned}
X(t) &= \zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X(\tau), Y(\tau), Z(\tau, s)) d\tau - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) \\
&- \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds), \ 0 < \rho \leq 1, \ a \leq t \leq T, \tag{3}
\end{aligned}
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, X is an adapted process, Y and Z are given control processes, $g : \Omega \times [a, T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R}) \rightarrow \mathbb{R}$ is predictable, $B(\cdot)$ is a given Brownian motion and $\tilde{N}$ is a compensated random measure.

Assumption 1. *(i) The generator $g : \Omega \times [a, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is predictable and Lipschitz in x and y*

$$|g(t, x, y, z) - g(t, x', y', z)| \leq K(|x - x'| + |y - y'|), \ \forall x, x', y, y' \in \mathbb{R},$$

where K is a positive constant.
(ii) For any $z, z' \in \mathbb{R}$, there exist constants $-1 < C_1 \leq 0$ and $C_2 \geq 0$ such that

$$g(t, x, y, z) - g(t, x, y, z') \leq \int_{\mathbb{R}} (z(t, s) - z'(t, s)) \delta^{x, y, z, z'}(t, s) Q(t, ds) \eta(t),$$

where $\delta^{x,y,z,z'}(t,s) : \Omega \times [a,T] \times \mathbb{R} \to \mathbb{R}$ is predictable and satisfies $C_1(1 \wedge |s|) \le \delta^{x,y,z,z'}(t,s) \le C_2(1 \wedge |s|)$.
(iii) For any $x \in \mathbb{R}$, $g(t,x,0,0) = 0$.

Notice that the comparison theorems in [21] follow from Definition 1. Hence a nonlinear expectation can be defined by conformable backward stochastic differential equations.

Definition 3. *A nonlinear expectation $\mathcal{E}_g[\cdot] : \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}) \to \mathbb{R}$ is called a g-expectation if the generator g of Equation (3) satisfies Assumption 1 and we define the g-expectation as $\mathcal{E}_g[\zeta] = X(a)$, where a triple (X,Y,Z) is a unique solution of Equation (3) and $X(a)$ denotes the initial value of the solution.*

Definition 4. *A nonlinear expectation $\mathcal{E}_g[\cdot|\mathcal{F}_t] : \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}) \to \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ is called a conditional g-expectation if for any $a \le t \le T$, the generator g of Equation (3) satisfies Assumption 1 and we define the conditional g-expectation as $\mathcal{E}_g[\zeta|\mathcal{F}_t] = X(t)$, where a triple (X,Y,Z) is a unique solution of Equation (3) and $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ denotes the terminal value of the solution.*

Proposition 1. *We have the following results:*
(i) For $a \le t \le T$, $A \in \mathcal{F}_t$ and $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $\mathcal{E}_g[\zeta 1_A|\mathcal{F}_t] = \mathcal{E}_g[\zeta|\mathcal{F}_t]1_A$.
(ii) For any $a \le s \le t \le T$ and $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $\mathcal{E}_g[\mathcal{E}_g[\zeta|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}_g[\zeta|\mathcal{F}_s]$.

Proof. Case (i). Let $A \in \mathcal{F}_t$. For any $a \le t \le T$ and $0 < \rho \le 1$, consider Equation (3) and

$$X_1(t) = \zeta 1_A + \int_t^T (\tau - a)^{\rho-1} g(\tau, X_1(\tau), Y_1(\tau), Z_1(\tau)) d\tau$$
$$- \int_t^T (\tau - a)^{\rho-1} Y_1(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_1(\tau, s) \tilde{N}(d\tau, ds), \quad (4)$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and the generator g satisfies Assumption 1. Multiplying by 1_A on both sides of (3) we get

$$X(t)1_A = \zeta 1_A + \int_t^T (\tau - a)^{\rho-1} 1_A g(\tau, X(\tau), Y(\tau), Z(\tau)) d\tau$$
$$- \int_t^T (\tau - a)^{\rho-1} Y(\tau) 1_A dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) 1_A \tilde{N}(d\tau, ds),$$

where $a \le t \le T$ and $0 < \rho \le 1$. Notice that $g(t, X(t), Y(t), Z(t))1_A = g(t, 1_A X(t), 1_A Y(t), 1_A Z(t))$, and then,

$$X(t)1_A = \zeta 1_A + \int_t^T (\tau - a)^{\rho-1} g(\tau, X(\tau)1_A, Y(\tau)1_A, Z(\tau)1_A) d\tau$$
$$- \int_t^T (\tau - a)^{\rho-1} Y(\tau) 1_A dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) 1_A \tilde{N}(d\tau, ds). \quad (5)$$

Let $(\bar{X}(t), \bar{Y}(t), \bar{Z}(t)) = (X(t)1_A, Y(t)1_A, Z(t)1_A)$, and (8) can be written as:

$$\bar{X}(t) = \zeta 1_A + \int_t^T (\tau - a)^{\rho-1} g(\tau, \bar{X}(\tau), \bar{Y}(\tau), \bar{Z}(\tau)) d\tau$$
$$- \int_t^T (\tau - a)^{\rho-1} \bar{Y}(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} \bar{Z}(\tau, s) \tilde{N}(d\tau, ds).$$

By the uniqueness of the conformable backward stochastic differential equation, we get $X_1(t) = \bar{X}(t) = X(t)1_A$, $a \le t \le T$. From Definition 4, we have $\mathcal{E}_g[\zeta 1_A|\mathcal{F}_t]X_1(t) = X(t)1_A = \mathcal{E}_g[\zeta|\mathcal{F}_t]1_A$.

Case (ii). For any $A \in \mathcal{F}_s$ and $a \le s \le t \le T$, we have $A \in \mathcal{F}_t$. From the result of (i), one has:

$$
\begin{aligned}
\mathcal{E}_g[\mathcal{E}_g[\mathcal{E}_g[\zeta|\mathcal{F}_t]|\mathcal{F}_s]1_A] &= \mathcal{E}_g[\mathcal{E}_g[\mathcal{E}_g[\zeta|\mathcal{F}_t]1_A|\mathcal{F}_s]] \\
&= \mathcal{E}_g[\mathcal{E}_g[\mathcal{E}_g[\zeta 1_A|\mathcal{F}_t]|\mathcal{F}_s]] \\
&= \mathcal{E}_g[\mathcal{E}_g[\zeta 1_A|\mathcal{F}_s]] \\
&= \mathcal{E}_g[\mathcal{E}_g[\zeta|\mathcal{F}_s]1_A],
\end{aligned}
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Let $\zeta_1 = \mathcal{E}_g[\mathcal{E}_g[\zeta|\mathcal{F}_t]|\mathcal{F}_s]$ and $\zeta_2 = \mathcal{E}_g[\zeta|\mathcal{F}_s]$. If we choose $A = \{\zeta_1 \ge \zeta_2\} \in \mathcal{F}_t$, from Definition 1, $\zeta_1 1_A \ge \zeta_2 1_A$ and $\mathcal{E}_g[\zeta_1 1_A] = \mathcal{E}_g[\zeta_2 1_A]$, we get $\zeta_1 1_A = \zeta_2 1_A$. Hence $\zeta_1 \le \zeta_2$. If we set $A = \{\zeta_1 \le \zeta_2\} \in \mathcal{F}_t$, we get $\zeta_1 \ge \zeta_2$ in the same way. Hence, we conclude that $\zeta_1 = \zeta_2$, that is, $\mathcal{E}_g[\mathcal{E}_g[\zeta|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}_g[\zeta|\mathcal{F}_s]$. $\quad\square$

Theorem 1. *The g-expectation is $\mathcal{F}$-consistent expectation.*

Proof. Let $A \in \mathcal{F}_t$. For any $a \le t \le T$ and $0 < \rho \le 1$, consider the following equations:

$$
\begin{aligned}
X_1(t) = \; & \zeta 1_A + \int_t^T (\tau - a)^{\rho-1} g(\tau, X_1(\tau), Y_1(\tau), Z_1(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y_1(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_1(\tau, s) \tilde{N}(d\tau, ds), \quad (6)
\end{aligned}
$$

$$
\begin{aligned}
X_2(t) = \; & X_1(u) 1_A + \int_t^T (\tau - a)^{\rho-1} g(\tau, X_2(\tau), Y_2(\tau), Z_2(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y_2(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_2(\tau, s) \tilde{N}(d\tau, ds), \quad (7)
\end{aligned}
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $X_1(u) = \mathcal{E}_g[\zeta 1_A|\mathcal{F}_t]$ and the generator g satisfies Assumption 1. Multiplying by 1_A on both sides of (7) we get:

$$
\begin{aligned}
X_2(t) 1_A = \; & \mathcal{E}_g[\zeta 1_A|\mathcal{F}_t] 1_A + \int_t^T (\tau - a)^{\rho-1} 1_A g(\tau, X_2(\tau), Y_2(\tau), Z_2(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y_2(\tau) 1_A dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_2(\tau, s) 1_A \tilde{N}(d\tau, ds)
\end{aligned}
$$

where $a \le t \le T$ and $0 < \rho \le 1$. Notice that $g(t, X(t), Y(t), Z(t)) 1_A = g(t, 1_A X(t), 1_A Y(t), 1_A Z(t))$, and then,

$$
\begin{aligned}
X_2(t) 1_A = \; & \mathcal{E}_g[\zeta 1_A|\mathcal{F}_t] 1_A + \int_t^T (\tau - a)^{\rho-1} g(\tau, X_2(\tau) 1_A, Y_2(\tau) 1_A, Z_2(\tau) 1_A) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y_2(\tau) 1_A dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_2(\tau, s) 1_A \tilde{N}(d\tau, ds). \quad (8)
\end{aligned}
$$

Let $(X_3(t), Y_3(t), Z_3(t)) = (X_2(t) 1_A, Y_2(t) 1_A, Z_2(t) 1_A)$, and (8) can be written as:

$$
\begin{aligned}
X_3(t) = \; & \mathcal{E}_g[\zeta 1_A|\mathcal{F}_t] 1_A + \int_t^T (\tau - a)^{\rho-1} g(\tau, X_3(\tau), Y_3(\tau), Z_3(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y_3(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_3(\tau, s) \tilde{N}(d\tau, ds).
\end{aligned}
$$

By the uniqueness of the conformable backward stochastic differential equation, we get $X_1(t) = X_3(t) = \mathcal{E}_g[\zeta 1_A|\mathcal{F}_t] 1_A$, $a \le t \le T$. From Definition 3 and Proposition 1, we have:

$$
\mathcal{E}_g[\zeta 1_A] = X_1(a) = X_3(a) = \mathcal{E}_g[\mathcal{E}_g[\zeta 1_A|\mathcal{F}_t] 1_A] = \mathcal{E}_g[\mathcal{E}_g[\zeta|\mathcal{F}_t] 1_A].
$$

Hence, there exists $\xi = \mathcal{E}_g[\zeta|\mathcal{F}_t]$ such that $\mathcal{E}_g[\zeta 1_A] = \mathcal{E}_g[\xi 1_A]$.

Next, we prove the uniqueness of ξ. Assume that there exists another random variable ξ' such that $\mathcal{E}_g[\zeta 1_A] = \mathcal{E}_g[\xi' 1_A]$ and $\xi \neq \xi'$. Choose $\xi > \xi'$. According to the comparison theorem in [21] and Definition 3, we have $\mathcal{E}_g[\xi 1_A] > \mathcal{E}_g[\xi' 1_A]$, which is contrary to $\mathcal{E}_g[\xi 1_A] = \mathcal{E}_g[\xi' 1_A]$. On the other hand, if we choose $\xi < \xi'$, the result $\mathcal{E}_g[\xi 1_A] = \mathcal{E}_g[\xi' 1_A]$ still does not hold. Hence $\xi = \xi'$.

Combining the existence and uniqueness of ξ, we conclude that the *g*-expectation is $\mathcal{F}$-consistent expectation. The proof is complete. $\square$

Next, we give two kinds of *g*-expectation with the special generators g_1 and g_2.

Proposition 2. *Let $\phi(\cdot)$ and $\kappa(\cdot)$ be $\mathcal{F}$-predictable processes. For any $\mu_1 \in \mathbb{R}^+$ and $-1 < C_1 \leq 0$, we define the following generators:*

$$g_1(t, x, y, z) = \mu_1|y| + C_1 \int_{\mathbb{R}} (1 \wedge |s|)|z(t,s)|Q(t,ds)\eta(t),$$

$$g_2(t, x, y, z) = -\mu_1|y| - C_1 \int_{\mathbb{R}} (1 \wedge |s|)|z(t,s)|Q(t,ds)\eta(t).$$

Then $\mathcal{E}_{g_1}[\zeta|\mathcal{F}_t] = \inf_{\mathbb{Q} \in \mathcal{D}} \mathbb{E}^{\mathbb{Q}}[\zeta|\mathcal{F}_t]$ and $\mathcal{E}_{g_2}[\zeta|\mathcal{F}_t] = \sup_{\mathbb{Q} \in \mathcal{D}} \mathbb{E}^{\mathbb{Q}}[\zeta|\mathcal{F}_t]$, where

$$\mathcal{D} = \left\{ \mathbb{Q} \sim \mathbb{P}, \left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|\mathcal{F}_t = M(t), \frac{dM(t)}{M(t)} = \phi(t)dB(t) + \int_{\mathbb{R}} \kappa(t,s)\tilde{N}(dt,ds), \right.$$

$$\left. M(a) = 1, |\phi(t)| \leq \mu_1, -1 < \kappa(t,s) \leq C_1(1 \wedge |s|), a \leq t \leq T \right\}.$$

Proof. Here we consider the case of $g_1(t, x, y, z) = \mu_1|y| + C_1 \int_{\mathbb{R}}(1 \wedge |s|)|z(t,s)|Q(t,ds)\eta(t)$. The proof of $g_2(t, x, y, z) = -\mu_1|y| - C_1 \int_{\mathbb{R}}(1 \wedge |s|)|z(t,s)|Q(t,ds)\eta(t)$ is similar. Consider the following equation

$$
\begin{aligned}
X(t) &= \zeta + \int_t^T (\tau - a)^{\rho-1} g_1(\tau, X(\tau), Y(\tau), Z(\tau))d\tau \\
&\quad - \int_t^T (\tau - a)^{\rho-1} Y(\tau)dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s)\tilde{N}(d\tau, ds) \\
&= \zeta + \int_t^T \mu_1(\tau - a)^{\rho-1}|Y(\tau)|d\tau + \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} C_1(1 \wedge |s|)|Z(\tau,s)|Q(\tau,ds)\eta(\tau)d\tau \\
&\quad - \int_t^T (\tau - a)^{\rho-1} Y(\tau)dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau, ds),
\end{aligned}
$$

where $a \leq t \leq T, 0 < \rho \leq 1, \mu_1 \in \mathbb{R}^+$ and $-1 < C_1 \leq 0$. Define

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|\mathcal{F}_t = M(t), \ M(a) = 1,$$

$$\frac{dM(t)}{M(t)} = \phi(t)dB(t) + \int_{\mathbb{R}} \kappa(t,s)\tilde{N}(dt,ds),$$

where $|\phi(t)| \leq \mu_1$ and $-1 < \kappa(t,s) \leq C_1(1 \wedge |s|)$. Let $\mathcal{D} = \{\mathbb{Q} \sim \mathbb{P}, \frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t = M(t)\}$, and from Lemma 3, we can get

$$
\begin{aligned}
X(t) \quad = \quad & \zeta + \int_t^T (\tau - a)^{\rho-1}[\mu_1|Y(\tau)| - \phi(\tau)Y(\tau)]d\tau \\
& + \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} [C_1(1 \wedge |s|)|Z(\tau,s)| - \kappa(\tau,s)Z(\tau,s)]Q(\tau,ds)\eta(\tau)d\tau \\
& - \int_t^T (\tau - a)^{\rho-1}Y(\tau)dB^{\mathbb{Q}}(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}^{\mathbb{Q}}(d\tau,ds) \\
\geq \quad & \zeta - \int_t^T (\tau - a)^{\rho-1}Y(\tau)dB^{\mathbb{Q}}(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}^{\mathbb{Q}}(d\tau,ds),
\end{aligned}
$$

where $a \leq t \leq T$ and $0 < \rho \leq 1$; note that $\phi(\tau)Y(\tau) = \mu_1 Y(\tau) > 0$ and $\kappa(\tau,s)Z(\tau,s) = C_1(1 \wedge |s|)Z(\tau,s) > 0$, we have

$$
X(t) = \zeta - \int_t^T (\tau - a)^{\rho-1}Y(\tau)dB^{\mathbb{Q}}(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}^{\mathbb{Q}}(d\tau,ds).
$$

Taking the conditional expectation under the probability measure $\mathbb{Q}$, we obtain $X(t) = \inf_{\mathbb{Q} \in \mathcal{D}} \mathbb{E}^{\mathbb{Q}}[\zeta|\mathcal{F}_t]$. Notice that Assumption 1 is satisfied for the generator $g_1(t,x,y,z)$. From Definition 4, we get $X(t) = \mathcal{E}_{g_1}[\zeta|\mathcal{F}_t]$, that is, $\mathcal{E}_{g_1}[\zeta|\mathcal{F}_t] = \inf_{\mathbb{Q} \in \mathcal{D}} \mathbb{E}^{\mathbb{Q}}[\zeta|\mathcal{F}_t]$.

The proof is complete. $\square$

Proposition 3. *Let $\mathcal{E}_g$ be a g-expectation and $\zeta, \zeta_1, \zeta_2 \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$.*
(i) Translation invariance: for any constant $c \in \mathbb{R}$ and $a \leq t \leq T$, we have

$$
\mathcal{E}_g[\zeta + c|\mathcal{F}_t] = \mathcal{E}_g[\zeta|\mathcal{F}_t] + c,
$$

where the generator g is independent of $X(\cdot)$.
(ii) Homogeneity: for any constant $c > 0$ and $a \leq t \leq T$, we have

$$
\mathcal{E}_g[c\zeta|\mathcal{F}_t] = c\mathcal{E}_g[\zeta|\mathcal{F}_t],
$$

where the generator g is positively homogenous.
(iii) Convexity: for any $c \in (0,1)$ and $a \leq t \leq T$, the g-expectation $\mathcal{E}_g$ is convex

$$
\mathcal{E}_g[c\zeta_1 + (1-c)\zeta_2|\mathcal{F}_t] \leq c\mathcal{E}_g[\zeta_1|\mathcal{F}_t] + (1-c)\mathcal{E}_g[\zeta_2|\mathcal{F}_t],
$$

if the generator g is convex, namely:

$$
\begin{aligned}
& g(t, cx_1 + (1-c)x_2, cy_1 + (1-c)y_2, cz_1 + (1-c)z_2) \\
\leq \quad & cg(t,x_1,y_1,z_1) + (1-c)g(t,x_2,y_2,z_2), \forall(x_1,y_1,z_1), (x_2,y_2,z_2) \in (\mathbb{R}, \mathbb{R}, L_Q^2).
\end{aligned}
$$

(iv) Sub-linearity and sub-additivity: the g-expectation $\mathcal{E}_g$ is sub-linear, sub-additive and positively homogenous

$$
\mathcal{E}_g[\zeta_1 + \zeta_2|\mathcal{F}_t] \leq \mathcal{E}_g[\zeta_1|\mathcal{F}_t] + \mathcal{E}_g[\zeta_2|\mathcal{F}_t],
$$

if the generator g is positively homogenous and satisfies:

$$
\begin{aligned}
& g(t, x_1 + x_2, y_1 + y_2, z_1 + z_2) \\
\leq \quad & g(t,x_1,y_1,z_1) + g(t,x_2,y_2,z_2), \forall(x_1,y_1,z_1), (x_2,y_2,z_2) \in (\mathbb{R}, \mathbb{R}, L_Q^2).
\end{aligned}
$$

Proof. Case (i). Consider the following conformable backward stochastic differential equations:

$$
\begin{aligned}
X(t) \;=\;& \zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X(\tau), Y(\tau), Z(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds),
\end{aligned}
$$

$$
\begin{aligned}
X'(t) \;=\;& \zeta + c + \int_t^T (\tau - a)^{\rho-1} g(\tau, X'(\tau), Y'(\tau), Z'(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y'(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z'(\tau, s) \tilde{N}(d\tau, ds),
\end{aligned}
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $c \in \mathbb{R}$, $a \leq t \leq T$ and $0 < \rho \leq 1$. Let

$$
\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|\mathcal{F}_t = M(t), \ M(a) = 1
$$

$$
\frac{dM(t)}{M(t)} = \tilde{\alpha}(t) dB(t) + \int_{\mathbb{R}} \tilde{\beta}(t) \tilde{N}(dt, ds), \ a \leq t \leq T, \tag{9}
$$

where $\tilde{\alpha}(\cdot)$ and $\tilde{\beta}(\cdot)$ are the predictable processes. If we choose a generator $g(t, X(t), Y(t), Z(t)) = \tilde{\alpha}(t) Y(t) + \int_{\mathbb{R}} \tilde{\beta}(t) Z(t, s) Q(t, ds) \eta(t)$ which does not depend on $X(\cdot)$, using Lemma 3, one has:

$$
X(t) = \zeta - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB^{\mathbb{Q}}(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}^{\mathbb{Q}}(d\tau, ds),
$$

$$
X'(t) = \zeta + c - \int_t^T (\tau - a)^{\rho-1} Y'(\tau) dB^{\mathbb{Q}}(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z'(\tau, s) \tilde{N}^{\mathbb{Q}}(d\tau, ds),
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $c \in \mathbb{R}$, $a \leq t \leq T$ and $0 < \rho \leq 1$. Hence, we get $X(t) = \mathbb{E}^{\mathbb{Q}}[\zeta|\mathcal{F}_t]$ and $X'(t) = \mathbb{E}^{\mathbb{Q}}[\zeta + c|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[\zeta|\mathcal{F}_t] + c$ under the probability measure $\mathbb{Q}$, that is, $X'(t) = X(t) + c$, $Y(t) = Y'(t)$ and $Z(t, s) = Z'(t, s)$.

On the other hand, since the generator $g(t, X(t), Y(t), Z(t)) = \tilde{\alpha}(t) Y(t) + \int_{\mathbb{R}} \tilde{\beta}(t) Z(t, s) Q(t, ds) \eta(t)$ satisfies Assumption 1, we have $X(t) = \mathcal{E}_g[\zeta|\mathcal{F}_t]$ and $X'(t) = \mathcal{E}_g[\zeta + c|\mathcal{F}_t]$. Hence, we conclude that $\mathcal{E}_g[\zeta + c|\mathcal{F}_t] = X'(t) = X(t) + c = \mathcal{E}_g[\zeta|\mathcal{F}_t] + c$.

Case (ii). Let $c > 0$, and using the same method as in case (i), consider the following equation:

$$
\begin{aligned}
X''(t) \;=\;& c\zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X''(\tau), Y''(\tau), Z''(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y''(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z''(\tau, s) \tilde{N}(d\tau, ds),
\end{aligned}
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $a \leq t \leq T$ and $0 < \rho \leq 1$. Choose

$$
g(t, X''(t), Y''(t), Z''(t)) = \tilde{\alpha}(t) Y''(t) + \int_{\mathbb{R}} \tilde{\beta}(t) Z''(t, s) Q(t, ds) \eta(t),
$$

which is positively homogenous. With the framework of (12), we have

$$
X''(t) = c\zeta - \int_t^T (\tau - a)^{\rho-1} Y''(\tau) dB^{\mathbb{Q}}(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z''(\tau, s) \tilde{N}^{\mathbb{Q}}(d\tau, ds),
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $a \leq t \leq T$ and $0 < \rho \leq 1$. Then we get $X''(t) = c\mathbb{E}^{\mathbb{Q}}[\zeta|\mathcal{F}_t]$. From Definition 4 and $X(t) = \mathbb{E}^{\mathbb{Q}}[\zeta|\mathcal{F}_t] = \mathcal{E}_g[\zeta|\mathcal{F}_t]$, we have $\mathcal{E}_g[c\zeta|\mathcal{F}_t] = X''(t) = c\mathbb{E}^{\mathbb{Q}}[\zeta|\mathcal{F}_t] = c\mathcal{E}_g[\zeta|\mathcal{F}_t]$.

Case (iii). Let $0 < c < 1$, and consider the following conformable backward stochastic differential equations

$$
\begin{aligned}
\bar{X}(t) \;=\; & c\zeta_1 + (1-c)\zeta_2 + \int_t^T (\tau - a)^{\rho-1} g(\tau, \bar{X}(\tau), \bar{Y}(\tau), \bar{Z}(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} \bar{Y}(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} \bar{Z}(\tau, s) \tilde{N}(d\tau, ds), \quad (10)
\end{aligned}
$$

$$
\begin{aligned}
X_i(t) \;=\; & \zeta_i + \int_t^T (\tau - a)^{\rho-1} g(\tau, X_i(\tau), Y_i(\tau), Z_i(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y_i(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_i(\tau, s) \tilde{N}(d\tau, ds),
\end{aligned}
$$

where $i = 1, 2$, $\zeta_i \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $0 < c < 1$, $a \le t \le T$ and $0 < \rho \le 1$.

Notice that:

$$
\begin{aligned}
& g(t, cX_1 + (1-c)X_2, cY_1 + (1-c)Y_2, cZ_1 + (1-c)Z_2) \\
\le\; & cg(t, X_1, Y_1, Z_1) + (1-c)g(t, X_2, Y_2, Z_2),
\end{aligned}
$$

and we see that:

$$
\begin{aligned}
& g(t, cX_1 + (1-c)X_2, cY_1 + (1-c)Y_2, cZ_1 + (1-c)Z_2) + f(t) \\
=\; & cg(t, X_1, Y_1, Z_1) + (1-c)g(t, X_2, Y_2, Z_2),
\end{aligned}
$$

where the nonnegative function $f(t)$ depends on X_i, Y_i and Z_i and $i = 1, 2$. Let $X_3(t) = cX_1(t) + (1-c)X_2(t)$, $Y_3(t) = cY_1(t) + (1-c)Y_2(t)$ and $Z_3(t) = cZ_1(t) + (1-c)Z_2(t)$, and we have

$$
\begin{aligned}
X_3(t) \;=\; & c\zeta_1 + (1-c)\zeta_2 + \int_t^T (\tau - a)^{\rho-1} [cg(\tau, X_1(\tau), Y_1(\tau), Z_1(\tau)) \\
& + (1-c)g(\tau, X_2(\tau), Y_2(\tau), Z_2(\tau))] d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} [cY_1(\tau) + (1-c)Y_2(\tau)] dB(\tau) \\
& - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} [cZ_1(\tau, s) + (1-c)Z_2(\tau, s)] \tilde{N}(d\tau, ds) \\
=\; & c\zeta_1 + (1-c)\zeta_2 + \int_t^T (\tau - a)^{\rho-1} [g(\tau, X_3(\tau), Y_3(\tau), Z_3(\tau)) \\
& + f(\tau, X_i(\tau), Y_i(\tau), Z_i(\tau))] d\tau - \int_t^T (\tau - a)^{\rho-1} Y_3(\tau) dB(\tau) \\
& - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_3(\tau, s) \tilde{N}(d\tau, ds), \quad (11)
\end{aligned}
$$

where $i = 1, 2$, $0 < c < 1$, $\zeta_i \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $a \le t \le T$ and $0 < \rho \le 1$. Using the comparison theorem to the Equations (10) and (11), we see that $\bar{X}(t) \le X_3(t)$. From Definition 4, one has

$$
\mathcal{E}_g[c\zeta_1 + (1-c)\zeta_2 | \mathcal{F}_t] = \bar{X}(t) \le X_3(t) = cX_1(t) + (1-c)X_2(t) \le c\mathcal{E}_g[\zeta_1 | \mathcal{F}_t] + (1-c)\mathcal{E}_g[\zeta_2 | \mathcal{F}_t],
$$

where $i = 1, 2$, $0 < c < 1$, $\zeta_i \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and $a \le t \le T$.

Case (iv). Similar to the proof process in case (iii), we get the result in case (iv).

The proof is complete. $\square$

Theorem 2. *Suppose $g(t, X(t), Y(t), Z(t)) = \tilde{\alpha}(t)Y(t) + \int_{\mathbb{R}} \tilde{\beta}(t)Z(t, s)Q(t, ds)\eta(t)$, $a \le t \le T$. Then the g-expectation is equivalent to the expectation under a probability measure $\mathbb{Q}$.*

Proof. Consider the following conformable backward stochastic differential equation

$$
\begin{aligned}
X(t) \;=\; & \zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X(\tau), Y(\tau), Z(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds),
\end{aligned}
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $a \le t \le T$ and $0 < \rho \le 1$.

It is clear that the generator $g(t, X(t), Y(t), Z(t)) = \tilde{\alpha}(t)Y(t) + \int_{\mathbb{R}} \tilde{\beta}(t)Z(t,s)Q(t,ds)\eta(t)$ satisfies Assumption 1. From Definition 4, we have $X(t) = \mathcal{E}_g[\zeta | \mathcal{F}_t]$.

Let

$$
\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = M(t), \ M(a) = 1
$$

$$
\frac{dM(t)}{M(t)} = \tilde{\alpha}(t)dB(t) + \int_{\mathbb{R}} \tilde{\beta}(t)\tilde{N}(dt, ds), \ a \le t \le T, \tag{12}
$$

where $\tilde{\alpha}(\cdot)$ and $\tilde{\beta}(\cdot)$ are the predictable processes. Using Lemma 3, one has:

$$
X(t) = \zeta - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB^{\mathbb{Q}}(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}^{\mathbb{Q}}(d\tau, ds),
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $a \le t \le T$ and $0 < \rho \le 1$. Hence, we get $X(t) = \mathbb{E}^{\mathbb{Q}}[\zeta | \mathcal{F}_t]$ under the probability measure $\mathbb{Q}$. From the uniqueness of the solution, we conclude that $\mathcal{E}_g[\zeta | \mathcal{F}_t] = X(t) = \mathbb{E}^{\mathbb{Q}}[\zeta | \mathcal{F}_t]$. The proof is complete. □

4. Doob–Meyer Decomposition Theorem

We first give some definitions.

Definition 5. *The process $X(\cdot)$ is called a g-martingale if for any $a \le s \le t \le T$, we have $\mathbb{E}[|X(t)|^2] < \infty$ and*

$$
\mathcal{E}_g[X(t)|\mathcal{F}_s] = X(s).
$$

Definition 6. *The process $X(\cdot)$ is called a g-supermartingale if for any $a \le s \le t \le T$, we have $\mathbb{E}[|X(t)|^2] < \infty$ and*

$$
\mathcal{E}_g[X(t)|\mathcal{F}_s] \le X(s).
$$

Theorem 3. *Assume that the generator g satisfies Assumption 1. If the process $X(\cdot) \in \mathbb{S}^2(\mathbb{R})$ is a g-supermartingale on $[a, T]$, then there exists a unique triple $(Y, Z) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ and a continuous increasing process $A(\cdot)$ such that:*

$$
\begin{aligned}
X(t) \;=\; & \zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X(\tau), Y(\tau), Z(\tau)) d\tau + A(T) - A(t) \\
& - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds),
\end{aligned}
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $a \le t \le T$, $0 < \rho \le 1$, $A(a) = 0$ and $\mathbb{E}[|A(T)|^2] < \infty$.

Proof. Consider the following conformable backward stochastic differential equation:

$$
\begin{aligned}
X^n(t) \;=\; & \zeta + \int_t^T (\tau - a)^{\rho-1} g^n(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau)) d\tau \\
& - \int_t^T (\tau - a)^{\rho-1} Y^n(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z^n(\tau, s) \tilde{N}(d\tau, ds), \tag{13}
\end{aligned}
$$

where $\zeta \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, $a \le t \le T$, $0 < \rho \le 1$ and $n = 1, 2, 3, \cdots$.

Assume that $g^n(t, x, y, z) = g(t, x, y, z) + n(X(t) - x)$, so Equation (13) can be written as:

$$
\begin{aligned}
X^n(t) = & \ \zeta + \int_t^T (\tau - a)^{\rho - 1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau)) d\tau + A^n(T) - A^n(t) \\
& - \int_t^T (\tau - a)^{\rho - 1} Y^n(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho - 1} \int_{\mathbb{R}} Z^n(\tau, s) \tilde{N}(d\tau, ds), \quad (14)
\end{aligned}
$$

where $A^n(t) = \int_a^t n(\tau - a)^{\rho - 1}(X(\tau) - X^n(\tau)) d\tau$. From Lemma 5 and a comparison theorem, we get that the sequence $(X^n(t))_{n \in \mathbb{N}^+}$ is increasing and monotonically converges. Hence the sequence $(A^n(t))_{n \in \mathbb{N}^+}$ is continuous and increasing. From Equation (14), we have

$$
\begin{aligned}
A^n(T) = & \ X^n(a) - \zeta - \int_a^T (\tau - a)^{\rho - 1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau)) d\tau \\
& + \int_a^T (\tau - a)^{\rho - 1} Y^n(\tau) dB(\tau) + \int_a^T (\tau - a)^{\rho - 1} \int_{\mathbb{R}} Z^n(\tau, s) \tilde{N}(d\tau, ds) \\
\leq & \ |X^n(a)| + |\zeta| + k \int_a^T (\tau - a)^{\rho - 1} [|X^n(\tau)| + |Y^n(\tau)| + \int_{\mathbb{R}} |Z^n(\tau, s)| Q(\tau, ds) \eta(\tau)] d\tau \\
& + \int_a^T (\tau - a)^{\rho - 1} |Y^n(\tau)| dB(\tau) + \int_a^T (\tau - a)^{\rho - 1} \int_{\mathbb{R}} |Z^n(\tau, s)| \tilde{N}(d\tau, ds),
\end{aligned}
$$

where k depends on K and $\delta^{x,y,z,z'}(t, s)$. Then,

$$
\begin{aligned}
\mathbb{E}[|A^n(T)|^2] \leq & \ \mathbb{E}[|X^n(a)|^2] + \mathbb{E}[\zeta^2] + k_1 \mathbb{E}[\sup_{a \leq t \leq T} |X^n(t)|^2] \\
& + k_2 \mathbb{E}\left[\int_a^T (\tau - a)^{2(\rho - 1)} \left(|Y^n(\tau)|^2 + \int_{\mathbb{R}} |Z^n(\tau, s)|^2 Q(\tau, ds) \eta(\tau) \right) d\tau \right],
\end{aligned}
$$

for a k_1 depending on T, ρ and a, and k_2 is a constant. Hence, there exists a constant l_1 such that:

$$
\mathbb{E}[|A^n(T)|^2] \leq l_1 + k_2 \mathbb{E}\left[\int_a^T (\tau - a)^{2(\rho - 1)} \left(|Y^n(\tau)|^2 + \int_{\mathbb{R}} |Z^n(\tau, s)|^2 Q(\tau, ds) \eta(\tau) \right) d\tau \right], \quad (15)
$$

where $a \leq t \leq T$, $0 < \rho \leq 1$ and $n = 1, 2, 3, \cdots$. Apply Lemma 2 to $|X^n(t)|^2$, and we have:

$$
\begin{aligned}
|X^n(t)|^2 = & \ \zeta^2 + 2 \int_t^T (\tau - a)^{\rho - 1} |X^n(\tau)| g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau)) d\tau \\
& + 2 \int_t^T |X^n(\tau)| dA^n(\tau) - 2 \int_t^T (\tau - a)^{\rho - 1} |X^n(\tau)| Y^n(\tau) dB(\tau) \\
& - 2 \int_t^T (\tau - a)^{\rho - 1} |X^n(\tau)| \int_{\mathbb{R}} Z^n(\tau, s) \tilde{N}(d\tau, ds) - \int_t^T (dX^n(\tau))^2,
\end{aligned}
$$

where $a \leq t \leq T$, $0 < \rho \leq 1$ and $n = 1, 2, 3, \cdots$. According to Lemma 4, Assumption 1 and Lemma 1, one has:

$$
\begin{aligned}
\mathbb{E}[|X^n(t)|^2] \;\leq\; & \mathbb{E}[\zeta^2] + 2\mathbb{E}\left[\int_t^T (\tau-a)^{\rho-1}|X^n(\tau)|g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau))d\tau\right] \\
& + 2\mathbb{E}\left[\int_t^T |X^n(\tau)|dA^n(\tau)\right] - \mathbb{E}\left[\int_t^T (\tau-a)^{2(\rho-1)}|Y^n(\tau)|^2 d\tau\right] \\
& - \mathbb{E}\left[\int_t^T \int_{\mathbb{R}} (\tau-a)^{2(\rho-1)}|Z^n(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right] \\
\leq\; & \mathbb{E}[\zeta^2] + 2K\mathbb{E}\left[\int_t^T (\tau-a)^{\rho-1}|X^n(\tau)|^2 d\tau\right] + 2K\mathbb{E}\left[\int_t^T (\tau-a)^{\rho-1}|X^n(\tau)||Y^n(\tau)|d\tau\right] \\
& + 2k\mathbb{E}\left[\int_t^T \int_{\mathbb{R}} (\tau-a)^{\rho-1}|X^n(\tau)||Z^n(\tau,s)|Q(\tau,ds)\eta(\tau)d\tau\right] + 2\mathbb{E}\left[\int_t^T |X^n(\tau)|dA^n(\tau)\right] \\
& - \mathbb{E}\left[\int_t^T (\tau-a)^{2(\rho-1)}|Y^n(\tau)|^2 d\tau\right] - \mathbb{E}\left[\int_t^T \int_{\mathbb{R}} (\tau-a)^{2(\rho-1)}|Z^n(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right] \\
\leq\; & \mathbb{E}[\zeta^2] + 2K\mathbb{E}\left[\int_t^T (\tau-a)^{\rho-1}|X^n(\tau)|^2 d\tau\right] + (\gamma K - 1)\mathbb{E}\left[\int_t^T (\tau-a)^{2(\rho-1)}|Y^n(\tau)|^2 d\tau\right] \\
& + \frac{1}{\gamma}(K+k+1)\mathbb{E}\left[\int_t^T |X^n(\tau)|^2 d\tau\right] + \gamma\mathbb{E}[|A^n(T)|^2] \\
& + (\gamma K - 1)\mathbb{E}\left[\int_t^T \int_{\mathbb{R}} (\tau-a)^{2(\rho-1)}|Z^n(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right],
\end{aligned}
$$

namely,

$$
\begin{aligned}
\mathbb{E}\left[\int_a^T (\tau-a)^{2(\rho-1)}\left(|Y^n(\tau)|^2 + \int_{\mathbb{R}} |Z^n(\tau,s)|^2 Q(\tau,ds)\eta(\tau)\right)d\tau\right] \\
\leq\; l_2 + \frac{\gamma}{1-\gamma K}\mathbb{E}[|A^n(T)|^2],
\end{aligned}
\tag{16}
$$

where l_2 is a constant and $\gamma \leq \frac{1}{K}$. Combining Equations (15) and (16), we conclude that there exists a constant C, independent of n, such that $\mathbb{E}[|A^n(T)|^2] \leq C$ and

$$
\mathbb{E}\left[\int_a^T (\tau-a)^{2(\rho-1)}|Y^n(\tau)|^2 d\tau\right] + \mathbb{E}\left[\int_a^T \int_{\mathbb{R}} (\tau-a)^{2(\rho-1)}|Z^n(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right] \leq C,
$$

where $a \leq t \leq T, 0 < \rho \leq 1$ and $n = 1,2,3,\cdots$. In addition, we also get

$$
\mathbb{E}\left[\int_a^T (\tau-a)^{2(\rho-1)}g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau))d\tau\right] \leq C.
$$

In other words, these sequences $(X^n(t))_{n\in\mathbb{N}^+}$, $(Y^n(t))_{n\in\mathbb{N}^+}$ and $(Z^n(t,\cdot))_{n\in\mathbb{N}^+}$ weakly converge in their spaces, and then for all stopping time ς, we have

$$
\lim_{n\to\infty} \int_a^\varsigma (\tau-a)^{\rho-1}X^n(\tau)d\tau = \int_a^\varsigma (\tau-a)^{\rho-1}X(\tau)d\tau,
$$

$$
\lim_{n\to\infty} \int_a^\varsigma (\tau-a)^{\rho-1}Y^n(\tau)dB(\tau) = \int_a^\varsigma (\tau-a)^{\rho-1}Y(\tau)dB(\tau),
$$

and

$$
\lim_{n\to\infty} \int_a^\varsigma (\tau-a)^{\rho-1}\int_{\mathbb{R}} Z^n(\tau,s)\tilde{N}(d\tau,ds) = \int_a^\varsigma (\tau-a)^{\rho-1}\int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds),
$$

and hence we have:

$$
\begin{aligned}
X(t) \;=\; & \zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X(\tau), Y(\tau), Z(\tau)) d\tau + A(T) - A(t) \\
& - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds),
\end{aligned}
$$

where a continuous increasing process $A(\cdot)$ satisfies $A(a) = 0$ and $\mathbb{E}[|A(T)|^2] < \infty$. $\quad\square$

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