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On the Semi-Group Property of the Perpendicular Bisector in a Normed Space

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Abstract: Let (X,d) be a metric linear space and $a \in X$. The point a divides the space into three sets: $H_a = \{x \in X: d(0,x) < d(x,a)\}$, $M_a = \{x \in X: d(0,x) = d(x,a)\}$ and $L_a = \{x \in X: d(0,x) > d(x,a)\}$. If the distance is generated by a norm, H_a is called *the Leibnizian halfspace of a*, M_a *is the perpendicular bisector of the segment* 0,a and L_a is the remaining set $L_a = X\backslash(H_a \cup M_a)$. It is known that the perpendicular bisector of the segment $[0,a]$ is an affine subspace of X for all $a \in X$ if, and only if, X is an inner product space, that is, if and only if the norm is generated by an inner product. In this case, it is also true that if $x,y \in L_a \cup M_a$, then $x + y \in L_a \cup M_a$. Otherwise written, the set $L_a \cup M_a$ is a semi-group with respect to addition. We investigate the problem: *for what kind of norms in X the pair* $(L_a \cup M_a,+)$ *is a semi-group for all* $a \in X$? In that case, we say that "$(X, \|.\|)$ *has the semi-group* property" or that "the norm $\|.\|$ has the semi-group property". This is a threedimensional property, meaning that if all the three-dimensional subspaces of X have it, then X also has it. We prove that for two-dimensional spaces, $(L_a,+)$ is a semi-group for any norm, that $(X, \|.\|)$ has the semi-group property if, and only if, the norm is strictly convex, and, in higher dimensions, the property fails to be true even if the norm is strictly convex. Moreover, studying the L^p norms in higher dimensions, we prove that the semi-group property holds if, and only if, $p = 2$. This fact leads us to the conjecture that in dimensions greater than three, the semi-group property holds if, and only if, X is an inner-product space.

Keywords: normed space; perpendicular bisector; inner product

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1. Introduction

Let (X, $\| \;\|$) be a real normed space.

In [1], the authors were interested in the following property of X

For every x,y,z $\in$ X, *the inequalities*

(**A**) *For every x,y,z* $\in$ X, *the inequalities*

$\|x\| \geq \|y + z\|$, $\|y\| \geq \|y + z\|$, $\|z\| \geq \|x + y\|$, *imply the equalities*

$\|x\| = \|y + z\|$, $\|y\| = \|y + z\|$, $\|z\| = \|x + y\|$

The typical examples of spaces with this property are the inner product spaces. Indeed, if there is an inner product $<\cdot,\cdot>$ such that $\|x\|^2 = <x,x>$, then the inequalities in (A) become $\|x\|^2 \geq \|y\|^2 + \|z\|^2 + 2<x,y>$, $\|y\|^2 \geq \|x\|^2 + \|z\|^2 + 2<x,z>$, $\|z\|^2 \geq \|y\|^2 + \|x\|^2 + 2<z,y>$; if we add them, we obtain $\|x + y + z\| \leq 0 \Leftrightarrow x + y + z = 0$. In this case, we obtain even more than the simple equalities $\|x\| = \|y + z\|$, $\|y\| = \|z + x\|$, $\|z\| = \|x + y\|$, namely, $x = -(y + z)$, $y = -(x + z)$, $z = -(y + x)$.

If we denote $a = x + y + z$, we can write property (**A**) as

(**A**) *For every x,y,a* $\in$ X, *the inequalities*

$\|x\| \geq \|a - x\|$, $\|y\| \geq \|a - y\|$, $\|a - x - y\| \geq \|x + y\|$ *imply the equalities*

$\|x\| = \|a - x\|$, $\|y\| = \|a - y\|$, $\|a - x - y\| = \|x + y\|$

Written in this form, the property (A) receives a geometric flavor.

Notation. *Let* $a \in X$. *Denote by* L_a, M_a, H_a *the sets*

$$L_a = \{x \in X \mid \|x - a\| < \|x\|\} \quad (1)$$

$$M_a = \{x \in X \mid \|x - a\| = \|x\|\} \quad (2)$$

$$H_a = \{x \in X \mid \|x - a\| > \|x\|\} \quad (3)$$

Note that M_a is **the perpendicular bisector** of the segment $[0,a]$, L_a is the set of points closer to a and H_a is the set of points from X closer to 0. Some authors [2] call the set H_a **the Leibnizian halfspace of *a*.** Horvath ([2,3]) writes $H_{0,a}$ instead of H_a and $H_{a,0}$ instead of M_a

In the sequel, we always assume that $a \neq 0$, since $a = 0$ is nonsensical.

Using these notations, the property (A) becomes

(A) *For any* $0 \neq a \in X$, *if* $x \in L_a \cup M_a$, $y \in L_a \cup M_a$, $x + y \in H_a \cup M_a$ *then* $x,y,x + y \in M_a$

A weaker form of this property is

(B) *For any* $0 \neq a \in X$, *if* $x \in L_a \cup M_a$, $y \in L_a \cup M_a$, $x + y \in H_a \cup M_a$ *then* $x + y \in M_a$

However, is it possible that $x,y,x + y \in M_a$? In an inner-product space, this is not possible. In this case, $x \in L_a \cup M_a$, $y \in L_a \cup M_a \Rightarrow x + y \in L_a$.

In other words, in an inner-product space, $(L_a \cup M_a,+)$ is a semigroup. Indeed, this is obvious: squaring the inequalities $\|x\| \geq \|a - x\|$, $\|y\| \geq \|a - y\|$and adding them, we obtain $\|x + y\|^2 \geq 2\|a\|^2 - 2<a,x + y> + \|x + y\|^2 \geq \|x + y - a\|^2$, meaning that $x + y$ is in L_a.

2. The Main Result

A norm space is ***strictly convex*** ([1–4]) if and only if the unity ball $B_1 = \{x \in X\colon \|x\| \leq 1\}$ is a strict convex set or, equivalently, the equality

$\|x + y\| = \|x\| + \|y\|$ can hold if and only if $x = 0$ or $y = 0$ or $y = \lambda x$ for some $\lambda > 0$. An equivalent definition is that the norm is a strict convex function, i.e., for $x \neq y$, the equality

$\|(1 - \lambda)x + \lambda y\| = (1 - \lambda)\|x\| + \lambda\|y\|$ can hold if, and only if, $\lambda \in \{0,1\}$. Here $\lambda \in [0,1]$.

Let us denote by $h_a : X \to \Re$ the mapping $h_a(x) = \|x - a\| - \|x\|$. Then

$H_a = \{h_a > 0\}$

$M_a = \{h_a = 0\}$

$L_a = \{h_a < 0\}$.

As the function h_a is continuous, M_a is a closed set and H_a, L_a are open.

Definition 1. *Say that* $(X, \|.\|)$ *has the* ***semigroup property*** (*or that the norm has the semigroup property*) *if it satisfies the condition*

(M) $(L_a \cup M_a,+)$ is a semi-group for every $a \in X$

or, equivalently,

(M) $(L_a \cup M_a) + (L_a \cup M_a) \subseteq (L_a \cup M_a)$.

Explicitly, the semi group property means that

$\|x\| \geq \|x - a\|,\|y\| \geq \|y - a\| \Rightarrow \|x + y\| \geq \|x + y - a\|$ or any $x,y,a \in X$

We shall also consider the following similar property

(M°) $(L_a,+)$ *is a semigroup*

Proposition 1.

(i) *The property* (M) *implies the property* (B).

(ii) *A space X has the properties* (A), (B), (M) *or* (M°) *if, and only if, all three-dimensional subspaces of X have it.*

Proof.

(i) Obvious. If $L_a \cup M_a$ is a semigroup, then $x \in L_a \cup M_a$, $y \in L_a \cup M_a \Rightarrow x + y \in L_a \cup M_a$. Thus, if $x \in L_a \cup M_a$, $y \in L_a \cup M_a$, $x + y \in H_a \cup M_a$, then $x + y \in (H_a \cup M_a) \cap (H_a \cup M_a) = M_a$.

(ii) For instance, if X has the property (A) and $Y = \text{span}(\{x,y,z\})$, then it obviously has the property (A) as well. Conversely, if we know that Y has the property (A), we know that X also has it. □

Remark 1. *Thus, the semigroup property is geometric; it deals with the shape of the unity ball in a three-dimensional subspace of X. In the case of inner product spaces, this is an ellipsoid. In fact, the inner product spaces satisfy a stronger assumption, namely*

(C) $\|x\| \geq \|x - a\|$, $\|y\| \geq \|y - a\|$, $\|x + y\| = \|x + y - a\| \Rightarrow a = 0$

Of course, (C) *implies* (A).

Any norm space has the following properties:

Proposition 2. *Let* X *be any norm space and* a ∈ X, a ≠ 0.

(i) *If* $\|x\| \geq \|x - a\|$ *and* $0 \leq t \leq 1$ *then* $\|x\| \geq \|x - ta\|$.
(ii) *If* $x \in L_a$ *and* $t \geq 1$ *then* $tx \in L_a$.
If $x \in M_a$ *and* $0 \leq t \leq 1$, *then* $tx \in M_a \cup H_a$.
If $x \in M_a$, $t \geq 1$ *then* $tx \in M_a \cup H_a$.
(iii) *If* $x \in L_a$, *there exists* $0 \leq t \leq 1$ *such that* $tx \in M_a$
(iv) *If, moreover, the function X is strictly convex, then* $x \in M_a$, $tx \in M_a$ *if, and only if,* $t = 1$. *In other words, if* $x \in M_a$ *and* $t > 1$, *then* $tx \notin M_a$.
(v) *If X is strictly convex, the closures are* $Cl(H_a) = H_a \cup M_a$ *and* $Cl(L_a) = L_a \cup M_a$.

Proof.

(i) Let $f: \Re \to \Re$ be defined by $f(t) = \|x\| - \|x - ta\|$. The function f is concave, $f(0) = 0$ and $f(1) \geq 0$. Thus $t \in [0,1] \Rightarrow f(t) \geq 0$ or, explicitly, $\|x\| \geq \|x - ta\|$
(ii) Obvious if we write the inequality $\|x\| \geq \|x - ta\|$ as $\|\frac{1}{t}x\| \geq \|\frac{1}{t}x - a\|$. If $t \in (0,1]$, then $\frac{1}{t} \geq 1$. If $x \in M_a$, then $f(0) = f(1) = 0 \Rightarrow f(s) \leq 0\ \forall\ s \in (-\infty,0) \cup (1,\infty) \Leftrightarrow \|x\| \leq \|x - ta\|$ $\forall\ t \in (0,1)$. Thus $\|tx\| \leq \|tx - a\|\ \forall\ t \in (0,1)$.
(iii) The function f from (i) has the property that $f(\infty) = -\infty$. As $f(1) \geq 0$, there must be $t \geq 1$ such that $f(t) = 0 \Leftrightarrow \|x\| = \|x - ta\| \Leftrightarrow \frac{1}{t}x \in M_a$
(iv) If the norm is strictly convex, then the function f from 1 is strictly concave. Let $s = \frac{1}{t}$. As $x \in M_a \Leftrightarrow f(0) = f(1) = 0$, the strict concavity implies $s \in (0,1) \Rightarrow f(s) > 0 \Leftrightarrow \|x\| > \|x - sa\|$ or $\|tx\| > \|tx - a\|$.
(v) The inclusion $\mathrm{Cl}(H_a) \subseteq H_a \cup M_a$ holds in any norm space. The problem is to check that $M_a \subseteq \mathrm{Cl}(H_a)$, and this is not true in general. However, if the norm $\|.\|$ is strictly convex, and $x \in M_a$, then the points $t_n x$ belong to H_a if $t_n > 1$. Now, it is obvious that if $t_n \downarrow 1$, then $t_n x \to x$ hence $x \in \mathrm{Cl}(H_a)$. The equality $\mathrm{Cl}(L_a) = L_a \cup M_a$ has the same proof. □

The following result also holds in any norm space and simplifies the issue:

Proposition 3. *If* $(X, \|\cdot\|)$ *has the property* (M) and T:X → X *is a bijective linear operator then* $(X, \|\cdot\|_T)$ *has the property* (M), *too, where the norm is defined by*

$\|x\|_T = \|Tx\|$

Moreover, if $(X, \|\cdot\|)$ *is strictly convex, then* $(X, \|\cdot\|_T)$ *is strictly convex, too.*

Proof. Let $H_a^{(T)} = \{x \in X: \|x\|_T > \|x - a\|_T\}$, $M_a^{(T)} = \{x \in X: \|x\|_T = \|x - a\|_T\}$ and $L_a^{(T)} = \{x \in X: \|x\|_T = \|x - a\|_T\}$.

We claim that $H_a^{(T)} = T^{-1}(H_{Ta})$, $M_a^{(T)} = T^{-1}(M_{Ta})$ and $L_a^{(T)} = T^{-1}(L_{Ta})$.

Indeed, $H_a^{(T)} = \{x \in X: \|Tx\| < \|Tx - Ta\|\} \Leftrightarrow T(H_a^{(T)}) = \{Tx: \|Tx\| < \|Tx - Ta\|\} = \{y \in X: \|y\| < \|y - a\|\} = H_a$ and the same holds for the other two sets. The property (M) says that $H_a \cup M_a$ is a semigroup for any $a \in X$. Then $H_{Ta} \cup M_{Ta}$ is also a semigroup. it is obvious that if $H \subseteq X$ is a semigroup and $U:X \to X$ is a linear operator, then $U(H)$ is a semigroup, too. Therefore $H_a^{(T)} \cup M_a^{(T)} = T^{-1}(H_{Ta} \cup M_{Ta})$ is a semigroup.

The last assertion is obvious. □

Example 1. *If* $\Re^2$ *endowed with some norm that has the property* (M), *the same holds for* $\Re^2$ *endowed with the norm* $\|(x_1, x_2)\|^* = \|(ax_1 + bx_2, cx_1 + dx_2)\|$ *provided that* $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Now we can state our main result.

Theorem 1. *Let* $X = \Re^2$ *endowed with some norm* $\|\cdot\|$. *Let* $a \in X$. *Then*

(i) $L_a + L_a \subseteq L_a$
(ii) *If X is strictly convex, then* $(L_a \cup M_a) + (L_a \cup M_a) \subseteq L_a \cup M_a$
(iii) *Conversely, if the norm has the semi-group property, then it is strictly convex.*

Thus, in the two-dimensional case, the semi-group property of a norm is equivalent with its strict convexity.

Proof. For the sake of a better understanding, the proof is divided into several steps. □

Step 1. There is no restriction on considering a = (0,1).
Indeed, if $a = (a_1, a_2)$, we can choose a linear operator T such that Ta = (0,1). Next, we replace the norm $\|.\|$ with the norm $\|\cdot\|_T$ and use Proposition 3.
Step 2. The two-dimensional norms have the following useful property (although it is well known, we prove it here since we did not find appropriate references to it in our research and it can be used to construct many norms on the plane):

Lemma 1. *For any norm on a two dimensional space there exists a convex function* $f\colon \Re \to (0,\infty)$, *such that*

$$\lim_{t\to\infty}\frac{f(t)}{t} = \lim_{t\to-\infty}\frac{f(t)}{-t} = m > 0 \text{ and} \tag{4}$$

$$\|(x,y)\| = \begin{cases} |x| f\left(\frac{y}{x}\right) & if\ x \neq 0 \\ m|y| & if\ x = 0 \end{cases}.\ \text{Notice that } m = \|(0,1)\| \tag{5}$$

Moreover, if X is strictly convex, then the function f is also strictly convex.
Conversely, for any convex function f satisfying (4), *the equality* (5) *defines a norm.*
All the norms in a two-dimensional space have this form. If f is strictly convex, then the norm given by (5) *is also strictly convex.*

For the proof of the Lemma, see Appendix A.
Combine Step 1 and Step 2. Thus a = (0,1) and f is a function, as in the above Lemma. Next, we drop the index a and write
$L = \{(x,y)\colon x \neq 0,\ f\left(\frac{y}{x}\right) > f\left(\frac{y-1}{x}\right) \text{ or } x = 0,\ m|y| > m|y-1| \Leftrightarrow y \in (\frac{1}{2}, \infty)\}$,
$M = \{(x,y)\colon x \neq 0,\ f\left(\frac{y}{x}\right) = f\left(\frac{y-1}{x}\right) \text{ or } x = 0,\ m|y| = m|y-1| \Leftrightarrow y = \frac{1}{2}\}$
$H = \{(x,y)\colon x \neq 0,\ f\left(\frac{y}{x}\right) < f\left(\frac{y-1}{x}\right) \text{ or } x = 0,\ m|y| < m|y-1| \Leftrightarrow y \in (-\infty, \frac{1}{2})\}$
Step 3. Describe the sets M, H, L.
Due to the property that $\lim_{t\to\infty}\frac{f(t)}{t} = \lim_{t\to-\infty}\frac{f(t)}{-t} = m > 0$, the convex function f behaves as follows: there exists $u_1 \leq u_2 \in \Re$, such that

$$f \text{ is decreasing on } (-\infty, u_1) \tag{6}$$

$$f \text{ is constant on } (u_1, u_2) \tag{7}$$

$$f \text{ is increasing on } (u_2, \infty) \tag{8}$$

Note that if f is strictly convex, then $u_1 = u_2$. Let $v = f(u_1) = f(u_2)$.

Lemma 2. *Let* $f\colon \Re \to \Re$ *be a continuous function which satisfies the assumptions* (6), (7), (8). *Suppose further that* $\lim_{t\to\infty}\frac{f(t)}{t} = m_2 > 0$, $\lim_{t\to-\infty}\frac{f(t)}{-t} = m_1 > 0$

Let $h(x,y) = f\left(\frac{y}{x}\right) - f\left(\frac{y-1}{x}\right)$, $h\colon \Re^* \times \Re \to \Re$. *The sets* $M = \{h = 0\}$, $L = \{h > 0\}$ *and* $H = \{h < 0\}$ *may then be characterized as follows:*
There are exist two functions $\varphi_1, \varphi_2\colon \Re^* \to \Re$, $\varphi_1 \leq \varphi_2$ *such that*

$$L = \{(x,y)\colon \varphi_1(x) \vee \varphi_2(x) < y\} \tag{9}$$

$$M = \{(x,y)\colon \varphi_1(x) \wedge \varphi_2(x) \leq y \leq \varphi_1(x) \vee \varphi_2(x)\} \tag{10}$$

$$H = \{(x,y):\ \varphi_1(x) \wedge \varphi_1(x) > y\} \tag{11}$$

Moreover,

$$\varphi_j(-x) = 1 - \varphi_j(x)\ \forall\ x \in \Re^*,\ j = 1,2 \tag{12}$$

$$\varphi_j(0+0) = \frac{m_1}{m_1+m_2},\ \varphi_j(0-0) = \frac{m_2}{m_1+m_2} \tag{13}$$

$$\varphi_1(x) = \varphi_2(x)\ \forall\ x\ \in\ (-\frac{1}{u_2-u_1}, \frac{1}{u_2-u_1})\backslash\{0\} \tag{14}$$

$$\textit{If } f \textit{ is strictly convex, then } \varphi_1 = \varphi_2 \tag{15}$$

$$\textit{if } (x,y)\ \in\ M \textit{ and } (tx,ty)\ \in\ M \textit{ for some } t\ >\ 1, \textit{ then either } x > \tfrac{1}{u_2-u_1} \text{ or } x < -\tfrac{1}{u_2-u_1}. \tag{16}$$

For the proof, see Appendix A.

In our case, $m_1 = m_2$ allows us to extend the mapping of φ_j to 0, defining $\varphi_j(0) = \frac{1}{2}$.

Recall our claim: we want to prove that

$$y > \varphi_1(x) \vee \varphi_2(x),\ y' > \varphi_1(x') \vee \varphi_2(x') \Rightarrow y + y' > \varphi_1(x + x') \vee \varphi_2(x + x') \tag{17}$$

This implication would surely hold if we could prove that φ_j are sub-additive.

Indeed, in this case, $y > \varphi_1(x)$, $y' > \varphi_1(x') \Rightarrow y + y' > \varphi_1(x) + \varphi_1(x') \geq \varphi_1(x + x')$ and, similarly, $y + y' > \varphi_2(x + x')$.

Step 4. The functions φ_j are sub-additive

We shall use the following criterion for sub-additivity:

Lemma 3.

(i) *Let $g:\Re \to \Re$ be such that $g(\lambda s) \leq \lambda g(s)\ \forall\ \lambda \geq 1\ \forall\ s \in \Re$. Next, $g(s + t) \leq g(s) + g(t)\ \forall\ s,t$ such that $st \geq 0$.*
If, moreover, $\lambda > 1 \Rightarrow g(\lambda s) < \lambda g(s)\ \forall\ s \in \Re$, then $g(s + t) < g(s) + g(t)\ \forall\ s,t$ such that $st \geq 0$.

(ii) *Let g be a function, as before. Suppose that g satisfies the following symmetry property:*

(S) *there is $a \in \Re$, $b > 0$, such that $g(a - s) + g(s) = b$.*
Consequently, g is sub-additive: $g(s + t) \leq g(s) + g(t)$.
If, moreover, $\lambda > 1 \Rightarrow g(\lambda s) < \lambda g(s)\ \forall\ s \in \Re$, then $g(s + t) < g(s) + g(t)\ \forall\ s,t$

The proof in Appendix A.

Proposition 4. *For $1 < p < \infty$, the functions φ_j defined by* Lemma 2 *are sub-additive.*
Moreover, if X is strictly convex, they coincide and are strictly sub-additive.

Proof. Because of the fact that $m_1 = m_2$, these two functions are defined on the whole real line ($\varphi_j(0) = \frac{1}{2}$) and they coincide on the interval $[-\frac{1}{u_2-u_1}, \frac{1}{u_2-u_1}]$. The equality (12) demonstrates that the symmetry property (S) is fulfilled with $a = 0$, $b = 1$. Thus, we only need to prove that

$$\varphi_j(\lambda x) \leq \lambda\varphi_j(x) \text{ for any } x. \tag{18}$$

Check that for φ_1. Indeed,

- if $x < -\frac{1}{u_2-u_1}$, then $\varphi_1(x) = 1 + xu_2$, $\varphi_1(tx) = 1 + txu_2 < t + txu_2 = t\varphi_1(x)$;
- if $x > \frac{1}{u_2-u_1}$, $\varphi_1(x) = xu_2 \Rightarrow \varphi_1(tx) = t\varphi_1(x)$.
- if $x \in [-\frac{1}{u_2-u_1}, \frac{1}{u_2-u_1}]$, then $y = \varphi_1(x) \Leftrightarrow (x,y) \in M$. Then $(tx,ty) \in H \cup M$ due to Proposition 2(ii). However, (tx,ty) does not belong to M because of (16); hence, in this case, $(tx,ty) \in H$. According to (9) this means that $ty > (\varphi_1 \vee \varphi_2)(tx) \Rightarrow t\varphi_1(x) > \varphi_1(tx)$.

In the same way, one proves that $\varphi_2(tx) \leq t\varphi_2(x)\ \forall\ t \geq 1$.

If X is strictly convex, then if $(x,y) \in M$ and $t > 1$, then $(tx, ty) \in M \Leftrightarrow \varphi(tx) < ty = t\varphi(x)$; hence, Lemma 3 points demonstrates that φ is strictly sub-additive.

This ends the proof. □

The proof of Theorem 1 immediately follows due to Lemma 4.

Step 5. End of proof of Theorem 1

(i) According to (9), $(x,y) \in L \Leftrightarrow y > \varphi(x)$ with $\varphi = \varphi_1 \vee \varphi_2$. The functions φ_j are sub-additive; hence, φ is also sub-additive. Therefore $y > \varphi(x)$, $y' > \varphi(x') \Rightarrow y + y' > \varphi(x) + \varphi(x') \geq \varphi(x + x') \Rightarrow (x + x', y + y') \in L$.

(ii) If the space X is strictly convex, $\mathrm{Cl}(L) = L \cup M$. However, obviously $L + L \subseteq L \Rightarrow \mathrm{Cl}(L) + \mathrm{Cl}(L) \subseteq \mathrm{Cl}(L)$.

(iii) Suppose that X is not strictly convex. The unity sphere $B = \{z \in X: \|z\| = 1\}$ contains a segment of line $I = \{(1-\lambda)z_1 + \lambda z_2: 0 \leq \lambda \leq 1\}$ for some $z_1 \neq z_2 \in B$. We suppose these two points to be extreme. Let $T:\Re^2 \to \Re^2$ a linear operator such that $Tz_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Tz_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Suppose that $(X, \|\cdot\|)$ has the semi-group property. Consequently, $(X, \|\cdot\|_T)$ also has it. The new unity sphere $B_T = \{z \in X: \|z\|_T = 1\}$ contains the segment $\{(1,t) \mid -1 \leq t \leq 1\}$. Let $f(t) = \left\| \begin{pmatrix} 1 \\ t \end{pmatrix} \right\|_T$. This function is convex and f is decreasing on $(-\infty,-1)$, constant on $(-1,1)$, and increasing on $(1,\infty)$. Thus, u_1 and u_2 from Lemma 2 are $u_1 = -1$, $u_2 = 1$. According to (10), $z = (x,y)$ belongs to M if, and only if, $(\varphi_1 \wedge \varphi_2)(x) \leq y \leq (\varphi_1 \vee \varphi_2)(x)$. Moreover, we see that $x \in [\frac{1}{2}, \infty) \Rightarrow \varphi_1(x) = x$, $\varphi_2(x) = 1 - x$ and $x \in (-\infty, \frac{1}{2}] \Rightarrow \varphi_1(x) = 1 + x$, $\varphi_2(x) = -x$. It follows that there are many pairs of z,z', such that $z,z' \in M$ but $z + z' = 0 \in L$ (for instance, $z = (1,0)$ and $z' = (-1,0)$ are both in M). This contradicts the semi-group property: $M + M$ should be included in $L \cup M$.

To conclude: if $a = T^{-1}\mathbf{e}_2$, then one can find $z \in M_a$ such that $-z \in M_a$, as well. Furthermore, we can prove the following.

Theorem 2. *The space $X = \Re^2$ endowed with a strict convex norm satisfies the assumption* (C)

Proof. We have to prove that the relations

$$\|z\| \geq \|z - a\|,\ \|z'\| \geq \|z' - a\|,\ \|z + z'\| = \|z + z' - a\| \tag{19}$$

for some $z, z' \in X$ can hold if, and only if, $a = 0$. Suppose, ad absurdum, that this is not true. Let $a = (a_1, a_2) \neq 0$. There is a one-to-one and onto linear operator $T: X \to X$, such that $Ta = \mathbf{e}_2 = (0, 1)$. Let $w = T^{-1}z$, $w' = T^{-1}z'$. Consequently, (19) becomes

$$\|w\|_T \geq \|w - \mathbf{e_2}\|_T, \|w\|_T \geq \|w' - \mathbf{e}_2\|_T, \|w + w'\|_T = \|w + w' - \mathbf{e}_2\|_T \tag{20}$$

where $\|\cdot\|$ is the norm defined at Proposition 3.

If X is strictly convex, the norm $\|\cdot\|_T$ is also strictly convex. Let $w = (x,y)$, $w' = (x',y')$ and φ the function defined by Lemma 2. Therefore, (20) becomes

$$y \geq \varphi(x),\ y' \geq \varphi(x'),\ y + y' = \varphi(x + x') \tag{21}$$

However, this is impossible, since $y + y' \geq \varphi(x) + \varphi(x') > \varphi(x + x')$, as φ is strictly sub-additive. □

3. Conjectures, Open Problems, and Counterexamples

At a first glance, a *possible conjecture* would be that a strict convex normed space should have the mediator property. However, this not true: all the spaces $L^p(\{1,2,3\})$ with $1 < p < \infty$ are strictly convex and fail to have the property (M).

Proposition 5. *Let* $X = \Re^3$ endowed with l^p norm, $\|x\| = \left(\sum_{j=1}^{3} |x_j|^p \right)^{\frac{1}{p}}$.

(i) *If $p \in (1,2) \cup (2,\infty)$, then X does **not** have the mediator property.*

(ii) *However, all its proper subspaces **have it** (due to Theorem 1).*

Proof.

(i) **Case 1**: $p \in (1,2)$. We choose $x = (3,t,-t)$, $y = (3, -t, t)$, $a = (4,4,4)$. Then

$x + y = (6, 0, 0)$, $x - a = (-1, t - 4, -t - 4)$, $y - a = (-1, -t - 4, t - 4)$, $x + y - a = (2, -4, -4)$; hence, if $t > 4$, $\|x\|^p = \|y\|^p = 3^p + 2t^p$, $\|x - a\|^p = \|y - a\|^p = 1 + (t - 4)^p + (t + 4)^p$, $\|x + y\|^p = 6^p$ and $\|x + y - a\|^p = 2^p + 2{\cdot}4^p$. We claim that for any $1 \le p < 2$, we can choose $t > 4$, such that $\|x\| > \|x - a\|$and $\|y\| > \|y - a\|$ but $\|x + y\| < \|x + y - a\|$. Indeed, this is an equivalent to $3^p - 1 > (t - 4)^p + (t + 4)^p - 2t^p$ but $6^p < 2^p + 2{\cdot}4^p$. The function $g(t) =(t - 4)^p + (t + 4)^p - 2t^p$ has the property that $g(\infty) = 0$ for any $1 \le p < 2$; hence, for any fixed $p \in [1,2)$, we can find a $t = t(p)$, such that $3^p - 1 > g(t)$. As with the second condition, $6^p < 2^p + 2{\cdot}4^p \Leftrightarrow 3^p < 1 + 2{\cdot}2^p$, it holds for any such $p \in (1,2)$, since the function $h(p) = 1 + 2{\cdot}2^p - 3^p$ is decreasing on (1,2) and positive.

(ii) **Case 2**: $p \in (2,\infty)$. Now, we choose $x = (1,-1,2)$, $y = (1,2,-1)$, $a = (-t,t,t)$ for some $t \in (0,1/2)$. Therefore, $x + y = (2,1,1)$, $x - a = (1 + t,-1 - t,2 - t)$, $y - a = (1 + t,2 - t,-1 - t)$, $x + y - a = (2 + t,1 - t, 1 - t)$; hence, $\|x\|^p = \|y\|^p = \|x + y\|^p = (2^p + 2)$,$\|x - a\|^p = \|y - a\|^p = 2(1 + t)^p + (2 - t)^p$ and $\|x + y - a\|^p = (2 + t)^p + 2(1 - t)^p$. We claim that for every $p > 2$, there is $t \in (0,1/2)$, such that $\|x\| > \|x - a\|$, $\|y\| > \|y - a\|$ but $\|x + y\| < \|x + y - a\|$. Indeed, let $p > 2$ be fixed and let f,g be defined by $f(t) = 2^p + 2 - 2(1 + t)^p - (2 - t)^p$, $g(t) = (2 + t)^p +2(1 - t)^p - 2^p - 2$. Note that $f(0) = g(0) = 0$ and that $f'(0) = g'(0) = p(2^{p-1} - 2)$. If $p > 2$, then $2^{p-1} > 2$; hence, the derivatives are positive. This means that for small t, we have $f(t) > 0$, $g(t) > 0$; this fact agrees with our claim. For $p = \infty$, it is even simpler, since now $\|x\| = \|y\| =\|x + y\| = 2$,$\|x - a\| = \|y - a\| = 2 - t$ and $\|x + y - t\| = 2 + t$.

Thus, the two-dimensional space $\Re^2$ endowed with the norm l^p, $1 < p < \infty$ has the mediator property, while $\Re^3$ endowed with the same norm **does not have it**. □

4. Open Problems

1. The only three-dimensional spaces that satisfy the mediator property (B) are inner-product spaces. Prove or disprove that if $\dim(X) \ge 3$, then property (M) implies the fact that X has an inner product. In other problems connected with perpendicular bisectors, this was indeed the case [5–9].
2. The only examples of spaces possessing the mediator property also satisfy the property C. Prove or disprove that (M) $\Leftrightarrow$ (C).
3. Prove or disprove that if $M + M \subseteq M \cup L$, then X has the property (M).

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Appendix A

Proof of Lemma 1. Let $f(t) = \|(1,t)\|$. Obviously f is convex, $\lim\limits_{t\to\infty} \frac{f(t)}{t} = \lim\limits_{t\to-\infty} \frac{f(t)}{-t} = \|(0,1)\| := m > 0$ and if $x \neq 0$ we can write $\|(x,y)\| = |x| \left\|\left(1, \frac{y}{x}\right)\right\|$. If $x = 0$ then $\|(0,y)\| = |y| \|(0,1)\| = m|y|$.

Conversely, let f: $\Re \to (0,\infty)$ be a convex function, such that $\lim\limits_{t\to\infty} \frac{f(t)}{t} = \lim\limits_{t\to-\infty} \frac{f(t)}{-t} = m > 0$. We claim that the equality

$$p(x,y) = \begin{cases} |x| f\left(\frac{y}{x}\right) & if\ x \neq 0 \\ m|y| & if\ x = 0 \end{cases} \tag{A1}$$

defines a norm: it is sub-additive, $p(x,y) = 0 \Leftrightarrow x = y = 0$ and $p(tx,ty) = |t|\,p(x,y)$. As the last two assertions are obvious, we shall focus on the sub-additivity. One must prove that

$$|x+x'|f\left(\frac{y+y'}{x+x'}\right) \le |x|f\left(\frac{y}{x}\right) + |x'|f\left(\frac{y'}{x'}\right) \text{ if } x, x' \text{ and } x+x' \neq 0 \tag{A2}$$

$$|x|f\left(\frac{y+y'}{x}\right) \le |x|f\left(\frac{y}{x}\right) + m|y'| \text{ if } x \neq 0,\ x'=0 \tag{A3}$$

$$|x'|f\left(\frac{y+y'}{x'}\right) \le |x'|f\left(\frac{y'}{x'}\right) + m|y| \text{ if } x' \neq 0,\ x=0 \tag{A4}$$

$$m|y+y'| \le |x|\left(f\left(\frac{y}{x}\right) + f\left(\frac{-y'}{x}\right)\right) \text{ if } x+x'=0,\ x \neq 0 \tag{A5}$$

Of course, the most important is (A2). If both x and x' are positive, this is easy. Indeed, $p(x + x', y + y') = (x+x')f\left(\frac{y+y'}{x+x'}\right) = (x+x')f\left(\frac{x}{x+x'}\cdot\frac{y}{x} + \frac{x'}{x+x'}\cdot\frac{y'}{x'}\right) \le |x|f\left(\frac{y}{x}\right) + |x'|f\left(\frac{y'}{x'}\right)$ due to convexity. However, if one of them is negative, this argument no longer holds. We shall use another.

We shall prove that any convex function $f\colon \Re \to (0,\infty)$, such that $\lim_{t\to\infty}\frac{f(t)}{t} = \lim_{t\to-\infty}\frac{f(t)}{-t} = m > 0$ admits the following representation

$$f(t) = a + \int |t-c|dv(c) \tag{A6}$$

where ν is a measure on the real line, such that $\nu(\Re) = m$ and $\mu_1 := \int |c|dv(c) < \infty$

It is well known (see, for instance, in [7], pg 96, formula (3.3)) that any convex function $g{:}[0,\infty) \to \Re$ that is differentiable at 0 admits a representation of the form

$$g(t) = a + bt + \int (t-c)_+ dv(c) \tag{A7}$$

where ν is a Stieltjes measure concentrated on $(0,\infty)$. If $b \ge 0$, we can replace ν with $\nu + b\delta_0$ (here, δ_0 is the Dirac measure concentrated at 0) to obtain a simpler relation

$$g(t) = a + \int (t-c)_+ dv(c),\ \mathrm{Supp}(\nu) \subseteq [0,\infty) \tag{A8}$$

Now, let us observe our function $f\colon \Re \to (0,\infty)$. It is convex and non-increasing on some intervals $(-\infty,u)$ and non-decreasing on (u,∞). Suppose that $u = 0$. There are, then, two Stieltjes measures, ν_1 and ν_2, $\mathrm{Supp}(\nu_1) \subseteq (-\infty,0)$, $\mathrm{Supp}(\nu_2) \subseteq (0,\infty)$ such that

$$f(t) = \begin{cases} a + \int (-t+c)_+ dv_1(c) & if\ t < 0 \\ a + \int (t-c)_+ dv_2(c) & if\ t \ge 0 \end{cases} \tag{A9}$$

As $\lim_{t\to\infty}\frac{f(t)}{t} = \lim_{t\to-\infty}\int \left(1-\frac{c}{t}\right)_+ dv_2(t) = \nu_2([0,\infty)) = \nu_2(\Re)$ (we applied Beppo Levi's theorem) and $\lim_{t\to-\infty}\frac{f(t)}{-t} = \lim_{t\to\infty}\int \left(1+\frac{c}{t}\right)_+ dv_1(t) = \nu_1((-\infty,0]) = \nu_1(\Re)$, we obtained $\nu_1(\Re) = \nu_2(\Re) := m$

Now, use the relation $2x_+ = |x| + x$. Consequently, (A9) becomes

$$2f(t) - 2a = \begin{cases} \int |t-c|dv_1(c) - tm + \int c\,dv_1(c) & if\ t < 0 \\ \int |t-c|dv_2(c) + tm - \int c\,dv_2(c) & if\ t \ge 0 \end{cases} \tag{A10}$$

Write (A10) as

$$2f(t)-2a=\begin{cases}\int|t-c|d\nu_1(c)+|t|m-\int|c|d\nu_1(c) & if\ t<0\\ \int|t-c|d\nu_2(c)+|t|m-\int|c|d\nu_2(c) & if\ t\ge 0\end{cases} = \begin{cases}f_1(t) & if\ t<0\\ f_2(t) & if\ t\ge 0\end{cases} \tag{A11}$$

Now, note that $t\ge 0 \Rightarrow f_1(t)=0$ and $t\le 0 \Rightarrow f_2(t)=0$ (this should be obvious if we look at (A9)). It means that we can write

$2f(t)-2a=f_1+f_2=\int|t-c|d\nu(c)+2m|t|-\int|c|d\nu(c)$.

Therefore, we arrived at a representation of the form

$$f(t)=a+m\left|t\right|+\int|t-c|d\mu(c) \tag{A12}$$

that holds for any convex function non-increasing on $(-\infty,0)$ and non-decreasing on $(0,\infty)$, such that $\lim\limits_{t\to\infty}\frac{f(t)}{t}=\lim\limits_{t\to-\infty}\frac{f(t)}{-t}=m>0$. If we replace the measure $\mu=\nu/2$ by $\mu+m\delta_0$, we arrive at the formula (A6). Subsequently, we can replace our assumption that f is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$ by "non-increasing on $(-\infty,u)$, non-decreasing on (u,∞)" for some $u\in\Re$; the proof remains the same.

Therefore, any convex function, as in Lemma 1, can be represented by a finite Stieltjes measure by formula (A6).

Let $C=\{f:\Re\to[0,\infty): f$ *satisfies conditions* (A2)−(A5)$\}$

Consequently, C contains all the positive constants (for them $m=0$) and all the functions $f(x)=|x-a|$ (indeed, now $m=1$: $|x+x'|f\left(\frac{y+y'}{x+x'}\right)=|(y-ax)+(y'-ax')|$, $|x|f\left(\frac{y}{x}\right)=|y-ax|$, $|x'|f\left(\frac{y'}{x'}\right)=|y'-ax'|$, the inequality $|x|f\left(\frac{y+y'}{x}\right)\le|x|f\left(\frac{y}{x}\right)+m|y'|$ becomes $|y+y'-ax|\le|y-ax|+|y'|$ hence C is a cone **:** $f,g\in C\Rightarrow af+bg\in C\ \forall\, a,b\ge 0$. Moreover, it is *closed* with respect to pointwise convergence. As any integral is a limit of finite sums, C contains all the functions of the form (A6). This ends the proof. □

Proof of Lemma 2. Let $f(\Re)=(v,\infty)$, and $f^{-1}(\{v\})=(u_1,u_2)$.

Let f_1: $(-\infty,u_1)\to(v,\infty)$ and f_2: $(u_2,\infty)\to(v,\infty)$ be defined by $f_j(x)=f(x)$. These functions are invertible. We know that $M=\{(x,y)\mid f\left(\frac{y}{x}\right)=f\left(\frac{y-1}{x}\right)\}$. There are two possibilities

- **Case 1.** $x>0$. Now $\frac{y-1}{x}<\frac{y}{x}$ and the equality $f\left(\frac{y}{x}\right)=f\left(\frac{y-1}{x}\right)$ is possible only if $u_1\le\frac{y-1}{x}<\frac{y}{x}\le u_2$ or if $\frac{y-1}{x}<u_1<\frac{y}{x}<u_2$.

  In the first case $x\ge\frac{1}{u_2-u_1}$ and $1+xu_1\le y\le xu_2$.

  In the second case, $f\left(\frac{y}{x}\right)=f_2\left(\frac{y}{x}\right)$ and $f\left(\frac{y-1}{x}\right)=f_1\left(\frac{y-1}{x}\right)$; hence, $(x,y)\in M$ means that $f_1\left(\frac{y-1}{x}\right)=f_2\left(\frac{y}{x}\right)$. Let $t>v$, $t=f\left(\frac{y}{x}\right)$. Therefore, $y=xf_2{}^{-1}(t)$ and $y-1=xf_1{}^{-1}(t)$. Consequently, the set M can be described by the parametric curve

$$x(t)=\frac{1}{f_2{}^{-1}(t)-f_1{}^{-1}(t)},\ y(t)=\frac{f_2{}^{-1}(t)}{f_2{}^{-1}(t)-f_1{}^{-1}(t)},\ t\in(v,\infty). \tag{A13}$$

This is the graph of the function ψ_2: $(0,\frac{1}{u_2-u_1}]\to\Re$ with the property that

$$\psi_2\left(\frac{1}{u_2-u_1}\right)=\frac{u_2}{u_2-u_1}\ \text{and}\ \varphi(0+0)=\frac{m_1}{m_1+m_2}. \tag{A14}$$

Indeed, $\lim\limits_{t\uparrow\infty}\frac{f_2{}^{-1}(t)}{f_2{}^{-1}(t)-f_1{}^{-1}(t)}=\lim\limits_{t\uparrow\infty}\frac{f_2{}^{-1}(t)/t}{f_2{}^{-1}(t)/t-f_1{}^{-1}(t)/t}$

However, $\lim\limits_{t\uparrow\infty}\frac{f_2^{-1}(t)}{t} = \lim\limits_{x\uparrow\infty}\frac{x}{f_2(x)} = \frac{1}{m_2}$ and $\lim\limits_{t\uparrow\infty}\frac{f_1^{-1}(t)}{t} = -\frac{1}{m_1}$

It follows that $\varphi(0+0) = \frac{1/m_2}{1/m_2+1/m_1} = \frac{m_1}{m_1+m_2}$ and we obtain

- **Case 2.** $x < 0$. Now, $\frac{y-1}{x} > \frac{y}{x}$ and the equality $f\left(\frac{y}{x}\right) = f\left(\frac{y-1}{x}\right)$ is possible only if $u_1 \le \frac{y}{x} < \frac{y-1}{x} \le u_2$ or if $\frac{y}{x} < u_1 < \frac{y-1}{x} < u_2$.

  In the first case, $x \le -\frac{1}{u_2-u_1}$ and $1 + xu_2 \le y \le xu_1$.

  In the second case, $(x,y) \in M \Rightarrow f_1\left(\frac{y}{x}\right) = f_2\left(\frac{y-1}{x}\right) = t$ for some $t > v$. Thus, $y = 1 + xf_2^{-1}(t) = xf_1^{-1}(t)$ and its parametric description is

$$x^*(t) = -\frac{1}{f_2^{-1}(t) - f_1^{-1}(t)}, y^*(t) = 1 - \frac{f_2^{-1}(t)}{f_2^{-1}(t) - f_1^{-1}(t)}, t \in (v, \infty) \tag{A15}$$

Comparing this to (A13), we see that the curve $(x^*(t),(y^*(t))_{t>v}$ is the graph of the function $\psi_1:[-\frac{1}{u_2-u_1}, 0) \to \Re$ which is the symmetric of the graph of ψ_2 with respect to the point $(0, \frac{1}{2})$:

$$\psi_1(-x) = 1 - \psi_2(x)\ \forall\ x \in \left(0, \frac{1}{u_2-u_1}\right]. \tag{A16}$$

Add all these facts together. Let φ_j: $\Re^* \to \Re$ be defined by

$$\varphi_1(x) = \begin{cases} 1 + xu_2 & if\ x < -\frac{1}{u_2-u_1} \\ 1 - \psi_2(-x) & -\frac{1}{u_2-u_1} \le x < 0 \\ \psi_2(x) & if\ 0 < x \le \frac{1}{u_2-u_1} \\ 1 + xu_1 & if\ x > \frac{1}{u_2-u_1} \end{cases},\ \varphi_2(x) = \begin{cases} xu_1 & if\ x < -\frac{1}{u_2-u_1} \\ 1 - \psi_2(-x) & -\frac{1}{u_2-u_1} \le x < 0 \\ \psi_2(x) & if\ 0 < x \le \frac{1}{u_2-u_1} \\ 1 + xu_1 & if\ x > \frac{1}{u_2-u_1} \end{cases}. \tag{A17}$$

We know that $M = \{h = 0\} = \{(x,y) \in \Re^* \times \Re: \varphi_1(x)\wedge\varphi_2(x) \le y \le \varphi_1(x)\vee\varphi_2(x)\}$

Note that if f $u_1 = u_2$, then $\varphi_1 = \varphi_2$. This happens, for instance, if f is strictly convex.

Now, let us determine the set $L = \{h > 0\} = \{(x,y)$: $f\left(\frac{y}{x}\right) > f\left(\frac{y-1}{x}\right)\}$. If $x > 0$, this set surely contains all the points from $\{(x,y) \in (0,\infty) \times \Re: \frac{y}{x} > \frac{y-1}{x} > u_2\}$ since on the interval (u_2,∞), the function f is increasing. Thus, H includes the set $\{(x,y) \in (0,\infty) \times \Re: y > 1 + xu_2\}$. If $x < 0$, it contains the set $\{(x,y) \in (-\infty,0) \times \Re: \frac{y}{x} < \frac{y-1}{x} < u_1\}$ since on $(-\infty,u_1)$, the function f is decreasing. To conclude, $L \supseteq \{(x,y) \in \Re^* \times \Re: y > \max(1+ xu_1, 1 + xu_2)\}$.

Similarly, $L \supseteq \{(x,y) \in \Re^* \times \Re: y < \min(xu_1, xu_2)\}$.

We found that some points from L are above the graph of $\varphi_1\vee\varphi_2$ and some points from L are below it. This means that ***all*** the points from L are in the set $\{(x,y)\in \Re^* \times \Re: y > \varphi_1(x)\vee\varphi_2(x)\}$ and ***all*** the points from H are in the set $\{(x,y) \in \Re^* \times \Re: y < \varphi_1(x)\vee\varphi_2(x)\}$. Suppose ad absurdum that L contains a point (x_0, y_0) such that $y_0 < (\varphi_1\wedge\varphi_2)(x_0)$. There are some instances of $c > 0$ such that $y_0 - c < \min(xu_1, xu_2) \Rightarrow (x_0, y_0 - c) \in L$. According to the Darboux theorem, on the segment joining the points (x_0, y_0) and $(x_0, y_0 - c)$, there must be at least one, let us say, (x_0, η), such that $h(x_0, \eta) = 0 \Leftrightarrow (x_0, \eta) \in M$, contradicting the fact that $(x,\eta) \in M \Leftrightarrow \varphi_1(x_0)\wedge\varphi_2(x_0) \le \eta \le \varphi_1(x_0)\vee\varphi_2(x_0)$.

To conclude, $L = \{(x,y): y > (\varphi_1\vee\varphi_2)(x)\}$ and $H = \{(x,y): y < (\varphi_1\wedge\varphi_2)(x)\}$.

The situation simplifies significantly if $u_1 = u_2$ (for instance, if f is strictly convex): in this case, $\varphi_1 = \varphi_2 = \varphi$, M is the graph of φ, L is the points above the graph, and H is the points below it.

Now, prove the claim (A16). Suppose that $(x,y) \in M$ and $(tx, ty) \in M$ for some $t > 1$. Thus $f\left(\frac{y}{x}\right) = f\left(\frac{y-1}{x}\right) = f\left(\frac{ty-1}{tx}\right)$. However, the only way in which the equation $f(x) = \alpha$ can have more than two solutions is if $t = v$; hence, these solutions are included in (u_1, u_2). It follows either that $x > 0$ and $\frac{y-1}{x} < \frac{ty-1}{tx} < \frac{y}{x}$, meaning $\Leftrightarrow x > \frac{1}{u_2-u_1}$, or that $x < 0$ and $\frac{y-1}{x} > \frac{ty-1}{tx} > \frac{y}{x}$; hence, $x < -\frac{1}{u_2-u_1}$. □

Proof of Lemma 3. This proof uses the following inequality, which is interesting in itself.

Lemma 4. (*An elementary inequality*)

Let a,b,s,t $\in \Re$ *such that st* > 0. *Consequently*, $\min[a(1+\frac{s}{t}), b(1+\frac{t}{s})] \le a+b$.

Proof. Let $\lambda = \frac{s}{t} > 0$. We want to check that $a(1+\lambda) \le a+b$ or $b(1+\frac{1}{\lambda}) \le a+b \Leftrightarrow a\lambda \le b$ or $\frac{b}{\lambda} \le a \Leftrightarrow a\lambda \le b$ or $b \le a\lambda$. This is obvious. □

The proof of Lemma 3 is as follows:

(i) Note that $g(0) = g(\lambda 0) \le \lambda g(0)\ \forall\ \lambda \ge 1$; hence, $g(0) \ge 0$, and if g satisfies the stronger condition $\lambda > 1 \Rightarrow g(\lambda x) > \lambda g(x)$, then $g(0) > 0$.

Let $s,t \in \Re$, such that $st \ge 0$. If $s = 0$, the claim becomes $g(t) \le g(t) + g(0) \Leftrightarrow g(0) \ge 0$, true.

If g satisfies the stronger condition $g(\lambda s) < \lambda g(s)\ \forall\ \lambda > 1$, then this implies $g(0) > 0$, true. Therefore, if $s = 0$ we have proven that $g(s+t) \le g(s) + g(t)$. If $t = 0$, the proof is the same. Suppose now that $st > 0$.

Consequently, $g(s+t) = g((1+\frac{t}{s})s) \le (1+\frac{t}{s})g(s)$ and $g(s+t) = g((1+\frac{s}{t})t) \le (1+\frac{s}{t})g(t)$.

Thus, by (i), $g(s+t) \le \min((1+\frac{t}{s})g(s), (1+\frac{t}{t})g(t)) \le g(s) + g(t)$.

Obviously, the stronger condition $\lambda > 1 \Rightarrow g(\lambda s) < \lambda g(s)\ \forall\ s$ implies the strict inequality $g(s+t) < g(s) + g(t)$.

(ii) It is obvious that if g is a sub-additive function, then the function $h(s) = \alpha g(\lambda s) + \beta$ is again sub-additive for any $\lambda \in \Re$, $\alpha,\beta > 0$. If we write $s = ax$, the symmetry condition (S) becomes

$$g(ax) + g(a(1-x)) = b \Leftrightarrow \frac{g(ax)}{b} + \frac{g(a(1-x))}{b} = 1.$$

It follows that the function g is sub-additive if, and only if, the function $\varphi(x) = \frac{g(ax)}{b}$ is sub-additive. This means that we can include the condition (S) $a = b = 1$.

If $st \ge 0$, the sub-additivity has been already proven at (ii).

Suppose s and t have opposite signs, for example, $s > 0$ and $t < 0$. We shall use the symmetry property (S).

Write $t = 1 - v$. Consequently, $v > 1$. The claimed inequality becomes

$\varphi(s+1-v) < \varphi(s) + \varphi(1-v) \Leftrightarrow 1 - \varphi(v-s) < \varphi(s) + 1 - \varphi(v) \Leftrightarrow \varphi(v) < \varphi(s) + \varphi(v-s)$.

This is true if $v - s \ge 0$, since $v = s + (v-s)$ and s and v-s are positive. This holds if $s + t \le 1$. If $s + t > 1$, we write $s = 1 - u$; note that $u - t < 0$ and write the claimed inequality as $\varphi(1-u+t) < \varphi(1-u) + \varphi(t) \Leftrightarrow 1 - \varphi(u-t) < 1 - \varphi(u) + \varphi(t) \Leftrightarrow \varphi(u-t) + \varphi(t) > \varphi(u)$, which is true again according to Lemma 1. This ends the proof. □

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