axioms

MDPI

Article

A Model in Which the Separation Principle Holds for a Given Effective Projective Sigma-Class

Vladimir Kanovei *,† and Vassily Lyubetsky *,†

Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute), 127051 Moscow, Russia

* Correspondence: kanovei@iitp.ru (V.K.); lyubetsk@iitp.ru (V.L.)

† These authors contributed equally to this work.

Abstract: In this paper, we prove the following: If $n \geq 3$, there is a generic extension of $\mathbf{L}$—the constructible universe—in which it is true that the Separation principle holds for both effective (lightface) classes Σ^1_n and Π^1_n of sets of integers. The result was announced long ago by Leo Harrington with a sketch of the proof for $n = 3$; its full proof has never been presented. Our methods are based on a countable product of almost-disjoint forcing notions independent in the sense of Jensen–Solovay.

Keywords: separation; projective hierarchy; forcing

MSC: 03E35; 03E15

check for updates

Citation: Kanovei, V.; Lyubetsky, V. A Model in Which the Separation Principle Holds for a Given Effective Projective Sigma-Class. *Axioms* **2022**, *11*, 122. https://doi.org/10.3390/axioms11030122

Academic Editor: Oscar Castillo

Received: 9 February 2022
Accepted: 7 March 2022
Published: 10 March 2022

1. Introduction

The separation problem was introduced in descriptive set theory by Luzin [1]. In modern terms, the separation principle—or simply **Separation**, for a given projective (boldface) class $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$—is the assertion that

Boldface Separation for $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$**:** any pair of disjoint $\mathbf{\Sigma}^1_n$, resp, $\mathbf{\Pi}^1_n$ sets X, Y of reals can be separated by a $\mathbf{\Delta}^1_n$ set.

Accordingly, *the classical separation problem* is a question of whether **Boldface Separation** holds for this or another projective class $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$. Luzin and then Novikov [2] underlined the importance and difficulty of this problem. (See [3–5] for details and further references).

Luzin [1,6] and Novikov [7] proved that **Boldface Separation** holds for $\mathbf{\Sigma}^1_1$ but fails for the dual class $\mathbf{\Pi}^1_1$. Somewhat later, it was established by Novikov [8] that the picture changes at the next projective level: **Boldface Separation** holds for $\mathbf{\Pi}^1_2$ but fails for $\mathbf{\Sigma}^1_2$.

As for the higher levels of projective hierarchy, all attempts made in classical descriptive set theory to solve the separation problem above the second level did not work, until some additional set theoretic axioms were introduced—in particular, those by Novikov [2] and Addison [9,10]. Gödel's *axiom of constructibility* $\mathbf{V} = \mathbf{L}$ implies that, for any $n \geq 3$, **Boldface Separation** holds for $\mathbf{\Pi}^1_n$ but fails for $\mathbf{\Sigma}^1_n$—pretty similar to second level.

In such a case, it is customary in modern set theory to look for models in which the separation problem is solved differently than under $\mathbf{V} = \mathbf{L}$ for at least some projective classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, $n \geq 3$. This goal is split into two different tasks:

(I) Prove the *independence* of the Π-side **Boldface Separation**—that is, given $n \geq 3$, find models in which **Boldface Separation** *fails* for the class $\mathbf{\Pi}^1_n$;

(II) Prove the *consistency* of the Σ-side **Boldface Separation**—that is, given $n \geq 3$, find models in which **Boldface Separation** *holds* for the class $\mathbf{\Sigma}^1_n$.

As for *models*, we focus here only on *generic extensions of the constructible universe* $\mathbf{L}$. Other set theoretic models, e.g., those based on strong determinacy or large cardinal hypotheses, are not considered in this paper. (We may only note in brackets that, by Addison and Moschovakis [11], and Martin [12], the *axiom of projective determinacy* **PD** implies that, for any $m \geq 1$, the separation problem is solved affirmatively for $\mathbf{\Sigma}^1_{2m+1}$ and $\mathbf{\Pi}^1_{2m+2}$ and negatively for $\mathbf{\Pi}^1_{2m+1}$ and $\mathbf{\Sigma}^1_{2m+2}$—similar to what happens at the first and second level corresponding to $n = 0$ in this scheme. See also Steel [13,14], and Hauser and Schindler [15] for some other relevant results.).

Problems (I) and (II) have been well-known since the early years of forcing, e.g., see problem P3030, and especially P3029 (= (II) for $n = 3$) in a survey [16] by Mathias.

Two solutions for part (I) are known so far. Harrington's two-page handwritten note ([Addendum A1] [17]) contains a sketch of a model, without going into details, defined by the technique of almost-disjoint forcing of Solovay and Jensen [18], in which indeed **Separation** fails for both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ for a given n. This research was cited in Moschovakis [3] (Theorem 5B.3), and Mathias [16] (Remark P3110 on page 166), but has never been published or otherwise detailed in any way. Some other models, with the same property of failure of **Separation** for different projective classes, were recently defined and studied in [5,19].

As for (II), the problem as it stands is open so far, and no conclusive achievement, such as a model (a generic extension of $\mathbf{L}$) in which **Boldface Separation** holds for $\mathbf{\Sigma}^1_n$ for some $n \geq 3$, is known. Yet, the following modification turns out to be easier to work with. The *effective* or *lightface* **Separation**, for a given lightface class Σ^1_n or Π^1_n (we give [3] as a reference on the lightface projective hierarchy), is the assertion that

Lightface Separation for Σ^1_n or Π^1_n**:** any pair of disjoint Σ^1_n, respectively, Π^1_n sets X, Y can be separated by a Δ^1_n set—here, unlike the **Boldface Separation** case, the sets X, Y can be either sets of reals or sets of integers.

Accordingly, the *effective* or *lightface* separation problem is a question of whether **Lightface Separation** holds for this or another class of the form Σ^1_n or Π^1_n, with specific versions for sets of reals and sets of integers. Addison [9,10] demonstrated that, similar to the above, **Lightface Separation** holds for Σ^1_1 and Π^1_2; fails for Π^1_1 and Σ^1_2; and under the axiom of constructibility $\mathbf{V} = \mathbf{L}$, it holds for Π^1_n and fails for Σ^1_n for all $n \geq 3$—both in the "real" and the "integer" versions. (See also [3]).

In this context, Harrington announced in [17] that there is a model in which **Lightface Separation** holds both for the class Σ^1_3 *for sets of integers*, and for the class Π^1_3 *for sets of integers*. A two-page handwritten sketch of a construction of such a model is given in ([Addendum A3] [17]) without much elaboration of arguments. **The goal of this paper** is to prove the next theorem, which generalizes the cited Harrington result and thereby is a definite advance in the direction of (II) in the context of **Lightface Separation** for sets of integers. This is the main result of this paper.

Theorem 1. *Let* $\mathbb{n} \geq 2$. *There is a generic extension of* $\mathbf{L}$ *in which*

(i) **Lightface Separation** *holds for* $\Sigma^1_{\mathbb{n}+1}$ *sets of integers, so that any pair of disjoint* $\Sigma^1_{\mathbb{n}+1}$ *sets* $X, Y \subseteq \omega$ *can be separated by a* $\Delta^1_{\mathbb{n}+1}$ *set;*

(ii) **Lightface Separation** *also holds for* $\Pi^1_{\mathbb{n}+1}$ *sets of integers, so that any pair of disjoint* $\Pi^1_{\mathbb{n}+1}$ *sets* $X, Y \subseteq \omega$ *can be separated by a* $\Delta^1_{\mathbb{n}+1}$ *set.*

Our proof of this theorem will follow a scheme that includes both some arguments outlined by Harrington in [17], Addendum A3 (mainly related to the most elementary case $\mathbb{n} = 2$) and some arguments absent in [17], in particular, those related to the generalization to the case $\mathbb{n} \geq 3$. (We may note here that [17] is neither a beta-version of a paper, nor a preprint of any sort, but rather handwritten notes to a talk in which omissions of even major details can be expected.) All this will require both a fairly sophisticated construction of the model itself and a fairly complex derivation of its required properties by rather new methods for modern set theoretic research. Thus, we are going to proceed with filling-in all necessary details left aside in [17]. We hope that the detailed acquaintance with the set theoretic methods first introduced by Harrington will serve to the benefits of the reader envisaged.

To prove Theorem 1, we make use of a generic extension of $\mathbf{L}$ defined in our earlier paper [20] (and before that in [17]—modulo some key details absent in [17]) in order to prove the consistency of the equality $\mathscr{P}(\omega) \cap \mathbf{L} = \mathscr{P}(\omega) \cap \Delta^1_{\mathfrak{n}+1}$ for a given $\mathfrak{n} \geq 2$. (The equality claims that the constructible reals are the same as $\Delta^1_{\mathfrak{n}+1}$ reals. Its consistency was a major problem posed by Harvey Friedman [21].) We present the construction of this generic model in all necessary detail.

This includes a version of almost-disjoint forcing considered in Section 2, the cone-homogeneity lemma in Section 3, the systems and product forcing construction in Section 4, and a Jensen–Solovay-style construction of the actual countable support forcing product $\mathbb{Q}$ in Section 5. Theorem 2 and Definition 2 in Section 5 present the construction of $\mathbb{Q}$ in $\mathbf{L}$ via the union of a 2^ω-long increasing sequence of systems $\mathbb{U}_\xi$, $\xi < \omega_2$, which satisfies suitable completeness and definability requirements (that depend on the choice of the value of an integer $\mathfrak{n}$ as in Theorem 1), and also follows the Jensen–Solovay idea of Cohen-generic extensions at each step $\xi < \omega_2$ of the inductive construction of $\mathbb{U}_\xi$.

Then, we consider corresponding $\mathbb{Q}$-generic extensions $\mathbf{L}[G]$ in Section 6, and their subextensions involved in the proof of Theorem 1 in Section 7 (Theorem 4). Two key lemmas are established in Section 8, and the proof of theorems 4 and 1 is finalized in Section 9 (the $\Sigma^1_{\mathfrak{n}+1}$ case) and in Section 10 (the $\Pi^1_{\mathfrak{n}+1}$ case).

The final section on conclusions and discussion completes the paper.

2. Almost-Disjoint Forcing

Almost-disjoint forcing was invented by Jensen and Solovay [18]. Here, we make use of a ω_1-version of this tool considered in ([Section 5] [18]). The version we utilize here exactly corresponds to the case $\Omega = \omega_1^{\mathbf{L}}$ developed in our earlier paper [20] (and to less extent in [22]). This will allow us to skip some proofs below. Assume the following:

- $\Omega = \omega_1^{\mathbf{L}}$, the first uncountable ordinal in $\mathbf{L}$.
- $\mathbf{Fun} = (\Omega^\Omega) \cap \mathbf{L}$ = all Ω-sequences of ordinals $< \Omega$ in $\mathbf{L}$.
- $\mathbf{Seq} = (\Omega^{<\Omega} \smallsetminus \{\Lambda\}) \cap \mathbf{L}$ is the set of all sequences $s \in \mathbf{L}$ of ordinals $< \Omega$, of length $0 < \mathtt{lh}\, s < \Omega$.

By definition, the sets $\mathbf{Fun}$, $\mathbf{Seq}$ belong to $\mathbf{L}$ and $\mathtt{card}\,(\mathbf{Seq}) = \Omega = \omega_1^{\mathbf{L}}$ whereas $\mathtt{card}\,(\mathbf{Fun}) = \omega_2^{\mathbf{L}}$ in $\mathbf{L}$. Note that Λ, the empty sequence, does not belong to $\mathbf{Seq}$.

- A set $X \subseteq \mathbf{Fun}$ is *dense* iff for any $s \in \mathbf{Seq}$ there is $f \in X$ such that $s \subset f$.
- If $S \subseteq \mathbf{Seq}$, $f \in \mathbf{Fun}$; then, let $S/f = \{\xi < \Omega : f \restriction \xi \in S\}$.
- If S/f is unbounded in Ω, then say that S *covers* f; otherwise, S *does not cover* f.

The general goal of the almost-disjoint forcing is the following: given a set $u \subseteq \mathbf{Fun}$ in the ground set universe $\mathbf{L}$, find a generic set $S \subseteq \mathbf{Seq}$ such that the equivalence "$f \in u \iff S$ does not cover f" holds for each $f \in \mathbf{Fun}$ in the ground universe.

Definition 1. *Q^* is the set of all pairs $p = \langle S_p; F_p \rangle \in \mathbf{L}$ of* finite *sets $F_p \subseteq \mathbf{Fun}$, $S_p \subseteq \mathbf{Seq}$. Note that $Q^* \in \mathbf{L}$. Elements of Q^* are called (forcing) conditions.*

If $p \in Q^*$, then put $F_p^\vee = \{f \restriction \xi : f \in F_p \wedge 1 \leq \xi < \Omega\}$; this is a tree in $\mathbf{Seq}$.

Let $p, q \in Q^*$. Define $q \leqslant p$ (that is, q is stronger *as a forcing condition) iff $S_p \subseteq S_q$, $F_p \subseteq F_q$, and the difference $S_q \smallsetminus S_p$ does not intersect $F_p^\vee$, i.e., $S_q \cap F_p^\vee = S_p \cap F_p^\vee$. Clearly, we have $q \leqslant p$ iff $S_p \subseteq S_q$, $F_p \subseteq F_q$, and $S_q \cap F_p^\vee = S_p \cap F_p^\vee$.*

If $u \subseteq$ $\mathbf{Fun}$, then put $Q[u] = \{p \in Q^* : F_p \subseteq u\}$.

Lemma 1 (Lemma 1 in [20]). *Conditions $p, q \in Q^*$ are compatible in Q^* iff (1) $S_q \smallsetminus S_p$ does not intersect $F_p^\vee$, and (2) $S_p \smallsetminus S_q$ does not intersect $F_q^\vee$.*

Thus, any conditions $p, q \in Q[u]$ are compatible in $Q[u]$ iff p, q are compatible in Q^* iff the condition $p \wedge q = \langle S_p \cup S_q ; F_p \cup F_q \rangle \in Q[u]$ satisfies $p \wedge q \leqslant p$ and $p \wedge q \leqslant q$.

3. The Almost-Disjoint Forcing Notions Are Homogeneous

We are going to show that forcing notions of the form $Q[u]$ are sufficiently homogeneous. This is not immediately clear here, unlike the case of many other homogeneity claims. Assume that conditions $p, q \in Q^*$ satisfy the next requirement:

$$F_p = F_q \quad \text{and} \quad S_p \cup S_q \subseteq F_p^\vee = F_q^\vee . \tag{1}$$

Then, a transformation h_{pq} acting on conditions is defined as follows.

If $p = q$, then define $h_{pq}(r) = r$ for all $r \in Q^*$, the identity.

Suppose that $p \neq q$. Then, p, q are incompatible by (1) and Lemma 1. Define $d_{pq} = \{r \in Q^* : r \leqslant p \vee r \leqslant q\}$, the *domain* of h_{pq}. Let $r \in d_{pq}$. We put $h_{pq}(r) = r' := \langle S_{r'}, F_{r'} \rangle$, where $F_{r'} = F_r$ and

$$S_{r'} = \begin{cases} (S_r \smallsetminus S_p) \cup S_q & \text{in case} \quad r \leqslant p , \\ (S_r \smallsetminus S_q) \cup S_p & \text{in case} \quad r \leqslant q . \end{cases} \tag{2}$$

In this case, the difference between S_r and $S_{r'}$ is located within the set $X = F_p^\vee = F_q^\vee$, so that $S_r \cap X = S_p$ and $S_{r'} \cap X = S_q$ whenever $r \leqslant p$, while $S_r \cap X = S_q$ and $S_{r'} \cap X = S_p$ whenever $r \leqslant q$. The next lemma is Lemma 6 in [20].

Lemma 2. (i) *If $u \subseteq \mathbf{Fun}$ is dense and $p_0, q_0 \in Q[u]$, then there exist conditions $p, q \in Q[u]$ with $p \leqslant p_0$, $q \leqslant q_0$, satisfying (1).*

(ii) *Let $p, q \in Q^*$ satisfy (1). If $p = q$, then h_{pq} is the identity transformation. If $p \neq q$, then h_{pq} is an order automorphism of $d_{pq} = \{r \in Q^* : r \leqslant p \vee r \leqslant q\}$, satisfying $h_{pq}(p) = q$ and $h_{pq} = (h_{pq})^{-1} = h_{qp}$.*

(iii) *If $u \subseteq \mathbf{Fun}$ and $p, q \in Q[u]$ satisfy (1), then h_{pq} maps the set $Q[u] \cap d_{pq}$ onto itself order-preserving.*

Proof (sketch). (i) By the density of u, there is a countable set $F \subseteq \mathbf{Fun}$ satisfying $F_p \cup F_q \subseteq F$ and $S_p \cup S_q \subseteq F^\vee = \{f \restriction \xi : f \in F \wedge 1 \leq \xi < \Omega\}$. Put $p = \langle S_p, F \rangle$ and $q = \langle S_q, F \rangle$. Claims (ii) and (iii) are routine. □

Corollary 1 (in $\mathbf{L}$). *If a set $u \subseteq \mathbf{Fun}$ is dense, then $Q[u]$ is* **cone homogeneous** *in the sense of [23], i.e., if $p_0, q_0 \in Q[u]$, then there exist conditions $p, q \in Q[u]$ with $p \leqslant p_0$, $q \leqslant q_0$, such that the cones $Q[u]_{\leqslant p} = \{p' \in Q[u] : p' \leqslant p\}$ and $Q[u]_{\leqslant q}$ are order-isomorphic.*

4. Systems and Product Almost-Disjoint Forcing

To prove Theorem 1, we make use of a forcing notion equal to the countable-support product of a collapse forcing $\mathbb{C}$ and $\omega_2^{\mathbf{L}}$-many forcing notions of the form $Q[u]$, $u \subseteq \mathbf{Fun}$.

We work in $\mathbf{L}$. Define $\mathbb{C} = \mathscr{P}(\omega) \cap \mathbf{L}^{<\omega}$, the set of all finite sequences of subsets of ω in $\mathbf{L}$, an ordinary forcing $\mathscr{P}(\omega) \cap \mathbf{L}$ to collapse down to ω.

Let $\mathcal{I} = \omega_2^{\mathbf{L}}$ and $\mathcal{I}^+ = \mathcal{I} \cup \{-1\}$, the *index set* of the mentioned product. Let a *system* be any map $U : |U| \to \mathscr{P}(\mathbf{Fun})$ such that $|U| \subseteq \mathcal{I}$, each set $U(\nu)$ $(\nu \in |U|)$ is dense in $\mathbf{Fun}$, and the *components* $U(\nu) \subseteq \mathbf{Fun}$ $(\nu \in |U|)$ are pairwise disjoint.

Given a system U in $\mathbf{L}$, we let $\mathbf{Q}[U]$ be the finite-support product of $\mathbb{C}$ and the sets $Q[U(\nu)]$, $\nu \in |U|$. That is, $\mathbf{Q}[U]$ consists of all maps p defined on a finite set $\mathtt{dom}\, p = |p|^+ \subseteq |U| \cup \{-1\}$ so that $p(\nu) \in Q[U(\nu)]$ for all $\nu \in |p| := |p|^+ \smallsetminus \{-1\}$, and if

$-1 \in |p|^+$, then $\boldsymbol{b}_p := p(-1) \in \mathbb{C}$. If $p \in \mathbf{Q}[U]$, then put $F_p(\nu) = F_{p(\nu)}$ and $S_p(\nu) = S_{p(\nu)}$ for all $\nu \in |p|$, so that $p(\nu) = \langle S_p(\nu); F_p(\nu)\rangle$.

We order $\mathbf{Q}[U]$ component-wise: $p \leqslant q$ (p is stronger as a forcing condition) iff $|q|^+ \subseteq |p|^+$, $\boldsymbol{b}_q \subseteq \boldsymbol{b}_p$ in case $-1 \in |q|^+$, and $p(\nu) \leqslant q(\nu)$ in $Q[U(\nu)]$ for all $\nu \in |q|$. Note that $\mathbf{Q}[U]$ contains *the empty condition* $\odot \in \mathbf{Q}[U]$ satisfying $|\odot|^+ = \varnothing$; obviously, $\odot$ is the $\leqslant$-least (and weakest as a forcing condition) element of $\mathbf{Q}[U]$.

Lemma 3 (in $\mathbf{L}$)**.** *If U is a system, then the forcing notion $\mathbf{Q}[U]$ satisfies $\omega_2^{\mathbf{L}}$-CC.*

Proof. We argue in $\mathbf{L}$. Assume towards the contrary that $X \subseteq \mathbf{Q}[U]$ is an antichain of cardinality $\mathtt{card}\, X = \omega_2$. As $\mathtt{card}\, \mathbb{C} = \omega_1$, we can assume that $\boldsymbol{b}_p = \boldsymbol{b}_q$ for all $p, q \in X$. Consider the set $S = \{|p| : p \in X\}$; it consists of finite subsets of ω_2.

Case 1: $\mathtt{card}\, S \le \omega_1$. Then, by the cardinality argument, there is a set $X' \subseteq X$ and some $a \in S$ such that $|p| = a$ for all $p \in X'$ and still $\mathtt{card}\, X' = \omega_2$. Note that if $p \neq q$ belongs to X', then $\boldsymbol{b}_p = \boldsymbol{b}_q$ by the above; therefore, as p, q are incompatible, we have $S_p \neq S_q$. Thus, $P = \{S_p : p \in X'\}$ still satisfies $\mathtt{card}\, P = \omega_2$. This is a contradiction since obviously the set $\{S_p : p \in \mathbf{Q}[U] \wedge |p| = a\}$ has cardinality ω_1.

Case 2: $\mathtt{card}\, S = \omega_2$. Then, by the Δ-system lemma (see e.g., Lemma 111.2.6 in Kunen [24]) there is a set $S' \subseteq S$ and a finite set $\delta \subseteq \omega_2$ (the root) such that $a \cap b = \delta$ for all $a \neq b$ in S', and still $\mathtt{card}\, S' = \omega_2$. For any $a \in S$, pick a condition $p_a \in X'$ with $|p| = a$; then, $X'' = \{p_a : a \in S'\}$ still satisfies $\mathtt{card}\, X'' = \omega_2$. By construction, if $p \neq q$ belong to X'', then $|p| \cap |q| = \delta$ and p, q are incompatible; hence, the restricted conditions $p \restriction \delta$, $q \restriction \delta$ are incompatible as well. Thus, the set $Y = \{p \restriction \delta : p \in X''\}$ still has cardinality $\mathtt{card}\, Y = \omega_2$ and is an antichain. On the other hand, $|q| = \delta$ for all $q \in Y$. Therefore, we have a contradiction as in Case 1. $\square$

5. Jensen–Solovay Construction

Our plan is to define a system $\mathbb{U} \in \mathbf{L}$ such that any $\mathbf{Q}[\mathbb{U}]$-generic extension of $\mathbf{L}$ has a subextension that witnesses Theorem 1. Such a system will be defined in the form of a component-wise union of a $\omega_2^{\mathbf{L}}$-long increasing sequence of *small* systems, where the smallness means that, in $\mathbf{L}$, the system involves only $\omega_1^{\mathbf{L}}$-many functions in $\mathbf{Fun}$.

We work in $\mathbf{L}$.

- A system U is *small*, if both $|U|$ and each set $U(\nu)$ $(\nu \in |U|)$ has cardinality $\le \omega_1^{\mathbf{L}}$.
- If U, V are systems, $|U| \subseteq |V|$, and $U(\nu) \subseteq V(\nu)$ for all $\nu \in |U|$, then say that V *extends* U, in symbol $U \preccurlyeq V$.
- If $\{U_\xi\}_{\xi<\lambda}$ is a $\preccurlyeq$-increasing sequence of systems, then define a system $U = \bigvee_{\xi<\lambda} U_\xi$ by $|U| = \bigcup_{\xi<\lambda} |U_\xi|$ and $U(\nu) = \bigcup_{\xi<\lambda,\, \nu \in |U_\xi|} U_\xi(\nu)$ for all $\nu \in |U|$.

We let $\mathbf{ZFC}^-$ be $\mathbf{ZFC}$ minus the Power Set axiom, with the schema of Collection instead of Replacement, with $\mathbf{AC}$ in the form of the well-orderability principle, and with the axiom: "ω_1 exists". See [25] on versions of $\mathbf{ZFC}$ sans the Power Set axiom in detail. Let $\mathbf{ZFC}_2^-$ be $\mathbf{ZFC}^-$ plus the axioms: $\mathbf{V} = \mathbf{L}$, and the axiom "every set x satisfies $\mathtt{card}\, x \le \omega_1$".

Let U, V be systems. Suppose that M is any transitive model of $\mathbf{ZFC}_2^-$ containing Ω. Define $U \preccurlyeq_M U'$ iff $U \preccurlyeq U'$ and the following holds:

(a) the set $\Delta(U, U') = \bigcup_{\nu \in |U|} (U'(\nu) \smallsetminus U(\nu))$ is *multiply* $\mathbf{Seq}$*-generic* over M, in the sense that every sequence $\langle f_1, \dots f_m\rangle$ of pairwise different functions $f_\ell \in \Delta(U, U')$ is generic over M in the sense of $\mathbf{Seq} = \omega_1{}^{<\omega_1}$ as the forcing notion in $\mathbf{L}$, and

(b) if $\nu \in |U|$, then the set $U'(\nu) \smallsetminus U(\nu)$ is dense in $\mathbf{Fun}$, and therefore uncountable.

Note a corollary of (a): $\Delta(U, U') \cap M = \varnothing$.

- Let $\mathbf{JS}$, *Jensen–Solovay pairs*, be the set of all pairs $\langle M, U\rangle$, where $M \models \mathbf{ZFC}_2^-$ is a transitive model containing Ω and $U \in M$ is a system. Then, the sets $\mathbf{Seq}$, $\mathbf{Q}[U]$ also belong to M.

- Let **sJS**, *small Jensen–Solovay pairs*, be the set of all pairs $\langle M, U\rangle \in \mathbf{JS}$ such that U is a small system in the sense above and $\mathtt{card}\, M \le \omega_1$ (in $\mathbf{L}$).
- $\langle M, U\rangle \preccurlyeq \langle M', U'\rangle$ ($\langle M', U'\rangle$ extends $\langle M, U\rangle$) iff $M \subseteq M'$ and $U \preccurlyeq_M U'$; $\langle M, U\rangle \prec \langle M', U'\rangle$ (strict) iff $\langle M, U\rangle \preccurlyeq \langle M', U'\rangle$ and $\forall\, \nu \in |U|\, (U(\nu) \subsetneqq U'(\nu))$.
- A *Jensen–Solovay sequence* of length $\lambda \le \Omega^{\oplus} = \omega_2$ is any strictly $\prec$-increasing λ-sequence $\{\langle M_\xi, U_\xi\rangle\}_{\xi<\lambda}$ of pairs $\langle M_\xi, U_\xi\rangle \in \mathbf{sJS}$, satisfying $U_\eta = \bigvee_{\xi<\eta} U_\xi$ on limit steps. Let $\overrightarrow{\mathbf{JS}}_\lambda$ be the set of all such sequences.
- A pair $\langle M, U\rangle \in \mathbf{sJS}$ *solves* a set $D \subseteq \mathbf{sJS}$ iff either $\langle M, U\rangle \in D$ or there is no pair $\langle M', U'\rangle \in D$ that extends $\langle M, U\rangle$.
- Let $D^{\mathtt{solv}}$ be the set of all pairs $\langle M, U\rangle \in \mathbf{sJS}$, which solve a given set $D \subseteq \mathbf{sJS}$.
- Let $n \ge 3$. A sequence $\{\langle M_\xi, U_\xi\rangle\}_{\xi<\omega_2} \in \overrightarrow{\mathbf{JS}}_{\omega_2}$ is *n-complete* iff it intersects every set of the form $D^{\mathtt{solv}}$, where $D \subseteq \mathbf{sJS}$ is a $\Sigma^{\mathrm{H}\omega_2}_{n-2}(\mathrm{H}\omega_2)$ set.

Recall that $\mathrm{H}\omega_2$ is the collection of all sets x whose transitive closure $\mathrm{TC}\,(x)$ has cardinality $\mathtt{card}\,(\mathrm{TC}\,(x)) < \omega_2$. Further, $\Sigma^{\mathrm{H}\omega_2}_{n-2}(\mathrm{H}\omega_2)$ means definability by a Σ_{n-2} formula of the $\in$-language, in which any definability parameters in $\mathrm{H}\omega_2$ are allowed, while $\Sigma^{\mathrm{H}\omega_2}_{n-2}$ means the parameter-free definability. Similarly, $\Delta^{\mathrm{H}\omega_2}_{n-1}(\{\Omega\})$ in the next theorem means that $\Omega = \omega_1^{\mathbf{L}}$ is allowed as a sole parameter. It is a simple exercise that sets $\{\mathbf{Seq}\}$ and $\mathbf{Seq}$ are $\Delta^{\mathrm{H}\omega_2}_1(\{\Omega\})$ under $\mathbf{V} = \mathbf{L}$. To account for Ω as a parameter, note that the set ω_1 is $\Sigma^{\mathrm{H}\omega_2}_1$; hence, the singleton $\{\omega_1\}$ is $\Delta^{\mathrm{H}\omega_2}_2$.

Generally, we refer to, e.g., [26], Part B, 5.4, or [27], Chap. 13, on the Lévy hierarchy of $\in$-formulas and definability classes Σ^H_n, Π^H_n, Δ^H_n for any transitive set H.

Theorem 2 (Theorem 3 in [20])**.** *It is true in* $\mathbf{L}$ *that if* $n \ge 2$*, then there is a sequence* $\{\langle M_\xi, U_\xi\rangle\}_{\xi<\omega_2} \in \overrightarrow{\mathbf{JS}}_{\omega_2}$ *of class* $\Delta^{\mathrm{H}\omega_2}_{n-1}(\{\Omega\})$*; hence,* $\Delta^{\mathrm{H}\omega_2}_{n-1}$ *in case* $n \ge 3$*, and in addition n-complete in case* $n \ge 3$*, such that* $\xi \in |U_{\xi+1}|$ *for all* $\xi < \omega_2$*.*

Similar theorems were established in [28–30] for different purposes.

Definition 2 (in $\mathbf{L}$)**.** ***Fix a number*** $\mathtt{n} \ge 2$ ***during the following proof of Theorem 1.***

Let $\{\langle \mathbb{M}_\xi, \mathbb{U}_\xi\rangle\}_{\xi<\omega_2} \in \overrightarrow{\mathbf{JS}}_{\omega_2}$ *be any Jensen–Solovay sequence as in Theorem 2—that is,*

(i) *the sequence is of class* $\Delta^{\mathrm{H}\omega_2}_{\mathtt{n}-1}$*;*

(ii) *we have* $\xi \in |\mathbb{U}_{\xi+1}|$ *for all* ξ*;*

(iii) *if* $\mathtt{n} \ge 3$*, then the sequence is* $\mathtt{n}$*-complete.*

Put $\mathbb{U} = \bigvee_{\xi<\omega_2} \mathbb{U}_\xi$*, so* $\mathbb{U}(\nu) = \bigcup_{\xi<\omega_2, \nu\in|\mathbb{U}_\xi|} \mathbb{U}_\xi(\nu)$ *for all* $\nu \in \mathcal{I}$*. Thus,* $\mathbb{U} \in \mathbf{L}$ *is a system and* $|\mathbb{U}| = \mathcal{I}$ *since* $\xi \in |\mathbb{U}_{\xi+1}|$ *for all* ξ*.*

We define $\mathbb{Q} = \mathbf{Q}[\mathbb{U}]$ *(the* **basic forcing notion***). Thus,* $\mathbb{Q} \in \mathbf{L}$ *is the finite-support product of the set* $\mathbb{C}$ *and sets* $\mathbb{Q}(\nu) = Q[\mathbb{U}(\nu)]$*,* $\nu \in \mathcal{I}$*.*

Lemma 4 (in $\mathbf{L}$)**.** *The binary relation* $f \in \mathbb{U}(\nu)$ *is* $\Sigma^{\mathrm{H}\omega_2}_{\mathtt{n}-1}(\{\Omega\})$*.*

Proof. Make use of (i) of Definition 2. □

6. Basic Generic Extension

We consider $\mathbb{Q}_{\mathtt{n}} := \mathbb{Q}_{\mathtt{n}} = \mathbf{Q}[\mathbb{U}]$ (see Definition 2) as a forcing notion in $\mathbf{L}$. Accordingly, we will study $\mathbb{Q}$-generic extensions $\mathbf{L}[G]$ of the ground universe $\mathbf{L}$. Define some elements of these extensions. Suppose that $G \subseteq \mathbb{Q}$. Let

$$\boldsymbol{b}_G = \bigcup_{p\in G} \boldsymbol{b}_p\,, \quad \text{and} \quad S_G(\nu) = S_{G(\nu)} = \bigcup_{p\in G} S_p(\nu)$$

for any $\nu \in |G|$, where $G(\nu) = \{p(\nu) : p \in G\} \subseteq Q[\mathbb{U}(\nu)]$. Thus, $S_G(\nu) \subseteq \mathbf{Seq}$.

Therefore, any $\mathbb{Q}$-generic set $G \subseteq \mathbb{Q}$ splits into the family of sets $G(\nu)$, $\nu \in \mathcal{I}$, and a separate map $\boldsymbol{b}_G : \omega \xrightarrow{\text{onto}} \mathscr{P}(\omega) \cap \mathbf{L}$. It follows from Lemma 3 by standard arguments that $\mathbb{Q}$-generic extensions of $\mathbf{L}$ satisfy $\omega_1 = \omega_2^{\mathbf{L}}$.

Lemma 5 (Lemma 9 in [20]). *Let $G \subseteq \mathbb{Q}$ be a set $\mathbb{Q}$-generic over $\mathbf{L}$. Then,*

(i) *$\boldsymbol{b}_G$ is a $\mathbb{C}$-generic map from ω onto $\mathscr{P}(\omega) \cap \mathbf{L}$;*

(ii) *if $\nu \in \mathcal{I}$, then the set $G(\nu) = \{p(\nu) : p \in G\} \in \mathbf{L}[G]$ is $P[\mathbb{U}(\nu)]$-generic over $\mathbf{L}$—hence, if $f \in \mathbf{Fun}$, then $f \in \mathbb{U}(\nu)$ iff $S_G(\nu)$ does not cover f.*

Now suppose that $c \subseteq \mathcal{I}^+$. If $p \in \mathbb{Q}$, then a *restriction* $p' = p \restriction c \in \mathbb{Q}$ is defined by $|p'| = c \cap |p|$ and $p'(\nu) = p(\nu)$ for all $\nu \in |p'|$. In particular, if $\nu \in \mathcal{I}^+$, then let

$$p \restriction_{\neq\nu} = p \restriction (|p|^+ \smallsetminus \{\nu\}) \quad \text{and} \quad p \restriction_\nu = p \restriction \{\nu\} \text{ (identified with } p(\nu)).$$

If $G \subseteq \mathbb{Q}$, then let $G \restriction c = \{p \in G : |p| \subseteq c\}$ ($=\{p \restriction c : p \in G\}$ in case $c \in \mathbf{L}$).

Put $G \restriction_{\neq\nu} = \{p \in G : \nu \notin |p|^+\} = G \restriction (\mathcal{I}^+ \smallsetminus \{\nu\})$.

Writing $p \restriction c$, it is not assumed that $c \subseteq |p|^+$.

The proof of Theorem 1 makes use of a generic extension of the form $\mathbf{L}[G \restriction c]$, where $G \subseteq \mathbb{Q}$ is a set $\mathbb{Q}$-generic over $\mathbf{L}$ and $c \subseteq \mathcal{I}^+$, $c \notin \mathbf{L}$.

Define formulas Γ_ν $(\nu \in \mathcal{I})$ as follows:

$$\Gamma_\nu(S) :=_{\text{def}} S \subseteq \mathbf{Seq} \wedge \forall f \in \mathbf{Fun} \left(f \in \mathbb{U}(\nu) \iff S \text{ does not cover } f \right).$$

Lemma 6 (Lemma 22 in [20]). *Suppose that a set $G \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over $\mathbf{L}$ and $\nu \in \mathcal{I}$, $c \in \mathbf{L}[G]$, $\varnothing \neq c \subseteq \mathcal{I}^+$. Then, $\omega_1^{\mathbf{L}[G \restriction c]} = \omega_2^{\mathbf{L}}$ and*

(i) *$\Gamma_\nu(S_G(\nu))$ holds;*

(ii) *$S_G(\nu) \notin \mathbf{L}[G \restriction_{\neq\nu}]$—generally, there are no sets $S \subseteq \mathbf{Seq}$ in $\mathbf{L}[G \restriction_{\neq\nu}]$ satisfying $\Gamma_\nu(S)$;*

(iii) *if $-1 \in c$, then $\boldsymbol{b}_G \in \mathbf{L}[G \restriction c]$, and if $\nu \in c$, then $S_G(\nu) \in \mathbf{L}[G \restriction c]$.*

The next key theorem is Theorem 4 in [20]. Note that if $\mathbb{n} = 2$, then the result is an easy corollary of the Shoenfield absoluteness theorem.

Theorem 3 (elementary equivalence theorem). *Assume that in $\mathbf{L}$, $-1 \in d \subseteq \mathcal{I}^+$, sets $Z', Z \subseteq \mathcal{I} \smallsetminus d$ satisfy $\mathtt{card}\,(\mathcal{I} \smallsetminus Z) \leq \omega_1$ and $\mathtt{card}\,(\mathcal{I} \smallsetminus Z') \leq \omega_1$, the symmetric difference $Z \,\Delta\, Z'$ is at most countable and the complementary set $\mathcal{I} \smallsetminus (d \cup Z \cup Z')$ is infinite.*

Let $G \subseteq \mathbb{Q}$ be $\mathbb{Q}$-generic over $\mathbf{L}$, and $x_0 \in \mathbf{L}[G \restriction d]$ be any real.

Then, any closed $\Sigma^1_{\mathbb{n}}$ formula φ, with real parameters in $\mathbf{L}[x_0]$, is simultaneously true in the models $\mathbf{L}[x_0, G \restriction Z]$ and $\mathbf{L}[x_0, G \restriction Z']$.

7. The Model

Here, we introduce a submodel of the basic $\mathbb{Q}$-generic extension $\mathbf{L}[G]$ defined in Section 6 that will lead to the proof of Theorem 1.

Recall that a number $\mathbb{n} \geq 2$ is fixed by Definition 2.

Under the assumptions and notation of Definition 2, consider a set $G \subseteq \mathbb{Q}$, $\mathbb{Q}$-generic over $\mathbf{L}$. Then, $\boldsymbol{b}_G = \bigcup G(-1)$ is a $\mathbb{C}$-generic map from ω onto $\mathscr{P}(\omega) \cap \mathbf{L}$ by Lemma 5 (i). We define

$$w[G] = \{\omega k + 2^j : k < \omega \wedge j \in \boldsymbol{b}_G(k)\} \cup \{\omega k + 3^j : j, k < \omega\} \subseteq \omega^2, \tag{3}$$

and $w^+[G] = \{-1\} \cup w[G]$. We also define, for any $m < \omega$,

$$w_{\geq m}[G] = \{\omega k + \ell \in w[G] : k \geq m\}, \quad w_{<m}[G] = \{\omega k + \ell \in w[G] : k < m\},$$

and accordingly, $w^+_{\geq m}[G] = \{-1\} \cup w_{\geq m}[G]$ and $w^+_{<m}[G] = \{-1\} \cup w_{<m}[G]$.

With these definitions, each kth slice

$$w_k[G] = \{\omega k + 2^j : j \in \boldsymbol{b}_G(k)\} \cup \{\omega k + 3^j : j < \omega\} \tag{4}$$

of $w[G]$ is necessarily infinite and coinfinite, and it codes the target set $\boldsymbol{b}_G(k)$ since

$$\boldsymbol{b}_G(k) = \{j < \omega : \omega k + 2^j \in w_k[G]\} = \{j < \omega : \omega k + 2^j \in w^+[G]\}. \tag{5}$$

Note that definition (3) is *monotone w.r.t.* $\boldsymbol{b}_G$, *i.e.*, if $\boldsymbol{b}_G(k) \subseteq \boldsymbol{b}_{G'}(k)$ for all k, then $w[G] \subseteq w[G']$ and $w^+[G] \subseteq w^+[G']$. Anyway, $w[G] \subseteq \omega^2$ (the ordinal product) is a set in the model $\mathbf{L}[\boldsymbol{b}_G] = \mathbf{L}[w^+[G]] = \mathbf{L}[w[G]] = \mathbf{L}[w_{\geq m}[G]]$ for each m, whereas $w_{<m}[G] \in \mathbf{L}$ for all m. Finally, let $W = [\omega^2, \omega_2) = \{\zeta : \omega^2 \leq \zeta < \omega_2\}$.

Recall that if $c \subseteq \mathcal{I}^+$, then $G \restriction c = \{p \in G : |p|^+ \subseteq c\}$.

To prove Theorem 1, we consider the model $\mathbf{L}[G \restriction (w^+[G] \cup W)] \subseteq \mathbf{L}[G]$.

Theorem 4. *If G is a $\mathbb{Q}$-generic set over $\mathbf{L}$, then the class $\mathbf{L}[G \restriction (w^+[G] \cup W)]$ suffices to prove Theorem 1. That is,* **Lightface Separation** *holds in $\mathbf{L}[G \restriction (w^+[G] \cup W)]$ both for $\Sigma^1_{\mathfrak{n}+1}$ sets of integers and for $\Pi^1_{\mathfrak{n}+1}$ sets of integers.*

The proof will include several lemmas.

For the next lemma, we let $\Vdash_{\mathbb{Q}}$ be the $\mathbb{Q}$-forcing notion defined in $\mathbf{L}$. If $p \in \mathbb{Q}$ and $-1 \in |p|^+$, then let $p \restriction_{-1} := p \restriction \{-1\}$. This can be identified with just $p(-1) \in \mathbb{C}$, of course, but formally $p \restriction_{-1} \in \mathbb{Q}$. If $-1 \notin |p|^+$, then let $p \restriction_{-1} := \odot$ (the empty condition). Let $\underline{G}$ be the canonical $\mathbb{Q}$-name for the generic set $G \subseteq \mathbb{Q}$, $\check{W}$ be a name for the set $W = [\omega^2, \omega_2^{\mathbf{L}}) \in \mathbf{L}$, and $\check{\boldsymbol{b}}$ be a canonical $\mathbb{Q}$-name for $\boldsymbol{b}_G$.

Lemma 7 (reduction to the $\mathbb{C}$-component)**.** *Let $p \in \mathbb{Q}$ and let $\Phi(\check{\boldsymbol{b}})$ be a closed formula containing only $\check{\boldsymbol{b}}$ and names for sets in $\mathbf{L}$ as parameters. Assume that*

$$p \Vdash_{\mathbb{Q}} \text{“}\Phi \text{ is true in } \mathbf{L}[\underline{G} \restriction (w^+[\underline{G}] \cup \check{W})]\text{”}.$$

Then, $p \restriction_{-1} \Vdash_{\mathbb{Q}}$ “Φ is true in $\mathbf{L}[\underline{G} \restriction (w^+[\underline{G}] \cup \check{W})]$” as well.

Proof. By the product forcing theorem, if $G \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over $\mathbf{L}$, then the model $\mathbf{L}[G \restriction (w^+[G] \cup W)]$ is a $\mathbb{Q}'$-generic extension of $\mathbf{L}[\boldsymbol{b}_G]$, where $\mathbb{Q}' = \prod_{\nu \in w^+[G] \cup W} Q[\mathbb{U}(\nu)]$ is a forcing in $\mathbf{L}[\boldsymbol{b}_G]$. However, it follows from Corollary 1 that $\mathbb{Q}'$ is a (finite-support) product of cone-homogeneous forcing notions. Therefore, $\mathbb{Q}'$ itself is a cone homogeneous forcing, and we are finished (see e.g., Lemma 3 in [23] or Theorem IV.4.15 in [24]). □

8. Two Key Lemmas

The following two lemmas present two key properties of models of the form $\mathbf{L}[G \restriction (w^+[G] \cup W)]$ involved in the proof of Theorem 4. The first lemma shows that all constructible reals are $\Delta^1_{\mathfrak{n}+1}$ in such a model.

Lemma 8. *Let a set $G \subseteq \mathbb{Q}$ be $\mathbb{Q}$-generic over $\mathbf{L}$. Then, it holds in $\mathbf{L}[G \restriction (w^+[G] \cup W)]$ that $w[G]$ is $\Sigma^1_{\mathfrak{n}+1}$ and each set $x \in \mathbf{L}$, $x \subseteq \omega$ is $\Delta^1_{\mathfrak{n}+1}$.*

Proof. Consider an arbitrary ordinal $\nu = \omega k + \ell$; $k, \ell < \omega$. We claim that

$$\nu \in w[G] \iff \exists S\, \mathbb{\Gamma}_\nu(S) \tag{6}$$

holds in $\mathbf{L}[G \restriction (w^+[G] \cup W)]$. Indeed, assume that $\nu \in w[G]$. Then, $S = S_G(\nu) \in \mathbf{L}[G \restriction w^+[G]]$, and we have $\mathbb{\Gamma}_\nu(S)$ in $\mathbf{L}[G \restriction (w^+[G] \cup W)]$ by Lemma 6 (iii),(i). Conversely, assume that $\nu \notin w[G]$. Then, we have $w^+[G] \in \mathbf{L}[\boldsymbol{b}_G] \subseteq \mathbf{L}[G\restriction w^+[G]] \subseteq \mathbf{L}[G\restriction_{\neq\nu}]$, but $\mathbf{L}[G\restriction_{\neq\nu}]$ contains no S with $\Gamma_\nu(S)$ by Lemma 6 (ii).

However, the right-hand side of (6) defines a $\Sigma^{\mathrm{H}\omega_2}_{\mathbb{n}}(\{\omega_1^{\mathbf{L}}, \mathbf{Seq}\})$ relation in the model $\mathbf{L}[G\restriction(w^+[G]\cup W)]$ by Lemma 4. (Indeed, we have $(\mathrm{H}\omega_2)^{\mathbf{L}} = \mathbf{L}_{\omega_2{}^{\mathbf{L}}} = \mathbf{L}_{\omega_1}$ in $\mathbf{L}[G\restriction(w^+[G]\cup W)]$; therefore, $(\mathrm{H}\omega_2)^{\mathbf{L}}$ is $\Sigma^{\mathrm{H}\omega_2}_1$ in $\mathbf{L}[G\restriction(w^+[G]\cup W)]$). On the other hand, the sets $\{\omega_1^{\mathbf{L}}\}$ and $\{\mathbf{Seq}\}$ remain $\Delta^{\mathrm{H}\omega_2}_2$ singletons in $\mathbf{L}[G\restriction(w^+[G]\cup W)]$; they can be eliminated since $\mathbb{n} \geq 2$. This yields $w[G] \in \Sigma^{\mathrm{HC}}_{\mathbb{n}}$ in $\mathbf{L}[G\restriction(w^+[G]\cup W)]$. It follows that $w[G] \in \Sigma^1_{\mathbb{n}+1}$ in $\mathbf{L}[G\restriction(w^+[G]\cup W)]$ by Lemma 25.25 in [27], as required.

Now, let $x \in \mathbf{L}$, $x \subseteq \omega$. By genericity, there exists $k < \omega$ such that $\boldsymbol{b}_G(k) = x$. Then, $x = \{j : \omega k + 2^j \in w[G]\}$ by (3); therefore, x is $\Sigma^1_{\mathbb{n}+1}$ as well. However, $\omega \smallsetminus x \in \Sigma^1_{\mathbb{n}+1}$ by the same argument. Thus, x is $\Delta^1_{\mathbb{n}+1}$ in $\mathbf{L}[G\restriction(w^+[G]\cup W)]$, as required. □

The proof of the next lemma involves Theorem 3 above as a key reference. The lemma holds for $\mathbb{n} = 2$ by Shoenfield.

Lemma 9. *Suppose that $G \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over $\mathbf{L}$, $m < \omega$, $c \subseteq w_{<m}[G]$, $c \in \mathbf{L}$. Then, any closed $\Sigma^1_{\mathbb{n}}$ formula Φ, with reals in $\mathbf{L}[G\restriction(c \cup w^+_{\geq m}[G] \cup W)]$ as parameters, is simultaneously true in $\mathbf{L}[G\restriction(c \cup w^+_{\geq m}[G] \cup W)]$ and in $\mathbf{L}[G\restriction(w^+[G] \cup W)]$.*

It follows that if $c' \subseteq c \subseteq w_{<m}[G]$ in $\mathbf{L}$, then any closed $\Sigma^1_{\mathbb{n}+1}$ formula Ψ, with parameters in $\mathbf{L}[G\restriction(c' \cup w^+_{\geq m}[G] \cup W)]$, true in $\mathbf{L}[G\restriction(c' \cup w^+_{\geq m}[G] \cup W)]$, is true in the model $\mathbf{L}[G\restriction(c \cup w^+_{\geq m}[G] \cup W)]$ as well.

Proof. There is an ordinal $\xi < \omega_2$ such that all parameters in φ belong to $\mathbf{L}[G\restriction Y]$, where $Y = c \cup w^+_{\geq m}[G] \cup X$ and $X = [\omega^2, \xi) = \{\gamma : \omega^2 \leq \gamma < \xi\}$. The set Y belongs to $\mathbf{L}[\boldsymbol{b}_G]$; in fact, $\mathbf{L}[Y] = \mathbf{L}[\boldsymbol{b}_G]$. Therefore, $G\restriction Y$ is equi-constructible with the pair $\langle \boldsymbol{b}_G, \{S_G(\nu)\}_{\nu\in Y}\rangle$. Here, $\boldsymbol{b}_G$ is a map from ω onto $\mathscr{P}(\omega) \cap \mathbf{L}$. It follows that there is a real x_0 with $\mathbf{L}[G\restriction Y] = \mathbf{L}[x_0]$. Then, all parameters of φ belong to $\mathbf{L}[x_0]$.

To prepare an application of Theorem 3 of Section 6, we put

$$
\begin{aligned}
Z' &= [\xi, \omega_2)\,, \\
Z &= e \cup Z'\,, \quad \text{where} \quad e = w_{<m}[G] \smallsetminus c\,, \\
d &= \{-1\} \cup \{\omega k + j : k \geq m \wedge j < \omega\} \cup X\,.
\end{aligned}
$$

It is easy to check that all requirements of Theorem 3 for these sets are fulfilled. Moreover, as $w^+_{\geq m}[G] \subseteq \{-1\} \cup \{\omega k + j : k \geq m \wedge j < \omega\}$, we have $Y = c \cup w^+_{\geq m}[G] \cup X \subseteq d$; hence, $x_0 \in \mathbf{L}[G\restriction d]$. Therefore, we conclude by Theorem 3 that the formula φ is simultaneously true in $\mathbf{L}[x_0, G\restriction Z]$ and in $\mathbf{L}[x_0, G\restriction Z']$. However,

$$
\mathbf{L}[x_0, G\restriction Z'] = \mathbf{L}[G\restriction(Y \cup Z')] = \mathbf{L}[G\restriction(c \cup w^+_{\geq m}[G] \cup W)]
$$

by construction, while $\mathbf{L}[x_0, G\restriction Z] = \mathbf{L}[G\restriction(w^+[G] \cup W)]$, and we are done. □

9. Finalization: $\Sigma^1_{\mathbb{n}+1}$ Case

Here, we finalize the proof of Theorems 4 and 1 w.r.t. $\Sigma^1_{\mathbb{n}+1}$ sets of integers. We generally follow the line of arguments sketched by Harrington ([Addendum A3] [17]) for the Σ^1_3 case, with suitable changes mutatis mutandis. We will fill in all details omitted in [17].

Recall that a number $\mathbb{n} \geq 2$ is fixed by Definition 2. We assume that

(∗) a set $G \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over $\mathbf{L}$, sets $x, y \subseteq \omega$ belong to $\mathbf{L}[G\restriction(w^+[G] \cup W)]$, and it holds in $\mathbf{L}[G\restriction(w^+[G] \cup W)]$ that x, y are disjoint $\Sigma^1_{\mathbb{n}+1}$ sets.

The goal is to prove that x, y can be separated by a set $Z \in \mathbf{L}$ and then argue that Z is $\Delta^1_{\mathbb{n}+1}$ by Lemma 8. Recall that $W = [\omega^2, \omega_2^{\mathbf{L}}) = \{\xi : \omega^2 \leq \xi < \omega_2^{\mathbf{L}}\}$. Suppose that

(†) $\varphi(\cdot)$ and $\psi(\cdot)$ are parameter-free $\Sigma^1_{\mathtt{n}+1}$ formulas that provide $\Sigma^1_{\mathtt{n}+1}$ definitions for the sets, respectively, x, y of (∗)—that is,

$$x = \{\ell < \omega : \varphi(\ell)\} \quad \text{and} \quad y = \{\ell < \omega : \psi(\ell)\}$$

in $\mathbf{L}[G \restriction (w^+[G] \cup W)]$. The assumed implication $\forall\,\ell\,(\varphi(\ell) \Longrightarrow \neg\,\psi(\ell))$ (as $x \cap y = \varnothing$) is forced to be true in $\mathbf{L}[\underline{G} \restriction (w^+[\underline{G}] \cup \check{W})]$ by a condition $p_0 \in G$.

Here, $\underline{G}$ is the canonical $\mathbb{Q}$-name for the generic set $G \subseteq \mathbb{Q}$ while $\check{W}$ is a name for the set $W = [\omega^2, \omega_2^{\mathbf{L}}) \in \mathbf{L}$.

We observe that $\forall\,\ell\,(\varphi(\ell) \Longrightarrow \neg\,\psi(\ell))$ is a parameter-free sentence. Therefore, it can w.l.o.g. be assumed that $|p_0|^+ = \{-1\}$, by Lemma 7. In this case, the condition $p_0 \in \mathbb{Q}$ can be identified with its only nontrivial component $s_0 = p_0(-1) \in \mathbb{C}$.

Lemma 10 (interpolation lemma)**.** *Under the assumptions of (†), if $\ell, \ell' < \omega$, conditions $p, p' \in \mathbb{Q}$ satisfy $p \le p_0$ and $p' \le p_0$, and we have*

$$p \Vdash_{\mathbb{Q}} \text{“}\mathbf{L}[\underline{G} \restriction (w^+[\underline{G}] \cup \check{W})] \models \varphi(\ell)\text{”} \quad \textit{and} \quad p' \Vdash_{\mathbb{Q}} \text{“}\mathbf{L}[\underline{G} \restriction (w^+[\underline{G}] \cup \check{W})] \models \psi(\ell')\text{”}.$$

Then, $\ell \neq \ell'$.

Proof (sketched in ([Addendum A3] [17]) for $\mathtt{n} = 2$)**.** First of all, by Lemma 7, we can w.l.o.g. assume that $|p|^+ = |p'|^+ = \{-1\}$; so, the components $s = p(-1)$ and $s' = p'(-1)$ satisfy $s_0 \subseteq s$ and $s_0 \subseteq s'$ in $\mathbb{C}$.

We w.l.o.g. assume that the tuples s, s' have the same length $\mathtt{lh}\,s = \mathtt{lh}\,s' = m$. (Otherwise, extend the shorted one by a sufficient number of new terms equal to $\varnothing$). Define another condition $t \in \mathbb{C}$ such that $\mathtt{dom}\,t = m$ and $t(j) = s(j) \cup s'(j)$ for all $j < m$. Accordingly, define $q \in \mathbb{Q}$ so that $|q|^+ = \{-1\}$ and $q(-1) = t$. Despite that q may well be incomparable with p, p' in $\mathbb{Q}$, we claim that

$$q \Vdash_{\mathbb{Q}} \text{“}\mathbf{L}[\underline{G} \restriction (w^+[\underline{G}] \cup \check{W})] \models \varphi(\ell) \wedge \psi(\ell')\text{”}. \tag{7}$$

To prove the φ-part of (7), let $H \subseteq \mathbb{Q}$ be a set $\mathbb{Q}$-generic over $\mathbf{L}$, and $q \in H$. Then, $t \subset \boldsymbol{b}_H$. We have to prove that $\varphi(\ell)$ holds in $\mathbf{L}[H \restriction (w^+[H] \cup W)]$.

Define another generic set $K \subseteq \mathbb{Q}$, slightly different from H, so that

(A) $K(\nu) = H(\nu)$ for all $\nu \in \mathcal{I} = \omega_2^{\mathbf{L}}$;

(B) $s \subset \boldsymbol{b}_K$; and

(C) if $m \le j < \omega$, then $\boldsymbol{b}_K(j) = \boldsymbol{b}_H(j)$.

In other words, the only difference between K and H is that $\boldsymbol{b}_K \restriction m = s$ but $\boldsymbol{b}_H \restriction m = t$.

It follows that $p \in K$; hence, $\varphi(\ell)$ holds in $\mathbf{L}[K \restriction (w^+[K] \cup W)]$ by the assumptions of the lemma. Now, we note that by definition,

$$w^+[K] \cup W = w_{<m}[K] \cup w^+_{\ge m}[K] \cup W, \quad w^+[H] \cup W = w_{<m}[H] \cup w^+_{\ge m}[H] \cup W,$$

Here, the sets $c_H = w_{<m}[H]$ and $c_K = w_{<m}[K]$ satisfy $c_K \subseteq c_H$ (since $\boldsymbol{b}_K(j) = s(j) \subseteq t(j) = \boldsymbol{b}_H(j)$ for all $j < m$). In addition, $w^+_{\ge m}[H] = w^+_{\ge m}[K]$ (since $\boldsymbol{b}_K(j)\boldsymbol{b}_H(j)$ for all $j \ge m$). To conclude,

$$w^+[K] \cup W = c_K \cup w^+_{\ge m}[H] \cup W, \quad w^+[H] \cup W = c_H \cup w^+_{\ge m}[H] \cup W, \tag{8}$$

and $c_K \subseteq c_H = w_{<m}[H]$. On the other hand, it follows from (A) that $K \restriction c = H \restriction c$ for any $c \subseteq \mathcal{I}$, whereas $\boldsymbol{b}_K$ and $\boldsymbol{b}_H$ are recursively reducible to each other by (B),(C). Therefore,

$$\mathbf{L}[K \restriction (w^+[K] \cup W)] = \mathbf{L}[H \restriction (w^+[K] \cup W)] = \mathbf{L}[H \restriction (c_K \cup w^+_{\ge m}[H] \cup W)]$$

by (8). However, $\varphi(\ell)$ holds in this model by the above. It follows by Lemma 9 that $\varphi(\ell)$ holds in $\mathbf{L}[H\restriction(w^+[H]\cup W)] = \mathbf{L}[c_H \cup w^+_{\geq m}[H]\cup W]$ as well. (Harrington circumvents Lemma 9 in [17] by a reference to the Shoenfield absoluteness theorem.) We are finished.

After (7) has been established, we recall that $q \leq p_0$ in $\mathbb{Q}$ by construction. It follows that $\ell \neq \ell'$ by the choice of p_0 (see $(\dagger)$ above). $\square$

Proof of Theorems 4 and 1: $\Sigma^1_{\mathbb{n}+1}$ case. We work under the assumptions of $(*)$ and $(\dagger)$ above. Consider the following sets in $\mathbf{L}$:

$$
\begin{aligned}
Z_x &= \{\ell < \omega : \exists\, p \in \mathbb{Q}\,(p \leq p_0 \wedge p \Vdash_{\mathbb{Q}} \text{“}\mathbf{L}[\underline{G}\restriction(w^+[\underline{G}]\cup \check{W})] \models \varphi(\ell)\text{”})\};\\
Z_y &= \{\ell' < \omega : \exists\, p' \in \mathbb{Q}\,(p' \leq p_0 \wedge p' \Vdash_{\mathbb{Q}} \text{“}\mathbf{L}[\underline{G}\restriction(w^+[\underline{G}]\cup \check{W})] \models \psi(\ell')\text{”})\}.
\end{aligned}
$$

Note that $Z_x \cap Z_y = \varnothing$ by Lemma 10. On the other hand, it is clear that $x \subseteq Z_x$ and $y \subseteq Z_y$ by $(\dagger)$. Thus, either of the two sets $Z_x, Z_y \in \mathbf{L}$ separates x from y. It remains to apply Lemma 8. $\square$

10. Finalization: $\Pi^1_{\mathbb{n}+1}$ Case

This will be a mild variation of the argument presented in the previous section.

Proof of Theorems 4 and 1: $\Pi^1_{\mathbb{n}+1}$ case, sketch. Emulating $(*)$ and $(\dagger)$ above, we assume that a set $G \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over $\mathbf{L}$, and $x, y \subseteq \omega$ are disjoint $\Pi^1_{\mathbb{n}+1}$ sets in $\mathbf{L}[G\restriction(w^+[G]\cup W)]$, defined by parameter-free $\Pi^1_{\mathbb{n}+1}$ formulas, respectively, $\varphi(\cdot)$ and $\psi(\cdot)$. The implication $\forall\,\ell\,(\varphi(\ell) \Longrightarrow \neg\,\psi(\ell))$ is forced to hold in $\mathbf{L}[\underline{G}\restriction(w^+[\underline{G}]\cup \check{W})]$ by a condition $p_0 \in G$ satisfying $|p_0|^+ = \{-1\}$. The proof of Lemma 10 goes on for $\Pi^1_{\mathbb{n}+1}$ formulas φ, ψ the same way, with the only difference that we define $t(j) = s(j) \cap s'(j)$ for $j < m$. Yet, this is compatible with the application of Lemma 9 because now, φ, ψ are $\Pi^1_{\mathbb{n}+1}$ formulas. $\square$

11. Conclusions and Discussion

In this study, the method of almost-disjoint forcing was employed to the problem of obtaining a model of **ZFC** in which the **Separation** principle holds for lightface classes Π^1_{n+1} and Σ^1_{n+1}, for a given $n \geq 2$, for sets of integers. The problem of obtaining such models has been generally known since the early years of modern set theory, see, e.g., problems 3029 and 3030 in a survey [16] by Mathias. Harrington ([17], Addendum A3) claimed the existence of such models; yet, a detailed proof has never appeared.

From our study, it is concluded that the technique developed in our earlier paper [20] solves the general case of the problem (an arbitrary $n \geq 2$) by providing a generic extension of $\mathbf{L}$ in which the **Lightface Separation** principle holds for classes Π^1_{n+1} and Σ^1_{n+1}, for a given $n \geq 2$, for sets of integers, for a chosen value $n \geq 2$.

From this result, we immediately come to the following problem.

Problem 1. *Define a generic extension of* $\mathbf{L}$ *in which the* **Lightface Separation** *principle holds for classes* Π^1_{n+1} *and* Σ^1_{n+1}*, for* ***all*** $n \geq 2$*, for sets of integers.*

The intended solution is expected to be obtained on the basis of a suitable product of the forcing notions $\mathbb{Q}_{\mathbb{n}}$, $\mathbb{n} \geq 2$, defined in Section 6.

And we recall the following major problem.

Problem 2. *Given* $n \geq 2$*, define a generic extension of* $\mathbf{L}$ *in which the* **Separation** *principle holds for the classes* Σ^1_{n+1} *and* $\mathbf{\Sigma}^1_{n+1}$ *for* ***sets of reals****.*

The case of *sets of reals* in the **Separation** problem is more general, and obviously much more difficult, than the case sets of integers.

Author Contributions: Conceptualization, V.K. and V.L.; methodology, V.K. and V.L.; validation, V.K.; formal analysis, V.K. and V.L.; investigation, V.K. and V.L.; writing original draft preparation, V.K.; writing review and editing, V.K. and V.L.; project administration, V.L.; funding acquisition, V.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially supported by Russian Foundation for Basic Research RFBR grant number 20-01-00670.

Data Availability Statement: Not applicable. The study does not report any data.

Acknowledgments: We thank the anonymous reviewers for their thorough review and highly appreciate the comments and suggestions, which significantly contributed to improving the quality of the publication.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Lusin, N. *Leçons sur les Ensembles Analytiques et Leurs Applications*; Gauthier-Villars: Paris, France, 1930; p. XVI+328.
2. Novikov, P.S. On the consistency of some propositions of the descriptive theory of sets. *Am. Math. Soc. Transl.* **1963**, *29*, 51–89.
3. Moschovakis, Y.N. *Descriptive Set Theory*; Studies in Logic and the Foundations of Mathematics; North-Holland: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK, 1980; Volume 100, p. XII+637
4. Kechris, A.S. *Classical Descriptive Set Theory*; Springer: New York, NY, USA, 1995; p. xviii+402.
5. Kanovei, V.; Lyubetsky, V. Models of set theory in which separation theorem fails. *Izv. Math.* **2021**, *85*, 1181–1219. [CrossRef]
6. Lusin, N. Sur les ensembles analytiques. *Fund. Math.* **1927**, *10*, 1–95. [CrossRef]
7. Novikoff, P. Sur les fonctions implicites mesurables B. *Fundam. Math.* **1931**, *17*, 8–25. [CrossRef]
8. Novikoff, P. Sur la séparabilité des ensembles projectifs de seconde classe. *Fundam. Math.* **1935**, *25*, 459–466. [CrossRef]
9. Addison, J.W. Some consequences of the axiom of constructibility. *Fundam. Math.* **1959**, *46*, 337–357. [CrossRef]
10. Addison, J.W. Separation principles in the hierarchies of classical and effective descriptive set theory. *Fundam. Math.* **1959**, *46*, 123–135. [CrossRef]
11. Addison, J.W.; Moschovakis, Y.N. Some consequences of the axiom of definable determinateness. *Proc. Natl. Acad. Sci. USA* **1968**, *59*, 708–712. [CrossRef]
12. Martin, D.A. The axiom of determinateness and reduction principles in the analytical hierarchy. *Bull. Am. Math. Soc.* **1968**, *74*, 687–689. [CrossRef]
13. Steel, J.R. Determinateness and the separation property. *J. Symb. Log.* **1981**, *46*, 41–44. [CrossRef]
14. Steel, J.R. *The Core Model Iterability Problem*; Lecture Notes in Logic; Springer: Berlin, Germay, 1996; Volume 8, p. v+112.
15. Hauser, K.; Schindler, R.D. Projective uniformization revisited. *Ann. Pure Appl. Logic* **2000**, *103*, 109–153. [CrossRef]
16. Mathias, A.R.D. Surrealist landscape with figures (a survey of recent results in set theory). *Period. Math. Hung.* **1979**, *10*, 109–175. (The original preprint of this paper is known in typescript since 1968 under the title "A survey of recent results in set theory".) [CrossRef]
17. Harrington, L. The Constructible Reals Can Be Anything. Preprint Dated May 1974 with Several Addenda Dated up to October 1975: (A1) Models Where Separation Principles Fail, May 74; (A2) Separation without Reduction, April 75; (A3) The Constructible Reals Can Be (almost) Anything, Part II, May 75. Available online: http://iitp.ru/upload/userpage/247/74harr.pdf (accessed on 9 February 2022).
18. Jensen, R.B.; Solovay, R.M. Some applications of almost disjoint sets. In *Studies in Logic and the Foundations of Mathematics*; Bar-Hillel, Y., Ed.; North-Holland: Amsterdam, The Netherlands; London, UK , 1970; Volume 59, pp. 84–104.
19. Kanovei, V.; Lyubetsky, V. Counterexamples to countable-section Π^1_2 uniformization and Π^1_3 separation. *Ann. Pure Appl. Logic* **2016**, *167*, 262–283. [CrossRef]
20. Kanovei, V.; Lyubetsky, V. On the Δ^1_n problem of Harvey Friedman. *Mathematics* **2020**, *8*, 1477. [CrossRef]
21. Friedman, H. One hundred and two problems in mathematical logic. *J. Symb. Log.* **1975**, *40*, 113–129. [CrossRef]
22. Kanovei, V.; Lyubetsky, V. On the 'definability of definable' problem of Alfred Tarski. *Mathematics* **2020**, *8*, 2214. [CrossRef]
23. Dobrinen, N.; Friedman, S.D. Homogeneous iteration and measure one covering relative to HOD. *Arch. Math. Logic* **2008**, *47*, 711–718. [CrossRef]
24. Kunen, K. *Set Theory*; Studies in Logic; College Publications: London, UK, 2011; Volume 34, p. viii+401.
25. Gitman, V.; Hamkins, J.D.; Johnstone, T.A. What is the theory ZFC without power set? *Math. Log. Q.* **2016**, *62*, 391–406. [CrossRef]
26. Barwise, J. (Ed.) *Handbook of Mathematical Logic*; Studies in Logic and the Foundations of Mathematics; North-Holland: Amsterdam, The Netherlands, 1977; Volume 90, p. 1165.
27. Jech, T. *Set Theory*; The third millennium revised and expanded ed.; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2003; p. xiii+769. [CrossRef]
28. Kanovei, V.; Lyubetsky, V. Definable E_0 classes at arbitrary projective levels. *Ann. Pure Appl. Logic* **2018**, *169*, 851–871. [CrossRef]

29. Kanovei, V.; Lyubetsky, V. Definable minimal collapse functions at arbitrary projective levels. *J. Symb. Log.* **2019**, *84*, 266–289. [CrossRef]
30. Kanovei, V.; Lyubetsky, V. Non-uniformizable sets with countable cross-sections on a given level of the projective hierarchy. *Fundam. Math.* **2019**, *245*, 175–215. [CrossRef]