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Splitting Extensions of Nonassociative Algebras and Modules with Metagroup Relations

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Abstract: A class of nonassociative algebras is investigated with mild relations induced from metagroup structures. Modules over nonassociative algebras are studied. For a class of modules over nonassociative algebras, their extensions and splitting extensions are scrutinized. For this purpose tensor products of modules and induced modules over nonassociative algebras are investigated. Moreover, a developed cohomology theory on them is used.

Keywords: algebra; nonassociative; separable; ideal; cohomology; tensor product; gradation

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1. Introduction

Extensions and splitting extensions of associative algebras play a very important role and have found many-sided applications (see, for example, refs. [1–5] and references therein). Studies of their structure are grounded on cohomology theory. In particular, great attention is paid to algebras with group identities. On the other hand, cohomology theory of associative algebras was investigated by Hochschild and other authors [6–8], but it is not applicable to nonassociative algebras. Cohomology theory of group algebras is an important and great part of algebraic topology. Nonassociative algebras with some identities in them, such as Cayley–Dickson algebras and their generalizations are wide spread not only in algebra but also in many-sided applications in physics, noncommutative geometry, quantum field theory, PDEs, and other sciences (see [9–18] and references therein). For example, the Klein–Gordon hyperbolic PDE of the second order with constant coefficients was solved by Dirac with the help of complexified quaternions [19]. Cayley–Dickson algebras were used for decompositions of higher-order PDEs into lower-order PDEs, which subsequently permitted integrating and analyzing them [15]. PDEs or their systems frequently possess groups of their symmetries [8]. Group algebras appearing over $\mathbf{C}$ in conjunction with Cayley–Dickson algebras lead to extensions that are generalized Cayley–Dickson algebras or even more general metagroup algebras. In their turn operator algebras over Cayley–Dickson algebras also induce the metagroup algebras. Besides algebras over $\mathbf{R}$ or $\mathbf{C}$, there are such algebras over other fields, that is important in non-archimedean quantum mechanics and quantum field theory. Analysis of PDEs and operators over Cayley–Dickson algebras provide generalized Cayley–Dickson algebras or metagroup algebras acting on modules of functions.

It is worth mentioning that in the 20th century it was demonstrated that a noncommutative geometry exists if there exists a corresponding quasi-group [20–22]. On the other hand, metagroups are quasigroups with some weak relations.

An extensive area of investigations of PDEs intersects with cohomologies and deformed cohomologies [8]. Therefore, it is important to develop this area over octonions, Cayley–Dickson algebras, and more general nonassociative algebras with metagroup relations. Examples of nonassociative algebras, modules, and homological complexes with metagroup relations are given in [12,13,23]. In these works, it is shown that generalized Cayley–Dickson algebras are particular cases of metagroup algebras. For related with them digital Hopf spaces cohomologies were investigated in [24].

In [14], smashed and twisted wreath products of metagroups or groups were studied. It permitted constructing wide families of metagroups even starting from groups. It also demonstrates that meta groups appear naturally in algebra and metagroup algebras compose an enormous class of nonassociative algebras.

This article is devoted to extensions and splitting extensions of nonassociative algebras. For this purpose cohomology theory on them was developed in [12]. In its turn, it demanded to impose some mild conditions on algebras to develop cohomology theory on them.

A class of nonassociative algebras is investigated with mild relations induced from metagroup structures. Modules over nonassociative metagroup algebras are studied. For a class of modules over nonassociative algebras their extensions, splitting extensions, and ideals are scrutinized in Section 3.

For this purpose tensor products of modules and induced modules over nonassociative metagroup algebras are investigated in Section 2. Smashly graded modules and algebras are studied. The smash gradation is considered over metagroups. The considered tensor products take into account smash nonassociative structures (see Theorem 1, Proposition 1, Remark 1). Moreover, a developed cohomology theory on them is used. Algebras satisfying these conditions are described (see Theorem 2). In Propositions 2 and 3 and Theorems 3 and 5, tensor products of morphisms are studied. The exactness of morphisms is investigated in Theorems 4 and 7 and Corollary 3. Generic morphisms are scrutinized in Theorem 6. Isomorphisms of such nonassociative algebras are studied in Proposition 4 and Corollary 6. Their meta-isomorphisms are investigated in Proposition 5 and Theorem 8.

The exactness of isomorphisms related with extensions and metagroup algebras is studied in Lemma 6 and Propositions 6 and 7 (see also Definitions 1 and 6). Extensions of smashly graded modules and algebras are investigated in Theorems 9 and 10 and Proposition 8.

Necessary definitions are provided in Appendix A (see also Formulas (A1)–(A19)).

All main results of this paper are obtained for the first time. They can be used for further studies of nonassociative algebras, their cohomologies, algebraic geometry, PDEs, their applications in the sciences, etc.

2. Modules over Nonassociative Algebras with Metagroup Relations

Definition 1. *Let $\mathcal{T}$ be a commutative associative unital ring, G be a metagroup and $A = \mathcal{T}[G]$ be a metagroup algebra. Let also B be a G-graded unital right A-module (see Definitions A2 and A3). Suppose in addition that*

(1) *there exists a mapping $B \times B \ni (x, y) \mapsto xy \in B$ such that*

$x(y+z) = xy + xz$ *and* $(y+z)x = yx + zx$ *and* $(bx)y = b(xy)$ *and* $(xb)y = x(by)$ *and* $(xy)b = x(yb)$

for all x, y, z in B, $b \in \mathcal{T}$;

(2) $(x_g y_h)z_s = t_3(g,h,s)x_g(y_h z_s)$ *and* $x_g y_h \in B_{gh}$

for every g, h, s in G, $x_g \in B_g$, $y_h \in B_h$, $z_s \in B_s$.

Then we call B a smashly G-graded algebra over A (or a smashly G-graded A-algebra). For short it will be written "an A-algebra" instead of "a smashly G-graded A-algebra". The algebra B is called unital if and only if

(3) *B has a unit element $1 = 1_B$ such that $1_B x = x$ and $x 1_B = x$ for each $x \in B$.*

Suppose that X is a G_1-graded left A_1-module and Y is a G_2-graded left A_2-module, $A_j = \mathcal{T}[G_j]$ is a metagroup algebra for each $j \in \{1, 2, 3, \dots\}$. Suppose also that $f : X \to Y$ is a map such that f is

(4) *a left $\mathcal{T}$-homomorphism and $f(X_g) \subseteq Y_{f_1(g)}$ for each $g \in G_1$, where $f_1 : G_1 \to G_2$ is a homomorphism of metagroups:*

(5) $f_1(gh) = f_1(g)f_1(h)$ *and* $f_1(g \setminus h) = f_1(g) \setminus f_1(h)$ *and*
$f_1(g/h) = f_1(g)/f_1(h)$ *for every g and h in G.*

The map $f : X \to Y$ *satisfying conditions* (4) *and* (5) *will be called a* (G_1, G_2)*-graded left* $\mathcal{T}$*-homomorphism of the left modules X and Y. If* $\mathcal{T}$ *is specified, it may be said a homomorphism instead of a* $\mathcal{T}$*-homomorphism. Symmetrically is defined a* (G_3, G_4)*-graded right homomorphism of a right* A_3*-module X and a right* A_4*-module Y. For a* (A_1, A_3)*-bimodule X and a* (A_2, A_4)*-bimodule Y if a map* $f : X \to Y$ *is* (G_1, G_2)*-graded left and* (G_3, G_4)*-graded right homomorphism, then f will be called a* $((G_1, G_2),(G_3, G_4))$*-graded homomorphism of bimodules X and Y.*

Assume that X is a left A_1*-module and Y is a left* A_2*-module and* $f : X \to Y$ *is a map, where* $A_j = \mathcal{T}[G_j]$ *for each j, such that*

(6) $f : X \to Y$ *is a left* $\mathcal{T}$ *homomorphism and* $f(ax + by) = f_1(a)f(x) + f_1(b)f(y)$*, where*
(7) f_1 *is a homomorphism from* A_1 *into* A_2*, that is*
$f_1 : A_1 \to A_2$ *and* $f_1 : G_1 \to G_2$ *and* $f_1 : \mathcal{T}[N(G_1)] \to \mathcal{T}[N(G_2)]$ *and*
$f_1(ab) = f_1(a)f_1(b)$ *and* $f_1(g \setminus h) = f_1(g) \setminus f_1(h)$ *and*
$f_1(g/h) = f_1(g)/f_1(h)$ *and* $f_1(pa + bs) = pf_1(a) + f_1(b)s$
for every g and h in G_1*, a and b in* A_1*, p and s in* $\mathcal{T}$*, where* G_j *is considered embedded into* $\mathcal{T}[G_j]$ *as* $G_j1_{\mathcal{T}}$ *identifying* $g1_{\mathcal{T}}$ *with g for simplicity of the notation, where* $1_{\mathcal{T}}$ *is the unit element of the ring* $\mathcal{T}$*,* $g \in G_j$*.*

If f satisfies conditions (6) *and* (7)*, then f will be called an* (A_1, A_2)*-generic left homomorphism of left modules X and Y. For right modules an* (A_1, A_2)*-generic right homomorphism is defined analogously. If X is an* (A_1, A_3)*-bimodule and Y is an* (A_2, A_4)*-bimodule and f is an* (A_1, A_2)*-generic left and* (A_3, A_4)*-generic right homomorphism, then f will be called an* $((A_1, A_2),(A_3, A_4))$*-generic homomorphism of bimodules X and Y.*

If additionally the homomorphism f_1 *is bijective and surjective in* (7) *and* $f_1^{-1} : A_2 \to A_1$ *is the homomorphism, then* f_1 *is called an isomorphism of* A_1 *with* A_2 *(or automorphism if* $A_1 = A_2$*).*

In particular, if $G_1 = G_2$*, then "*(G_1, G_1)*-graded" or "*(A_1, A_1)*-generic" will be shortened to "*G_1*-graded" or "*A_1*-generic" correspondingly, etc. If* f_1 *is an automorphism of* G_1 *(or of* A_1 *correspondingly), then a* G_1*-graded (or an* A_1*-generic) left homomorphism from X into Y will be called* G_1*-exact (or* A_1*-exact correspondingly). Similarly* G_3*-exact or* A_3*-exact right homomorphisms of right modules and* (G_1, G_3)*-exact or* (A_1, A_3)*-exact homomorphisms of bimodules are defined.*

If X and Y are A-algebras and f is a G-graded (or G-exact or A-generic or A-exact) homomorphism from X into Y considered as A-bimodules and in addition the following condition is satisfied

(8) $f(vx) = f(v)f(x)$ *for each x and v in X,*
then f will be called a G-graded (or G-exact or A-generic or A-exact correspondingly) homomorphism of the A-algebras.

We consider the Cartesian product $X \times Y$ *of A-bimodules X and Y. Let* $X \times_A Y$ *be an A-bimodule generated from* $X \times Y$ *using finite additions of elements* $(x, y) \in X \times Y$ *and the left and right multiplications on elements* $a \in A$ *such that*

(9) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ *and*
(10) $a(x, y) = (ax, ay)$ *and* $(x, y)a = (xa, ya)$ *and*
(11) $g(X_e, Y_e) = (X_g, Y_g)$ *and* $(X_e, Y_e)g = (X_g, Y_g)$ *(see also* (A16)*) for each x and* x_1 *in X, y, and* y_1 *in Y,* $a \in A$*,* $g \in G$*.*

Suppose that X, Y and Z are A-bimodules.

(12) *Let* $\Lambda : X \times Y \to Z$ *be a* $\mathcal{T}[\mathcal{C}(G)]$*-bilinear map. Let also* Λ *satisfy the following identities:*
(13) $\Lambda(x_g b_h, y_s) = \mathrm{t}_3(g, h, s)\Lambda(x_g, b_h y_s)$ *and* $\Lambda(cx, y) = c\Lambda(x, y)$ *and*
$\Lambda(x, yc) = \Lambda(x, y)c$ *for each* $c \in N(G)$*,* $x \in X$*,* $y \in Y$*, g and h and s in G. If* Λ *fulfills conditions* (12) *and* (13)*, then it will be said that the map* Λ *is G-balanced.*

Definition 2. *Let X and Y be A-bimodules (see Definitions A3 and 1), where* $A = \mathcal{T}[G]$ *is the metagroup algebra. Let C be an A-bimodule supplied with a* $\mathcal{T}[\mathcal{C}(G)]$*-bilinear map* $\xi : X \times Y \to C$ *denoted by* $\xi(x, y) = x \otimes y$ *for each* $x \in X$ *and* $y \in Y$ *such that*

(14) *C is generated by a set* $\{x \otimes y : x \in X, y \in Y\}$ *and*

(15) *if* $\Lambda : X \times Y \to Z$ *is a G-balanced map of A-bimodules X, Y, and Z, and for each fixed* $x \in X$ *the map* $\Lambda(x, \cdot) : Y \to Z$ *and for each fixed* $y \in Y$ *the map* $\Lambda(\cdot, y) : X \to Z$ *are G-graded homomorphisms of A-bimodules, then there exists a G-graded homomorphism* $\psi : C \to Z$ *of A-bimodules such that* $\psi(x \otimes y) = \Lambda(x, y)$ *for each* $x \in X$ *and* $y \in Y$.

If conditions (14) *and* (15) *are satisfied, then the A-bimodule C is called a tensor product of X with Y over A and denoted by* $X \otimes_A Y$.

Remark 1. *Definition 2 implies the following identities in* $X \otimes_A Y$:
(16) $(cx + dx_1) \otimes y = c(x \otimes y) + d(x_1 \otimes y)$;
(17) $x \otimes (yc + y_1 d) = (x \otimes y)c + (x \otimes y_1)d$;
(18) $(x_g b_h) \otimes y_s = \mathsf{t}_3(g, h, s) x_g \otimes (b_h y_s)$;
(19) $(b_h x_g) \otimes y_s = \mathsf{t}_3(h, g, s) b_h (x_g \otimes y_s)$;
(20) $(x_g \otimes y_s) b_h = \mathsf{t}_3(g, s, h) x_g \otimes (y_s b_h)$;
(21) $\gamma(x \otimes y) = (x \otimes y)\gamma$;
(22) $x \otimes 0 = 0$ *and* $0 \otimes y = 0$
for each x and x_1 *in X; y and* y_1 *in Y; c and d in* $\mathcal{T}[N(G)]$, *g and h and s in G,* $b_h \in A_h$, $x_g \in X_g$, $y_s \in Y_s$, $\gamma \in \mathcal{T}[\mathcal{C}(G)]$.

If $p : G \to H$ *is a homomorphism of metagroups, then* $p(G)$ *is a submetagroup in H such that* $p(N(G)) \subseteq N(p(G))$ *and* $p(Com(G)) \subseteq Com(p(G))$, *hence* $p(\mathsf{C}(G)) \subseteq \mathsf{C}(p(G))$.

We remind the reader that a submetagroup Q in G is called normal if it satisfies the following conditions: $gQ = Qg$, $(hg)Q = h(gQ)$, $(hQ)g = h(Qg)$ *and* $Q(hg) = (Qh)g$ *for each g and h in G. There exists a quotient metagroup* $G/\cdot/Q$ *consisting of classes gQ in G with* $g \in G$, *since*

$$[((gQ)(hQ))(sQ)]/[(gQ)((hQ)(sQ))] = ([(gh)s]/[g(hs)])Q \in \mathsf{C}(G/\cdot/Q)$$

for each g, h, and s in G. Moreover, there exists a quotient homomorphism $\pi_{G,Q}$ *from G onto* $G/\cdot/Q$. *In particular,* $p^{-1}(e_H) =: Ker(p)$ *is a normal submetagroup in G, where* e_H *denotes the unit element in H, such that the image* $p(G)$ *is isomorphic with* $G/\cdot/Ker(p)$.

As usually id_X *denotes the identity homomorphism on X,* $id_X(x) = x$ *for each* $x \in X$. *Note that there are natural embeddings of G and* $\mathcal{T}$ *in* $A = \mathcal{T}[G]$ *as G1 and* $e\mathcal{T}$ *correspondingly, where* $1_{\mathcal{T}} = 1$ *is a unit of the ring* $\mathcal{T}$ *and* $e = e_G$ *is a unit of the metagroup G. In particular, there may be a case when G and* $\mathcal{T}$ *are contained in a nonassociative ring* $\mathcal{R}$ *such that* $G \cap \mathcal{T} \neq \emptyset$. *This may induce algebraic identities in the metagroup algebra.*

Lemma 1. *Let X, Y, Z, and A be as in Definition 2. Let also* ϕ *and* θ *be G-graded homomorphisms from* $X \otimes_A Y$ *into Z such that* $\phi(x \otimes y) = \theta(x \otimes y)$ *for each* $x \in X$ *and* $y \in Y$. *Then* ϕ *and* θ *coincide on* $X \otimes_A Y$.

Proof. Certainly $ker(\phi - \theta)$ is a submodule in $X \otimes_A Y$. From conditions (14) and (15) it follows that $\phi = \theta$. □

Corollary 1. *The homomorphism* ξ *is uniquely defined by condition* (15).

Corollary 2. *If* $C = X \otimes_A Y$ *and* $C' = X \otimes'_A Y$ *are two tensor products of X and Y over A (see Definition 2), then there exists a unique G-graded isomorphism* $\phi : C \to C'$ *such that*
(23) $\phi(x \otimes y) = x \otimes' y$ *for each* $x \in X$ *and* $y \in Y$, *where* $x \otimes y \in C$ *and* $x \otimes' y \in C'$.

Proof. An existence of ϕ satisfying condition (23) follows from identities (16)–(22). Similarly, there exists a G-graded homomorphism $\eta : C' \to C$ such that $\eta(x \otimes' y) = x \otimes y$ for each $x \in X$ and $y \in Y$. In view of Lemma 1 $\phi\eta = id$ and $\eta\phi = id$, where id denotes the identity map. □

Theorem 1. *Assume that X and Y are A-bimodules, where* $A = \mathcal{T}[G]$ *is a metagroup algebra. Then the tensor product* $X \otimes_A Y$ *exists and it is an A-bimodule.*

Proof. We consider an A-bisubmodule H in the A-bimodule $K := X \times_A Y$ (see Definition 1). Let H be generated by elements of the following types:

(24) $(cx + dx_1, y) - c(x, y) - d(x_1, y)$;
(25) $(x, yc + y_1 d) - (x, y)c - (x, y_1)d$;
(26) $(x_g b_h, y_s) - (\mathrm{t}_3(g, h, s)x_g, b_h y_s)$;
(27) $(b_h x_g, y_s) - \mathrm{t}_3(h, g, s)b_h(x_g, y_s)$;
(28) $(x_g, y_s)b_h - \mathrm{t}_3(g, s, h)(x_g, y_s b_h)$
(29) $\gamma(x, y) - (x, y)\gamma$;
(30) $(x, 0)$ and $(0, y)$
for each x and x_1 in X; y and y_1 in Y; c and d in $\mathcal{T}[N(G)]$, g and h and s in G, $b_h \in A_h$, $x_g \in X_g$, $y_s \in Y_s$, $\gamma \in \mathcal{T}[\mathcal{C}(G)]$.

There exists the quotient module $Q := K/\cdot/H$ of K by H, since K and H have a structure of commutative groups relative to the addition. One can put $x \otimes y = (x, y) + H$. Note that all pairs $x \otimes y$ generate the quotient module Q, because all pairs $x \times y$ generate K. Let $\Lambda : X \times Y \to Z$ be a G-balanced map of A-bimodules X, Y and Z such that for each fixed $x \in X$ the map $\Lambda(x, \cdot) : Y \to Z$ and for each fixed $y \in Y$ the map $\Lambda(\cdot, y) : X \to Z$ are G-graded homomorphisms (see Definition 1). Evidently there exists a $\mathcal{T}[\mathcal{C}(G)]$-bilinear extension η of Λ on K, $\eta : K \to Z$. From the properties of the map Λ it follows that all elements of the types (24)–(30) belong to the kernel $ker(\eta)$ of η. This induces a homomorphism $\psi : Q \to Z$ such that $\psi((x \otimes y)) = \psi((x, y) + H) = \eta((x, y)) = \Lambda(x, y)$ for each $x \in X$ and $y \in Y$. $\square$

Proposition 1. *Let X_1, X_2, Y_1, and Y_2 be A-bimodules, where $A = \mathcal{T}[G]$ is a metagroup algebra. Let $\phi : X_1 \to X_2$ and $\psi : Y_1 \to Y_2$ be their G-graded homomorphisms such that $\phi_1(g) = \psi_1(\theta_1(g))$ for each $g \in G$, where θ_1 is an automorphism of G. Then there exists a unique G-graded homomorphism*
(31) $\phi \otimes \psi : X_1 \otimes_A Y_1 \to X_2 \otimes_A Y_2$ *such that* $(\phi \otimes \psi)(x \otimes y) = \phi(x) \otimes \psi(y)$
for each $x \in X_1$ and $y \in Y_1$. Moreover, the following identities are accomplished:
(32) $\phi \otimes (\psi a + \xi b) = (\phi \otimes \psi)a + (\phi \otimes \xi)b$,
(33) $(a\phi + b\nu) \otimes \psi = a(\phi \otimes \psi) + b(\nu \otimes \psi)$
for each a and b in $\mathcal{T}[N(G)]$, G-graded homomorphisms $\nu : X_1 \to X_2$, $\xi : Y_1 \to Y_2$ such that $\phi_1(g) = \xi_1(\theta_1(g))$ and $\nu_1(g) = \psi_1(\theta_1(g))$ for each $g \in G$;
(34) $(\phi a) \otimes \psi = \phi \otimes (a\psi)$, *for each a in $\mathcal{T}[N(G)]$;*
(35) $(\gamma\phi) \otimes \psi = \phi \otimes (\psi\gamma)$ *for each $\gamma \in \mathcal{T}[\mathcal{C}(G)]$;*
(36) $id_{X_1} \otimes id_{Y_1} = id_{X_1 \otimes_A Y_1}$;
(37) $(\mu \otimes \eta)(\phi \otimes \psi) = (\mu\phi) \otimes (\eta\psi)$
for each G-graded homomorphisms $\mu : X_2 \to X_3$, $\eta : Y_2 \to Y_3$ of A-bimodules with $\mu_1(g) = \eta_1(\theta_2(g))$ for each $g \in G$, where θ_2 is an automorphism of G;
(38) $\phi \otimes 0 = 0$ *and* $0 \otimes \psi = 0$.

Proof. The homomorphisms ϕ and ψ are G-graded. Therefore,
(39) $(\phi \otimes \psi)(x_g \otimes y_h) \in X_{1,\phi_1(g)} \otimes_{\mathcal{T}} Y_{1,\psi_1(h)}$
for each g and h in G, $x_g \in X_{1,g}$, $y_h \in Y_{1,h}$. On the other hand, by the conditions of this proposition ϕ_1 and ψ_1 are homomorphisms from G into G such that $\phi_1(g) = \psi_1(\theta_1(g))$ for each $g \in G$. Therefore, from $(A16)$ it follows that
(40) $X_{1,\phi_1(g)} \otimes_{\mathcal{T}} Y_{1,\psi_1(h)}$ is isomorphic as the $\mathcal{T}$-bimodule with
$X_{1,s} \otimes_{\mathcal{T}} Y_{1,v}$ for each s in G and $v = s \setminus \psi_1(\theta_1(g)h)$.

Thus the properties (39) and (40) and Definition 2 of $X_1 \otimes_A Y_1$ imply that there exists a unique G-graded homomorphism $(\phi \otimes \psi)$. Identities (32)–(37) follow from Lemma 1, Theorem 1 and (16)–(22). $\square$

Theorem 2. *Assume that X and Y are A-bimodules, where $A = \mathcal{T}[G]$ is a metagroup algebra. Assume also that P is a quotient metagroup $P := G/\cdot/S$ of a metagroup G by a subgroup S such that $S \subseteq \mathcal{C}(G)$ and $X_g \neq X_e$ and $Y_g \neq Y_e$ for each $g \in G - S$. Then there exist $\mathcal{T}[S]$-bimodules X^σ_g such that X considered as a $\mathcal{T}[S]$-bimodule is isomorphic with the direct sum $\bigoplus_{g \in P} X^\sigma_g$ and $(X \otimes_A Y)^\sigma_r$ is isomorphic with $\bigoplus_{h \in P, s \in P,\ hs = r}(X^\sigma_h \otimes_{\mathcal{T}[S]} Y^\sigma_s)$ for each $r \in P$.*

Proof. Notice that S is a normal commutative subgroup in G, because S is contained in $\mathcal{C}(G)$. On the other hand, $t_3(\gamma_1 a, \gamma_2 b, \gamma_3 c) = t_3(a, b, c) \in \mathcal{C}(G)$ for each a, b, and c in G, γ_j in $\mathcal{C}(G)$, $j \in \{1, 2, 3\}$. Therefore, the quotient P of G by S is a metagroup, where elements of P are classes bS, $b \in G$. We denote by θ a quotient map from G onto P such that $\theta(b) = bS$ for each $b \in G$.

Apparently $\mathcal{T}[S]$ is an algebra over the ring $\mathcal{T}$ and $\mathcal{T}[S]$ also has a structure of an associative commutative unital ring, because S is a commutative group and $\mathcal{T}$ is the commutative associative unital ring, $1_{\mathcal{T}[S]} = 1_{\mathcal{T}} e_G$ (see Definitions A1 and A2). We put $X^\sigma_g = \sum_{a \in \theta^{-1}(g)} X_a$ (see also Definition A3). This implies that X^σ_g is a $\mathcal{T}[S]$-bimodule for each $g \in P$, since $\theta^{-1}(g) = S$, while X_a is the $\mathcal{T}$-bimodule. From $X_g \neq X_e$ for each $g \in G - S$ and $(A16)$–$(A19)$ it follows that for each $g \neq h$ in P the intersection $X^\sigma_g \cap X^\sigma_h$ is null. Therefore X, considered as the $\mathcal{T}[S]$-bimodule, is isomorphic with the direct sum $\bigoplus_{g \in P} X^\sigma_g$ of $\mathcal{T}[S]$-bimodules. From Theorem 1 and Remark 1 it follows that $\mathcal{T}[S]$-bimodules $(X \otimes_A Y)^\sigma_r$ and $\bigoplus_{h \in P, s \in P,\ hs = r} (X^\sigma_h \otimes_{\mathcal{T}[S]} Y^\sigma_s)$ are isomorphic for each $r \in P$. □

Proposition 2. *If the conditions of Proposition 1 are satisfied, ϕ and ψ are G-exact, then $\phi \otimes \psi$ is G-exact.*

Proof. Since $\phi_1 : G \to G$ is an automorphism and $\phi_1(g) = \psi_1(\theta_1(g))$ for each $g \in G$, then

(41) $\phi_1 \times \psi_1 : G \times G \to G \times G$ is an automorphism of the direct product $G \times G$ of the metagroup G.

From Theorem 1 and Definition A3 we deduce that

(42) $(X_j \otimes_A Y_j)_g$ is isomorphic as the $\mathcal{T}$-bimodule with
$\sum_{h \in G,\ s = h \backslash g} X_{j,h} \otimes_{\mathcal{T}} Y_{j,s}$ for each j.

Thus (41), (42), conditions $(A16)$, $(A17)$, (14), and (15) induce a bijective surjective map $(\phi \otimes \psi)_1 : G \to G$. Since $\phi_1 \times \psi_1 : G \times G \to G \times G$ is an automorphism and ϕ and ψ also possess properties of $\mathcal{T}$-homomorphisms, then $(\phi \otimes \psi)_1$ is an automorphism of G. From this and Proposition 1 it follows that $\phi \otimes \psi$ is a G-exact homomorphism. □

Proposition 3. *Assume that X_1, X_2, Y_1, and Y_2 are A-bimodules, where $A = \mathcal{T}[G]$ is a metagroup algebra. Assume also that P is a quotient metagroup $P := G/\cdot/S$ of a metagroup G by a subgroup S such that $S \subseteq \mathcal{C}(G)$ and $X_{k,g} \neq X_{k,e}$ and $Y_{k,g} \neq Y_{k,e}$ for each $g \in G - S$, $k \in \{1, 2\}$. Assume also that $\phi : X_1 \to X_2$ and $\psi : Y_1 \to Y_2$ are their P-graded $\mathcal{T}[S]$-homomorphisms such that $\phi_1(g) = \psi_1(\theta_1(g))$ for each $g \in P$, where θ_1 is an automorphism of P. Then a unique P-graded $\mathcal{T}[S]$-homomorphism exists*

*(43) $\phi \otimes \psi : X_1 \otimes_A Y_1 \to X_2 \otimes_A Y_2$ such that $(\phi \otimes \psi)(x \otimes y) = \phi(x) \otimes \psi(y)$
for each $x \in X_1$ and $y \in Y_1$. Moreover, if ϕ and ψ are P-exact, then $\phi \otimes \psi$ is P-exact.*

Proof. In view of Theorem 2 X_1, X_2, Y_1, Y_2 also have structures of P-graded $\mathcal{T}[S]$-bimodules. Note that $\mathcal{T}[S] X^\sigma_{1,p} = X^\sigma_{1,p}$ for each $p \in P$, since $\mathcal{T}[S]$ is a commutative associative unital ring. Therefore Theorem 1, Proposition 1 and 2 and Remark 1 imply the assertion of this proposition. □

Theorem 3. *Let X, Y, X_1, and Y_1 be A-bimodules, where $A = \mathcal{T}[G]$ is a metagroup algebra. Then there exist A-exact $\mathcal{T}$-isomorphisms*

$\eta : X \otimes_A (Y \oplus Y_1) \to (X \otimes_A Y) \oplus (X \otimes_A Y_1)$ of A-bimodules and
$\rho : (X \oplus X_1) \otimes_A Y \to (X \otimes_A Y) \oplus (X_1 \otimes_A Y)$ of A-bimodules
such that

(44) $\eta(x \otimes (y, y_1)) = (x \otimes y, x \otimes y_1)$ and
$\rho((x, x_1) \otimes y) = (x \otimes y, x_1 \otimes y)$ for each $x \in X$, $x_1 \in X_1$, $y \in Y$ and $y_1 \in Y_1$;
there exists a G-exact $\mathcal{T}$-isomorphism

$\nu : (X \otimes_A X_1) \otimes_A Y \to X \otimes_A (X_1 \otimes_A Y)$ of A-bimodules such that

(45) $\nu((x_g \otimes x_{1,h}) \otimes y_s) = t_3(g, h, s) x_g \otimes (x_{1,h} \otimes y_s)$ for each g, h, and s in G, $x_g \in X_g$, $x_{1,h} \in X_{1,h}$ and $y_s \in Y_s$;

(46) *for each g and h in G there exists an isomorphism of $\mathcal{T}$-bimodules $\mu_{g,h} : X_g \otimes_{\mathcal{T}} Y_h \rightarrow Y_g \otimes_{\mathcal{T}} X_h$;*

(47) *there are A-exact $\mathcal{T}$-isomorphisms*
$X \otimes_A A \cong X$ and $A \otimes_A X \cong X$ of A-bimodules.

Proof. Note that $X \otimes_A (Y_1 \oplus Y_2)$ and $(X \otimes_A Y_1) \oplus (X \otimes_A Y_2)$ are isomorphic as $\mathcal{T}$-bimodules, because the ring $\mathcal{T}$ is associative and commutative. On the other hand, $\eta((bx_e) \otimes (y, y_1)) = b(x_e \otimes y, x_e \otimes y_1)$ and $\eta(x \otimes ((y_e, y_{1,e})b)) = (x \otimes y_e, x \otimes y_{1,e})b$ for each $b \in A$, $x \in X$, $x_e \in X_e$, $y \in Y$, $y_e \in Y_e$, $y_1 \in Y_1$ and $y_{1,e} \in Y_{1,e}$. In view of Proposition 1 $X \otimes_A (Y_1 \oplus Y_2)$ and $(X \otimes_A Y_1) \oplus (X \otimes_A Y_2)$ are isomorphic as A-bimodules. Using Definition A3, conditions (14) and (15) it is sufficient to take in (6) and (7) a homomorphism $\eta_1(a) = a$ for each $a \in A$, since $(bx_e) \otimes (y, y_1) = b(x_e \otimes (y, y_1))$ and $x \otimes ((y_e, y_{1,e})b) = (x \otimes (y_e, y_{1,e}))b$ and $(x_e g) \otimes (y, y_1) = x_e \otimes (gy, gy_1)$ and $x \otimes (gy_e, gy_{1,e}) = (xg) \otimes (y_e, y_{1,e})$ for each $g \in G$. Thus the isomorphism η is A-exact. Symmetrically it is proved that the isomorphism ρ is A-exact.

Since $\mathcal{T} \subset \mathcal{T}[\mathcal{C}(G)] \subset A$, then $(X \otimes_A X_1) \otimes_A Y$ and $X \otimes_A (X_1 \otimes_A Y)$ are isomorphic as $\mathcal{T}$-bimodules. Note that (42) implies

(48) $((X \otimes_A X_1) \otimes_A Y)_v = \sum_{g\in G,\, h\in G} (X_g \otimes_{\mathcal{T}} X_{1,h}) \otimes_{\mathcal{T}} Y_{(gh)\setminus v}$

for each v in G. We put $\nu_1(g) = g$ for each $g \in G$. Therefore, the latter formula and (48) induce a G-exact isomorphism ν satisfying (45).

Assertion (46) follows from $(A16)$–$(A19)$ and the conditions of this proposition, since $X_e \otimes_{\mathcal{T}} Y_e$ and $Y_e \otimes_{\mathcal{T}} X_e$ are isomorphic $\mathcal{T}$-bimodules.

There exists a left and right $\mathcal{T}$-linear homomorphism ξ from $X \otimes_A A$ onto X such that $\xi(x \otimes a) = xa$ for each $x \in X$ and $a \in A$, since A is the unital algebra and $x1 = x$ for each $x \in X$, because for unital rings modules are supposed to be unital according to Definition A3. On the other side, there exists a homomorphism $\omega : X \rightarrow X \otimes_A A$ such that $\omega(x) = x \otimes 1$ for each $x \in X$. Therefore, $\omega(\xi(x \otimes a)) = (xa) \otimes 1 = x \otimes a$ by (18) and $\xi(\omega(x)) = x$ for each $x \in X$ and $a \in A$. It is sufficient to put $\xi_1(a) = a$ and $\omega_1(a) = a$ for each $a \in A$. Thus ξ is A-exact. Similarly there exists an A-exact isomorphism $A \otimes_A X \cong X$ of A-bimodules. $\square$

Theorem 4. *Assume that X_1, X_2, X_3, and Y are A-bimodules, where $A = \mathcal{T}[G]$ is a metagroup algebra. Assume also that $X_1 \underset{f^1}{\rightarrow} X_2 \underset{f^2}{\rightarrow} X_3 \rightarrow 0$ is an exact sequence with G-graded homomorphisms f^1 and f^2. Then a sequence $X_1 \bigotimes_A Y \underset{s^1}{\rightarrow} X_2 \bigotimes_A Y \underset{s^2}{\rightarrow} X_3 \bigotimes_A Y \rightarrow 0$ is exact with G-graded homomorphisms s^1 and s^2 such that $s^j|_{X_j \otimes_{\mathcal{T}} Y_e} = f^j \otimes id_{Y_e}$ and $s^j_1 = f^j_1$, where $j \in \{1, 2\}$, $id_{Y_e}(y_e) = y_e$ for each $y_e \in Y_e$.*

Proof. Apparently the formulas $s^j|_{X_j \otimes_{\mathcal{T}} Y_e} = f^j \otimes id_{Y_e}$, $s^j_1 = f^j_1$, $(A16)$–$(A19)$, (4), (5) and (16)–(21) induce G-graded homomorphisms $s^j : X_j \bigotimes_A Y \rightarrow X_{j+1} \bigotimes_A Y$, where $j \in \{1, 2\}$. Notice that $s^2(X_2 \bigotimes_A Y)$ contains all tensors of rank 1, that is $x_3 \otimes y$ with $x_3 \in X_3$ and $y \in Y$. Then $f^1_1(G)$ is a submetagroup in G isomorphic with $G/\cdot/Ker(f^1_1)$ (see Remark 1). Evidently $f^2 \circ f^1 = 0$ implies $s^2 \circ s^1 = 0$, consequently, $Im(s^1) \subseteq Ker(s^2)$. Therefore we get

$Ker(f^2) \bigotimes_A Y = Im(f^1) \bigotimes_A Y \subseteq Im(f^1 \otimes id_Y)$,

where $Im(f^1) = f^1(X_1)$, $Ker(f^2) = (f^2)^{-1}(0)$. For $X_2 \bigotimes_A Y$ considered as the $\mathcal{T}$-bimodule there exists a natural projection π of $X_2 \bigotimes_A Y$ onto a quotient $\mathcal{T}$-bimodule $V := (X_2 \bigotimes_A Y)/Im(s^1)$, because $Im(s^1)$ is a $\mathcal{T}$-bimodule contained in $X_2 \bigotimes_A Y$. Therefore the map $\Phi(x_3, y) := \pi((f^2)^{-1}x_3 \otimes y)$ is $\mathcal{T}$-bilinear from $X_3 \times Y$ into V, where $x_3 \in X_3$ and $y \in Y$. This induces a homomorphism μ from $X_3 \bigotimes_A Y$ onto V considered as $\mathcal{T}$-bimodules such that $\mu(x_3 \otimes y) = \pi(((f^2)^{-1}x_3) \otimes y)$, hence $\mu(f^2(x_2) \otimes y) = \pi(x_2 \otimes y)$ for each $x_2 \in X_2$ and $y \in Y$. In view of Lemma 1 $\mu \circ s^2 = \pi$. Therefore $Ker(s^2) \subseteq Ker(\pi) = Im(s^1)$. Thus $Ker(s^2) = Im(s^1)$. $\square$

Corollary 3. *Suppose that X_1, X_2, X_3, and Y are A-bimodules, where $A = \mathcal{T}[G]$ is a metagroup algebra. Suppose also that $0 \to X_1 \underset{f^1}{\to} X_2 \underset{f^2}{\to} X_3 \to 0$ is a splitting exact sequence with G-exact homomorphisms f^1 and f^2. Then a sequence*

$0 \to X_1 \bigotimes_A Y \underset{s^1}{\to} X_2 \bigotimes_A Y \underset{s^2}{\to} X_3 \bigotimes_A Y \to 0$ is exact with G-exact homomorphisms s^1 and s^2 such that $s^1 = f^1 \otimes id_Y$ and $s^2 = f^2 \otimes id_Y$.

Proof. There exists a homomorphism $\nu : X_2 \to X_1$ such that $\nu \circ f^1 = id_{X_1}$, because the sequence $0 \to X_1 \underset{f^1}{\to} X_2 \underset{f^2}{\to} X_3 \to 0$ splits. Therefore, the homomorphism ν is also G-exact such that $\nu_1 = (f^1_1)^{-1}$, since the homomorphism f^j is G-exact for each j. In view of Theorem 4 the sequence $X_1 \bigotimes_A Y \underset{s^1}{\to} X_2 \bigotimes_A Y \underset{s^2}{\to} X_3 \bigotimes_A Y \to 0$ is exact with G-exact homomorphisms s^1 and s^2 and $(\nu \otimes id_Y)(f^1 \otimes id_Y) = id_{X_1 \bigotimes_A Y}$, consequently, s^1 is injective. □

Theorem 5. *Let X and Y be A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra. Then on the A-bimodule $X \bigotimes_A Y$ there exists a multiplication satisfying the following conditions:*

(49) $(ax_1 \otimes y_1)(x_2 \otimes y_2 b) = a(x_1x_2) \otimes (y_1y_2)b$

for each x_1 and x_2 in X, y_1 and y_2 in Y, a and b in $\mathcal{T}[N(G)]$;

(50) $((x_{1,g_1} \otimes y_{1,h_1})(x_{2,g_2} \otimes y_{2,h_2}))(x_{3,g_3} \otimes y_{3,h_3}) =$

$\mathrm{t}_3(g_1, g_2, g_3)(x_{1,g_1} \otimes y_{1,h_1})((x_{2,g_2} \otimes y_{2,h_2})(x_{3,g_3} \otimes y_{3,h_3}))\mathrm{t}_3(h_1, h_2, h_3)$

for each $x_{j,g_j} \in X_{g_j}$, $y_{j,h_j} \in Y_{h_j}$, g_j and h_j in G, $j \in \{1, 2, 3\}$;

(51) $1_X \otimes 1_Y = 1_{X \otimes_A Y}$ *for unital algebras X and Y.*

Proof. By virtue of Proposition 1 $(aL_{x_{1,g_1}} \otimes L_{y_{1,h_1}})(x_2 \otimes y_2 b) = a(x_{1,g_1}x_2) \otimes (y_{1,h_1}y_2)b$ for each $x_{1,g_1} \in X_{g_1}$ and x_2 in X, $y_{1,h_1} \in Y_{h_1}$ and y_2 in Y, a and b in $\mathcal{T}[N(G)]$, where $L_{x_1}x_2 = x_1x_2$, $aL_{x_1} = (aI_X)L_{x_1}$, $L_{x_1}a = L_{x_1}(aI_X)$, $I_X x_1 = x_1$ for each $x_1 \in X$. Evidently, $L_{ax_1+x_2b} = aL_{x_1} + L_{x_2}b$ for each x_1 and x_2 in X, a and b in $\mathcal{T}[N(G)]$. Therefore $aL_{x_{1,g_1}} \otimes L_{y_{1,h_1}}$ is a G-graded endomorphism of $X \bigotimes_A Y$. This induces a $\mathcal{T}$-bilinear map $X \times Y \ni (x_1, y_1) \mapsto L_{x_1} \otimes L_{y_1} \in Hom_{\mathcal{T}}(U, U)$, where $Hom_{\mathcal{T}}(V, W)$ denotes the family of all $\mathcal{T}$-bilinear homomorphisms from V into W, where V and W are $\mathcal{T}$-bimodules, U is $X \bigotimes_A Y$ considered as the $\mathcal{T}$-bimodule. Thus there exists a $\mathcal{T}$-bilinear homomorphism of $\mathcal{T}$-bimodules $f : U \to Hom_{\mathcal{T}}(U, U)$ such that $f(x_1 \otimes y_1) = L_{x_1} \otimes L_{y_1}$ for each $x_1 \in X$ and $y_1 \in Y$. We define a map $\mu : U \times U \to U$ by the following formula $\mu(u, v) = f(u)(v)$ for each u and v in U. The construction of μ and identities (16)–(22) imply that μ is $\mathcal{T}$-bilinear and satisfies (49) and (50). Thus μ is the multiplication on $X \bigotimes_A Y$. From Lemma 1 it follows that $1_X \otimes 1_Y$ is the unit element in $X \bigotimes_A Y$, if X and Y are unital algebras. □

Corollary 4. *Let W, X, and Y be A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra. Then*

(52) *the A-algebra $(W \oplus X) \bigotimes_A Y$ is isomorphic with*

$(W \bigotimes_A Y) \oplus (X \bigotimes_A Y)$;

(53) $(w_g \otimes x_h) \otimes y_s = \mathrm{t}_3(g, h, s)w_g \otimes (x_h \otimes y_s)$ *for every $w_g \in W_g$, $x_h \in X_h$, $y_s \in Y_s$, g, h, and s in G;*

(54) *the $\mathcal{T}$-algebra $W_e \bigotimes_{\mathcal{T}} X_e$ is isomorphic with $X_e \bigotimes_{\mathcal{T}} W_e$;*

(55) *the A-algebras $W \bigotimes_A A$ and $A \bigotimes_A W$ are isomorphic with W.*

Proof. This follows from Theorems 3 and 5 and Lemma 1. □

Lemma 2. *Suppose that X and Y are unital A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra. Suppose also that $f_X : X \to X \bigotimes_A Y$ and $f_Y : Y \to X \bigotimes_A Y$ are maps such that $f_X(x) = x \otimes 1_Y$ and $f_Y(y) = 1_X \otimes y$ for each $x \in X$ and $y \in Y$. Then f_X and f_Y are A-homomorphisms such that $f_X(X) \cup f_Y(Y)$ generates $X \bigotimes_A Y$ and $f_X(x)f_Y(y) = f_Y(y)f_X(x)$ for each $x \in X$ and $y \in Y$. Particularly, if $\mathcal{T} = F$ is a field, then f_X and f_Y are injective.*

Proof. From (49) and (50) it follows that f_X and f_Y are A-homomorphisms such that $f_X(x)f_Y(y) = x \otimes y$ and $f_Y(y)f_X(x) = x \otimes y$ for each $x \in X$ and $y \in Y$, since $f_X(ax) = (ax) \otimes 1_Y = a(x \otimes 1_Y)$ and $f_X(xa) = (xa) \otimes 1_Y = x \otimes (1_Y a) = (x \otimes 1_Y)a$ for each $x \in X$ and $a \in A$. Therefore $f_X(X) \cup f_Y(Y)$ generates $X \bigotimes_A Y$, because the set $\{x \otimes y : x \in X, y \in Y\}$ generates the tensor product $X \bigotimes_A Y$ of algebras X and Y over A.

If $\mathcal{T} = F$ is a field, then the restriction of f_X on X_e is injective, $f_X : X_e \to X_e \bigotimes_F Y_e$. On the other hand, $X_e \bigotimes_F Y_e \subset X \bigotimes_A Y$ and $f_X(\sum_{g \in G} x_g) = \sum_{g \in G} f_X(x_g)$ for each $x = \sum_{g \in G} x_g$ in X, where $x_g \in X_g$ for each $g \in G$. For each $g \in G$ the F-bimodule X_g is F-isomorphic with $X_e g$, consequently, the homomorphism $f_X : X \to X \bigotimes_A Y$ is injective. $\square$

Definition 3. *Assume that X is an A-algebra and $B \subseteq X$, where $A = \mathcal{T}[G]$ is a metagroup algebra. We put*

(56) $Com_X(B) := \{x \in X : \forall b \in B, xb = bx\}$;

(57) $N_{X,l}(B) := \{x \in X : \forall b \in B, \forall c \in B, (xb)c = x(bc)\}$;

(58) $N_{X,m}(B) := \{x \in X : \forall b \in B, \forall c \in B, (bx)c = b(xc)\}$;

(59) $N_{X,r}(B) := \{x \in X : \forall b \in B, \forall c \in B, (bc)x = b(cx)\}$;

(60) $N_X(B) := N_{X,l}(B) \cap N_{X,m}(B) \cap N_{X,r}(B)$ *and*

(61) $\mathsf{C}_X(B) := Com_X(B) \cap N_X(B)$.

Then $Com_X(B)$, $N_X(B)$, and $\mathsf{C}_X(B)$ are called a commutant, a nucleus and a centralizer correspondingly of the algebra X relative to a subset B in X. Instead of $Com_X(X)$, $N_X(X)$, or $\mathsf{C}_X(X)$ it will be also written shortly $Com(X)$, $N(X)$, or $\mathsf{C}(X)$ correspondingly. We put $B_e = B \cap X_e$ and $B_{\mathsf{C}} := B \cap X_{\mathsf{C}}$, where $X_{\mathsf{C}} = \sum_{g \in \mathsf{C}(G)} X_g$.

Lemma 3. *Let X and Y be A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra. Let also $Y \subseteq X$, $B \subseteq X$, $D \subseteq X$ and $E \subset X$ such that $B_e A \subseteq B \subseteq B_e \bigotimes_{\mathcal{T}} A$ and $D_e A \subseteq D \subseteq D_e \bigotimes_{\mathcal{T}} A$. Then*

(62) $\mathsf{C}_X(B)$ *is a $\mathcal{T}$-subalgebra in X;*

(63) $N_X(B)$ *is a $\mathcal{T}[N(G)]$-subalgebra in X;*

(64) *if $D \subseteq B$, then $\mathsf{C}_X(B) \subseteq \mathsf{C}_X(D)$ and $N_X(B) \subseteq N_X(D)$;*

(65) $B_{\mathsf{C}} \subseteq \mathsf{C}_X(D)$ *if and only if* $D_{\mathsf{C}} \subseteq \mathsf{C}_X(B)$;

(66) $Y \cap \mathsf{C}_X(Y) = \mathsf{C}_Y(Y)$;

(67) $Com_X(E) = X$ *if and only if* $E \subseteq Com_X(X)$;

(68) *if $E \subseteq N_X(X)$, then $N_X(E) = X$ and*

(69) *if $E \subseteq \mathsf{C}_X(X)$, then $\mathsf{C}_X(E) = X$.*

Proof. From Formulas (1), (60), (61), and $(A13)$–$(A19)$, it follows that $a_1x_1 + a_2x_2 \in \mathsf{C}_X(B)$ and $a_1(x_1x_2) \in \mathsf{C}_X(B)$ for each x_1 and x_2 in $\mathsf{C}_X(B)$, a_1 and a_2 in $\mathcal{T}$, since $B_e A \subseteq B \subseteq B_e \bigotimes_{\mathcal{T}} A$. This implies (62).

If x_1 and x_2 in $N_X(B)$, a_1 and a_2 in $\mathcal{T}[N(G)]$, then $a_1x_1 + a_2x_2$, $a_1(x_1x_2)$ and $x_1a_1 + x_2a_2$, $(x_1x_2)a_1$ belong to $N_X(B)$ by Formulas (57)–(60), $(A2)$–$(A5)$ and Definition A2, since $B_e A \subseteq B \subseteq B_e \bigotimes_{\mathcal{T}} A$ and $D_e A \subseteq D \subseteq D_e \bigotimes_{\mathcal{T}} A$. This implies (63). Then (64) evidently follows from (60) and (61).

Let $B_{\mathsf{C}} \subseteq \mathsf{C}_X(D)$. If $b \in B_{\mathsf{C}}$, then there exist $g_1,\dots,g_n$ in $\mathsf{C}(G)$ such that $b = b_{g_1} + \dots + b_{g_n}$, since $B_e A \subseteq B \subseteq B_e \bigotimes_{\mathcal{T}} A$, where $b_{g_i} \in B_{g_i}$ for each i, $B_g := B \cap X_g$. Therefore $bc = cb$, $b(cd) = (bc)d$, $(cb)d = c(bd)$ and $(cd)b = c(db)$ for each c and d in D. From $B_e A \subseteq B \subseteq B_e \bigotimes_{\mathcal{T}} A$ and $D_e A \subseteq D \subseteq D_e \bigotimes_{\mathcal{T}} A$ it follows that

$span_{\mathcal{T}} B_{\mathsf{C}} D = span_{\mathcal{T}} B D_{\mathsf{C}}$ and $span_{\mathcal{T}} D_{\mathsf{C}} B = span_{\mathcal{T}} D B_{\mathsf{C}}$,

where $span_{\mathcal{T}} S$ denotes the family of all elements $v_1 s_1 + \dots + v_m s_m$ in X with $s_j \in S$ and $v_j \in \mathcal{T}$ for each j; $BD := \{bd : b \in B, d \in D\}$. This together with $(A16)$–$(A19)$ and (1) implies that $D_{\mathsf{C}} \subseteq \mathsf{C}_X(B)$. Symmetrically, $D_{\mathsf{C}} \subseteq \mathsf{C}_X(B)$ implies $B_{\mathsf{C}} \subseteq \mathsf{C}_X(D)$ and hence (65).

From (56)–(60) it follows that $Y \cap Com_X(Y) = Com_Y(Y)$ and $Y \cap N_X(Y) = N_Y(Y)$, consequently, $Y \cap \mathsf{C}_X(Y) = \mathsf{C}_Y(Y)$ giving (66).

The equality $Com_X(E) = X$ is equivalent to $ax = xa$ for each $x \in X$ and $a \in E$. That is $Com_X(E) = X \Leftrightarrow E \subseteq Com_X(X)$. Thus assertion (67) is proven.

Then (68) follows from (57)–(60). Finally (61), (67), and (68) imply (69). □

Theorem 6. *Let W, X, and Y be A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra. Let $\xi : X \to W$ and $\eta : Y \to W$ be left and right A-generic homomorphisms correspondingly such that $\eta(Y_e) \subseteq N(W)$, $\xi(X_e) \subseteq N(W)$, and $\eta(Y_e) \subseteq Com_{N(W)}(\xi(X_e))$. Then there exists a unique homomorphism $\nu : X \bigotimes_A Y \to W$ such that $\nu : X \bigotimes_{\mathcal{T}} Y_e \to W$ and $\nu : X_e \bigotimes_{\mathcal{T}} Y \to W$ are left and right A-generic homomorphisms of a left and right A-modules correspondingly and $\nu(x \otimes y) = \xi(x)\eta(y)$ for each $x \in X$ and $y \in Y$.*

Proof. From the conditions of this proposition and Definition 1 it follows that
$\xi((ab)x_e) = \xi_1(a)\xi_1(b)\xi(x_e) = \xi_1(a)\xi(bx_e)$ and
$\eta(y_e(ab)) = \eta(y_e)\eta_1(a)\eta_1(b) = \eta(y_e a)\eta_1(b)$
for each a and b in A, $x_e \in X_e$, $y_e \in Y_e$, consequently,
$\xi((ab)x_e)\eta(y_e) = \xi_1(a)\xi_1(b)\xi(x_e)\eta(y_e) = \xi_1(a)\xi(bx_e)\eta(y_e)$ and
$\xi(x_e)\eta(y_e(ab)) = \xi(x_e)\eta(y_e)\eta_1(a)\eta_1(b) = \xi(x_e)\eta(y_e a)\eta_1(b)$.
Then we put $\nu(x \otimes y) = \xi(x)\eta(y)$ for each $x \in X$ and $y \in Y$. Therefore, we infer that
$\nu((ab)(x_e \otimes y_e)) = \xi_1(a)\nu((bx_e) \otimes y_e)$ and
$\nu((x_e \otimes y_e)(ab)) = \nu(x_e \otimes (y_e a))\eta_1(b)$
for each a and b in A, $x_e \in X_e$, $y_e \in Y_e$. Then we deduce that
$\nu(((ab)x_{1,e} \otimes y_{1,e})(x_{2,e} \otimes y_{2,e})) = \xi_1(a)\xi_1(b)\nu((x_{1,e} \otimes y_{1,e})(x_{2,e} \otimes y_{2,e})) =$
$\xi_1(a)\xi_1(b)\nu(x_{1,e}x_{2,e} \otimes y_{1,e}y_{2,e}) =$
$\xi_1(a)\nu((bx_{1,e}x_{2,e}) \otimes y_{1,e}y_{2,e})$ and
$\nu((x_{1,e} \otimes y_{1,e})(x_{2,e} \otimes y_{2,e}(ab))) =$
$\nu((x_{1,e} \otimes y_{1,e})(x_{2,e} \otimes y_{2,e}))\eta_1(a)\eta_1(b) = \nu(x_{1,e}x_{2,e} \otimes (y_{1,e}y_{2,e}a))\eta_1(b)$.
From $\mathcal{T}$-linearity of homomorphisms ξ and η it follows that ν is also $\mathcal{T}$-linear. The latter together with the identities given above lead to the conclusion that $\nu : X \bigotimes_{\mathcal{T}} Y_e \to W$ and $\nu : X_e \bigotimes_{\mathcal{T}} Y \to W$ are left and right A-generic homomorphisms of a left and right A-modules correspondingly. □

Corollary 5. *Let V, W, X, and Y be A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra. Let $\xi : X \to V$ and $\eta : Y \to W$ be left and right A-generic homomorphisms correspondingly such that $\xi(X_e) \subseteq N(V)$, $\eta(Y_e) \subseteq N(W)$. Then there exists a left and right A-generic homomorphism $\xi \otimes \eta : X \bigotimes_A Y \to V \bigotimes_A W$. If $\mathcal{T} = F$ is field and homomorphisms ξ and η are injective, then the homomorphism $\xi \otimes \eta$ is also injective.*

Proof. Note that there exists an isomorphism of $\mathcal{T}$-algebras:

$$N(V \bigotimes_A W) \simeq \sum_{g \in G,\, h \in G,\, gh \in N(G)} (V_g \bigotimes_{\mathcal{T}} W_h).$$

Therefore $N(V) \bigotimes_{\mathcal{T}} N(W)$ is contained in $N(V \bigotimes_A W)$. This inclusion together with Lemma 2 and Theorems 5, 6 imply the assertion of this corollary. □

Proposition 4. *Assume that W, X, and Y are A-algebras, where $A = F[G]$ is a metagroup algebra over a field F. Then there exists an isomorphism of A-algebras $W \simeq (X \bigotimes_A Y)$ if and only if the following conditions are fulfilled:*

(70) *W contains A-subalgebras U and V such that there exist A-exact isomorphisms $\xi : X \to U$ and $\eta : Y \to V$;*

(71) *$V_e \subseteq Com_{N(W)}(U_e)$;*

(72) *there exist bases $\{u_i : i \in \Omega\}$ and $\{v_j : j \in \Upsilon\}$ of the F-algebras U_e and V_e such that $\{(su_i)((s \setminus e)v_j) : i \in \Omega, j \in \Upsilon, s \in S\}$ is a basis of W_e, where Ω and Υ are sets, where S is a coset of representatives of classes $q(G \cap F)$ in G, $q \in G$.*

If W_e is finite dimensional over F, then condition (72) can be substituted with:

(73) *W_e as the F-algebra is generated by $U_e \cup V_e$ and $dim_F W_e = dim_F U_e\ dim_F V_e$ card $(G/\cdot/(F \cap G))$, where $dim_F U_e$ denotes a dimension of U_e over the field F.*

Proof. In view of Lemma 2 conditions (70)–(73) are necessary (see also Remark 1). Vice versa if conditions (70) and (71) are satisfied, then there exists a homomorphism of A-algebras $\nu : X \bigotimes_A Y \to W$ such that $\nu(x \otimes y) = \xi(x)\eta(y)$ for each $x \in X$ and $y \in Y$ by Theorem 6. From condition (72) and Lemma 2 it follows that ν is an isomorphism.

If $dim_F W_e < \aleph_0$ and condition (73) is fulfilled, then ν is surjective, since $\nu(X \bigotimes_A Y)$ is a A-subalgebra in W containing $U \cup V$. By the conditions of this proposition

$G = \bigcup_{s \in S} s(G \cap F)$ such that $s(G \cap F) \cap q(G \cap F) = \emptyset$

for each $s \neq q$ in S. Notice that

$dim_F W_e = dim_F U_e \, dim_F V_e \, card(G/\cdot/(F \cap G)) =$

$dim_F(X_e \bigotimes_F Y_e) \, card(G/\cdot/(F \cap G)) = dim_F(X \bigotimes_A Y)_e$ and $W = \sum_{g \in G} W_g$

with $W_g = \sum_{s \in G}(X_s \bigotimes_F Y_{s \setminus g})$, where W_g is isomorphic with $W_e g$ and with gW_e as the F-algebra. By virtue of Lemma 2 the homomorphism ν is injective. □

Corollary 6. *Let G_1 and G_2 be two metagroups and F be a field such that $card(G_j/\cdot/(F \cap G_j)) < \aleph_0$ for each $j \in \{1,2\}$. Let $G = G_1 \times G_2$ be the direct product of metagroups and let X and Y be associative algebras over F. Then a metagroup algebra $A = F[G]$ is isomorphic with $A_1 \bigotimes_F A_2$, where $A_j = F[G_j]$ for each $j \in \{1,2\}$. Moreover, $(X \bigotimes_F A) \bigotimes_F Y$ and $(X \bigotimes_F A_1) \bigotimes_F (A_2 \bigotimes_F Y)$ can be supplied with A-exact F-isomorphic structures of A-algebras.*

Proof. Since $ab = ba$ for each $a \in G_1$ and $b \in G_2$, then $A_{1,e} \subseteq Com_{N(A)}(A_{2,e})$. The union $A_1 \cup A_2$ generates A as the F-algebra. In view of Proposition 4, A and $A_1 \bigotimes_F A_2$ are isomorphic as the F-algebras. We put $(a_1, b_1)((x \otimes (a_2, b_2)) \otimes y) = (x \otimes (a_1 a_2, b_1 b_2)) \otimes y$ and $((x \otimes (a_2, b_2)) \otimes y)(a_1, b_1) = (x \otimes (a_2 a_1, b_2 b_1)) \otimes y$ and $(a_1, b_1)((x \otimes a_2) \otimes (b_2 \otimes y)) = (x \otimes (a_1 a_2)) \otimes ((b_1 b_2) \otimes y)$ and $((x \otimes a_2) \otimes (b_2 \otimes y))(a_1, b_1) = (x \otimes (a_2 a_1)) \otimes ((b_2 b_1) \otimes y)$ for each $a_j \in A_1$ and $b_j \in A_2$, $j \in \{1,2\}$, $x \in X$ and $y \in Y$. By F-linearity we extend the latter formulas from $(a_j, b_j) \in A$ on any $a \in A$, because A and $A_1 \bigotimes_F A_2$ are isomorphic as the F-algebras. This supplies $(X \bigotimes_F A) \bigotimes_F Y$ and $(X \bigotimes_F A_1) \bigotimes_F (A_2 \bigotimes_F Y)$ with A-algebras structures, which are A-exact isomorphic by Theorem 3, Proposition 4, and Corollary 4. □

Lemma 4. *Let X and Y be A-algebras, U be a right X-module, and V be a right Y-module, let also U and V be A-bimodules, where $A = \mathcal{T}[G]$ is a metagroup algebra. Then $U \bigotimes_A V$ can be supplied with a structure of a right $X \bigotimes_A Y$-module such that*

(74) $(u \otimes v)(xa \otimes yb) = (ux)a \otimes (vy)b$ *for each* $u \in U, \ v \in V, \ x \in X, \ y \in Y, \ a$ *and* b *in* $N(A)$*;*

(75) $((u_{g_1} \otimes v_{h_1})(x_{2,g_2} \otimes y_{2,h_2}))(x_{3,g_3} \otimes y_{3,h_3}) =$

$(u_{g_1} \otimes v_{h_1})((x_{2,g_2} \otimes y_{2,h_2})(x_{3,g_3} t_3(g_1, g_2, g_3) \otimes y_{3,h_3} t_3(h_1, h_2, h_3))$

for each $u_{g_1} \in U_{g_1}, \ v_{h_1} \in V_{h_1}, \ x_{j,g_j} \in X_{g_j}, \ y_{j,h_j} \in Y_{h_j}, \ g_j$ *and* h_j *in* G*,* $j \in \{1,2,3\}$*;*

(76) *if the algebras X and Y are unital and the modules U and V are unital, then the right $X \bigotimes_A Y$-module $U \bigotimes_A V$ is also unital.*

Proof. By virtue of Theorem 5 $X \bigotimes_A Y$ is a A-algebra. In view of Proposition 1 $(R_a R_x \otimes R_b R_y)(u \otimes v) = (ux)a \otimes (vy)b$ for each $u \in U, \ v \in V, \ x \in X, \ y \in Y, \ a$ and b in $N(A)$, where $R_x u = ux, \ R_a R_x u = (ux)a = u(xa) = R_{xa} u$ for each $u \in U, \ x \in X$, $a \in N(A)$. This implies that $R_{ax_1 + x_2 b} = R_{x_1} R_a + R_b R_{x_2}$ for each x_1 and x_2 in X, a and b in $N(A)$. Therefore $R_a R_{x_{1,g_1}} \otimes R_{y_{1,h_1}}$ is a G-graded $\mathcal{T}$-endomorphism of $U \bigotimes_A V$ as the right A-module. This provides a $\mathcal{T}$-bilinear map $X \times Y \ni (x_1, y_1) \mapsto R_{x_1} \otimes R_{y_1} \in Hom_{\mathcal{T}}(W, W)$, where $W = U \bigotimes_A V$. Therefore there exists a $\mathcal{T}$-bilinear homomorphism of $\mathcal{T}$-bimodules $f : W \to Hom_{\mathcal{T}}(W, W)$ such that $f(x_1 \otimes y_1) = R_{x_1} \otimes R_{y_1}$ for each $x_1 \in X$ and $y_1 \in Y$. The latter property and Proposition 4 give (74).

From Proposition 1 it follows that W is the A-bimodule. Then identity (75) follows from (74) and Remark 1. Thus identities (74) and (75) supply W with a right $X \bigotimes_A Y$-module structure.

By virtue of Theorem 5 $1_X \otimes 1_Y$ is the unit element in $X \bigotimes_A Y$, if X and Y are unital algebras. Therefore if modules U and V are unital, then W is also unital. □

Theorem 7. *Let X_1, X_2, Y_1, and Y_2 be A-bimodules and X_1, X_2 be right D-modules, Y_1 and Y_2 be right B-modules, where B and D are A-algebras, $A = \mathcal{T}[G]$ is a metagroup algebra, $\mathcal{T}$ is a commutative associative unital ring. Then*

(77) *if $\xi \in Hom_D(X_1, X_2)$ and $\eta \in Hom_B(Y_1, Y_2)$, then*
$\xi \otimes \eta \in Hom_{D \bigotimes_A B}(X_1 \bigotimes_A Y_1, X_2 \otimes_A Y_2)$,
where $Hom_D(X_1, X_2)$ denotes a $\mathcal{T}$-bimodule of all right D-homomorphism from X_1 into X_2 of right D-modules;

(78) *a map $(\xi, \eta) \mapsto \xi \otimes \eta$ induces a $\mathcal{T}$-homomorphism of $\mathcal{T}$-bimodules:*
$\omega : Hom_D(X_1, X_2) \bigotimes_{\mathcal{T}} Hom_B(Y_1, Y_2) \to Hom_{D \bigotimes_A B}(X_1 \bigotimes_A Y_1, X_2 \otimes_A Y_2)$;

(79) $\omega : E_D(X_1) \bigotimes_{\mathcal{T}} E_B(Y_1) \to E_{D \bigotimes_A B}(X_1 \bigotimes_A Y_1)$ *is a $\mathcal{T}$-homomorphism of $\mathcal{T}$-algebras, where $E_D(X_1) := Hom_D(X_1, X_1)$ and*

(80) *if D is unital, then each $\xi \in Hom_D(X_1, X_2)$ is A-exact.*

Proof. Let $\xi \in Hom_D(X_1, X_2)$ and $\eta \in Hom_B(Y_1, Y_2)$. Then
$(\xi \otimes \eta)((x \otimes y)(d \otimes b)) = \xi(xd) \otimes \eta(yb)$
$= (\xi(x)d) \otimes (\eta(y)b) = ((\xi \otimes \eta)(x \otimes y))(d \otimes b)$
for each $x \in X_1$, $y \in Y_1$, $d \in D$, $b \in B$. From Lemmas 1 and 4 it follows that $\xi \otimes \eta \in Hom_{D \bigotimes_A B}(X_1 \bigotimes_A Y_1, X_2 \otimes_A Y_2)$.

The map $(\xi, \eta) \mapsto \xi \otimes \eta$ is $\mathcal{T}$-bilinear. In view of Proposition 1 the $\mathcal{T}$-homomorphism ω exists.

Let $\xi_j \in E_D(X_1)$ and $\eta_j \in E_B(Y_1)$ for each $j \in \{1, 2\}$. Let also $(\xi_j \otimes' \eta_j)$ denotes a tensor of rank 1 in $E_D(X_1) \bigotimes_{\mathcal{T}} E_B(Y_1)$. Then the assertion (79) follows from Theorem 5 and (78), since
$\omega((\xi_1 \otimes' \eta_1)(\xi_2 \otimes' \eta_2)) = \omega(\xi_1\xi_2 \otimes' \eta_1\eta_2) = (\xi_1\xi_2 \otimes \eta_1\eta_2) = (\xi_1 \otimes \eta_1)(\xi_2 \otimes \eta_2) = \omega(\xi_1 \otimes' \eta_1)\omega(\xi_2 \otimes' \eta_2)$.

Note that if the algebra D is unital, then there exists an embedding of A into D as $A1_D$, where 1_D is a unit element in D. Therefore in this case each $\xi \in Hom_D(X_1, X_2)$ is A-exact by (6), (7), $(A16)$. □

Lemma 5. *Let B and D be unital A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra, $\mathcal{T}$ is a commutative associative unital ring. Let also X be a right B-module and Y be a (B, D)-bimodule. Then their tensor product $X \bigotimes_B Y$ is a right D-module such that*

(81) $(x_g \otimes y_h)d_s = x_g \otimes (y_h(\mathrm{t}_3(g, h, s)d_s))$ *for each $x_g \in X_g$, $y_h \in Y_h$, $d_s \in D_s$, g, h, and s in G.*

Proof. We put $\Phi_d(x, y) = x \otimes (yd)$ for each $x \in X$, $y \in Y$, $d \in D$. The latter formula defines a map $\Phi_d : X \times Y \to X \bigotimes_B Y$. This map is $\mathcal{T}[\mathcal{C}(G)]$-bilinear and

(82) $\Phi_{d_s}(x_g b_q, y_h) = (x_g b_q) \otimes (y_h d_s) =$
$x_g \otimes ((b_q y_h)[\mathrm{t}_3(g, q, hs)/\mathrm{t}_3(q, h, s)]d_s) = \Phi_z(x_g, (b_q y_h))$ with
$z = [\mathrm{t}_3(g, q, hs)/\mathrm{t}_3(q, h, s)]d_s$ for each $x_g \in X_g$, $y_h \in Y_h$, $b_q \in B_q$, $d_s \in D_s$, g, h, q and s in G. Then we deduce that

(83) $\Phi_{d\alpha + f\beta} = \Phi_d\alpha + \Phi_f\beta$ and

(84) $x \otimes ((y_h d_s)f_q) = x \otimes (y_h(\mathrm{t}_3(h, s, q)d_s f_q))$
for each α and β in $\mathcal{T}[\mathcal{C}(G)]$, d and f in D, $x \in X$, $y_h \in Y_h$, $d_s \in D_s$, $f_q \in D_q$, h, s, q in G. Note that $(X \bigotimes_B Y)_v$ is a $\mathcal{T}$-bimodule generated by sums of elements of the form $(x_g b_q) \otimes y_h$ with $x_g \in X_g$, $b_q \in B_q$, $y_h \in Y_h$, g, q, and h in G such that $(gq)h = v$, where $v \in G$. Thus Formulas (81)–(84) supply $X \bigotimes_B Y$ with a structure of a right D-module. On the other hand, the metagroup algebra A has embeddings $A1_B$ in B and $A1_D$ in D, where 1_B denotes the unit element in B. Hence the tensor product $X \bigotimes_B Y$ is G-graded by Theorem 1. □

Remark 2. *Similarly to Lemma 5, if X is a (B, D)-bimodule and Y is a left D-module, then $X \bigotimes_D Y$ is a left B-module. Then if X is a (B, D)-bimodule and Y is a (D, J)-module, where B, D and J are unital A-algebras, then $X \bigotimes_D Y$ is a (B, J)-bimodule such that*

(85) $((b_r x_g) \otimes y_h)j_s = \mathrm{t}_3(r, g, h)\mathrm{t}_3(r, gh, s)\mathrm{t}_3(g, h, s)b_r(x_g \otimes (y_h j_s))$

for each $b_r \in B_r$, $x_g \in X_g$, $y_h \in Y_h$, $j_s \in J_s$, r, g, h, and s in G. Moreover, if $a \in \mathcal{T}[N(G)]$, $x \in X$, $y \in Y$, then

(86) $a(x \otimes y) = (ax) \otimes y$ *and* $(x \otimes y)a = x \otimes (ya)$.

In particular, if D is a subalgebra in B, then B can be considered as a (D, B)-bimodule. For a right D-module X then $X \bigotimes_D B$ is a right B-module, which is called a module induced from X and it is denoted by X^B.

Definition 4. *Assume that B is a unital A-algebra, where $A = \mathcal{T}[G]$ is a metagroup algebra, $\mathcal{T}$ is a commutative associative unital ring. If X and Y are right B-modules and isomorphic as right $\mathcal{T}[\mathcal{C}(G)]$-modules and such that $X_e \bigotimes_{\mathcal{T}} \mathcal{T}[\mathcal{C}(G)]$ and $Y_e \bigotimes_{\mathcal{T}} \mathcal{T}[\mathcal{C}(G)]$ are isomorphic right $\mathcal{T}[\mathcal{C}(G)]$-modules, then we say that X and Y are meta-isomorphic. A meta-isomorphism for left or two-sided D-modules is similarly defined.*

Proposition 5. *Suppose that B and D are unital A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra over a commutative associative unital ring $\mathcal{T}$. Suppose also that X is a right B-module, Y is a (B, D)-bimodule, Z is a left D-module such that X, Y and Z also have structures of A-bimodules. Then $(X \bigotimes_B Y) \bigotimes_D Z$ and $X \bigotimes_B (Y \bigotimes_D Z)$ are A-bimodules which are meta-isomorphic.*

Proof. By virtue of Lemma 5 and Remark 2 $(X \bigotimes_B Y) \bigotimes_D Z$ and $X \bigotimes_B (Y \bigotimes_D Z)$ are A-bimodules. From identities (85), (86) and the inclusion $t_3(g, h, s) \in \mathcal{C}(G)$ for each g, h, and s in G it follows that $(X \bigotimes_B Y) \bigotimes_D Z$ and $X \bigotimes_B (Y \bigotimes_D Z)$ are isomorphic as $\mathcal{T}[\mathcal{C}(G)]$-bimodules, also $((X \bigotimes_B Y) \bigotimes_D Z) \bigotimes_{\mathcal{T}} \mathcal{T}[\mathcal{C}(G)]$ and $(X \bigotimes_B (Y \bigotimes_D Z)) \bigotimes_{\mathcal{T}} \mathcal{T}[\mathcal{C}(G)]$ are isomorphic $\mathcal{T}[\mathcal{C}(G)]$-bimodules. □

Theorem 8. *Let B, C, and D be unital A-algebras, let D be a subalgebra in B and B be a subalgebra in C, where $A = \mathcal{T}[G]$ is a metagroup algebra, $\mathcal{T}$ is an associative commutative unital ring. Let also X and Y be right D-modules. Then*

(87) $(X \oplus Y)^B$ *and* $X^B \oplus Y^B$ *are isomorphic as right B-modules;*

(88) $(X^B)^C$ *and* X^C *are meta-isomorphic right C-modules and*

(89) X^D *is a right D-module meta-isomorphic with X.*

Proof. Assertion (87) follows from Lemma 5 and the proof is similar to that of Theorem 3 and Corollary 4, since $(x \oplus (yd))b = (xb) \oplus ((yd)b)$ and $((xd) \oplus y)b = ((xd)b) \oplus (yb)$ for each $x \in X$, $y \in Y$, $d \in D$ and $b \in B$.

In view of Proposition 5 and Remark 2

(90) $(x_g \otimes b_h) \otimes c_s = x_g \otimes (b_h \otimes c_s)t_3(g, h, s)$ and $(x_g \otimes (b_h d_r)) \otimes c_s = x_g \otimes ((b_h d_r) \otimes c_s)t_3(g, hr, s)$ for each $x_g \in X_g$, $b_h \in B_h$, $d_r \in D_r$, $c_s \in C_s$, g, h, r, s in G. Consider the map $w : b_h \otimes c_s \mapsto 1 \otimes (b_h c_s) \in 1 \otimes C_{hs}$, consequently,

$w((a_g b_h) \otimes c_s) = 1 \otimes ((a_g b_h)c_s) = 1 \otimes (a_g(b_h c_s)t_3(g, h, s)) = w(a_g \otimes (b_h c_s t_3(g, h, s)))$

for each $a_g \in B_g$, $b_h \in B_h$, $c_s \in C_s$, g, h, and s in G. Therefore $B \otimes_B C$ is meta-isomorphic with C, because the algebras B and C are unital (see Definition 4). The latter meta-isomorphism and identities (90) imply assertion (88). Then $x \otimes (dv) = (x(dv)) \otimes 1$ for each $x \in X$, d and v in D; $(x_g \otimes d_h) \otimes v_s = x_g \otimes (d_h \otimes v_s)t_3(g, h, s)$ for each $x_g \in X_g$, $d_h \in D_h$, $v_s \in D_s$, g, h and s in G. On the other hand, $(x1)d = x(1d) = xd$ for each $x \in X$ and $d \in D$. Thus X^D is a right D-module meta-isomorphic with X. □

3. Extensions of Nonassociative Algebras with Metagroup Relations

Definition 5. *Let B and D be A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra and $\mathcal{T}$ is a commutative associative unital ring. Let also $f : B \to D$ be an A-exact homomorphism (see Definition 1). For a right D-module X we put $x_g b_h := \xi(x_g) f(b_h)$ for each $x_g \in X_g$ and $b_h \in B_h$, g and h in G, where $\xi(x_g) := ((x_g)/g) f_1(g)$, $(x_g)/g = x_e$ with $x_e g = x_g$, $x_e \in X_e$, because $X_g = X_e g$. This provides on X a structure of a right B-module denoted by X_B, because $f_1 : G \to G$ is an automorphism of the metagroup G. In particular, if B is a subalgebra in D and f is an embedding, then the correspondence $X \to X_B$ is called a restriction functor.*

Lemma 6. *Let B, C, and D be unital A-algebras and let D be a subalgebra in B, where $A = \mathcal{T}[G]$ is a metagroup algebra and $\mathcal{T}$ is a commutative associative unital ring. Let also X be a right D-module and Y be a right B-module. Then there exist A-exact homomorphisms $\nu_X : X \to (X^B)_D$ of right D-modules and $\mu_Y : (Y_D)^B \to Y$ of right B-modules such that*

(91) $\nu_X(x) = x \otimes 1_B$ for each $x \in X$ and

(92) $\mu_Y(y \otimes 1_B) = y$ for each $y \in Y$.

Moreover, if X is a (C, D)-bimodule, then ν_X is an A-exact homomorphism of left C-modules; if Y is a (C, B)-bimodule, then μ_Y is an A-exact homomorphism of left C-modules.

Proof. The A-algebras B, C, and D are unital, consequently, there are natural embeddings of A into them $A \hookrightarrow A1_B$, etc. Evidently formula (91) defines a homomorphism of $\mathcal{T}$-modules. Then $\nu_X(xd) = (xd) \otimes 1_B = x \otimes d = (x \otimes 1_B)d = \nu_X(x)d$ for each $x \in X$ and $d \in D$ by Lemma 5 and Theorem 8. Thus ν_X is an A-exact homomorphism of right D-modules.

For a (C, D)-bimodule X formula (91) means that ν_X is an A-exact homomorphism of left C-modules.

Let $\eta : Y_D \times B \to Y$ be a mapping such that $\eta(y, b) = yb$ for each $y \in Y$ and $b \in B$. Apparently the mapping η is $\mathcal{T}$-bilinear and G-balanced (see Definition 1). Therefore there exists a homomorphism of $\mathcal{T}$-modules $\mu_Y : Y_D \bigotimes_D B \to Y$ such that $\mu_Y(y \otimes b) = yb$ for each $y \in Y$ and $b \in B$. By virtue of Lemma 5 and Remark 2 μ_Y is an A-exact homomorphism of right B-modules. If Y is a (C, B)-bimodule, then evidently μ_Y is an A-exact homomorphism of left C-modules, since $\mu_{Y,1}|_{A1_C} = id|_{A1_C}$. □

Definition 6. *Let X be a (B, B)-bimodule, where B is an A-algebra, $A = \mathcal{T}[G]$ (see Definitions A3 and A5). Let $X \times X \ni (u, v) \mapsto uv \in X$ be a mapping such that*

(93) $u(v + w) = uv + uw$ and $(v + w)u = vu + wu$ and $(bu)v = b(uv)$ and $(ub)v = u(bv)$ and $(uv)b = u(vb)$
for all u, v, w in X, $b \in \mathcal{T}$;

(94) $(u_g v_h)w_s = \mathrm{t}_3(g, h, s)u_g(v_h w_s)$ and $u_g v_h \in X_{gh}$ and

(95) $(x_s v_h)u_g = \mathrm{t}_3(s, h, g)x_s(v_h u_g)$ and $(u_g x_s)v_h = \mathrm{t}_3(g, s, h)u_g(x_s v_h)$ and
$(u_g v_h)x_s = \mathrm{t}_3(g, h, s)u_g(v_h x_s)$ and $u_g x_s \in X_{gs}$ and $x_s u_g \in X_{sg}$ and

(96) $(x_s y_h)u_g = \mathrm{t}_3(s, h, g)x_s(y_h u_g)$ and
$(u_g x_s)y_h = \mathrm{t}_3(g, s, h)u_g(x_s y_h)$ and
$(y_h u_g)x_s = \mathrm{t}_3(h, g, s)y_h(u_g x_s)$
for every g, h, s in G, $u_g \in X_g$, $v_h \in X_h$, $w_s \in X_s$, $x_s \in B_s$, $y_h \in B_h$. Then X will be called a multiplicative two-sided B-module.

We use the following notation for left ordered products: $\{u_1\}_{l(1)} = u_1$, $\{u_1 u_2\}_{l(2)} = u_1 u_2$, $\{u_1 \dots u_{k+1}\}_{l(k+1)} = \{u_1 \dots u_k\}_{l(k)} u_{k+1}$ for each $k \geq 2$ and $u_1, \dots, u_{k+1}$ in X. If $\{u_1 \dots u_k\}_{l(k)} = 0$ for each $u_1, \dots, u_k$ in X, where $k \geq 2$, then the multiplicative two-sided B-module X will be called k-nilpotent.

If M and X are two multiplicative two-sided B-modules and $\phi : M \to X$ is an additive mapping $\phi(u + v) = \phi(u) + \phi(v)$ which is left $\mathcal{T}$-homogeneous $\phi(bu) = b\phi(u)$ or right $\mathcal{T}$-homogeneous $\phi(ub) = \phi(u)b$ for each $b \in \mathcal{T}$, u and v in M, then ϕ is called left $\mathcal{T}$-linear or right $\mathcal{T}$-linear respectively. If ϕ is left and right $\mathcal{T}$-linear, then it is called $\mathcal{T}$-linear. If ϕ is $\mathcal{T}$-linear and $\phi(u_g v_h) = \zeta(g, h)\phi(u_g)\phi(v_h)$ with $\zeta(g, h) \in \mathcal{T}[\mathcal{C}(G)] - \{0\}$ for each g and h in G, $u_g \in M_g$, $v_h \in M_h$, then ϕ will be called a metahomomorphism.

Remark 3. *A multiplication in a multiplicative two-sided B-module X induces a homomorphism $\mu \in Hom_{\mathcal{T}}(X \otimes_B X, X)$ satisfying the following conditions:*

(97) $\mu(u \otimes (v + w)) = \mu(u \otimes v) + \mu(u \otimes w)$ and
$\mu((v + w) \otimes u) = \mu(v \otimes u) + \mu(w \otimes u)$ and
$\mu((bu) \otimes v) = b\mu(u \otimes v)$ and $\mu((ub) \otimes v) = \mu(u \otimes (bv))$ and
$\mu(u \otimes (vb)) = \mu(u \otimes v)b$ for all u, v, w in X, $b \in \mathcal{T}$;

(98) $\mu(\mu(u_g \otimes v_h) \otimes w_s) = \mu((\mathrm{t}_3(g, h, s)u_g) \otimes \mu(v_h \otimes w_s))$ and

(99) $\mu((x_s v_h) \otimes u_g) = \mu((\mathsf{t}_3(s,h,g)x_s)(v_h \otimes u_g))$ *and*
$\mu((u_g x_s) \otimes v_h) = \mu((\mathsf{t}_3(g,s,h)u_g) \otimes (x_s v_h))$ *and*
$\mu((u_g \otimes v_h)x_s) = \mu((\mathsf{t}_3(g,h,s)u_g) \otimes (v_h x_s))$
for every g, h, s *in* G, $u_g \in X_g$, $v_h \in X_h$, $w_s \in X_s$, $x_s \in B_s$.

Vice versa, each homomorphism satisfying conditions (97)–(99) *induces on the two-sided* B*-module* X *a multiplicative structure. As* $B^e := B \otimes_{\mathcal{T}} B^{op}$ *usually denotes the enveloping algebra of* B, B^{op} *notates the opposite algebra of* B.

Next we take an A*-algebra* C *and its* A*-subalgebra* B *and an ideal* X *in* C*, where* $A = \mathcal{T}[G]$ *is a metagroup algebra. Then a multiplication in* C *provides a structure of a two-sided* B*-module* $X \times B \ni (u,b) \mapsto ub$, $B \times X \ni (b,u) \mapsto bu$ *and a multiplication* $(X \times X) \ni (u,v) \mapsto uv \in X$ *in* X *for each* u *and* v *in* X, $b \in B$. *This construction makes of* X *a multiplicative two-sided* B*-module.*

Proposition 6. *Assume that* X *is a multiplicative two-sided* B*-module with* B *being a unital* A*-algebra, where* $A = \mathcal{T}[G]$ *is a metagroup algebra. Put* $X \uplus B$ *to be* $X \oplus B$ *as an* A*-module and define a multiplication in it by:*

(100) $(w,b)(v,a) = (wv + bv + wa, ba)$ *for each* w, v *in* X, a *and* b *in* B. *Then*

(101) $X \uplus B$ *is the* A*-algebra with a unit element* $(0,1)$;

(102) $B' := \{(0,b) : b \in B\}$ *is a subalgebra in* $X \uplus B$ *and a mapping* $(0,b) \mapsto b$ *is an* A*-exact isomorphism of* A*-algebras* B' *with* B;

(103) $X' := \{(w,0) : w \in X\}$ *is an ideal in* $X \uplus B$ *and a mapping* $(w,0) \mapsto w$ *is an* A*-exact isomorphism of multiplicative two-sided* B*-modules* X' *with* X;

(104) $X \uplus B = X' \oplus B'$.

Proof. Since B is the A-algebra and $1 \in B$ (see Definition 6), then X also has a structure of a two-sided A-module. Evidently, A is embedded into $X \uplus B$ as $(0,A)$. Therefore, the multiplication rule (100) shows $X \uplus B$ is the A-algebra with the unit element $(0,1)$. Then from (100) and (101) assertions (102) and (103) evidently follow, since $(0,b)(0,a) = (0,ba)$; $(w,0)(v,a) = (wv + wa, 0) \in X'$ and $(w,b)(v,0) = (wv + bv, 0) \in X'$ for each w, v in X and a, b in B. The assertion (104) follows from (100), (102), and (103). □

Proposition 7. *Let* C *be an* A*-algebra and let* B *be its unital* A*-subalgebra, where* $A = \mathcal{T}[G]$ *is a metagroup algebra. Let* X *be an ideal in* C *and* $C = X \oplus B$ *as two-sided* A*-modules. Then* X *is a multiplicative two-sided* B*-module relative to operations induced from* C *and* $C = X \uplus B$.

Proof. From the conditions of this lemma and Remark 3 it follows that X is a multiplicative two-sided B-module. A mapping $p : (w,b) \mapsto w + b$ for each $w \in X$ and $b \in B$ provides an isomorphism between two-sided A-modules $X \uplus B$ and C. Indeed, $C = X \oplus B$ and $p((w,b)(v,a)) = p(wv + bv + wa, ba) = wv + bv + wa + ba = (w+b)(v+a) = p(w,b)p(v,a)$ for each w and v in X, a and b in B. Then $p((w_g,b_g)(v_h,a_h)) = (w_g + b_g)(v_h + a_h) = p(w_g,b_g)p(v_h,a_h)$ for each $w_g \in X_g$, $v_h \in X_h$, $b_g \in B_g$, $a_h \in B_h$, g, and h belonging to G. Therefore this mapping p is an isomorphism of C with $X \uplus B$. □

Definition 7. *If the conditions of Proposition 7 are satisfied, then* C *is called a splitting extension of* X *with the help of* B.

If an algebra C *over a metagroup algebra* $A = \mathbf{F}[G]$ *is such that* C_e *is finite dimensional over a field* $\mathbf{F}$*, then it will be said that* C *is finite dimensional over* A*, where* e *is a unit element in* G.

Remark 4. *A direct sum* $X \oplus B$ *of two* A*-algebras may usually be not a splitting extension, because* B *may usually be not a subalgebra in* $X \oplus B$*, since* $1_{X \oplus B} \neq 1_B$ *if* $X \neq 0$.

Proposition 8. *Let* C *be an* A*-algebra finite dimensional over* A*, where* $A = \mathbf{F}[G]$ *is a metagroup algebra over a field* $\mathbf{F}$.

(105) *If* $B = C/J(C)$ *is separable, then a radical* $J(C)$ *is a nilpotent multiplicative two-sided* B*-module and* C *is isomorphic with* $J(C) \uplus B$.

(106) *If B is a semisimple A-algebra, X is a nilpotent multiplicative two-sided A-module and $C = X \uplus B$, then X is isomorphic with a radical $J(C)$ and B is isomorphic with $C/J(C)$.*

Proof. This follows from Theorem 3 in [13] and Propositions 6 and 7 above. □

Theorem 9. *Let B and D be two unital A-algebras such that B and D also have structure of separable algebras over a field $\mathbf{F}$, where $A = \mathbf{F}[G]$ is a metagroup algebra. Let X and M be a multiplicative two-sided B-module and D-module respectively.*

(i). If $X \uplus B$ is isomorphic with $M \uplus D$ as $\mathbf{F}$-algebras and two-sided A-modules, then there exists an A-exact isomorphism of A-algebras $\theta : B \to D$ and an A-exact isomorphism of two-sided A-modules $\psi : X \to M$ such that
(107) $\psi(uy) = \psi(u)\psi(y)$,
(108) $\psi(bu) = \theta(b)\psi(u)$,
(109) $\psi(ub) = \psi(u)\theta(b)$,
(110) $(\psi(v_g)\psi(u_h))\psi(y_s) = t_3(g,h,s)\psi(v_g)(\psi(u_h)\psi(y_s))$ *and*
(111) $(\theta(a_g)\theta(b_h))\theta(c_s) = t_3(g,h,s)\theta(a_g)(\theta(b_h)\theta(c_s))$
for each u and y in X and $b \in B$; g, h, and s in G; $v_g \in X_g$, $u_h \in X_h$, $y_s \in X_s$; $a_g \in B_g$, $b_h \in B_h$, $c_s \in B_s$.

(ii). If $\theta : B \to D$ is an A-exact isomorphism of A-algebras and $\psi : X \to M$ is an A-exact isomorphism of two-sided A-modules such that conditions (107)–(111) *are satisfied, then $X \uplus B$ is A-exact isomorphic with $M \uplus D$.*

Proof. (i). The metagroup algebra A is embedded into the unital A-algebras B and D as $A1_B$ and $A1_D$ correspondingly, where 1_B is the unit element in B. Let $f : X \uplus B \to M \uplus D$ be an isomorphism of $X \uplus B$ with $M \uplus D$ as $\mathbf{F}$-algebras and A-exact as two-sided A-modules. By virtue of Proposition 8 $f(X) = f(J(X \uplus B)) = J(M \uplus D) = M$ and $M \uplus D = J(M \uplus D) \oplus f(B)$. In view of Proposition 8 $(1-w)D = f(A)(1-w)$ for some $w \in M$ such that $(1-w)$ has a left inverse $(1-w)_l$ and a right inverse $(1-w)_r$, that is $(1-w)_l(1-w) = 1$ and $(1-w)(1-w)_r = 1$.

On the other hand, we have
$X \uplus B = \sum_{g \in G}(X \uplus B)_g$
(see Definition A3). Note that X_e and M_e is a multiplicative two-sided B_e-module and D_e-module respectively, since X, M, B and D are G-graded, B_e and D_e are F-algebras, X_e and M_e are F-bimodules. From Definition 6 it follows that X_e, M_e, B_e, and D_e are associative relative to the multiplication and addition. In view of Proposition 7 $(X \uplus B)_e$ is isomorphic with $(M \uplus D)_e$ as $\mathbf{F}$-algebras. From Proposition 8 it follows that that there exists $v \in M_e$ with left and right invertible $1-v$ such that $(1-v)D_e = f_e(B_e)(1-v)$, where $f_e : (X \uplus B)_e \to (M \uplus D)_e$ denotes an isomorphism of the $\mathbf{F}$-algebras. From $f(gu) = gf(u)$ and $f(ug) = f(u)g$ for each $u \in (X \uplus B)_e$ and $g \in G$ it follows that f_e is the restriction of f on $(X \uplus B)_e$. Since $v \in M_e$, then the left and right inverses of $1-v$ coincide, because M_e is the associative multiplicative two-sided B_e-module. Therefore $((1-v)^{-1}g)(1-v) = (1-v)^{-1}(g(1-v))$ for each $g \in G$. We put $H = (1-v)^{-1}G(1-v) := \{h : h = (1-v)^{-1}g(1-v),\ g \in G\}$. Hence H and G are isomorphic metagroups. This implies that $(1-v)D(1-v)^{-1}$ is isomorphic with $f(B)$ as the $\mathbf{F}$-algebras and two-sided A-modules, since

$$f((a_g b_h)c_s) = (f(a_g)f(b_h))f(c_s) = t_3(g,h,s)f(a_g)(f(b_h)f(c_s)) = t_3(g,h,s)f(a_g(b_h c_s))$$

for each $a_g \in B_g$, $b_h \in B_h$, $c_s \in B_s$ and g, h, s in G. Hence there exists and isomorphism $\phi : X \uplus B \to M \uplus D$ as $\mathbf{F}$-algebras and as two-sided A-modules, ϕ is A-exact such that $(1-v)D(1-v)^{-1} = \phi(B)$, because $v \in M_e$ and $e \in \mathcal{C}(G) \subseteq \mathcal{C}(A)$.

We put a morphism $\theta : B \to D$ as algebras over $\mathbf{F}$ and two-sided A-modules to be $(1-v)\theta(b) = \phi(b)(1-v)$ for each $b \in B$. Let a morphism $\psi : X \to M$ as $\mathbf{F}$-linear spaces and two-sided A-modules be given by the following equality $(1-v)\psi(u) = \phi(u)(1-v)$ for each $u \in X$. Then $(1-v)(X \uplus B)_e$ and $(X \uplus B)_e$ are isomorphic as $\mathbf{F}$-linear spaces, since

$$(1-v)^{-1}[(1-v)(X \uplus B)_e] = [(1-v)^{-1}(1-v)](X \uplus B)_e.$$

Therefore $(1-v)(X \uplus B)$ and $X \uplus B$ are isomorphic as **F**-linear spaces and as two-sided A-modules they are A-exact isomorphic, since $[(1-v)(X \uplus B)_e]g = (1-v)(X \uplus B)_g$ for each $g \in G$, because $t_3(h,e,g) = e$ for each h and g in G. Hence ψ is the isomorphism of X and M as **F**-linear spaces and as two-sided A-modules ψ is A-exact. Then

$(1-v)\psi(uy) = (\phi(u)\phi(y))(1-v)$ and

$(1-v)(\psi(u)\psi(y)) = (1-v)(\psi(u)(1-v)^{-1}((1-v)\psi(y))) = (\phi(u)\phi(y))(1-v)$

for each u and y in X. This implies Identity (107). Then from

$\theta(\alpha a + \beta b) = \alpha(1-v)^{-1}\phi(a)(1-v) + \beta(1-v)^{-1}\phi(b)(1-v)$ and

$\theta(ab) = (1-v)^{-1}(\phi(a)\phi(b))(1-v) = ((1-v)^{-1}\phi(a)(1-v))((1-v)^{-1}\phi(b)(1-v)) = \theta(a)\theta(b)$

for each a and b in B and α and β in **F** it follows that $\theta : B \to D$ is the isomorphism of them as the **F**-algebras and as two-sided A-modules the isomorphism θ is A-exact, because ϕ is A-exact and $v \in M_e$. Then

$\psi(bu) = (1-v)^{-1}\phi(bu)(1-v) = ((1-v)^{-1}\phi(b)(1-v))((1-v)^{-1}\phi(u)(1-v))$

for each $b \in B$ and $u \in X$, consequently, Identity (108) is satisfied. Similarly is verified Identity (109).

For each v_g, u_h, y_s, a_g, b_h, c_s satisfying the conditions of this theorem we infer that

$(\psi(v_g)\psi(u_h))\psi(y_s) = [((1-v)^{-1}\phi(v_g)(1-v))((1-v)^{-1}\phi(u_h)(1-v))]$
$((1-v)^{-1}\phi(y_s)(1-v)) = (1-v)^{-1}\phi((v_g u_h)y_s)(1-v)$
$= t_3(g,h,s)((1-v)^{-1}\phi(v_g)(1-v))[((1-v)^{-1}\phi(u_h)(1-v))$
$((1-v)^{-1}\phi(y_s)(1-v))] = t_3(g,h,s)\psi(v_g)(\psi(u_h)\psi(y_s))$ and
$(\theta(a_g)\theta(b_h))\theta(c_s) = [((1-v)^{-1}\phi(a_g)(1-v))((1-v)^{-1}\phi(b_h)(1-v))]$
$((1-v)^{-1}\phi(c_s)(1-v)) = (1-v)^{-1}\phi((a_g b_h)c_s)(1-v) =$
$t_3(g,h,s)((1-v)^{-1}\phi(a_g)(1-v))[((1-v)^{-1}\phi(b_h)(1-v))$
$((1-v)^{-1}\phi(c_s)(1-v))] = t_3(g,h,s)\theta(a_g)(\theta(b_h)\theta(c_s))$,

since $v \in M_e$. Thus identities (110) and (111) are satisfied.

(ii). Vice versa if $\theta : B \to D$ is an A-exact isomorphism of A-algebras and $\psi : X \to M$ is an A-exact isomorphism of two-sided A-modules such that conditions (107)–(111) are satisfied, then $\phi(u \uplus b) = \psi(u) \uplus \theta(b)$ for each $u \in X$ and $b \in B$ provides an A-exact isomorphism of the A-algebras $X \uplus B$ and $M \uplus D$. □

Theorem 10. *Assume that B and D are A-algebras, where $A = \mathcal{T}[G]$ is a metagroup algebra, $\mathcal{T}$ is an associative commutative unital ring. Assume also that D is a subalgebra of B. Then the following conditions are equivalent:*

(112) *$B = D \oplus Y$, where Y is a (D,D)-bisubmodule in B,*

(113) *for each A-algebra C and each (C,D)-bimodule X a homomorphism $\nu_X : X \to (X^B)_D$ is a splitting injective A-exact homomorphism of (C,D)-bimodules.*

Proof. (112) $\Rightarrow$ (113). From (112) it follows that there exists an A-exact homomorphism $f : B \to D$ such that $f|_D = id_D$, where $id_D(d) = d$ for each $d \in D$. Therefore a mapping $p : X \times B \to X$ such that $p(x,b) = xf(b)$ for each $x \in X$ and $b \in B$ is A-exact and G-balanced (see Definition 1). Thus there exists a homomorphism $w : X^B \to X$ such that $w(x \otimes b) = xf(b)$ for each $x \in X$ and $b \in B$. Evidently w is an A-exact homomorphism of left D-modules and an A-exact homomorphism of right D-modules, since f is the A-exact homomorphism of right D-modules. On the other hand, $1_B = 1_D \in D$, consequently, $w\nu_X(x) = w(x \otimes 1_B) = x1_B = x$ for each $x \in X$ (see Lemma 6). This means that ν_X is an A-exact splitting injective homomorphism.

Vice versa (113) implies (112) by taking in particular $C = D$ and $X = D$ considered as (D,D)-bimodules. □

4. Conclusions

In this article new specific tensor products of nonassociative algebras and modules induced over nonassociative algebras with metagroup relations are investigated (see also

Formulas (1)–(113) above). In particular, their splitting extensions also are scrutinized. Their radicals and separability are studied. For this purpose, their cohomologies are used.

The obtained results can be used for further studies of the structure of nonassociative algebras, modules, and homological complexes over them, their tensor products, ideals, extensions, homomorphisms. On the other side, smashed twisted products and smashed twisted wreath products of metagroups or groups were described in [14]. They also provide tools for the construction of a wide class of nonassociative algebras, modules, and homological complexes over them with metagroup relations. With the help of the results presented above it also is possible to continue investigations of nonassociative generalized Cayley–Dickson algebras cohomologies, noncommutative geometry, algebraic geometry, operator theory, spectral theory, PDEs, their applications in the sciences, etc. [6,8,10,15,18,20,22,24–26]. It can also be applied in information technologies for the classification of flows of information [27,28].

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Appendix A

For the convenience of readers we review definitions from previous works [12,13,20]. Though a reader familiar with them may skip these definitions.

Definition A1. *Let G be a set with a single-valued binary operation (multiplication) $G^2 \ni (a,b) \mapsto ab \in G$. Then we can define the following:*

$(A1)$ $Com(G) := \{a \in G : \ \forall b \in G, \ ab = ba\}$;

$(A2)$ $N_l(G) := \{a \in G : \ \forall b \in G, \ \forall c \in G, \ (ab)c = a(bc)\}$;

$(A3)$ $N_m(G) := \{a \in G : \ \forall b \in G, \ \forall c \in G, \ (ba)c = b(ac)\}$;

$(A4)$ $N_r(G) := \{a \in G : \ \forall b \in G, \ \forall c \in G, \ (bc)a = b(ca)\}$;

$(A5)$ $N(G) := N_l(G) \cap N_m(G) \cap N_r(G)$;

$(A6)$ $\mathsf{C}(G) := Com(G) \cap N(G)$.

We will say that G is a metagroup if it satisfies conditions $(A7)$–$(A11)$:

$(A7)$ *for each a and b in G there is a unique $x \in G$ with $ax = b$ and*

$(A8)$ *a unique $y \in G$ exists satisfying $ya = b$.*

These elements are denoted by $x = a \setminus b$ and $y = b/a$ correspondingly.

$(A9)$ *There exists a neutral (i.e., unit) element $e_G \in G$, which will be shortly denotes by e instead of e_G;*

$(A10)$ $\forall a \in G, \forall b \in G, \forall c \in G$ *one has* $(ab)c = \mathsf{t}_3(a,b,c)a(bc)$,
where $\mathsf{t}_3(a,b,c) \in \mathbf{\Psi}, \ \mathbf{\Psi} \subset \mathsf{C}(G)$;

$(A11)$ $\mathbf{\Psi}$ *is a subgroup of* $\mathsf{C}(G)$.

If moreover the following is satisfied:

$(A12)$ $\forall a \in G, \ \forall b \in G, \ ab = \mathsf{t}_2(a,b)ba$, *where* $\mathsf{t}_2(a,b) \in \mathbf{\Psi}$,
we will say that G is a central metagroup.

Definition A2. *Let $\mathcal{T}$ be an associative unital ring and A a $(\mathcal{T},\mathcal{T})$-bimodule. Then A is said to be a $\mathcal{T}$-algebra if it is endowed with a map $A \times A \to A$ which is right and left distributive $a(b+c) = ab + ac$, $(b+c)a = ba + ca$ and satisfies the following identities $r(ab) = (ra)b$, $(ar)b = a(rb)$, $(ab)r = a(br)$, $s(ra) = (sr)a$, and $(ar)s = a(rs)$ for any a, b, and c in A, r, and s in $\mathcal{T}$.*

Let G be a metagroup. Let also the algebra A consist of all formal sums $s_1a_1 + \ldots + s_na_n$, where $s_1,\ldots,s_n$ are in $\mathcal{T}$ and $a_1,\ldots,a_n$ belong to G, where n is an arbitrary natural number, $n \in \mathbf{N} = \{1,2,3,\ldots\}$. Suppose that A satisfies the following conditions $(A13)$–$(A15)$:

$(A13)$ $sa = as$ *for each* $s \in \mathcal{T}$ *and* $a \in G$;

$(A14)$ $s(ra) = (sr)a$ *for each s and r in* $\mathcal{T}$, *and* $a \in G$;

$(A15)$ $r(ab) = (ra)b, \ (ar)b = a(rb), \ (ab)r = a(br)$ *for each a and b in G,* $r \in \mathcal{T}$.

Then A will be denoted by $\mathcal{T}[G]$ and called a metagroup algebra over $\mathcal{T}$.

Henceforth, the ring $\mathcal{T}$ will be supposed commutative, if something other will not be specified.

Definition A3. *Suppose that $\mathcal{R}$ is a ring, which may be nonassociative relative to the multiplication. Suppose also that M is an additive commutative group. If there exists a mapping $\mathcal{R} \times M \to M$, $\mathcal{R} \times M \ni (a, m) \mapsto am \in M$ such that $a(m+k) = am + ak$ and $(a+b)m = am + bm$ for each a and b in $\mathcal{R}$, m and k in M, then M will be called a generalized left $\mathcal{R}$-module or shortly: left $\mathcal{R}$-module or left module over $\mathcal{R}$.*

If $\mathcal{R}$ is a unital ring and $1m = m$ for each $m \in M$, then M is called a left unital module over $\mathcal{R}$, where 1 denotes the unit element in the ring $\mathcal{R}$. Symmetrically is defined a right $\mathcal{R}$-module.

If M is a left and right $\mathcal{R}$-module, then it is called a two-sided $\mathcal{R}$-module or a $(\mathcal{R}, \mathcal{R})$-bimodule. If M is a left $\mathcal{R}$-module and a right $\mathcal{S}$-module, then it is called a $(\mathcal{R}, \mathcal{S})$-bimodule. For the unital ring the module will be supposed unital, if something else will not be outlined.

A two-sided module M over $\mathcal{R}$ is called cyclic, if an element $y \in M$ exists such that $M = \mathcal{R}(y\mathcal{R}) = \{s(yp) : \ s, p \in \mathcal{R}\}$ and

$M = (\mathcal{R}y)\mathcal{R} = \{(sy)p : \ s, p \in \mathcal{R}\}$.

A non null module (left or right or two-sided) M over $\mathcal{R}$ is called simple if it does not contain proper nontrivial (left or right or two-sided respectively) submodules over $\mathcal{R}$. A module (left or right or two-sided) M over $\mathcal{R}$ is called semisimple, if it is a direct sum of its simple (left or right or two-sided respectively) submodules over $\mathcal{R}$.

Take a metagroup algebra $A = \mathcal{T}[G]$ and a two-sided A-module M, where $\mathcal{T}$ is a commutative associative unital ring (see Definition A2). Let M have the decomposition $M = \sum_{g \in G} M_g$ as a two-sided $\mathcal{T}$-module, where M_g is a two-sided $\mathcal{T}$-module for each $g \in G$, G is a metagroup, and let M satisfy the following conditions:

$(A16)$ $hM_g = M_{hg}$ and $M_g h = M_{gh}$,

$(A17)$ $(bh)x_g = b(hx_g)$ and $x_g(bh) = (x_g h)b$ and $bx_g = x_g b$,

$(A18)$ $(hs)x_g = \mathrm{t}_3(h, s, g)h(sx_g)$ and $(hx_g)s = \mathrm{t}_3(h, g, s)h(x_g s)$ and

$(x_g h)s = \mathrm{t}_3(g, h, s)x_g(hs)$;

$(A19)$ $(bc)x = b(cx)$, $(bx)c = b(xc)$, $(xb)c = x(bc)$

for every h, g, s in G, b and c in $\mathcal{T}$ and $x_g \in M_g$.

Then a two-sided A-module M satisfying the above conditions will be said to be smashly G-graded. Shortly it will be said that M is G-graded. Henceforth for a nonzero module with the nontrivial metagroup G we consider a nontrivial G-gradation for which there exists $g \neq e$ in G such that $M_g \neq M_e$, if something other will not be specified. In particular, if the sum is direct $M = \bigoplus_{g \in G} M_g$, then we will say that M is directly G-graded.

Similarly are defined G-graded left and right A-modules. Henceforward, it will be said shortly "an A-module" instead of "a G-graded A-module", if $A = \mathcal{T}[G]$ is the metagroup algebra.

If P and N are left $\mathcal{R}$-modules and a homomorphism $\gamma : P \to N$ is such that $\gamma(ax) = a\gamma(x)$ for each $a \in \mathcal{R}$ and $x \in P$, then γ is called a left $\mathcal{R}$-homomorphism. Right $\mathcal{R}$-homomorphisms for right $\mathcal{R}$-modules are defined analogously. For two-sided $\mathcal{R}$ modules a left and right $\mathcal{R}$-homomorphism is called an $\mathcal{R}$-homomorphism.

For left $\mathcal{R}$-modules M and N by $Hom_{\mathcal{R}}(M, N)$ is denoted a family of all left $\mathcal{R}$-homomorphisms from M into N. A similar notation is used for a family of all $\mathcal{R}$-homomorphisms (or right $\mathcal{R}$-homomorphisms) of two-sided $\mathcal{R}$-modules (or right $\mathcal{R}$-modules correspondingly). If a ring $\mathcal{R}$ is specified it may be written shortly as a homomorphism instead of an $\mathcal{R}$-homomorphism.

References

1. Bourbaki, N. *Algèbre*; Springer: Berlin/Heidelberg, Germany, 2007; Chapters 1–3.
2. Georgantas, G.T. Derivations in central separable algebras. *Glasg. Math. J.* **1978**, *19*, 75–77. [CrossRef]
3. Van Oystaeyen, F. Separable algebras. In *Handbook of Algebra*; Hazewinkel, M., Ed.; Elsevier: Amsterdam, The Netherlands, 2000; Volume 2, pp. 463–505.

4. Pierce, R.S. *Associative Algebras*; Springer: New York, NY, USA, 1982.
5. Rumynin, D.A. Cohomomorphisms of separable algebras. *Algebra Log.* **1994**, *33*, 233–237. [CrossRef]
6. Cartan, H.; Eilenberg, S. *Homological Algebra*; Princeton Univerity Press: Princeton, NJ, USA, 1956.
7. Hochschild, G. On the cohomology theory for associative algebras. *Ann. Math.* **1946**, *47*, 568–579. [CrossRef]
8. Pommaret, J.F. *Systems of Partial Differential Equations and Lie Pseudogroups*; Gordon and Breach Science Publishers: New York, NY, USA, 1978.
9. Dickson, L.E. *The Collected Mathematical Papers*; Chelsea Publishing Co.: New York, NY, USA, 1975; Volumes 1–5.
10. Gürsey, F.; Tze, C.-H. *On the Role of Division, Jordan and Related Algebras in Particle Physics*; World Scientific Publishing Co.: Singapore, 1996.
11. Kantor, I.L.; Solodovnikov, A.S. *Hypercomplex Numbers*; Springer: Berlin/Heidelberg, Germany, 1989.
12. Ludkowski, S.V. Cohomology theory of nonassociative algebras with metagroup relations. *Axioms* **2019**, *8*, 78. [CrossRef]
13. Ludkowski, S.V. Separability of nonassociative algebras with metagroup relations. *Axioms* **2019**, *8*, 139. [CrossRef]
14. Ludkowski, S.V. Smashed and twisted wreath products of metagroups. *Axioms* **2019**, *8*, 127. [CrossRef]
15. Ludkowski, S.V. Integration of vector Sobolev type PDE over octonions. *Complex Variab. Elliptic Equ.* **2016**, *61*, 1014–1035.
16. Ludkovsky, S.V. Normal families of functions and groups of pseudoconformal diffeomorphisms of quaternion and octonion variables. *J. Math. Sci.* **2008**, *150*, 2224–2287. [CrossRef]
17. Nichita, F.F. Unification theories: New results and examples. *Axioms* **2021**, *8*, 60. [CrossRef]
18. Schafer, R.D. *An Introduction to Nonassociative Algebras*; Academic Press: New York, NY, USA, 1966.
19. Dirac, P.A.M. *Die Prinzipen der Quantenmechanik*; Hirzel: Leipzig, Germany, 1930.
20. Bruck, R.H. *A Survey of Binary Systems*; Springer: Berlin/Heidelberg, Germany, 1971.
21. Pickert, G. *Projektive Ebenen*; Springer: Berlin/Heidelberg, Geramany, 1955.
22. Pickert, G. Doppelebenen und loops. *J. Geom.* **1991**, *41*, 133–144. [CrossRef]
23. Ludkowski, S.V. Torsion for homological complexes of nonassociative algebras with metagroup relations. *Axioms* **2021**, *10*, 319. [CrossRef]
24. Lee, D.-W. On the digital cohomology modules. *Mathematics* **2020**, *8*, 1451. [CrossRef]
25. Bourbaki, N. Algèbre homologique. In *Algèbre*; Springer: Berlin/Heidelberg, Germany, 2007; Chapter 10.
26. Bredon, G.E. *Sheaf Theory*; McGraw-Hill: New York, NY, USA, 2012.
27. Shang, Y. Lie algebraic discussion for affinity based information diffusion in social networks. *Open Phys.* **2017**, *15*, 705–711. [CrossRef]
28. Tuong, N.M.; Nefedov, V.I.; Chekalkin, N.S.; Kozlovsky, I.V.; Malafeev, A.V.; Mirolyubova, N.A.; Nazarenko, M.A. On the integration of the methods of forming and research of images of objects against the background of noises and interference. *Russ. Technol. J.* **2020**, *8*, 33–42.