axioms

MDPI

Article

On Some New Ostrowski–Mercer-Type Inequalities for Differentiable Functions

Ifra Bashir Sial [1], Nichaphat Patanarapeelert [2], Muhammad Aamir Ali [3,*], Hüseyin Budak [4] and Thanin Sitthiwirattham [5,*]

1 School of Mathematics Science, Jiangsu University, Zhenjiang 212114, China; ifrabashir92@gmail.com
2 Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; nichaphat.p@sci.kmutnb.ac.th
3 Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
4 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 86120, Turkey; hsyn.budak@gmail.com
5 Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand
* Correspondence: mahr.muhammad.aamir@gmail.com (M.A.A.); thanin_sit@dusit.ac.th (T.S.)

Abstract: In this paper, we establish a new integral identity involving differentiable functions, and then we use the newly established identity to prove some Ostrowski–Mercer-type inequalities for differentiable convex functions. It is also demonstrated that the newly established inequalities are generalizations of some of the Ostrowski inequalities established inside the literature. There are also some applications to the special means of real numbers given.

Keywords: Ostrowski inequality; Jensen–Mercer inequality; convex functions

Citation: Sial, I.B.; Patanarapeelert, N.; Ali, M. A.; Budak, H.; Sitthiwirattham, T. On Some New Ostrowski–Mercer-Type Inequalities for Differentiable Functions. *Axioms* **2022**, *11*, 132. https://dx.doi.org/10.3390/axioms11030132

Academic Editor: Christophe Chesneau

Received: 6 February 2022
Accepted: 8 March 2022
Published: 14 March 2022

1. Introduction

The study of different forms of fundamental inequalities has been the subject of great interest for well over a century. A variety of mathematicians, interested in both pure and applied mathematics, have carried out studies regarding this topic. One of the various mathematical basic discoveries of A. M. Ostrowski [1] is the following classical integral inequality:

Theorem 1. *REf. [1] Let $f : [1, \infty) \to \mathbb{R}$ as a differentiable function on $(1, \infty)$ and $f \in L[a, b]$, where $a, b \in [1, \infty)$ with $a < b$. If $|f'(x)| \leq M$, then we have following inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{M}{(b-a)}\left[\frac{(x-a)^2 + (b-x)^2}{2} \right]. \quad (1)$$

The Ostrowski inequality has applications in quadrature, the theory of probability and optimization, stochastics, statistics, information and the theory of integral operators. During the last few years, many researchers have obtained new results on Ostrowski-type inequalities for bounded variation functions; see, for example, [2–6]. To date, a significant number of research papers and books have been published on Ostrowski inequalities and their numerous applications.

In the literature, the well-known Jensen inequality [7] states that if f is a convex function on an interval containing points $x_1, x_2, ..., x_n$, then

$$f\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j f(x_j). \quad (2)$$

In convex functions theory, the Hermite–Hadamard inequality is very important and was discovered by C. Hermite and J. Hadamard independently (see also [8], and [9] (p.137))

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{3}$$

where $f : I \to \mathbb{R}$ is a convex function over I and $a, b \in I$ with $a < b$. In the case of concave mappings, the above inequality is reversed.

The following variant of the Jensen inequality, known as the Jensen–Mercer inequality, was demonstrated by Mercer [10]:

Theorem 2. *Ref. [10] states that if f is a convex function on $[a, b]$, then the following inequality is true:*

$$f\left(a+b-\sum_{j=1}^{n}\lambda_j x_j\right) \leq f(a)+f(b)-\sum_{j=1}^{n}\lambda_j f(x_j) \tag{4}$$

for all $x_j \in [a, b]$ and $\lambda_j \in [0, 1]$ with $\sum_{j=1}^{n} \lambda_j = 1$.

In [11], the idea of the Jensen–Mercer inequality is used by Kian and Moslehian, and the following Hermite–Hadamard–Mercer inequality was demonstrated:

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{y-x}\int_x^y f(a+b-t)dt \\ &\leq \frac{f(a+b-x)+f(a+b-y)}{2} \\ &\leq f(a)+f(b)-\frac{f(x)+f(y)}{2} \end{aligned} \tag{5}$$

where f is a convex function on $[a, b]$. After that, in [12–14], the authors used the fractional integral operators and proved some Hermite–Hadamard–Mercer-type inequalities for convex functions. Niezgoda gave the generalized Mercer's results for convex functions in [15]. In [16], Wang et al. used the well-known Riemann–Liouville fractional integrals and established some new Hermite–Hadamard–Mercer-type inequalities. In [17], Butt et al. proved some new Hermite–Hadamard–Mercer inequalities for harmonically convex functions and in [18], the authors used h-convexity and proved some new Hermite–Hadamard-Mercer inequalities for Caputo–Fabrizio fractional integrals. For generalized fractional Hermite–Hadamard–Mercer inequalities, one can consult [19].

Inspired by these ongoing studies, we develop some new Ostrowski-type inequalities by using the Jensen–Mercer inequalities for differentiable convex functions.

2. Ostrowski–Mercer Inequalities

New Ostrowski–Mercer inequalities are obtained for differentiable convex functions in this section. For this, we first give a new integral identity that will serve as an auxiliary to produce subsequent results for advancement.

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $f \in L[a, b]$, then for all $x \in [u_1, u_2]$, $u_1, u_2 \in [a, b]$ and $t \in [0, 1]$, the following equality is satisfied:*

$$\begin{aligned} &(x-u_1)^2\int_0^1 tf'(x+a-(tu_1+(1-t)x))dt-(u_2-x)^2\int_0^1 tf'(x+b-(tu_2+(1-t)x))dt \\ =\;&(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]. \end{aligned} \tag{6}$$

Proof. It is enough to remember that

$$\begin{aligned} I &= (x-u_1)^2\int_0^1 tf'(x+a-(tu_1+(1-t)x))dt \\ &\quad -(u_2-x)^2\int_0^1 tf'(x+b-(tu_2+(1-t)x))dt \\ &= (x-u_1)^2 I_1-(u_2-x)^2 I_2. \end{aligned} \tag{7}$$

Using the integration by parts, we obtain the equalities

$$\begin{aligned} I_1 &= \int_0^1 tf'(x+a-(tu_1+(1-t)x))dt \\ &= \frac{f(x+a-u_1)}{x-u_1}-\frac{1}{(x-u_1)^2}\int_a^{x+a-u_1} f(t)dt \end{aligned} \tag{8}$$

and

$$\begin{aligned} I_2 &= \int_0^1 tf'(x+b-(tu_2+(1-t)x))dt \\ &= -\frac{f(x+b-u_2)}{(u_2-x)}+\frac{1}{(u_2-x)^2}\int_{x+b-u_2}^{b} f(t)dt. \end{aligned} \tag{9}$$

We obtain the resulting equality (6) by placing the equalities (8) and (9) in (7). □

Remark 1. *If we set $u_1=a$ and $u_2=b$ and in Lemma 1, then we obtain Lemma 1, as proved in [20].*

Theorem 3. *We assume that the conditions of Lemma 1 hold. If the mapping $|f'|$ is convex on $[a,b]$, then we have the following inequality*

$$\begin{aligned} &\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right| \\ \leq\ & \frac{1}{6}\Big[(x-u_1)^2\{2|f'(x)|+3|f'(a)|-2|f'(u_1)|\} \\ &+(u_2-x)^2\{2|f'(x)|+3|f'(b)|-2|f'(u_2)|\}\Big]. \end{aligned} \tag{10}$$

Proof. Taking the modulus in Lemma 1 and from the Jensen–Mercer inequality, we have the inequality

$$\begin{aligned} &\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right| \\ \leq\ & (x-u_1)^2\int_0^1 t|f'(x+a-(tu_1+(1-t)x))|dt \\ &+(u_2-x)^2\int_0^1 t|f'(x+b-(tu_2+(1-t)x))|dt \\ \leq\ & (x-u_1)^2\int_0^1 t[|f'(x)|+|f'(a)|-t|f'(u_1)|-(1-t)|f'(x)|]dt \\ &+(u_2-x)^2\int_0^1 t[|f'(x)|+|f'(b)|-t|f'(u_2)|-(1-t)|f'(x)|]dt \\ =\ & \frac{1}{6}\Big[(x-u_1)^2\{2|f'(x)|+3|f'(a)|-2|f'(u_1)|\} \\ &+(u_2-x)^2\{2|f'(x)|+3|f'(b)|-2|f'(u_2)|\}\Big] \end{aligned}$$

which ends the proof. □

Corollary 1 (Ostrowski–Mercer Inequality)**.** *In Theorem 3, if we choose $|f'(t)| \leq M$ for all $t \in [a,b]$, then we have the following Ostrowski–Mercer inequality*

$$\begin{aligned} &\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right| \quad (11)\\ \leq\ & \frac{M}{2}\left((x-u_1)^2+(u_2-x)^2\right). \end{aligned}$$

Proof. The result can be easily obtained using $|f'(x+a-(tu_1+(1-t)x))| \leq M$ and $|f'(x+b-(tu_2+(1-t)x))| \leq M$. □

Remark 2. *If we consider $u_1 = a$ and $u_2 = b$ in Corollary 1, then inequality (11) is reduced to (1).*

Remark 3. *If we consider $u_1 = a$ and $u_2 = b$ in Theorem 3, then we obtain Theorem 3, established in [20].*

Theorem 4. *We assume that the conditions of Lemma 1 hold. If the mapping $|f'|^q$, $q > 1$ is convex on $[a,b]$, then we have the following inequality*

$$\begin{aligned} &\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right| \quad (12)\\ \leq\ & \frac{1}{2(1+p)^{\frac{1}{p}}}\left[(x-u_1)^2\left(|f'(x)|^q+2|f'(a)|^q-|f'(u_1)|^q\right)^{\frac{1}{q}}\right.\\ &\left.+(u_2-x)^2\left(|f'(x)|^q+2|f'(b)|^q-|f'(u_2)|^q\right)^{\frac{1}{q}}\right] \end{aligned}$$

where $\frac{1}{r}+\frac{1}{p}=1$.

Proof. From Lemma 1 and Hölder's inequality, we have the inequality

$$\begin{aligned} &\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right| \quad (13)\\ \leq\ & (x-u_1)^2\int_0^1 t|f'(x+a-(tu_1+(1-t)x))|dt\\ &+(u_2-x)^2\int_0^1 t|f'(x+b-(tu_2+(1-t)x))|dt\\ \leq\ & (x-u_1)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(x+a-(tu_1+(1-t)x))|^q dt\right)^{\frac{1}{q}}\\ &+(u_2-x)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(x+b-(tu_2+(1-t)x))|^q dt\right)^{\frac{1}{q}}. \end{aligned}$$

From the Jensen–Mercer inequality, we have the inequality

$$\begin{aligned} &\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right|\\ \leq\ & (x-u_1)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[|f'(x)|^q+|f'(a)|^q-t|f'(u_1)|^q-(1-t)|f'(x)|^q\right]dt\right)^{\frac{1}{q}}\\ &+(u_2-x)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[|f'(x)|^q+|f'(b)|^q-t|f'(u_2)|^q-(1-t)|f'(x)|^q\right]dt\right)^{\frac{1}{q}}\\ =\ & \frac{1}{2(1+p)^{\frac{1}{p}}}\left[(x-u_1)^2\left(|f'(x)|^q+2|f'(a)|-|f'(u_1)|^q\right)^{\frac{1}{q}}\right. \end{aligned}$$

$$+(u_2-x)^2\Big(|f'(x)|^q+2|f'(b)|-|f'(u_2)|^q\Big)^{\frac{1}{q}}\Bigg]$$

which finishes the proof. □

Corollary 2. *In Theorem 4, if we choose $|f'(t)| \le M$ for all $t \in [a,b]$, then we have the following Ostrowski–Mercer inequality*

$$\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right|$$
$$\le \frac{M}{(p+1)^{\frac{1}{p}}}\Big((x-u_1)^2+(u_2-x)^2\Big).$$

Proof. The result can be easily obtained by using $|f'(x+a-(tu_1+(1-t)x))| \le M$ and $|f'(x+b-(tu_2+(1-t)x))| \le M$. □

Remark 4. *If we consider $u_1=a,\ u_2=b$ and $x=x$ in Corollary 2, then we obtain Theorem 3, as proved in [21] for $s=1$.*

Remark 5. *If we consider $u_1=a,\ u_2=b$ and $x=x$ in Theorem 4, then we have the following inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le \frac{1}{2(b-a)(1+p)^{\frac{1}{p}}}\left[(x-a)^2\Big(|f'(x)|^q+|f'(a)|^q\Big)^{\frac{1}{q}}+(b-x)^2\Big(|f'(x)|^q+|f'(b)|^q\Big)^{\frac{1}{q}}\right].$$

Theorem 5. *We assume that the conditions of Lemma 1 hold. If the mapping $|f'|^q$, $q \ge 1$ is convex on $[a,b]$, then we have the following inequality*

$$\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right| \tag{14}$$
$$\le \frac{1}{2}\left[(x-u_1)^2\left(\frac{2|f'(x)|^q+3|f'(a)|^q-2|f'(u_1)|^q}{3}\right)^{\frac{1}{q}}\right.$$
$$\left.+(u_2-x)^2\left(\frac{2|f'(x)|^q+3|f'(b)|^q-2|f'(u_2)|}{3}\right)^{\frac{1}{q}}\right].$$

Proof. From Lemma 1 and the well-known power mean inequality, we obtain the inequality

$$\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right|$$
$$\le (x-u_1)^2\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|f'(x+a-(tu_1+(1-t)x))|^q dt\right)^{\frac{1}{q}}$$
$$+(u_2-x)^2\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|f'(x+b-(tu_2+(1-t)x))|^q dt\right)^{\frac{1}{q}}.$$

From the Jensen–Mercer inequality, we obtain

$$\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^{b} f(t)dt\right]\right|$$

$$\leq (x-u_1)^2\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[|f'(x)|^q+|f'(a)|^q-t|f'(u_1)|^q-(1-t)|f'(x)|^q\right]dt\right)^{\frac{1}{q}}$$
$$+(u_2-x)^2\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[|f'(x)|^q+|f'(b)|^q-t|f'(u_2)|^q-(1-t)|f'(x)|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{1}{2}\left[(x-u_1)^2\left(\frac{2|f'(x)|^q+3|f'(a)|^q-2|f'(u_1)|^q}{3}\right)^{\frac{1}{q}}\right.$$
$$\left.+(u_2-x)^2\left(\frac{2|f'(x)|^q+3|f'(b)|^q-2|f'(u_2)|^q}{3}\right)^{\frac{1}{q}}\right]$$

which finishes the proof. □

Remark 6. *In Theorem 5, if we choose $|f'(t)| \leq M$ for all $t \in [a,b]$, then we recapture the inequality (11).*

Remark 7. *If we consider $u_1 = a$ and $u_2 = b$ in Theorem 5, then we have the following inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{1}{2(b-a)}\left[(x-a)^2\left(\frac{2|f'(x)|^q+|f'(a)|^q}{3}\right)^{\frac{1}{q}}+(b-x)^2\left(\frac{2|f'(x)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right].$$

Remark 8. *In the previous inequalities, by setting $x = \frac{a+b}{2}$, one can acquire multiple midpoint-type inequalities. Furthermore, it leaves the specifics to the interested reader.*

Theorem 6. *We assume that the conditions of Lemma 1 hold. If the mapping $|f'|^q$, $q > 1$ is concave on $[a,b]$, then we have the following inequality*

$$\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^b f(t)dt\right]\right| \tag{15}$$
$$\leq \frac{1}{(1+p)^{\frac{1}{p}}}\left[(x-u_1)^2\left|f'\left(\frac{2a+x-u_1}{2}\right)\right|+(u_2-x)^2\left|f'\left(\frac{2b+x-u_2}{2}\right)\right|\right]$$

where $\frac{1}{r}+\frac{1}{p}=1$.

Proof. From Lemma 1 and Hölder's inequality, we have the inequality

$$\left|(x-u_1)f(x+a-u_1)+(u_2-x)f(x+b-u_2)-\left[\int_a^{x+a-u_1} f(t)dt+\int_{x+b-u_2}^b f(t)dt\right]\right| \tag{16}$$
$$\leq (x-u_1)^2\int_0^1 t|f'(x+a-(tu_1+(1-t)x))|dt$$
$$+(u_2-x)^2\int_0^1 t|f'(x+b-(tu_2+(1-t)x))|dt$$
$$\leq (x-u_1)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(x+a-(tu_1+(1-t)x))|^q dt\right)^{\frac{1}{q}}$$
$$+(u_2-x)^2\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(x+b-(tu_2+(1-t)x))|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is concave mapping, therefore, from inequality (5), we have

$$\int_0^1 |f'(x+a-(tu_1+(1-t)x))|^q dt \leq \left|f'\left(x+a-\frac{u_1+x}{2}\right)\right|^q = \left|f'\left(\frac{2a+x-u_1}{2}\right)\right| \tag{17}$$

and

$$\int_0^1 |f'(x+b-(tu_2+(1-t)x))|^q dt \leq \left|f'\left(x+b-\frac{u_2+x}{2}\right)\right|^q = \left|f'\left(\frac{ba+x-u_2}{2}\right)\right|. \tag{18}$$

We obtain the resulting inequality (15) by placing the inequalities (17) and (18) in (16). □

Remark 9. *If we consider $u_1 = a$ and $u_2 = b$ in Theorem 6, then we obtain Theorem 5, as proved in [21] for $s = 1$.*

3. Application to Special Means

For arbitrary positive numbers a, b $(a \neq b)$, we consider the means as follows:

1. The arithmetic mean
$$A(a,b) = \frac{a+b}{2}.$$
2. The generalized logarithmic mean
$$L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)}\right]^{\frac{1}{p}}, \; p \in \mathbb{R}\backslash\{-1,0\}.$$
3. The identric mean
$$I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b, \\ a, & \text{if } a = b, \end{cases} \quad a,b > 0.$$

Proposition 1. *Let $a, b > 0$, then we have the following inequality*

$$\begin{aligned} & \left|(x-u_1)(2A(x,a)-u_1)^n + (u_2-x)(2A(x,b)-u_2)^n\right. \\ & \left. -[(x-u_1)L_n^n(x+a-u_1,a) + (u_2-x)L_n^n(b,x+b-u_2)]\right| \\ \leq\; & \frac{M}{2}\left((x-u_1)^2 + (u_2-x)^2\right). \end{aligned}$$

Proof. The result can be directly obtained by applying Corollary 1 to the convex function $f(x) = x^n$, $x > 0$. There, some information is omitted. □

Proposition 2. *Let $a, b > 0$, then we have the following inequality*

$$\begin{aligned} & \Big|\ln(2A(x,a)-u_1)^{(x-u_1)} + \ln(2A(x,b)-u_2)^{(u_2-x)} \\ & -\Big[\ln I(x+a-u_1,a)^{(x-u_1)} + \ln I(b,x+b-u_2)^{(u_2-x)}\Big]\Big| \\ \leq\; & \frac{1}{(1+p)^{\frac{1}{p}}}\left[\frac{2(x-u_1)^2}{2a+x-u_1} + \frac{2(u_2-x)^2}{2b+x-u_2}\right]. \end{aligned}$$

Proof. The result can be directly obtained by applying Theorem 6 to the concave function $f(x) = \ln x$. There, some information is omitted. □

4. Conclusions

In this work, we established some new Ostrowski–Mercer-type inequalities for differentiable convex functions. Moreover, we proved that the newly established inequalities are strong generalizations of comparable results in the literature. Finally, we gave some applications to the special means of real numbers using the newly established inequalities. It is an interesting and novel problem that upcoming researchers may prove some new inequalities for co-ordinated convex functions using the techniques of this paper in their future work.

Author Contributions: Conceptualization, I.B.S., N.P., M.A.A., H.B. and T.S.; methodology, I.B.S., N.P., M.A.A., H.B. and T.S.; software, I.B.S., N.P., M.A.A., H.B. and T.S.; validation, I.B.S., N.P., M.A.A., H.B. and T.S.; formal analysis, I.B.S., N.P., M.A.A., H.B. and T.S.; investigation, I.B.S., N.P., M.A.A., H.B. and T.S.; resources, I.B.S., N.P., M.A.A., H.B. and T.S.; data curation, I.B.S., N.P., M.A.A., H.B. and T.S.; writing—original draft preparation, I.B.S., N.P., M.A.A., H.B. and T.S.; writing—review and editing, I.B.S., N.P., M.A.A., H.B. and T.S.; visualization, I.B.S., N.P., M.A.A., H.B. and T.S.; supervision, M.A.A., H.B. and T.S.; project administration, H.B. and T.S.; funding acquisition, T.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-63-KNOW-20.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ostrowski, A.M.; Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert. *Comment. Math. Helv.* **1938**, *10*, 226–227. [CrossRef]
2. Budak, H.; Sarikaya, M.Z.; Dragomir, S.S. Some perturbed Ostrowski type inequalities for twice differentiable functions. In *Advances in Mathematical Inequalities and Applications*; Birkhä user: Singapore, 2018; pp. 279–294.
3. Cerone, P.; Dragomir, S.S.; Roumeliotis, J. An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications. *East Asian Math. J.* **1999**, *15*, 1–9.
4. Dragomir, S.S. The Ostrowski integral inequality for Lipschitzian mappings and applications. *Comput. Math. Appl.* **1999**, *38*, 33–37. [CrossRef]
5. Sarikaya, M.Z.; Budak, H. Generalized Ostrowski type inequalities for local fractional integrals. *Proc. The American Math. Soc.* **2017**, *145*, 1527–1538. [CrossRef]
6. Set, E. New inequalities of Ostrowski type for mappings whose derivatives are *s*-convex in the second sense via fractional integrals. *Comput. Math. Appl.* **2012**, *63*, 1147–1154. [CrossRef]
7. Mitrinović, D.S.; Pexcxarixcx, J.E.; Fink, A.M. In *Classical and New Inequalities in Analysis*; Kluwer Academic Publishers Group: Dordrecht, The Netherlands, 1993.
8. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*; RGMIA Monographs; Victoria University: Melbourne, Austrila, 2000.
9. Pećarixcx, J.E.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Orderings and Statistical Applications*; Academic Press: Boston, MA, USA, 1992.
10. Mercer, A.M. A Variant of Jensenís Inequality. *J. Ineq. Pure and Appl. Math* **2003**, *4*, Art 73.
11. Kian, M.; Moslehian, M.S. Refinements of the operator Jensen-Mercer inequality. *Electron. J. Linear Algebra* **2013**, *26*, 742–753. [CrossRef]
12. Abdeljawad, T.; Ali, M.A.; Mohammed, P.O.; Kashuri, A. On inequalities of Hermite-Hadamard-Mercer type involving Riemann-Liouville fractional integrals. *AIMS Math.* **2021**, *6*, 712–725. [CrossRef]
13. Ali, M.M.; Khan, A.R. Generalized integral Mercer's inequality and integral means. *J. Inequal. Spec. Funct.* **2019**, *10*, 60–76.
14. Chu, H.H.; Rashid, S.; Hammouch, H.; Chu, Y-. New fractional estimates for Hermite-Hadamard-Mercer's type inequalities. *Alexadria Eng. J.* **2020**, *59*, 3079–3089 doi: 10.1016/j.aej.2020.06.040. [CrossRef]
15. Niezgoda, M. A generalization of Mercer's result on convex functions. *Nonlinear Anal.* **2009**, *71*, 277. [CrossRef]
16. Wang, H.; Khan, J.; Khan, M.A.; Khalid, S.; Khan, R. The Hermite–Hadamard-Jensen-Mercer Type Inequalities for Riemann-Liouville Fractional Integral. *J. Math.* **2021**, *2021*, 5516987. [CrossRef]
17. Butt, S.I.; Yousaf, S.; Asghar, A.; Khan, K.A.; Moradi, H.R. New Fractional Hermite–Hadamard–Mercer Inequalities for Harmonically Convex Function. *J. Funct. Spaces* **2021**, *2021*, 5868326. [CrossRef]

18. Vivas-Cortez, M.; Saleem, M.S.; Sajid, S.; Zahoor, M.S.; Kashuri, A. Hermite-Jensen-Mercer-Type Inequalities via Caputo-Fabrizio Fractional Integral for h-Convex Function. *Fractal Fract.* **2021**, *5*, 269. [CrossRef]
19. Set, E.; Celik, B.; Ozdemir, M.E.; Aslan, M. Some New Results on Hermite–Hadamard-Mercer-Type Inequalities Using a General Family of Fractional Integral Operators. *Fractal Fract.* **2021**, *5*, 68. [CrossRef]
20. Cerone, P.; Dragomir, S.S. Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions. *Demonstratio Math.* **2004**, *37*, 299–308. [CrossRef]
21. Alomari, M.; Darus, M.; Dragomir, S.S.; Cerone, P. Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense. *Appl. Math. Lett.* **2010**, *23*, 1071–1076. [CrossRef]