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Forward Order Law for the Reflexive Inner Inverse of Multiple Matrix Products †

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Abstract: The generalized inverse has numerous important applications in aspects of the theoretic research of matrices and statistics. One of the core problems of generalized inverse is finding the necessary and sufficient conditions for the reverse (or the forward) order laws for the generalized inverse of matrix products. In this paper, by using the extremal ranks of the generalized Schur complement, some necessary and sufficient conditions are given for the forward order law for $A_1\{1,2\}A_2\{1,2\}\ldots A_n\{1,2\} \subseteq (A_1A_2\ldots A_n)\{1,2\}$.

Keywords: generalized inverse; reflexive inner inverse; forward order law; maximal and minimal ranks; generalized schur complement

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1. Introduction

Throughout this paper, all matrices will be over the complex number field C. $C^{m\times n}$ and C^m denote the set of $m \times n$ complex matrices and m-dimensional complex vectors, respectively. For a matrix A in the set $C^{m\times n}$ of all $m \times n$ matrices over C, the symbols $r(A)$ and A^* denote the rank and the conjugate transpose of the matrix A, respectively. As usual, the identity matrix of order k is denoted by I_k, and the $m \times n$ matrix of all zero entries is denoted by $O_{m\times n}$ (if no confusion occurs, we will drop the subscript).

For various applications, we will introduce some generalized inverses of matrices. Let $A \in C^{m\times n}$ and $\eta \subset \{1,2,3,4\}$ be nonempty sets. If $X \in C^{n\times m}$ satisfies the following equations (i) for all $i \in \eta$:

$$(1)\ AXA = A;\ \ (2)\ XAX = X;\ \ (3)\ (AX)^* = AX;\ \ (4)\ (XA)^* = XA,$$

then X is said to be an η-inverse of A, which is denoted by $X = A^\eta$. The set of all η-inverses of A is denoted by $A\{\eta\}$. For example, X is called a $\{1\}$-inverse or an inner inverse of A if it satisfies Equation (1), which is always denoted by $X = A^{(1)} \in A\{1\}$. An $n \times m$ matrix X of the set $A\{1,2\}$ is called a $\{1,2\}$-inverse or a reflexive inner inverse of A and is denoted by $X = A^{(1,2)} \in A\{1,2\}$. The unique $\{1,2,3,4\}$-inverse of A is denoted by $X = A^{(1,2,3,4)} = A^\dagger$, which is also called the Moore Penrose inverse of A. As is well-known, each kind of η-inverse has its own properties and functions; see [1–4].

Theories and computations of the reverse (or the forward) order laws for generalized inverse are important in many branches of applied sciences, such as in non-linear control theory [2], matrix analysis [1,4], statistics [5,6], and numerical linear algebra [1,5,7]. Suppose that $A_i \in C^{m\times m}$, $i = 1, 2, \ldots, n$, and $b \in C^m$, the least squares problems (LS):

$$\min_{x\in C^m} \|(A_1A_2\ldots A_n)x - b\|_2$$

is used in many practical scientific problems, see [4–9]. If the above LS is consistent, then any solution x for the above LS can be expressed as $x = (A_1A_2\ldots A_n)^{(1,j,k)}b$, where $\{j,k\} \subseteq \{2,3,4\}$. For example, the minimum norm solution x has the form $x = (A_1A_2\ldots A_n)^{(1,4)}b$. The unique minimal norm least square solution x of the LS above is $x = (A_1A_2\ldots A_n)^{\dagger}b$.

One of the core problems with the LS above is identifying the conditions under which the following reverse order laws hold:

$$A_n^{(1,j,\ldots,k)}A_{n-1}^{(1,j,\ldots,k)}\ldots A_1^{(1,j,\ldots,k)} \subseteq (A_1A_2\ldots A_n)^{(1,j,\ldots,k)} \quad (1)$$

Another core problem with the LS above is identifying the conditions under which the following forward order laws hold:

$$A_1^{(1,j,\ldots,k)}A_2^{(1,j,\ldots,k)}\ldots A_n^{(1,j,\ldots,k)} \subseteq (A_1A_2\ldots A_n)^{(1,j,\ldots,k)} \quad (2)$$

The reverse order laws for the generalized inverse of multiple matrix products (1) yield a class of interesting problems that are fundamental in the theory of the generalized inverse of matrices; see [1,4–6]. As a hot topic in current matrix research, the necessary and sufficient conditions for the reverse order laws for the generalized inverse of matrix products are useful in both theoretical study and practical scientific computing; hence, this has attracted considerable attention and several interesting results have been obtained; see [10–23].

The forward order law for the generalized inverse of multiple matrix products (2) originally arose in the study of the inverse of multiple matrix Kronecker products; see [1,4]. Recently, Xiong et al. studied the forward order laws for some generalized inverses of multiple matrix products by using the maximal and minimal ranks of the generalized Schur complement; see [24–27]. To our knowledge, the forward order law for the reflexive inner inverse of multiple matrix products has not yet been studied in the literature. In this paper, by using the extremal ranks of the generalized Schur complement, we will provide some necessary and sufficient conditions for the forward order law:

$$A_1\{1,2\}A_2\{1,2\}\ldots A_n\{1,2\} \subseteq (A_1A_2\ldots A_n)\{1,2\}. \quad (3)$$

As we all know, the most widely used generalized inverses of matrices, such as M-P inverses, Drazin inverses, group inverses, etc., are some special $\{1,2\}$-inverses. Therefore, the forward order law for the $\{1,2\}$-inverse of a multiple matrix product studied in this paper is broad and general and contains the forward order laws for the above-mentioned generalized inverses.

The main tools of the later discussion are the following lemmas.

Lemma 1 ([1]). *Let $A \in C^{m\times n}$ and $X \in C^{n\times m}$. Then,*

$$X \in A\{1,2\} \Leftrightarrow AXA = A \ \text{and} \ r(X) \le r(A).$$

Lemma 2 ([28]). *Let $A \in C^{m\times n}$, $B \in C^{m\times k}$, $C \in C^{l\times n}$, and $D \in C^{l\times k}$. Then,*

$$\max_{A^{(1,2)}} r(D - CA^{(1,2)}B) = \min\left\{r(A) + r(D),\ r(C,\ D),\ r\begin{pmatrix} B \\ D \end{pmatrix},\ r\begin{pmatrix} A & B \\ C & D \end{pmatrix} - r(A)\right\},$$

where $A^{(1,2)} \in A\{1,2\}$.

Lemma 3 ([27]). *Let $A_i \in C^{m\times m}$, $i = 1, 2, \ldots, n$. Then,*

$$(n-1)m + r(A_1A_2\ldots A_n) \geq r(A_1) + r(A_2) + \ldots + r(A_n).$$

Lemma 4 ([29]). *Let A, B have suitable sizes. Then,*

$$r(A,\ B) \leq r(A) + r(B) \ \ and \ \ r(A,\ B) \geq max\{r(A), r(B)\}.$$

2. Main Results

In this section, by using the extremal ranks of the generalized Schur complement, we will give some necessary and sufficient conditions for the forward order law for the reflexive inner inverse of multiple matrix products (3).

Let

$$S_{A_1A_2\ldots A_n} = A_1A_2\ldots A_n - A_1A_2\ldots A_nX_1X_2\ldots X_nA_1A_2\ldots A_n, \tag{4}$$

where $A_i \in C^{m\times m}$, $X_i \in A_i\{1,2\}$, $i = 1, 2, \ldots, n$. From Lemma 1, we know that (3) holds if and only if:

$$S_{A_1A_2\ldots A_n} = 0 \ \ and \ \ r(X_1X_2\ldots X_n) \leq r(A_1A_2\ldots A_n), \tag{5}$$

hold for any $X_i \in A_i\{1,2\}$, $i = 1, 2, \ldots, n$, which are respectively equivalent to the following two identities:

$$\max_{X_1,X_2,\ldots,X_n} r(S_{A_1A_2\ldots A_n}) = 0 \tag{6}$$

and

$$\max_{X_1,X_2,\ldots,X_n} r(X_1X_2\ldots X_n) \leq r(A_1A_2\ldots A_n). \tag{7}$$

Hence, we can present the equivalent conditions for the forward order law (3) if the concrete expression of the maximal ranks involved in the identities in (6) and (7) are derived. The relative results are included in the following three theorems.

Theorem 1. *Let $A_i \in C^{m\times m}$, $X_i \in A_i\{1,2\}$, $i = 1, 2, \ldots, n$ and $S_{A_1A_2\ldots A_n}$ be as in (4). Then,*

$$\begin{aligned} & \max_{X_1,X_2,\ldots,X_n} r(S_{A_1A_2\ldots A_n}) \\ = \ & \min\{r(A_1A_2\ldots A_n), \\ & r(A_nA_{n-1}\ldots A_1 - A_1A_2\ldots A_n) + r(A_1A_2\ldots A_n) + (n-1)m - \sum_{l=1}^{n} r(A_l)\}. \end{aligned} \tag{8}$$

Proof. Suppose that $X_0 = I_m$. For $1 \leq i \leq n-1$, we first prove the following:

$$\begin{aligned} & \max_{\substack{X_{n-i} \\ 1\leq i\leq n-1}} r(A_nA_{n-1}\ldots A_{n-i+1} - A_1A_2\ldots A_nX_1X_2\ldots X_{n-i}) \\ = \ & \min\{r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1},\ \ A_nA_{n-1}\ldots A_{n-i+1}), \\ & \quad r(A_nA_{n-1}\ldots A_{n-i} - A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1}) + m - r(A_{n-i})\}. \end{aligned} \tag{9}$$

In fact, by Lemma 2, we have the equations below:

$$\begin{aligned} & \max_{X_{n-i}} r(A_nA_{n-1}\ldots A_{n-i+1} - A_1A_2\ldots A_nX_1X_2\ldots X_{n-i}) \\ = \ & \min\{r(A_{n-i}) + r(A_nA_{n-1}\ldots A_{n-i+1}), \\ & \quad r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1},\ \ A_nA_{n-1}\ldots A_{n-i+1}), \end{aligned}$$

$$\begin{aligned}
& r\begin{pmatrix} I_m \\ A_nA_{n-1}\ldots A_{n-i+1}\end{pmatrix}, \\
& r\begin{pmatrix} A_{n-i} & I_m \\ A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1} & A_nA_{n-1}\ldots A_{n-i+1}\end{pmatrix} - r(A_{n-i})\} \\
= \; & \min\{r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1},\ A_nA_{n-1}\ldots A_{n-i+1}), \\
& r(A_nA_{n-1}\ldots A_{n-i} - A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1}) + m - r(A_{n-i})\},
\end{aligned}$$

where the second equality holds, since by Lemma 4, we have:

$$\begin{aligned}
& r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1},\ A_nA_{n-1}\ldots A_{n-i+1}) \\
& \le r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1}) + r(A_nA_{n-1}\ldots A_{n-i+1}) \\
& \le r(A_{n-i}) + r(A_nA_{n-1}\ldots A_{n-i+1})
\end{aligned}$$

and

$$r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1},\ A_nA_{n-1}\ldots A_{n-i+1}) \le m = r\begin{pmatrix} I_m \\ A_nA_{n-1}\ldots A_{n-i+1}\end{pmatrix}.$$

More specifically, when $i = n-1$, we have the following:

$$\begin{aligned}
& \max_{X_1} r(A_nA_{n-1}\ldots A_2 - A_1A_2\ldots A_nX_1) \\
= \; & \min\{r(A_1A_2\ldots A_n,\ A_nA_{n-1}\ldots A_2), \\
& r(A_nA_{n-1}\ldots A_1 - A_1A_2\ldots A_n) + m - r(A_1)\}.
\end{aligned} \tag{10}$$

We now prove (8). Again, by Lemma 2, we have the following equations:

$$\begin{aligned}
& \max_{X_n} r(S_{A_1A_2\ldots A_n}) \\
= \; & \max_{X_n} r(A_1A_2\ldots A_n - A_1A_2\ldots A_nX_1X_2\ldots X_nA_1A_2\ldots A_n) \\
= \; & \min\{r(A_n) + r(A_1A_2\ldots A_n), \\
& r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-1},\ A_1A_2\ldots A_n), \\
& r\begin{pmatrix} A_1A_2\ldots A_n \\ A_1A_2\ldots A_n\end{pmatrix}, \\
& r\begin{pmatrix} A_n & A_1A_2\ldots A_n \\ A_1A_2\ldots A_nX_1X_2\ldots X_{n-1} & A_1A_2\ldots A_n\end{pmatrix} - r(A_n)\} \\
= \; & \min\{r(A_1A_2\ldots A_n), \\
& r(A_n - A_1A_2\ldots A_nX_1X_2\ldots X_{n-1}) + r(A_1A_2\ldots A_n) - r(A_n)\},
\end{aligned} \tag{11}$$

where the third equality holds, since by Lemma 4, we have:

$$r\begin{pmatrix} A_1A_2\ldots A_n \\ A_1A_2\ldots A_n\end{pmatrix} = r(A_1A_2\ldots A_n) \le r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-1},\ A_1A_2\ldots A_n)$$

and

$$r\begin{pmatrix} A_1A_2\ldots A_n \\ A_1A_2\ldots A_n\end{pmatrix} = r(A_1A_2\ldots A_n) \le r(A_n) + r(A_1A_2\ldots A_n).$$

Combining (9) with (11) , we obtain the following equations:

$$\begin{aligned}
& \max_{X_{n-1},X_n} r(S_{A_1A_2\ldots A_n}) \\
= \; & \min\{r(A_1A_2\ldots A_n), \\
& \max_{X_{n-1}} r(A_n - A_1A_2\ldots A_nX_1X_2\ldots X_{n-1}) + r(A_1A_2\ldots A_n) - r(A_n)\} \\
= \; & \min\{r(A_1A_2\ldots A_n), \\
& r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-2},\ A_n) + r(A_1A_2\ldots A_n) - r(A_n), \\
& r(A_nA_{n-1} - A_1A_2\ldots A_nX_1X_2\ldots X_{n-2}) + r(A_1A_2\ldots A_n) + m - r(A_{n-1}) - r(A_n)\}
\end{aligned}$$

$$= \min\{r(A_1A_2\ldots A_n),\ r(A_nA_{n-1} - A_1A_2\ldots A_nX_1X_2\ldots X_{n-2}) + r(A_1A_2\ldots A_n) + m - r(A_{n-1}) - r(A_n)\},$$

where the third equality holds, since by Lemma 4, we have:

$$r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-2},\ \ A_n) \geq r(A_n).$$

In general, for $1 \leq i \leq n-2$, we have the equations below:

$$\begin{aligned}&\max_{\substack{X_{n-i},X_{n-i+1},\ldots,X_n\\1\leq i\leq n-2}} r(S_{A_1A_2\ldots A_n})\\ =\ &\min\{r(A_1A_2\ldots A_n),\\ &r(A_nA_{n-1}\ldots A_{n-i} - A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1}) + r(A_1A_2\ldots A_n) + im - \sum_{l=n-i}^{n} r(A_l)\}.\end{aligned} \tag{12}$$

Equation (12) can be proved by using induction on i. In fact, for $i = 1$, the statement in (12) is proved. Assuming the statement (12) is true for $i-1$, that is:

$$\begin{aligned}&\max_{X_{n-i+1},X_{n-i+2},\ldots,X_n} r(S_{A_1A_2\ldots A_n})\\ =\ &\min\{r(A_1A_2\ldots A_n),\\ &r(A_nA_{n-1}\ldots A_{n-i+1} - A_1\ldots A_nX_1\ldots X_{n-i}) + r(A_1\ldots A_n) + (i-1)m - \sum_{l=n-i+1}^{n} r(A_l)\}.\end{aligned} \tag{13}$$

We now prove that (12) is also true for i. By (9) and (13), we have the equations below:

$$\begin{aligned}&\max_{X_{n-i},X_{n-i+1},\ldots,X_n} r(S_{A_1A_2\ldots A_n})\\ =\ &\min\{r(A_1A_2\ldots A_n),\\ &\ \max_{X_{n-i}} r(A_n\ldots A_{n-i+1} - A_1\ldots A_nX_1\ldots X_{n-i}) + r(A_1\ldots A_n) + (i-1)m - \sum_{l=n-i+1}^{n} r(A_l)\}\\ =\ &\min\{r(A_1A_2\ldots A_n),\\ &\quad r(A_1\ldots A_nX_1\ldots X_{n-i-1},\ \ A_n\ldots A_{n-i+1}) + r(A_1\ldots A_n) + (i-1)m - \sum_{l=n-i+1}^{n} r(A_l),\\ &\quad r(A_n\ldots A_{n-i} - A_1\ldots A_nX_1\ldots X_{n-i-1}) + m + r(A_1\ldots A_n) - r(A_{n-i}) + (i-1)m\\ &\quad - \sum_{l=n-i+1}^{n} r(A_l)\}.\end{aligned}$$

From Lemma 4, we have the following:

$$r(A_1A_2\ldots A_nX_1X_2\ldots X_{n-i-1},\ \ A_nA_{n-1}\ldots A_{n-i+1}) \geq r(A_nA_{n-1}\ldots A_{n-i+1})$$

and from Lemma 3, we have:

$$r(A_nA_{n-1}\ldots A_{n-i+1}) + (i-1)m \geq r(A_{n-i+1}) + r(A_{n-i+2}) + \ldots + r(A_n).$$

Then, we recognize that (12) holds, that is, for $1 \leq i \leq n-2$:

$$\begin{aligned}&\max_{X_{n-i},X_{n-i+1},\ldots,X_n} r(S_{A_1A_2\ldots A_n})\\ =\ &\min\{r(A_1A_2\ldots A_n),\\ &\quad r(A_n\ldots A_{n-i} - A_1\ldots A_nX_1\ldots X_{n-i-1}) + r(A_1A_2\ldots A_n) + im - \sum_{l=n-i}^{n} r(A_l)\}.\end{aligned}$$

When $i = n-2$, we get the following from (12):

$$\begin{aligned}
& \max_{X_2,X_3,\cdots,X_n} r(S_{A_1A_2\ldots A_n}) \\
= \; & \min\{r(A_1A_2\ldots A_n), \\
& r(A_nA_{n-1}\ldots A_2 - A_1\ldots A_nX_1) + r(A_1\ldots A_n) + (n-2)m - \sum_{l=2}^{n} r(A_l)\}. \qquad (14)
\end{aligned}$$

Hence, by (10) and (14), we have:

$$\begin{aligned}
& \max_{X_1,X_2,\cdots,X_n} r(S_{A_1A_2\ldots A_n}) \\
= \; & \min\{r(A_1A_2\ldots A_n), \\
& \quad \max_{X_1} r(A_n\ldots A_2 - A_1\ldots A_nX_1) + r(A_1\ldots A_n) + (n-2)m - \sum_{l=2}^{n} r(A_l)\} \\
= \; & \min\{r(A_1A_2\ldots A_n), \\
& \quad r(A_1A_2\ldots A_n,\;\; A_nA_{n-1}\ldots A_2) + r(A_1\ldots A_n) + (n-2)m - \sum_{l=2}^{n} r(A_l), \\
& \quad r(A_n\ldots A_1 - A_1\ldots A_n) - r(A_1) + m + r(A_1\ldots A_n) + (n-2)m - \sum_{l=2}^{n} r(A_l)\} \\
= \; & \min\{r(A_1A_2\ldots A_n), \\
& \quad r(A_nA_{n-1}\ldots A_1 - A_1A_2\ldots A_n) + r(A_1A_2\ldots A_n) + (n-1)m - \sum_{l=1}^{n} r(A_l)\},
\end{aligned}$$

where the third equality holds, since by Lemma 4, we have:

$$r(A_1A_2\ldots A_n,\;\; A_nA_{n-1}\ldots A_2) \geq r(A_nA_{n-1}\ldots A_2)$$

and

$$r(A_nA_{n-1}\ldots A_2) + (n-2)m \geq \sum_{l=2}^{n} r(A_l).$$

□

The next theorem gives the expression in the ranks of the known matrices for:

$$\max_{X_n,X_{n-1},\cdots,X_1} r(X_1X_2\ldots X_n),$$

where X_i varies over $A_i\{1,2\}$ for $i = 1,2,\cdots,n$.

Theorem 2. *Let $A_i \in C^{m\times m}$, $X_i \in A_i\{1,2\}$, $i = 1,2,\cdots,n$. Then,*

$$\max_{X_n,X_{n-1},\cdots,X_1} r(X_1X_2\ldots X_n) = \min\{r(A_1), r(A_2), \cdots, r(A_n)\}. \qquad (15)$$

Proof. We will divide the proof of Theorem 2 into two parts: first, $n = 2$; second, $n \geq 3$. When $n = 2$, according to Lemma 2, with $A = A_1$, $B = X_2$, $C = I_m$, and $D = O$, we have the following equations:

$$\begin{aligned}
& \max_{X_1} r(X_1X_2) \\
= \; & \min\{r(A_1),\;\; r(I_m,\;\; O),\;\; r\begin{pmatrix} X_2 \\ O \end{pmatrix},\;\; r\begin{pmatrix} A_1 & X_2 \\ I_m & O \end{pmatrix} - r(A_1)\} \\
= \; & \min\{r(A_1),\;\; m,\;\; r(X_2),\;\; r(X_2) + m - r(A_1)\} \\
= \; & \min\{r(A_1),\;\; r(X_2)\}. \qquad (16)
\end{aligned}$$

Since $X_2 \in A_2\{1,2\}$, then $r(X_2) = r(A_2)$. Thus, by (16), we have the equation below:

$$\max_{X_2, X_1} r(X_1 X_2) = \min\{r(A_1), r(A_2)\}, \tag{17}$$

i.e., Theorem 2 is true when $n = 2$.

When $n \geq 3$, by Lemma 2, with $A = A_1$, $B = X_2 X_3 \ldots X_n$, $C = I_m$, and $D = O$, we have:

$$\begin{aligned}
&\max_{X_1} r(X_1 X_2 \ldots X_n) \\
= \; &\min\{r(A_1),\; r(I_m,\, O),\; r\begin{pmatrix} X_2 X_3 \ldots X_n \\ O \end{pmatrix},\; r\begin{pmatrix} A_1 & X_2 X_3 \ldots X_n \\ I_m & O \end{pmatrix} - r(A_1)\} \\
= \; &\min\{r(A_1),\; m,\; r(X_2 X_3 \ldots X_n),\; r(X_2 X_3 \ldots X_n) + m - r(A_1)\} \\
= \; &\min\{r(A_1),\; r(X_2 X_3 \ldots X_n)\}.
\end{aligned} \tag{18}$$

Again, by Lemma 2, with $A = A_2$, $B = X_3 X_4 \ldots X_n$, $C = I_m$, and $D = O$, we have:

$$\begin{aligned}
&\max_{X_2, X_1} r(X_1 X_2 \ldots X_n) \\
= \; &\min\{r(A_1),\; \max_{X_2} r(X_2 X_3 \ldots X_n)\} \\
= \; &\min\{r(A_1), \\
&\quad \min\{r(A_2),\; r(I_m,\, O),\; r\begin{pmatrix} X_3 X_4 \ldots X_n \\ O \end{pmatrix},\; r\begin{pmatrix} A_2 & X_3 X_4 \ldots X_n \\ I_m & O \end{pmatrix} - r(A_2)\}\} \\
= \; &\min\{r(A_1),\; \min\{r(A_2),\; m,\; r(X_3 X_4 \ldots X_n),\; r(X_3 X_4 \ldots X_n) + m - r(A_2)\}\} \\
= \; &\min\{r(A_1),\; r(A_2),\; r(X_3 X_4 \ldots X_n)\}.
\end{aligned} \tag{19}$$

We claim that for $2 \leq i \leq n-1$:

$$\begin{aligned}
&\max_{X_i, X_{i-1}, \ldots, X_1} r(X_1 X_2 \ldots X_n) \\
= \; &\min\{r(A_1),\; r(A_2),\; \ldots,\; r(A_i),\; r(X_{i+1} X_{i+2} \ldots X_n)\}.
\end{aligned} \tag{20}$$

Equation (20) can be proved by using induction on i. In fact, for $i = 2$, the statement in (20) has been proved. Assuming the statement in (20) is true for $i - 1$, that is:

$$\max_{X_{i-1}, X_{i-2}, \ldots, X_1} r(X_1 X_2 \ldots X_n) = \min\{r(A_1),\, r(A_2),\; \ldots,\; r(A_{i-1}),\; r(X_i X_{i+1} \ldots X_n)\}. \tag{21}$$

We now prove that (20) is also true for i. By (21) and Lemma 2, with $A = A_i$, $B = X_{i+1} X_{i+2} \ldots X_n$, $C = I_m$, and $D = O$, we have the following:

$$\begin{aligned}
&\max_{X_i, X_{i-1}, \ldots, X_1} r(X_1 X_2 \ldots X_n) \\
= \; &\min\{r(A_1),\, r(A_2),\; \ldots,\; r(A_{i-1}),\; \max_{X_i} r(X_i X_{i+1} \ldots X_n)\} \\
= \; &\min\{r(A_1),\, r(A_2),\; \ldots,\; r(A_{i-1}), \\
&\quad \min\{r(A_i),\, r(I_m,\, O),\, r\begin{pmatrix} X_{i+1} X_{i+2} \ldots X_n \\ O \end{pmatrix},\, r\begin{pmatrix} A_i & X_{i+1} X_{i+2} \ldots X_n \\ I_m & O \end{pmatrix} - r(A_i)\}\} \\
= \; &\min\{r(A_1),\, r(A_2),\; \ldots,\; r(A_{i-1}), \\
&\quad \min\{r(A_i),\, m,\, r(X_{i+1} X_{i+2} \ldots X_n),\, r(X_{i+1} X_{i+2} \ldots X_n) + m - r(A_i)\}\} \\
= \; &\min\{r(A_1),\, r(A_2),\; \ldots,\; r(A_{i-1}),\; \min\{r(A_i),\, r(X_{i+1} X_{i+2} \ldots X_n)\}\} \\
= \; &\min\{r(A_1),\, r(A_2),\; \ldots,\; r(A_{i-1}),\, r(A_i),\; r(X_{i+1} X_{i+2} \ldots X_n)\}.
\end{aligned}$$

When $i = n - 1$, from (20), we have the following equations:

$$\max_{X_{n-1}, X_{n-2}, \ldots, X_1} r(X_1 X_2 \ldots X_n) = \min\{r(A_1),\, r(A_2),\; \ldots,\; r(A_{n-1}),\, r(X_n)\}. \tag{22}$$

Since $X_n \in A_n\{1,2\}$, then $r(X_n) = r(A_n)$. Thus, by (22), we have the equation below:

$$\max_{X_n, X_{n-1}, \dots, X_1} r(X_1 X_2 \dots X_n) = \min\{r(A_1),\ r(A_2),\ \dots,\ r(A_n)\}, \tag{23}$$

i.e., Theorem 2 is true when $n \geq 3$. $\square$

Based on Theorem 1 and 2, we can immediately obtain the main result of this paper.

Theorem 3. *Let $A_i \in C^{m\times m}$, $i = 1, 2, \dots, n$. Then, the following statements are equivalent:*

(1) $A_1\{1,2\}A_2\{1,2\}\dots A_n\{1,2\} \subseteq (A_1A_2\dots A_n)\{1,2\}$;

(2) $\min\{r(A_1\dots A_n),\ r(A_n\dots A_1 - A_1\dots A_n) + r(A_1\dots A_n) + (n-1)m - \sum_{l=1}^{n} r(A_l)\} = 0$ *and* $\min\{r(A_1),\ r(A_2),\ \dots,\ r(A_n)\} \leq r(A_1A_2\dots A_n)$;

(3) $A_1A_2\dots A_n = O$ *or* $r(A_1\dots A_n - A_n\dots A_1) + r(A_1\dots A_n) + (n-1)m - \sum_{l=1}^{n} r(A_l) = 0$ *and* $\min\{r(A_1),\ r(A_2),\ \dots,\ r(A_n)\} \leq r(A_1A_2\dots A_n)$;

(4) $A_1A_2\dots A_n = O$ *or* $A_1\dots A_n = A_n\dots A_1$ *and* $r(A_1\dots A_n) + (n-1)m = \sum_{l=1}^{n} r(A_l)$ *and* $\min\{r(A_1),\ r(A_2),\ \dots,\ r(A_n)\} \leq r(A_1A_2\dots A_n)$;

(5) $A_1A_2\dots A_n = O$ *or* $A_1\dots A_n = A_n\dots A_1$ *and* $r(A_1\dots A_{n-i}) + (n-i-1)m = \sum_{l=1}^{n-i} r(A_l), i = 0, \dots, n-2,$ *and* $\min\{r(A_1),\ r(A_2),\ \dots,\ r(A_n)\} \leq r(A_1A_2\dots A_n)$.

Proof. $(1) \Leftrightarrow (2)$. From Lemma 1, we know that (3) holds if and only if Equations (6) and (7) hold. Then, according to Equation (8) in Theorem 1 and Equation (15) in Theorem 2, we have $(1) \Leftrightarrow (2)$ in Theorem 3.

$(2) \Leftrightarrow (3)$. In fact, $r(A) = 0$ if and only if $A = O$, so $(2) \Leftrightarrow (3)$ is obvious.

$(3) \Leftrightarrow (4)$. Since

$$r(A_1A_2\dots A_n - A_nA_{n-1}\dots A_1) \geq 0$$

and from Lemma 3, we have:

$$r(A_1A_2\dots A_n) + (n-1)m \geq \sum_{l=1}^{n} r(A_l),$$

it is easy to obtain $(3) \Leftrightarrow (4)$.

$(4) \Leftrightarrow (5)$. In fact, $(5) \Rightarrow (4)$ is obvious. We now show $(4) \Rightarrow (5)$. In fact, for the case of $i = 0$, the results in (4) are actually for (5). Assuming (5) holds for $i-1$, where $1 \leq i \leq n-2$, i.e.,:

$$r(A_1A_2\dots A_{n-i+1}) + (n-i)m = \sum_{l=1}^{n-i+1} r(A_l). \tag{24}$$

We now prove that (5) is also true for i. Based on Lemma 3, we know that:

$$r(A_1A_2\dots A_{n-i+1}) + m \geq r(A_1A_2\dots A_{n-i}) + r(A_{n-i+1}). \tag{25}$$

From (24) and (25), we have the following:

$$\sum_{l=1}^{n-i} r(A_l) \geq r(A_1A_2\dots A_{n-i}) + (n-i-1)m. \tag{26}$$

On the other hand, again by Lemma 3, we know the following:

$$r(A_1A_2\ldots A_{n-i}) + (n-i-1)m \geq \sum_{l=1}^{n-i} r(A_l). \tag{27}$$

Hence, from (26) and (27), we have:

$$\sum_{l=1}^{n-i} r(A_l) = r(A_1A_2\ldots A_{n-i}) + (n-i-1)m. \tag{28}$$

This means that (4) $\Rightarrow$ (5) hold. □

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