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Pólya–Szegö Integral Inequalities Using the Caputo–Fabrizio Approach

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Abstract: In this article, we establish some of the Pólya–Szegö and Minkowsky-type fractional integral inequalities by considering the Caputo–Fabrizio fractional integral. Moreover, we give some special cases of Pólya–Szegö inequalities.

Keywords: Pólya–Szegö inequality; Minkowsky inequality; Caputo–Fabrizio fractional integrals

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1. Introduction

Mathematical integral inequalities plays a very important role in classical differential and integral equations, which have many applications in many fields.

In 1925, Pólya–Szegö proved the following inequality (see [1]):

$$\frac{\int_{c_1}^{c_2}\varphi^2(x)dx\int_{c_1}^{c_2}\psi^2(x)dx}{\left(\int_{c_1}^{c_2}\varphi(x)dx\int_{c_1}^{c_2}\psi(x)dx\right)^2}\leq\frac{1}{4}\left(\sqrt{\frac{\mathcal{S}\mathcal{T}}{st}}+\sqrt{\frac{st}{\mathcal{S}\mathcal{T}}}\right)^2, \tag{1}$$

and, in [2], Dragomir and Diamond proved the following inequality:

$$\begin{aligned}&\left|\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\varphi(x)\psi(x)dx-\left(\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\varphi(x)dx\right)\left(\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\psi(x)dx\right)\right|\\&\leq\frac{(\mathcal{S}-s)(\mathcal{T}-t)}{4(c_2-c_1)^2\sqrt{s\mathcal{S}t\mathcal{T}}}\int_{c_1}^{c_2}\varphi(x)dx\int_{c_1}^{c_2}\psi(x)dx,\end{aligned} \tag{2}$$

provided that φ and ψ are two integrable functions on $[c_1,c_2]$ and satisfy the condition

$$0<s\leq\varphi(x)\leq\mathcal{S}<\infty,\ \ 0<t\leq\psi(x)\leq\mathcal{T}<\infty;\ \ s,\mathcal{S},t,\mathcal{T}\in\mathbb{R},\ x\in[c_1,c_2]. \tag{3}$$

In 1935, G. Grüss proved the following classical integral inequality (see [1,3,4]):

$$\begin{aligned}&\left|\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\varphi(x)\psi(x)dx-\left(\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\varphi(x)dx\right)\left(\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\psi(x)dx\right)\right|\\&\leq\frac{(\mathcal{S}-s)(\mathcal{T}-t)}{4},\end{aligned} \tag{4}$$

provided that φ and ψ are two integrable functions on $[c_1,c_2]$ and satisfy the conditions

$$s\leq\varphi(x)\leq\mathcal{S},\ \ t\leq\psi(x)\leq\mathcal{T};\ \ s,\mathcal{S},t,\mathcal{T}\in\mathbb{R},\ x\in[c_1,c_1]. \tag{5}$$

Recently, many researchers in several fields have found different results about some known fractional calculus and applications by means of the Riemann–Liouville [5–11], k-Riemann Liouville [12,13], Caputo [5,12,14], Hadamard [15,16], Marichev–Saigo–Maeda [17], Saigo [18–20], Katugamapola [21], Atangana—Baleanu [22] and some other fractional integral operators. Many mathematicians have worked on the Pólya–Szegö inequalities using various fractional integral operators in recent years (see [23–26]). Caputo and Fabrizio [27,28] obtained new fractional derivatives and integrals without a singular kernel, which apply to time and spatial fractional derivatives. In [29], the authors defined the weighted Caputo–Fabrizio fractional derivative and studied related linear and nonlinear fractional differential equations. In the literature, very little work has been reported on fractional integral inequalities using Caputo and Caputo–Fabrizio integral operators. Wang et al. [30] obtained the Hermite–Hadamard inequalities by employing the Caputo–Fabrizio fractional operator. In [31], Chinchane et al. dealt with the Caputo—Fabrizio fractional integral operator with a nonsingular kernel and established some new integral inequalities for the Chebyshev functional, in the case of synchronous function, by employing the fractional integral. Jain et al. [24] established some new Pólya–Szegö inequality fractional integral inequalities by considering Riemann–Liouville-type fractional integral operators. In [32], Tariq et al. improved integral inequalities of the Hermite–Hadamard and Pachpatte types by incorporating the concept of preinvexity by considering the Caputo–Fabrizio fractional integral operator. Saad et al. [33,34] proved some new integral inequalities by using generalized fractional integral operators and some classical inequalities for integrable functions and their applications to the Zipf–Mandelbrot law. Motivated by the above work, the main objective of this article is to establish some new results for the Pólya–Szegö inequality and some other inequalities using the Caputo–Fabrizio fractional integrals. The paper is organized into the following sections: Section 2 gives some basic definitions of fractional calculus. Section 3 is devoted to the proof of some Pólya–Szegö and Minkowsky-type fractional inequalities by considering the Caputo–Fabrizio fractional operator. Finally, conclusion are given in Section 4.

2. Preliminaries

First, the definitions of the Caputo–Fabrizio fractional integrals are reviewed.

Definition 1 ([28,31,35])**.** *Let $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq 1$. The Caputo–Fabrizio fractional integral of order α of a function f is defined by*

$$\mathcal{I}_{0,x}^{\alpha}[\phi(x)] = \frac{1}{\alpha}\int_0^x e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-s)}\phi(s)ds. \tag{6}$$

For $\alpha = 1$, it is reduced to

$$\mathcal{I}_{0,x}^{1}[\phi(x)] = \int_0^x \phi(s)ds.$$

This integral operator will be at the center of our main results.

3. Fractional Pólya–Szegö Inequality

In this section, we investigate some new fractional Pólya–Szegö inequalities by considering the Caputo–Fabrizio integral operator.

Theorem 1. *Let h_1 and h_2 be two integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{R}_1$ and $\mathcal{R}_2$ on $[0,\infty)$ such that*

$$0 < \mathcal{P}_1(\eta) \leq h_1(\eta) \leq \mathcal{P}_2(\eta),\ 0 < \mathcal{R}_1(\eta) \leq h_2(\eta) \leq \mathcal{R}_2(\eta),\ (\eta \in (0,x), x > 0).$$

Then for $x > 0$ and $\alpha > 0$, the following inequality holds:

$$\frac{\mathcal{I}_{0,x}^{\alpha}[\mathcal{R}_1\mathcal{R}_2h_1^2(x)]\mathcal{I}_{0,x}^{\alpha}[\mathcal{P}_1\mathcal{P}_2h_2^2(x)]}{\left(\mathcal{I}_{0,x}^{\alpha}[(\mathcal{R}_1\mathcal{P}_1+\mathcal{R}_2\mathcal{P}_2)h_1h_2(x)]\right)^2} \leq \frac{1}{4}. \tag{7}$$

Proof. To prove (7), since $\eta \in (0,x)$ and $x > 0$, we have

$$\left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\eta)} - \frac{h_1(\eta)}{h_2(\eta)}\right) \geq 0. \tag{8}$$

Furthermore, we have

$$\left(\frac{h_1(\eta)}{h_2(\eta)} - \frac{\mathcal{P}_1(\eta)}{\mathcal{R}_2(\eta)}\right) \geq 0. \tag{9}$$

Multiplying (8) and (9), we have

$$\left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\eta)} - \frac{h_1(\eta)}{h_2(\eta)}\right)\left(\frac{h_1(\eta)}{h_2(\eta)} - \frac{\mathcal{P}_1(\eta)}{\mathcal{R}_2(\eta)}\right) \geq 0,$$

which implies that

$$\left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\eta)} - \frac{h_1(\eta)}{h_2(\eta)}\right)\frac{h_1(\eta)}{h_2(\eta)} - \left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\eta)} - \frac{h_1(\eta)}{h_2(\eta)}\right)\frac{\mathcal{P}_1(\eta)}{\mathcal{R}_2(\eta)} \geq 0,$$

so

$$\left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\eta)} + \frac{\mathcal{P}_1(\eta)}{\mathcal{R}_2(\eta)}\right)\frac{h_1(\eta)}{h_2(\eta)} \geq \frac{h_1^2(\eta)}{h_2^2(\eta)} + \frac{\mathcal{P}_1(\eta)\mathcal{P}_2(\eta)}{\mathcal{R}_1(\eta)\mathcal{R}_2(\eta)},$$

and

$$[\mathcal{P}_1(\eta)\mathcal{R}_1(\eta) + \mathcal{P}_2(\eta)\mathcal{R}_2(\eta)]h_1(\eta)h_2(\eta) \geq \mathcal{R}_1(\eta)\mathcal{R}_2(\eta)h_1^2(\eta) + \mathcal{P}_1(\eta)\mathcal{P}_2(\eta)h_2^2(\eta). \tag{10}$$

Multiplying (10) by $\frac{1}{\alpha}e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}$, we obtain

$$\begin{aligned}&\frac{1}{\alpha}e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}[\mathcal{P}_1(\eta)\mathcal{R}_1(\eta) + \mathcal{P}_2(\eta)\mathcal{R}_2(\eta)]h_1(\eta)h_2(\eta)\\ &\geq \frac{1}{\alpha}e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}\mathcal{R}_1(\eta)\mathcal{R}_2(\eta)h_1^2(\eta) + \frac{1}{\alpha}e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}\mathcal{P}_1(\eta)\mathcal{P}_2(\eta)h_2^2(\eta).\end{aligned} \tag{11}$$

Integrating (11) with respect to η from 0 to x, we obtain

$$\mathcal{I}_{0,x}^{\alpha}[(\mathcal{P}_1\mathcal{R}_1+\mathcal{P}_2\mathcal{R}_2)h_1h_2(x)] \geq \mathcal{I}_{0,x}^{\alpha}[\mathcal{R}_1\mathcal{R}_2h_1^2(x)] + \mathcal{I}_{0,x}^{\alpha}[\mathcal{P}_1\mathcal{P}_2h_2^2(x)].$$

By considering inequality $c_1 + c_2 \geq 2\sqrt{c_1c_2}$, where $c_1, c_2 \in [0,\infty)$, we have

$$\mathcal{I}_{0,x}^{\alpha}[(\mathcal{P}_1\mathcal{R}_1+\mathcal{P}_2\mathcal{R}_2)h_1h_2(x)] \geq 2\sqrt{\mathcal{I}_{0,x}^{\alpha}[\mathcal{R}_1\mathcal{R}_2h_1^2(x)]\mathcal{I}_{0,x}^{\alpha}[\mathcal{P}_2\mathcal{P}_1h_2^2(x)]},$$

so

$$\left(\mathcal{I}_{0,x}^{\alpha}[(\mathcal{P}_1\mathcal{R}_1+\mathcal{P}_2\mathcal{R}_2)h_1h_2(x)]\right)^2 \geq 4\left(\mathcal{I}_{0,x}^{\alpha}[\mathcal{R}_1\mathcal{R}_2h_1^2(x)]\mathcal{I}_{0,x}^{\alpha}[h_1\mathcal{P}_2h_2^2(x)]\right),$$

it follows that

$$\mathcal{I}_{0,x}^{\alpha}[\mathcal{R}_1\mathcal{R}_2h_1^2(x)]\mathcal{I}_{0,x}^{\alpha}[\mathcal{P}_1\mathcal{P}_2h_2^2(x)] \leq \frac{1}{4}\left(\mathcal{I}_{0,x}^{\alpha}[(\mathcal{P}_1\mathcal{R}_1+\mathcal{P}_1\mathcal{P}_2)h_1h_2(x)]\right)^2,$$

which gives the required inequality (7). □

Theorem 2. *Let h_1 and h_2 be two integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions $\mathcal{P}_1$,$\mathcal{P}_2$, $\mathcal{R}_1$ and $\mathcal{R}_2$ on $[0,\infty)$ such that*

$$0<\mathcal{P}_1(\eta)\leq h_1(\eta)\leq \mathcal{P}_2(\eta),\ 0<\mathcal{R}_1(\theta)\leq h_2(\theta)\leq \mathcal{R}_2(\theta),\ (\eta,\theta\in(0,x],x>0).$$

Then for $x>0$ and $\alpha>0$, $\beta>0$, the following inequality holds:

$$\frac{\mathcal{I}^{\alpha}_{0,x}[\mathcal{P}_1\mathcal{P}_2(x)]\mathcal{I}^{\beta}_{0,x}[\mathcal{R}_1\mathcal{R}_2(x)]\mathcal{I}^{\alpha}_{0,x}[h_1^2(x)]\mathcal{I}^{\beta}_{0,x}[h_2^2(x)]}{\left(\mathcal{I}^{\alpha}_{0,x}[\mathcal{P}_1h_1(x)]\mathcal{I}^{\beta}_{0,x}[\mathcal{R}_1h_2(x)]+\mathcal{I}^{\alpha}_{0,x}[\mathcal{P}_2h_1(x)]\mathcal{I}^{\beta}_{0,x}[\mathcal{R}_2h_2(x)]\right)^2}\leq\frac{1}{4}. \tag{12}$$

Proof. To prove (12), since $\eta,\theta\in(1,x]$ and $x>0$, we have

$$\frac{h_1(\eta)}{h_2(\theta)}\leq\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\theta)},$$

which implies that

$$\left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\theta)}-\frac{h_1(\eta)}{h_2(\theta)}\right)\geq 0. \tag{13}$$

Furthermore, we have

$$\left(\frac{h_1(\eta)}{h_2(\theta)}-\frac{\mathcal{P}_1(\eta)}{\mathcal{R}_2(\theta)}\right)\geq 0. \tag{14}$$

Multiplying (13) and (14), we have

$$\left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\theta)}-\frac{h_1(\eta)}{h_2(\theta)}\right)\left(\frac{h_1(\eta)}{h_2(\theta)}-\frac{\mathcal{P}_1(\eta)}{\mathcal{R}_2(\theta)}\right)\geq 0,$$

which implies that

$$\left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\theta)}-\frac{h_1(\eta)}{h_2(\theta)}\right)\frac{h_1(\eta)}{h_2(\theta)}-\left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\theta)}-\frac{h_1(\eta)}{h_2(\theta)}\right)\frac{\mathcal{P}_1(\eta)}{\mathcal{R}_2(\theta)}\geq 0,$$

and it follows that

$$\left(\frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\theta)}+\frac{\mathcal{P}_1(\eta)}{\mathcal{R}_2(\theta)}\right)\frac{h_1(\eta)}{h_2(\theta)}\geq\frac{h_1^2(\eta)}{h_2^2(\theta)}+\frac{\mathcal{P}_1(\eta)\mathcal{P}_2(\eta)}{\mathcal{R}_1(\theta)\mathcal{R}_2(\theta)}. \tag{15}$$

Multiplying both sides of inequality (15) by $\mathcal{R}_1(\theta)\mathcal{P}_2(\theta)h_2^2(\theta)$, we obtain

$$\mathcal{P}_1(\eta)h_1(\eta)\mathcal{R}_1(\theta)h_2(\theta)\mathcal{P}_2(\eta)h_1(\eta)\mathcal{R}_1(\theta)h_2(\theta)\geq\mathcal{R}_1(\theta)\mathcal{R}_2(\theta)h_1^2(\eta)+\mathcal{P}_1(\eta)\mathcal{P}_2(\eta)h_2^2(\theta). \tag{16}$$

Multiplying both sides of (16) by $\frac{1}{\alpha}e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}$, then integrating with respect to η from 0 to x, we get

$$\begin{aligned}&\mathcal{R}_1(\theta)h_2(\theta)\mathcal{I}^{\alpha}_{0,x}[\mathcal{P}_1h_1(x)]+\mathcal{R}_1(\theta)h_2(\theta)\mathcal{I}^{\alpha}_{0,x}[\mathcal{P}_2h_1(x)]\\&\geq\mathcal{R}_1(\theta)\mathcal{R}_2(\theta)\mathcal{I}^{\alpha}_{0,x}[h_1^2(x)]+h_2^2(\theta)\mathcal{I}^{\alpha}_{0,x}[\mathcal{P}_1\mathcal{P}_2(x)].\end{aligned} \tag{17}$$

Multiplying both sides of (17) by $\frac{1}{\beta}e^{-\left(\frac{1-\beta}{\beta}\right)(x-\theta)}$, then integrating with respect to θ from 0 to x, we have

$$\begin{aligned}&\mathcal{I}^{\beta}_{0,x}[\mathcal{R}_1h_2(x)]\mathcal{I}^{\alpha}_{0,x}[\mathcal{P}_1h_1(x)]+\mathcal{I}^{\beta}_{0,x}[\mathcal{R}_1h_2(x)]\mathcal{I}^{\alpha}_{0,x}[\mathcal{P}_2h_2(x)]\\&\geq\mathcal{I}^{\beta}_{0,x}[\mathcal{R}_1\mathcal{R}_2(x)]\mathcal{I}^{\alpha}_{0,x}[h_1^2(x)]+\mathcal{I}^{\beta}_{0,x}[h_2^2(x)]\mathcal{I}^{\alpha}_{0,x}[\mathcal{P}_1\mathcal{P}_2(x)].\end{aligned}$$

By $c_1 + c_2 \geq 2\sqrt{c_1 c_2}$, where $c_1, c_2 \in [0, \infty)$, we have

$$\begin{aligned}&\mathcal{I}_{0,x}^{\beta}[\mathcal{R}_1 h_2(x)]\mathcal{I}_{0,x}^{\alpha}[\mathcal{P}_1 h_1(x)] + \mathcal{I}_{0,x}^{\beta}[\mathcal{R}_1 h_2(x)]\mathcal{I}_{0,x}^{\alpha}[\mathcal{P}_2 h_1(x)]\\ &\geq 2\sqrt{\mathcal{I}_{0,x}^{\beta}[\mathcal{R}_1\mathcal{R}_2(x)]\mathcal{I}_{0,x}^{\alpha}[h_1^2(x)]\mathcal{I}_{0,x}^{\beta}[h_2^2(x)]\mathcal{I}_{0,x}^{\alpha}[\mathcal{P}_1\mathcal{P}_2(x)]},\end{aligned}$$

which gives the required inequality (12). □

Theorem 3. *Let h_1 and h_2 be two integrable functions on $[0, \infty)$. Assume that there exist four positive integrable functions $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{R}_1$ and $\mathcal{R}_2$ on $[0, \infty)$ such that*

$$0 < \mathcal{P}_1(\eta) \leq h_1(\eta) \leq \mathcal{P}_2(\eta),\ 0 < \mathcal{R}_1(\theta) \leq h_2(\theta) \leq \mathcal{R}_2(\theta),\ (\eta, \theta \in (0, x], x > 0).$$

Then for $x > 0$ and $\alpha, \beta > 0$, the following inequality holds:

$$\mathcal{I}_{0,x}^{\alpha}[h_1^2(x)]\mathcal{I}_{0,x}^{\beta}[h_2^2(x)] \leq \mathcal{I}_{0,x}^{\alpha}\left[\frac{\mathcal{P}_2 h_1 h_2}{\mathcal{R}_1}(x)\right]\mathcal{I}_{0,x}^{\beta}\left[\frac{\mathcal{R}_2 h_1 h_2}{\mathcal{P}_1}(x)\right]. \tag{18}$$

Proof. Multiplying (8) by $h_1(\eta)$, we obtain

$$h_1^2(\eta) \leq \frac{\mathcal{P}_2(\eta)}{\mathcal{R}_1(\eta)} h_1(\eta) h_2(\eta). \tag{19}$$

Multiplying the inequality (19) by $\frac{1}{\alpha} e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}$, which is positive because $\eta \in (0, x)$, $x > 0$ and then integrating with respect to η from 0 to x, we get

$$\mathcal{I}_{0,x}^{\alpha}[h_1^2(x)] \leq \mathcal{I}_{0,x}^{\alpha}\left[\frac{\mathcal{P}_2 h_1 h_2}{\mathcal{R}_1}(x)\right]. \tag{20}$$

Analogously, we obtain

$$\mathcal{I}_{0,x}^{\beta}[h_2^2(x)] \leq \mathcal{I}_{0,x}^{\beta}\left[\frac{\mathcal{R}_2 h_1 h_2}{\mathcal{P}_1}(x)\right], \tag{21}$$

multiplying the inequalities (20) and (21), we establish the required inequality (18). This completes the proof. □

Hereafter, we present some special cases of the above theorem.

Proposition 1. *Let h_1 and h_2 be two integrable functions on $[0, \infty)$ such that*

$$0 < \gamma_1 \leq h_1(\eta) \leq \Gamma_1 < \infty,\ 0 < \gamma_2 \leq h_2(\eta) \leq \Gamma_2 < \infty,\ (\eta \in (0, x], x > 0).$$

Then for $x > 0$ and $\alpha > 0$, the following inequality holds:

$$\frac{\mathcal{I}_{0,x}^{\alpha}[h_1^2(x)]\mathcal{I}_{0,x}^{\alpha}[h_2^2(x)]}{\left(\mathcal{I}_{0,x}^{\alpha}[h_1 h_2(x)]\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{\gamma_1\gamma_2}{\Gamma_1\Gamma_2}} + \sqrt{\frac{\Gamma_1\Gamma_2}{\gamma_1\gamma_2}}\right)^2.$$

Proposition 2. *Let h_1 and h_2 be two integrable functions on $[0, \infty]$ such that*

$$0 < \gamma_1 \leq h_1(\eta) \leq \Gamma_1 < \infty,\ 0 < \gamma_2 \leq h_2(\theta) \leq \Gamma_2 < \infty,\ (\eta, \theta \in [0, x], x > 0).$$

Then for $x > 0$ and $\alpha, \beta > 0$, the following inequality holds:

$$\frac{\left[\left[1 - e^{-\left(\frac{1-\alpha}{\alpha}\right)x}\right]\left[1 - e^{-\left(\frac{1-\beta}{\beta}\right)x}\right]\right]}{(1-\alpha)(1-\beta)} \frac{\mathcal{I}_{0,x}^{\alpha}[h_1^2(x)]\mathcal{I}_{0,x}^{\beta}[h_2^2(x)]}{(\mathcal{I}_{0,x}^{\alpha}[h_1(x)]\mathcal{I}_{0,x}^{\beta}[h_2(x)])^2} \leq \frac{1}{4}\left(\sqrt{\frac{\gamma_1\gamma_2}{\Gamma_1\Gamma_2}} + \sqrt{\frac{\Gamma_1\Gamma_2}{\gamma_1\gamma_2}}\right)^2.$$

Proposition 3. *Let h_1 and h_2 be two integrable functions on $[0, \infty)$ that satisfies condition (1). Then for $x > 0$ and $\alpha, \beta > 0$, we have*

$$\frac{\mathcal{I}_{0,x}^{\alpha}[h_1^2(x)]\mathcal{I}_{0,x}^{\beta}[h_2^2(x)]}{\left(\mathcal{I}_{0,x}^{\alpha}[h_1h_2(x)]\mathcal{I}_{0,x}^{\beta}[h_1h_2(x)]\right)^2} \leq \frac{\Gamma_1\Gamma_2}{\gamma_1\gamma_2}.$$

Now, we establish the Minkowsky-type inequality using the Caputo–Fabrizio integral operator.

Theorem 4. *Let h_1 and h_2 be two integrable functions on $[0, \infty]$ such that $\frac{1}{c_1} + \frac{1}{c_2} = 1$, $c_1 > 1$, and $0 \leq \gamma_1 \leq \frac{h_1(\eta)}{h_2(\eta)} \leq \Gamma$, $\eta \in (0, x)$, $x > 0$. Then for all $\alpha > 0$, we have*

$$\mathcal{I}_{0,x}^{\alpha}[h_1h_2(x)] \leq \frac{2^{c_1-1}\Gamma^{c_1}}{c_1(\Gamma+1)^{c_1}}\mathcal{I}_{0,x}^{\alpha}\left[(h_1^{c_1} + h_2^{c_1})(x)\right] + \frac{2^{c_2-1}}{c_2(\gamma+1)^{c_2}}\mathcal{I}_{0,x}^{\alpha}\left[(h_1^{c_2} + h_2^{c_2})(x)\right]. \quad (22)$$

Proof. Since, $\frac{h_1(\eta)}{h_2(\eta)} < \Gamma$, $\eta \in (0, x)$, $x > 0$, we have

$$(\Gamma + 1)h_1(\eta) \leq \Gamma(h_1 + h_2)(\eta). \quad (23)$$

Taking the c_1th power of both sides and multiplying the resulting inequality by $\frac{1}{\alpha}e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}$, we obtain

$$\frac{1}{\alpha}e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}(\Gamma+1)^{c_1}h_1^{c_1}(\eta) \leq \frac{1}{\alpha}e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}\Gamma^{c_1}(h_1+h_2)^{c_1}(\eta), \quad (24)$$

integrating (24) with respect to η from 0 to x, we get

$$\frac{1}{\alpha}\int_0^x e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}(\Gamma+1)^{c_1}h_1^{c_1}(\eta)d\eta \leq \frac{1}{\alpha}\int_0^x e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}\Gamma^{c_1}(h_1+h_2)^{c_1}(\eta)d\eta, \quad (25)$$

therefore

$$\mathcal{I}_{0,x}^{\alpha}[h_1^{c_1}(x)] \leq \frac{\Gamma^{c_1}}{(\Gamma+1)^{c_1}}\mathcal{I}_{0,x}^{\alpha}[(h_1+h_2)^{c_1}(x)]. \quad (26)$$

On the other hand, $0 \leq \gamma \leq \frac{h_1(\eta)}{h_2(\eta)}$, $\eta \in (0, x)$, $x > 0$, so

$$(\gamma + 1)h_2(\eta) \leq \gamma(h_1 + h_2)(\eta),$$

therefore

$$\frac{1}{\alpha}\int_0^x e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}(\gamma+1)^{c_2}h_2^{c_2}(\eta)d\eta \leq \frac{1}{\alpha}\int_0^x e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}\gamma^{c_2}(h_1+h_2)^{c_2}(\eta)d\eta$$

and we have

$$\mathcal{I}_{0,x}^{\alpha}[h_2^{c_2}(x)] \leq \frac{1}{(\gamma+1)^{c_2}}\mathcal{I}_{0,x}^{\alpha}[(h_1+h_2)^{c_2}(x)]. \quad (27)$$

Using the Young inequality, we obtain

$$h_1(\eta)h_2(\eta) \leq \frac{h_1^{c_1}(\eta)}{c_1} + \frac{h_2^{c_2}(\eta)}{c_2}. \quad (28)$$

Multiplying (28) by $\frac{1}{\alpha}e^{-\left(\frac{1-\alpha}{\alpha}\right)(x-\eta)}$, then integrating the resulting inequality with respect to η from 0 to x, we get

$$\mathcal{I}_{0,x}^{\alpha}[h_1(x)h_2(x)] \leq \frac{1}{c_1}\mathcal{I}_{0,x}^{\alpha}[h_1^{c_1}(x)] + \frac{1}{c_2}\mathcal{I}_{0,x}^{\alpha}[h_2^{c_2}(x)] \tag{29}$$

and from the equations (26), (27) and (29), we obtain

$$\mathcal{I}_{0,x}^{\alpha}[h_1(x)h_2(x)] \leq \frac{\Gamma^{c_1}}{c_1(\Gamma+1)^{c_1}}\mathcal{I}_{0,x}^{\alpha}[(h_1+h_2)^{c_1}(x)] + \frac{1}{c_2(\gamma+1)^{c_2}}\mathcal{I}_{0,x}^{\alpha}[(h_1+h_2)^{c_2}(x)]. \tag{30}$$

Now, using the inequality $(x+y)^m \leq 2^{m-1}(x^m+y^m)$, $m>1$, $x,y\geq 0$, we have

$$\mathcal{I}_{0,x}^{\alpha}[(h_1+h_2)^{c_1}(x)] \leq 2^{c_1-1}\mathcal{I}_{0,x}^{\alpha}[(h_1^{c_1}+h_2^{c_1})(x)] \tag{31}$$

and

$$\mathcal{I}_{0,x}^{\alpha}[(h_1+h_2)^{c_{21}}(x)] \leq 2^{c_2-1}\mathcal{I}_{0,x}^{\alpha}[(h_1^{c_2}+h_2^{c_2})(x)]. \tag{32}$$

Inserting (31), (32) in (30) we get the required inequality (22). This completes the proof. □

4. Conclusions

Nchama et al. [35] investigated some integral inequalities by considering the Caputo–Fabrizio fractional integral operator. In [28], Caputo and Fabrizio introduced a new fractional differential and integral operator. In the above work, we have applied the Caputo–Fabrizo fractional integral operator to establish some Pólya–Szegö and Minkowsky-type fractional integral inequalities. With the help of this study, we have established more general inequalities than in the classical cases due to the nonsingularity of the kernel. We believe that the Caputo–Fabrizio fractional integral is a formalism due to its nonsingularity of the kernel, which may provide an alternative way to solve many problems. The obtained fractional integral inequalities are very general and can be specialized to discover numerous interesting fractional integral inequalities. The inequalities investigated in this paper bring some contributions to the fields of fractional calculus and Caputo–Fabrizio fractional integral operator. These inequalities should lead to some applications for determining bounds and uniqueness of solutions in fractional differential equations.

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