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Constant-Sign Green's Function of a Second-Order Perturbed Periodic Problem

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Abstract: In this paper, we were interested in obtaining the exact expression and studying the regions of constant sign of Green's function related to a second-order perturbed periodic problem coupled with integral boundary conditions at the extremes of the interval of the definition. To obtain the expression of Green's function related to this problem, we used the theory presented in a previous paper of the authors for general non-local perturbed boundary-value problems. Moreover, we characterized the parameter set where such a Green's function has a constant sign. To this end, we needed to consider first a related second-order problem without integral boundary conditions, obtaining the properties of its Green's function and then using them to compute the sign of the one related to the main problem.

Keywords: Green's function; periodic boundary-value problem; integral boundary conditions

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1. Introduction

In this paper, we studied the regions of constant sign of Green's function related to the following perturbed second-order periodic problem, coupled with integral conditions on the boundary:

$$\begin{cases} u''(t) + Mu(t) = \sigma(t), \quad t \in I := [0,1], \\ u(0) - u(1) = \delta_1 \displaystyle\int_0^1 u(s)ds, \\ u'(0) - u'(1) = \delta_2 \displaystyle\int_0^1 u(s)ds, \end{cases} \tag{1}$$

where $M, \delta_1, \delta_2 \in \mathbb{R}$ and σ is a continuous function. In particular, we considered separately the cases $M = 0$, $M > 0$, and $M < 0$, and we analyzed each of them and give the optimal values of $M, \delta_1, \delta_2 \in \mathbb{R}$ for which Green's function (denoted by G_{M,δ_1,δ_2}) has a constant sign.

The interest of this study relies on the fact that the constant sign of Green's function is a fundamental tool to ensure the existence of constant-sign solutions of the related nonlinear problems, since it is a basic assumption to apply some classical methods as, for instance, lower and upper solutions, monotone iterative techniques, Leray–Schauder degree theory, or fixed-point theorems on cones.

Furthermore, the solvability of differential equations coupled with different types of boundary-value conditions is a topic that has awakened interest in recent and classical literature (see [1,2] and references therein). In particular, integral boundary conditions have been widely considered in many works in the recent literature. For more information on this topic, we refer the reader to [3–8] (for integral boundary conditions in second- and fourth-order ODEs or systems) or [9–13] (for fractional equations) and the references therein.

In a recent paper [14], the authors proved the existence of a relation between the Green's function of a differential problem coupled with some functional boundary conditions (where the functional is given by a linear operator) and the Green's function of the same differential problem coupled with homogeneous boundary conditions. Such a formula was used now to compute the exact expression of Green's function related to Problem (1) for the cases $M > 0$ and $M < 0$. In such cases, the very well-known properties of the periodic Green's function will help to study the constant sign of the Green's function of Problem (1). For the case $M = 0$, this technique cannot be applied, as $M = 0$ is an eigenvalue of the periodic problem, and consequently, we need to compute the expression of the Green's function of (1) by means of direct integration.

The paper is organized as follows: In Section 2, we compile the preliminary results that are used later. In Section 3, we prove some properties of Green's function, which allow us to simplify the study of the general case. Section 4 considers the particular case of considering parameter $\delta_1 = 0$ in Problem (1). Finally, Section 5 includes the complete study of the case $\delta_1 \neq 0$, which is related to the study developed in Section 4 by means of the general properties proven in Section 3.

2. Preliminaries

In this section, we compile the main results of [14] that are used to compute the exact expression of Green's function related to (1). We include also the definition and main properties of Green's function, for the reader's convenience.

Consider the following n-th order linear boundary-value problem with parameter dependence:

$$\begin{cases} T_n[M]u(t) = \sigma(t), & t \in J := [a,b], \\ B_i(u) = \delta_i\, C_i(u), & i = 1, \ldots, n, \end{cases} \tag{2}$$

where $T_n[M]u(t) := L_n u(t) + M\, u(t)$, $t \in J$, with:

$$L_n u(t) := u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \cdots + a_n(t)u(t), \quad t \in J.$$

Here, σ and a_k are continuous functions for all $k = 0, \ldots, n-1$, $M \in \mathbb{R}$ and $\delta_i \in \mathbb{R}$ for all $i = 1, \ldots, n$. Moreover, $C_i : C(J) \to \mathbb{R}$ is a linear continuous operator, and B_i covers the general two-point linear boundary conditions, i.e.,

$$B_i(u) = \sum_{j=0}^{n-1} \left(\alpha_j^i u^{(j)}(a) + \beta_j^i u^{(j)}(b) \right), \quad i = 1, \ldots, n,$$

α_j^i, β_j^i being real constants for all $i = 1, \ldots, n$, $j = 0, \ldots, n-1$.

We note that Problem (1) is a particular case of (2).

Definition 1. *Given a Banach space X, operator $T_n[M]$ is said to be nonresonant in X if and only if the homogeneous equation:*

$$T_n[M]\, u(t) = 0, \quad t \in J, \qquad u \in X, \tag{3}$$

has only the trivial solution.

Let us consider in this case:

$$X = \{u \in \mathcal{C}(J),\ B_i(u) = \delta_i\, C_i(u),\ i = 1, \ldots, n\}.$$

It is very well known that if $\sigma \in \mathcal{C}(J)$ and operator $T_n[M]$ is nonresonant in X, then the non-homogeneous problem:

$$T_n[M]\, u(t) = \sigma(t), \quad t \in J, \quad u \in X,$$

has a unique solution given by:

$$u(t) = \int_a^b G_{M,\delta_1,\dots,\delta_n}(t,s)\,\sigma(s)\,ds, \qquad \forall\, t \in J,$$

where $G_{M,\delta_1,\dots,\delta_n}$ denotes the so-called Green's function related to operator $T_n[M]$ on X and it is uniquely determined (see [1] for the details).

Green's function can be defined axiomatically in the following way.

Definition 2. *We say that $G_{M,\delta_1,\dots,\delta_n}$ is a Green's function for Problem (3) if it satisfies the following properties:*

(G1) $G_{M,\delta_1,\dots,\delta_n}$ is defined on the square $J \times J$ (except at the points with $t = s$ if $n = 1$);

(G2) For $k = 0,\dots,n-2$, the partial derivatives $\frac{\partial^k G_{M,\delta_1,\dots,\delta_n}}{\partial t^k}$ exist and are continuous on $J \times J$;

(G3) Both $\frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}$ and $\frac{\partial^n G_{M,\delta_1,\dots,\delta_n}}{\partial t^n}$ exist and are continuous on the triangles $a \le s < t \le b$ and $a \le t < s \le b$;

(G4) For each $s \in (a,b)$, the function $G_{M,\delta_1,\dots,\delta_n}(\cdot,s)$ is a solution of the differential equation $T_n[M]\,y = 0$ a. e. on $[a,s) \cup (s,b]$, that is,

$$\frac{\partial^n G_{M,\delta_1,\dots,\delta_n}}{\partial t^n}(t,s) + a_1(t)\,\frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}(t,s) + \cdots + a_n(t)\,G_{M,\delta_1,\dots,\delta_n}(t,s) = 0,$$

for all $t \in J \setminus \{s\}$;

(G5) For each $t \in (a,b)$, there exist the lateral limits:

$$\frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}(t^-,t) = \frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}(t,t^+)$$

and:

$$\frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}(t,t^-) = \frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}(t^+,t).$$

Moreover,

$$\begin{aligned} &\frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}(t^+,t) - \frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}(t^-,t) \\ = &\frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}(t,t^-) - \frac{\partial^{n-1} G_{M,\delta_1,\dots,\delta_n}}{\partial t^{n-1}}(t,t^+) = 1; \end{aligned}$$

(G6) For each $s \in (a,b)$, the function $G_{M,\delta_1,\dots,\delta_n}(\cdot,s)$ satisfies the boundary conditions $B_i(G_{M,\delta_1,\dots,\delta_n}(\cdot,s)) = \delta_i\,C_i(G_{M,\delta_1,\dots,\delta_n}(\cdot,s))$, $i = 1,\dots,n$.

Lemma 1 ([14], Lemma 1)**.** *There exists a unique Green's function related to the homogeneous problem:*

$$\begin{cases} T_n[M]u(t) = 0, & t \in J, \\ B_i(u) = 0, & i = 1,\dots n, \end{cases} \tag{4}$$

if and only if for any $i \in \{1,\cdots,n\}$, the following problem:

$$\begin{cases} T_n[M]u(t) = 0, & t \in J, \\ B_j(u) = 0, & j \ne i, \\ B_i(u) = 1, \end{cases} \tag{5}$$

has a unique solution, which we denote as $\omega_i(t)$, $t \in J$.

The following result shows the existence and uniqueness of the solution of Problem (2), and it is a direct consequence of [14], Theorem 2.

Theorem 1 ([14] Corollary 1)**.** *Assume that the homogeneous Problem* (4) *has $u = 0$ as its unique solution, and let $G_{M,0,\dots,0}$ be its unique Green's function. Let $\sigma \in C(J)$ and δ_i, $i = 1,\dots,n$ be such that $\sum_{i=1}^{n} \delta_i\, C(\omega_i) \neq 1$. Then, Problem* (2) *has a unique solution $u \in C^n(J)$, given by the expression:*

$$u(t) = \int_a^b G_{M,\delta_1,\dots,\delta_n}(t,s)\,\sigma(s)ds,$$

where:

$$G_{M,\delta_1,\dots,\delta_n}(t,s) := G_{M,0,\dots,0}(t,s) + \frac{\sum_{i=1}^{n} \delta_i\, \omega_i(t)}{1 - \sum_{j=1}^{n} \delta_j\, C(\omega_j)}\, C(G_{M,0,\dots,0}(\cdot,s)). \tag{6}$$

3. First Results

This section is devoted to deducing some preliminary results that will be fundamental in the development of the paper. First, we prove that Green's function satisfies the following symmetry.

Lemma 2. *Assume that Problem* (1) *has a unique solution, and let G_{M,δ_1,δ_2} be its related Green's function. Then, it holds that:*

$$G_{M,\delta_1,\delta_2}(t,s) = G_{M,-\delta_1,\delta_2}(1-t,1-s), \quad t,\, s \in I. \tag{7}$$

Proof. Let:

$$u(t) = \int_0^1 G_{M,\delta_1,\delta_2}(t,s)\,\sigma(s)\,ds$$

be the unique solution of Problem (1).

In such a case, it is immediate to verify that $v(t) := u(1-t)$ is the unique solution of problem:

$$\begin{cases} v''(t) + Mv(t) = \sigma(1-t), \quad t \in I, \\ v(0) - v(1) = -\delta_1 \int_0^1 v(s)ds, \\ v'(0) - v'(1) = \delta_2 \int_0^1 v(s)ds, \end{cases}$$

and as a direct consequence, we deduce that:

$$v(t) = \int_0^1 G_{M,-\delta_1,\delta_2}(t,s)\,\sigma(1-s)\,ds.$$

On the other hand, we have that:

$$v(t) = u(1-t) = \int_0^1 G_{M,\delta_1,\delta_2}(1-t,s)\,\sigma(s)\,ds = \int_0^1 G_{M,\delta_1,\delta_2}(1-t,1-s)\,\sigma(1-s)\,ds.$$

By identifying the two previous equalities, we obtain that:

$$\int_0^1 \big(G_{M,-\delta_1,\delta_2}(t,s) - G_{M,\delta_1,\delta_2}(1-t,1-s)\big)\,\sigma(1-s)\,ds = 0$$

and since σ is arbitrary, from the regularity properties $(G1)$–$(G3)$ on Definition 2, we deduce (7). □

Let us now characterize the points where a constant-sign Green's function may vanish.

Lemma 3. *Let $M < \pi^2$. If G_{M,δ_1,δ_2} has a constant sign on $I \times I$ and vanishes at some point (t_0,s_0), then either $t_0 = 0$, $t_0 = 1$ or $t_0 = s_0$.*

Proof. Let us suppose that $(t_0, s_0) \in (0,1) \times (0,1)$, with $t_0 > s_0$. In such a case, $u_1(t) := G_{M,\delta_1,\delta_2}(t, s_0)$ solves the problem:

$$\begin{cases} u_1''(t) + M\,u_1(t) = 0, & t \in (s_0, 1], \\ u_1(s_0) = u_1'(s_0) = 0, \end{cases}$$

and so, $G_{M,\delta_1,\delta_2}(t, s_0) = 0$ for all $t \in (s_0, 1]$.

Now, using Condition $(G5)$ in Definition 2, we may extend u_1 to the interval $[0,1]$ as the unique solution of the problem:

$$\begin{cases} u_1''(t) + M\,u_1(t) = 0, & t \in [0, s_0), \\ u_1(s_0) = 0, \quad u_1'(s_0) = -1, \end{cases}$$

and from Condition $(G6)$ in Definition 2, this function must satisfy that:

$$u_1(0) = \delta_1 \int_0^1 u_1(t)\,dt = \delta_1 \int_0^{s_0} u_1(t)\,dt.$$

It can be easily seen that previous equality is only true for a particular value of δ_1:

$$\delta_1 = \begin{cases} \frac{\sqrt{M}\,\sin\left(\sqrt{M}\,s_0\right)}{1-\cos\left(\sqrt{M}\,s_0\right)}, & M > 0, \\ \frac{2}{s_0}, & M = 0, \\ \frac{\sqrt{-M}\,\sinh\left(\sqrt{-M}\,s_0\right)}{-1+\cosh\left(\sqrt{-M}\,s_0\right)}, & M < 0. \end{cases} \tag{8}$$

Now, let us fix $t_1 \in (s_0, 1)$. As was proven in [15], Section 3.2, it occurs that $v(s) = G_{M,\delta_1,\delta_2}(t_1, s)$ solves the problem:

$$\begin{cases} v''(s) + M\,v(s) = 0, & s \in [0, t_1), \\ v(s_0) = v'(s_0) = 0, \end{cases} \tag{9}$$

and as a consequence, $G_{M,\delta_1,\delta_2}(t_1, s) = 0$ for all $s \in [0, t_1)$.

Now, if we choose some $s_1 \in (0, t_1)$, $s_1 \neq s_0$, defining $u_2(t) = G_{M,\delta_1,\delta_2}(t, s_1)$ and reasoning analogously to the case with u_1, we deduce that $G_{M,\delta_1,\delta_2}(t, s_1) = 0$ for all $t \in (s_1, 1]$. As we have already seen, this is only possible if:

$$\delta_1 = \begin{cases} \frac{\sqrt{M}\,\sin\left(\sqrt{M}\,s_1\right)}{1-\cos\left(\sqrt{M}\,s_1\right)}, & M > 0, \\ \frac{2}{s_1}, & M = 0, \\ \frac{\sqrt{-M}\,\sinh\left(\sqrt{-M}\,s_1\right)}{-1+\cosh\left(\sqrt{-M}\,s_1\right)}, & M < 0, \end{cases}$$

which contradicts (8).

In the case that Green's function has a constant sign and vanishes at some point $(t_0, 0)$, with $t_0 \in (0,1)$, with the same arguments, we conclude that $G_{M,\delta_1,\delta_2}(t, 0) = 0$ for all $t \in [0,1]$. Now, for any $t_1 \in (0,1)$, we have that $G_{M,\delta_1,\delta_2}(t_1, s)$ solves (9) with $v(0) = v'(0) = 0$. As a direct consequence $G_{M,\delta_1,\delta_2}(t_1, s) = 0$ for all $s \in [0, t_1)$, and we arrive at a contradiction in a similar way to $s_0 \in (0,1)$.

Finally, we note that the case $(t_0, s_0) \in (0,1) \times (0,1]$, with $t_0 < s_0$, can also be discarded as Lemma 2 implies that if G_{M,δ_1,δ_2} has a constant sign and vanishes at $(t_0, s_0) \in (0,1) \times (0,1]$, with $t_0 < s_0$, then $G_{M,-\delta_1,\delta_2}$ will also have a constant sign and will vanish at the point $(1 - t_0, 1 - s_0)$ (which satisfies that $1 > 1 - t_0 > 1 - s_0 \geq 0$). $\square$

Let us now compute the value of the Green's function of Problem (1). For $M \in \mathbb{R} \setminus \{0\}$, according to (6), it holds that:

$$G_{M,\delta_1,\delta_2}(t,s) = G_{M,0,0}(t,s) + \frac{\delta_1\,\omega_1(t) + \delta_2\,\omega_2(t)}{1 - \left(\delta_1 \int_0^1 \omega_1(s)\,ds + \delta_2 \int_0^1 \omega_2(s)\,ds\right)} \int_0^1 G_{M,0,0}(t,s)\,dt, \tag{10}$$

where ω_1 is the unique solution to the problem:

$$\begin{cases} u''(t) + Mu(t) = 0, & t \in I, \\ u(0) - u(1) = 1, \\ u'(0) - u'(1) = 0, \end{cases}$$

and ω_2 is the unique solution to the problem:

$$\begin{cases} u''(t) + Mu(t) = 0, & t \in I, \\ u(0) - u(1) = 0, \\ u'(0) - u'(1) = 1. \end{cases}$$

It is immediate to see that and $\omega_1(t) = \omega_2'(t)$, for all $t \in I$, and so, as a consequence, $\int_0^1 \omega_1(s)\,ds = 0$. Moreover, it is very well known (see [1]) that $\omega_2(t) = G_{M,0,0}(t,0)$, and this functions satisfies that:

$$G_{M,0,0}(t,s) = \begin{cases} G_{M,0,0}(t-s,0), & 0 \le s \le t \le 1, \\ \\ G_{M,0,0}(1+t-s,0), & 0 \le t < s \le 1. \end{cases}$$

Thus, it holds that:

$$\int_0^1 G_{M,0,0}(t,s)dt = \int_0^1 G_{M,0,0}(t,0)dt = \int_0^1 \omega_2(t)dt = -\frac{1}{M}\int_0^1 \omega_2''(t)dt = \frac{1}{M}, \quad \forall\, s \in I.$$

Therefore, (10) can be rewritten as:

$$G_{M,\delta_1,\delta_2}(t,s) = G_{M,0,0}(t,s) + \frac{\delta_1\,\omega_1(t) + \delta_2\,\omega_2(t)}{M - \delta_2} = G_{M,0,\delta_2}(t,s) + \frac{\delta_1\,\omega_1(t)}{M - \delta_2}. \tag{11}$$

Taking into account the previous expression, we shall start with the study of $G_{M,0,\delta_2}$ (that is, the particular case in which $\delta_1 = 0$), and later on, we use (11) to study the general case from the previous one.

4. Study of Case $\delta_1 = 0$

As we mentioned before, in this section, we study the regions of constant sign of Green's function related to the following perturbed periodic problem:

$$\begin{cases} u''(t) + Mu(t) = \sigma(t), & t \in I, \\ u(0) - u(1) = 0, \\ u'(0) - u'(1) = \delta_2 \displaystyle\int_0^1 u(s)ds, \end{cases} \tag{12}$$

for $M, \delta_2 \in \mathbb{R}$. We recall that this problem is the particular case of considering $\delta_1 = 0$ in (1).

First of all, we note that the spectrum of Problem (12) is given by:

$$(\delta_2, M) \in \left\{ (4k^2\pi^2, \delta_2),\ \delta_2 \in \mathbb{R},\ k = 1,2,\dots \right\} \cup \{(M,M),\ M \in \mathbb{R}\}.$$

On the other hand, the spectrum of the homogeneous periodic problem ($\delta_1 = \delta_2 = 0$):

$$\begin{cases} u''(t) + Mu(t) = \sigma(t), \quad t \in I, \\ u(0) - u(1) = 0, \\ u'(0) - u'(1) = 0, \end{cases} \tag{13}$$

is given by $4k^2\pi^2$, $k = 0, 1, 2 \ldots$, that is $G_{M,0,0}$ exists and is unique if and only if $M \neq 4k^2\pi^2$, $k = 0, 1, 2 \ldots$.

Thus, Formula (10) is valid to compute $G_{M,0,\delta_2}$ for all $M \neq 4k^2\pi^2$, $k = 0, 1, \ldots$ and $\delta_2 \neq M$. Green's function $G_{0,0,\delta_2}$, with $\delta_2 \neq 0$, exists, but it cannot be calculated using (10), so we need to do this by means of direct integration.

Let us now characterize the points where a constant-sign Green's function $G_{M,0,\delta_2}$ may vanish.

Lemma 4. *Let $M < \pi^2$. If $\delta_2 < 0$, $G_{M,0,\delta_2}$ is non-negative on $I \times I$ and vanishes at some point $(t_0, s_0) \in I \times I$, then $t_0 = s_0$.*

Proof. From Lemma 3, we only need to discard the cases $(0, s_0)$ and $(1, s_0)$ with $s_0 \in (0, 1)$. We note that, since $G_{M,0,\delta_2}(0, s_0) = G_{M,0,\delta_2}(1, s_0)$, both cases are equivalent. Suppose then that:

$$G_{M,0,\delta_2}(0, s_0) = G_{M,0,\delta_2}(1, s_0) = 0.$$

In such a case, it would occur that $\frac{\partial G_{M,0,\delta_2}}{\partial t}(0, s_0) \geq 0$ and $\frac{\partial G_{M,0,\delta_2}}{\partial t}(1, s_0) \leq 0$, which contradicts the fact that:

$$\frac{\partial G_{M,0,\delta_2}}{\partial t}(0, s) - \frac{\partial G_{M,0,\delta_2}}{\partial t}(1, s) = \frac{\delta_2}{M - \delta_2} < 0 \quad \forall s \in (0, 1).$$

As a consequence, the only possibility is that $t_0 = s_0$. □

Lemma 5. *Let $M < \pi^2$. If $\delta_2 > M$, $G_{M,0,\delta_2}$ is non-positive on $I \times I$ and vanishes at some point $(t_0, s_0) \in I \times I$, then either $t_0 = 0$ or $t_0 = 1$.*

Proof. From Lemma 3, we only need to discard the case $t_0 = s_0$. In such a case, since $G_{M,0,\delta_2}$ is non-positive, it must occur that $\frac{\partial G_{M,0,\delta_2}}{\partial t}(t_0^-, t_0) \geq 0$ and $\frac{\partial G_{M,0,\delta_2}}{\partial t}(t_0^+, t_0) \leq 0$, which contradicts the fact that:

$$\frac{\partial G_{M,0,\delta_2}}{\partial t}(t^+, t) - \frac{\partial G_{M,0,\delta_2}}{\partial t}(t^-, t) = 1 \quad \forall t \in (0, 1).$$

As a consequence, the only possibility is that either $t_0 = 0$ or $t_0 = 1$. □

4.1. Expression of Green's Function

We obtain now the exact expression of Green's function related to Problem (12) by considering the different situations of the parameters M and δ_2. We start with $M \neq 0$, that is the situation in which Problem (12) is uniquely solvable for $\delta_2 = 0$.

We must point out that the exact expression of $G_{M,0,0}$ can be consulted in [1] and references therein. More concisely, on such references (coauthored by the first author of this work) it was developed an algorithm that calculates the exact expression of Green's function related to any n-th order differential equation with constant coefficients coupled to arbitrary homogeneous ($\delta_i = 0$, $i = 0, \ldots, n-1$) two-point linear boundary conditions. Such an algorithm was developed in a Mathematica package and is free available at the web page of Wolfram.

4.1.1. $M \neq 0$

In this case, using Expression (11) and taking into account that, as we already mentioned, $\omega_2(t) = G_{M,0,0}(t,0)$, the expression of Green's function related to Problem (12) is given by:

$$G_{M,0,\delta_2}(t,s) = G_{M,0,0}(t,s) + \frac{\delta_2}{M-\delta_2}\, G_{M,0,0}(t,0). \tag{14}$$

We shall consider two different cases:

$M = m^2 > 0$, with $m \in (0,\infty)$:

In such a case, $G_{M,0,0}$ is given by the expression:

$$G_{M,0,0}(t,s) = \frac{\csc\left(\frac{m}{2}\right)}{2m} \begin{cases} \cos\left(\frac{m}{2}(1+2s-2t)\right), & 0 \le s \le t \le 1, \\ \cos\left(\frac{m}{2}(1+2t-2s)\right), & 0 \le t < s \le 1, \end{cases}$$

and so, (14) implies that:

$$G_{M,0,\delta_2}(t,s) = \frac{\delta_2 \cos\left(\frac{m}{2}(1-2t)\right)}{m^2-\delta_2} + \frac{\csc\left(\frac{m}{2}\right)}{2m} \begin{cases} \cos\left(\frac{m}{2}(1+2s-2t)\right), & 0 \le s \le t \le 1, \\ \cos\left(\frac{m}{2}(1+2t-2s)\right), & 0 \le t < s \le 1. \end{cases}$$

$M = -m^2 < 0$, with $m \in (0,\infty)$:

In this case, $G_{M,0,0}$ is given by:

$$G_{M,0,0}(t,s) = \frac{1}{2m(1-e^m)} \begin{cases} e^{m(1+s-t)} + e^{m(t-s)}, & 0 \le s \le t \le 1, \\ e^{m(1+t-s)} + e^{m(s-t)}, & 0 \le t < s \le 1, \end{cases}$$

and thus:

$$G_{M,0,\delta_2}(t,s) = -\frac{\delta_2}{m^2+\delta_2}\left(e^{m(1-t)} + e^{mt}\right) + \frac{1}{2m(1-e^m)} \begin{cases} e^{m(1+s-t)} + e^{m(t-s)}, & 0 \le s \le t \le 1, \\ e^{m(1+t-s)} + e^{m(s-t)}, & 0 \le t < s \le 1. \end{cases}$$

4.1.2. $M = 0$

In this case, Formula (6) is not valid to calculate the expression of Green's function, so we shall compute it by direct integration.

Since the solution of equation $u''(t) = \sigma(t)$ is given by:

$$u(t) = c_1 + c_2 t + \int_0^t (t-s)\sigma(s)ds,$$

then $u'(t) = c_2 + \int_0^t \sigma(s)ds$. Imposing now the condition $u(0) = u(1)$, we have that $c_2 = -\int_0^1 (1-s)\sigma(s)ds$. Therefore, $u'(0) - u'(1) = -\int_0^1 \sigma(s)ds$, and since $u'(0) - u'(1) = \delta_2 \int_0^1 u(s)ds$, we deduce that:

$$c_1 = -\frac{1}{\delta_2}\int_0^1 \sigma(s)ds - \frac{1}{2}\int_0^1 \left(1-s^2\right)\sigma(s)ds + \int_0^1 \left(s-s^2\right)\sigma(s)ds + \frac{1}{2}\int_0^1 (1-s)\sigma(s)ds.$$

Therefore,

$$\begin{aligned} u(t) &= -\frac{1}{\delta_2}\int_0^1 \sigma(s)ds - \frac{1}{2}\int_0^1 \left(1-s^2\right)\sigma(s)ds + \int_0^1 s(1-s)\sigma(s)ds \\ &\quad + \left(\frac{1}{2}-t\right)\int_0^1 (1-s)\sigma(s)ds + \int_0^t (t-s)\sigma(s)ds, \\ &= \int_0^1 G_{0,0,\delta_2}(t,s)\,\sigma(s)ds, \end{aligned}$$

where:

$$G_{0,0,\delta_2}(t,s) = \begin{cases} -\frac{s}{2} - \frac{s^2}{2} + st - \frac{1}{\delta_2}, & 0 \le s \le t \le 1, \\ \frac{s}{2} - \frac{s^2}{2} - t + st - \frac{1}{\delta_2}, & 0 \le t < s \le 1. \end{cases}$$

4.2. Regions of Constant Sign of Green's Function

We shall study now the regions in which the functions that we have just calculated have a constant sign. To begin with, we note that we can bound these regions in the following way.

Lemma 6. *$G_{M,0,\delta_2}$ will never have a constant sign on $I \times I$ for any $M > \pi^2$.*

Proof. From Expression (14) and the fact that:

$$G_{M,0,\delta_2}(t,0) = \left(1 + \frac{\delta_2}{M-\delta_2}\right) G_{M,0,0}(t,0) = \frac{m^2}{m^2-\delta_2}\,\frac{\csc\left(\frac{m}{2}\right)\cos\left(\frac{m}{2}(1-2t)\right)}{2m},$$

it is immediately deduced that $G_{M,0,\delta_2}(t,0)$ is sign-changing on I for any $m > \pi$. □

Lemma 7. *The following properties are fulfilled:*

- *If $M < \delta_2 \le 0$, then $G_{M,0,\delta_2}$ is negative on $I \times I$;*
- *If $0 \le \delta_2 < M \le \pi^2$, then $G_{M,0,\delta_2}$ is positive on $I \times I$;*
- *If $M = \pi^2$ and $0 \le \delta_2 < M$, then $G_{M,0,\delta_2}$ vanishes at the set $A := \{(0,0),(0,1),(1,0),(1,1)\}$ and is positive on $(I \times I)\backslash A$.*

Proof. This is immediately deduced from (14) and the fact that $G_{M,0,0}$ is negative on $I \times I$ for $M < 0$, positive on $I \times I$ for $0 < M < \pi$, and positive on $(I \times I)\backslash A$, vanishing at the set A, for $M = \pi^2$. □

Moreover, since:

$$\frac{\partial G_{M,0,\delta_2}}{\partial \delta_2}(t,s) = \frac{M}{(M-\delta_2)^2}\,G_{M,0,0}(t,0) > 0, \quad \forall\, M \in \left(-\infty,\pi^2\right) \setminus \{0\},\ \delta_2 \ne M,\ t,s \in I$$

and:

$$\frac{\partial G_{0,0,\delta_2}}{\partial \delta_2}(t,s) = \left(\frac{1}{\delta_2}\right)^2 > 0 \quad \forall\, t,s \in I, \tag{15}$$

we deduce that, for any fixed $M < \pi^2$, $G_{M,0,\delta_2}$ is strictly increasing with respect to δ_2. As a consequence, we deduce the following facts:

- Since $G_{M,0,0} > 0$ on $I \times I$ for $M \in (0,\pi^2)$, we know that $G_{M,0,\delta_2}$ will be positive for some values of $\delta_2 < 0$. In particular, $G_{M,0,\delta_2}$ will be positive for $\delta_2 \in (\delta_2(M), 0]$, where the optimal value $\delta_2(M)$ will be either $-\infty$ or the biggest negative real value for which $G_{M,0,\delta_2(M)}$ attains the value of zero at some point $(t_0,s_0) \in I \times I$;
- Since $G_{M,0,0} < 0$ on $I \times I$ for $M < 0$, we know that $G_{M,0,\delta_2}$ will be negative for some values of $\delta_2 > 0$. In particular, $G_{M,0,\delta_2}$ will be negative for $\delta_2 \in [0, \delta_2(M))$, where the

optimal value $\delta_2(M)$ will be either $+\infty$ or the smallest positive real value for which $G_{M,0,\delta_2(M)}$ attains the value of zero at some point $(t_0, s_0) \in I \times I$.

Let us study now the range of values $\delta_2 < 0$ for which $G_{M,0,\delta_2}$ is positive.

Theorem 2. *If $M = m^2$ with $m \in (0, \pi)$ and $\delta_2 \leq 0$, then $G_{M,0,\delta_2}(t,s) > 0$ for all $(t,s) \in I \times I$ if and only if:*

$$-\frac{m^2 \cos\left(\frac{m}{2}\right)}{1-\cos\left(\frac{m}{2}\right)} < \delta_2 \leq 0.$$

Proof. From Lemma 4, we only need to study the values of function $G_{M,0,\delta_2}$ at the diagonal of its square of definition, where we obtain the function:

$$h(t) = G_{M,0,\delta_2}(t,t) = \frac{\coth\left(\frac{m}{2}\right)}{2m} + \frac{\delta_2 \cos\left(\frac{m}{2}(1-2t)\right)\csc\left(\frac{m}{2}\right)}{2m(m^2-\delta_2)}, \quad t \in I,$$

whose minimum is attained at $t = \frac{1}{2}$. Therefore, h has a positive sign on I if and only if $h\left(\frac{1}{2}\right)$ is positive, that is $\delta_2 > -\frac{m^2\cos\left(\frac{m}{2}\right)}{1-\cos\left(\frac{m}{2}\right)}$. □

Let us analyze now the range of values $\delta_2 > 0$ for which $G_{M,0,\delta_2}$ is negative.

Theorem 3. *Let $M = -m^2$ with $m \in (0, \infty)$ and $\delta_2 \geq 0$, then $G_{M,0,\delta_2}$ is strictly negative on $I \times I$ if and only if:*

$$0 \leq \delta_2 < \frac{2m^2 e^{\frac{m}{2}}}{1+e^m-2e^{\frac{m}{2}}}.$$

Proof. From Lemma 5, we only need to study the values of function $G_{M,0,\delta_2}$ at the points of the form $(0,s)$ and $(1,s)$. Therefore, we have to study the function:

$$r(s) = G_{M,0,\delta_2}(0,s) = G_{M,0,\delta_2}(1,s) = \frac{1}{2m(1-e^m)}\left(e^{m(1-s)} + e^{ms} - \frac{\delta_2}{\delta_2+m^2}(1+e^m)\right),$$

whose maximum value is attained at $s = \frac{1}{2}$. Therefore, r is negative if and only if $r\left(\frac{1}{2}\right) < 0$, that is $\delta_2 < \frac{2m^2e^{\frac{1}{2}}}{1+e^m-2e^{\frac{m}{2}}}$. □

From the previous results and (15), we deduce the following facts:

- Since $G_{M,0,\delta_2} > 0$ for $M \in (0, \pi^2)$ and $-\frac{m^2\cos\left(\frac{m}{2}\right)}{1-\cos\left(\frac{m}{2}\right)} < \delta_2 \leq 0$, we know that $G_{0,0,\delta_2}$ will be positive for some values of $\delta_2 < 0$. In particular, $G_{0,0,\delta_2}$ will be positive for $\delta_2 \in (\delta_2(0), 0)$, where the optimal value $\delta_2(0)$ will be either $-\infty$ or the biggest negative real value for which $G_{0,0,\delta_2(0)}$ attains the value of zero at some point $(t_0, s_0) \in I \times I$;
- Since $G_{M,0,\delta_2} < 0$ for $M < 0$ and $0 \leq \delta_2 < \frac{2m^2e^{\frac{m}{2}}}{1+e^m-2e^{\frac{m}{2}}}$, we know that $G_{0,0,\delta_2}$ will be negative for some values of $\delta_2 > 0$. In particular, $G_{0,0,\delta_2}$ will be negative for $\delta_2 \in (0, \delta_2(0))$, where the optimal value $\delta_2(0)$ will be either $+\infty$ or the smallest positive real value for which $G_{0,0,\delta_2(0)}$ attains the value of zero at some point $(t_0, s_0) \in I \times I$.

Let us study the sign of function $G_{0,0,\delta_2}$ according to the value of $\delta_2 \in \mathbb{R} \setminus \{0\}$.

Theorem 4. *$G_{0,0,\delta_2}$ is strictly negative on $I \times I$ if and only if $\delta_2 \in (0, 8)$.*

Proof. For $\delta_2 > 0$, using Lemma 4, the function to study in this case is:

$$r(s) = G_{0,0,\delta_2}(0,s) = G_{0,0,\delta_2}(1,s) = \frac{s}{2} - \frac{s^2}{2} - \frac{1}{\delta_2}, \quad s \in I,$$

which reaches its maximum at $s = \frac{1}{2}$. As a consequence, $G_{0,0,\delta_2}$ is negative if and only if $r\left(\frac{1}{2}\right) < 0$, that is $0 < \delta_2 < 8$. $\square$

Using the same arguments, by means of Lemma 5, we arrive at the following result for the negative sign of δ_2.

Theorem 5. *$G_{0,0,\delta_2}$ is strictly positive on $I \times I$ if and only if $\delta_2 \in (-8, 0)$.*

Finally, we have that:

- Since $G_{0,0,\delta_2} > 0$ on $I \times I$ for $\delta_2 \in (-8, 0)$, we know that $G_{M,0,\delta_2}$ will be positive for some values of $\delta_2 < M < 0$. In particular, $G_{M,0,\delta_2}$ will be positive for $\delta_2 \in (\delta_2(M), M)$, where the optimal value $\delta_2(M)$ will be either $-\infty$ or the biggest negative real value for which $G_{M,0,\delta_2(M)}$ attains the value of zero at some point $(t_0, s_0) \in I \times I$;
- Since $G_{0,0,\delta_2} < 0$ on $I \times I$ for $\delta_2 \in (0, 8)$, we know that $G_{M,0,\delta_2}$ will be negative for some values of $\delta_2 > M > 0$. In particular, $G_{M,0,\delta_2}$ will be negative for $\delta_2 \in (M, \delta_2(M))$, where the optimal value $\delta_2(M)$ will be either $+\infty$ or the smallest positive real value for which $G_{M,0,\delta_2(M)}$ attains the value of zero at some point $(t_0, s_0) \in I \times I$.

Theorem 6. *If $M = m^2$ with $m \in (0, \pi)$, then $G_{M,0,\delta_2}(t,s) < 0$ for all $(t,s) \in I \times I$ if and only if:*

$$m^2 < \delta_2 < \frac{m^2}{1 - \cos\left(\frac{m}{2}\right)}.$$

Proof. Arguing as in the previous results, using Lemma 5, we only need to consider the function $G_{M,0,\delta_2}$ at the points of the form $(0,s)$ and $(1,s)$, where the corresponding function to study is:

$$r(s) = G_{M,0,\delta_2}(0,s) = G_{M,0,\delta_2}(1,s) = \frac{\csc\left(\frac{m}{2}\right)}{2m}\left[\cos\left(\frac{m}{2}(1-2t)\right) + \frac{\delta_2}{m^2 - \delta_2}\cos\left(\frac{m}{2}\right)\right].$$

In this case, r has an absolute maximum at $s = \frac{1}{2}$. Thus, $r(s) < 0$ for all $s \in I$ if and only if $r\left(\frac{1}{2}\right) < 0$, that is $\delta_2 < \frac{m^2}{1-\cos\left(\frac{m}{2}\right)}$. $\square$

We now perform a study of the positive sign of $G_{M,0,\delta_2}$ for $m \in (0, \infty)$ and $\delta_2 < -m^2 < 0$.

Theorem 7. *Let $M = -m^2$ with $m \in (0, \infty)$, then Green's function related to Problem (12) is strictly positive on $I \times I$ if and only if $\delta_2 > -\dfrac{m^2(1+e^m)}{1+e^m-2e^{\frac{m}{2}}}$.*

Proof. From Lemma 4, we only must study the behavior of Green's function at the points of its diagonal. In this case,

$$h(t) = G_{M,0,\delta_2}(t,t) = \frac{e^m + 1}{2m(1-e^m)} - \frac{\delta_2}{\delta_2 + m^2}\,\frac{e^{m(1-t)} + e^{mt}}{2m(1-e^m)}$$

has in this case an absolute minimum at $t = \frac{1}{2}$. Therefore, h is positive on I if and only if $h\left(\frac{1}{2}\right) > 0$, that is $\delta_2 > -\dfrac{m^2(1+e^m)}{1+e^m-2e^{\frac{m}{2}}}$. $\square$

Figure 1 shows the regions where the function $G_{M,0,\delta_2}$ maintains a constant sign.

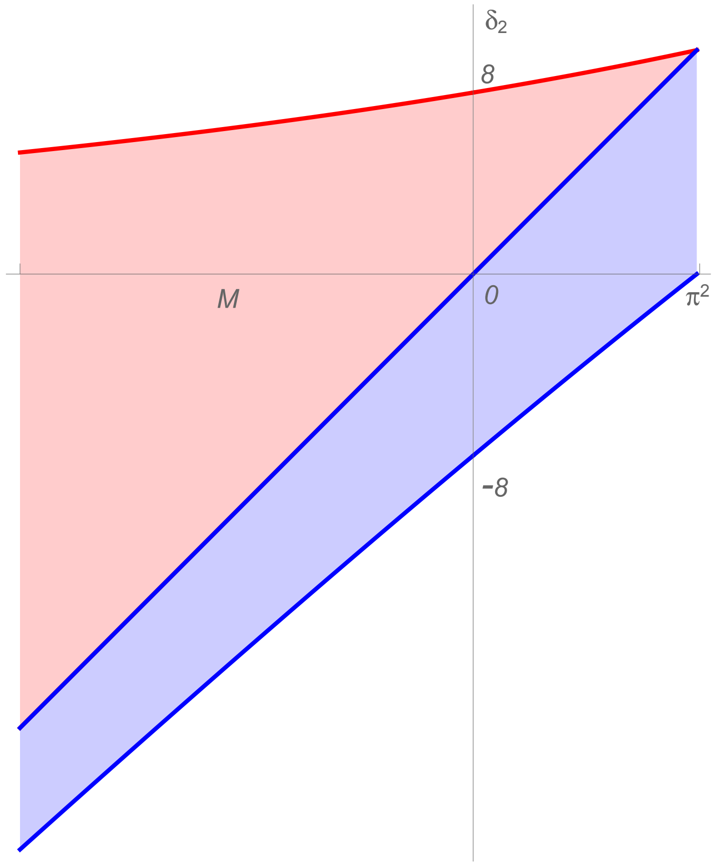

Figure 1. The blue region represents the positive sign of $G_{M,0,\delta_2}$, while the red region corresponds to the negative sign of $G_{M,0,\delta_2}$ at the (M,δ_2)-plane.

5. Case $\delta_1 \neq 0$

In this last section, we consider the general situation of $\delta_1 \neq 0$ and calculate the regions of constant sign of Green's function related to Problem (1). We divide this study into two different situations, depending on the fact that the parameter M is or is not equal to zero.

First of all, we obtain the expression of Green's function in each of the aforementioned cases.

5.1. Expression of Green's Function

In this subsection, we obtain the expression of Green's function related to Problem (1) as a function of the real parameter M.

5.1.1. $M \neq 0$

Using Formula (11) and the fact that $\omega_1(t) = \omega_2'(t)$, it is obtained that the expression of $G_{\delta_1,\delta_2,M}$ is given by:

$$G_{M,\delta_1,\delta_2}(t,s) = G_{M,0,0}(t,s) + \frac{\delta_1\,\omega_2'(t) + \delta_2\,\omega_2(t)}{M-\delta_2}, \tag{16}$$

where:

$$\omega_2(t) = \frac{1}{2m}\begin{cases} \cos\left(\frac{m}{2}\,(2\,t-1)\right)\csc\left(\frac{m}{2}\right), & M = m^2 > 0,\ M \neq 4k^2\pi^2,\ k = 0,1,\dots \\ -\cosh\left(\frac{m}{2}\,(2\,t-1)\right)\operatorname{csch}\left(\frac{m}{2}\right), & M = -m^2 < 0. \end{cases}$$

Thus, for $M = m^2 > 0$, $m > 0$, $M \neq 4k^2\pi^2$, $k = 0,1,\dots$, $G_{\delta_1,\delta_2,M}$ follows the expression:

$$G_{M,\delta_1,\delta_2}(t,s) = \frac{\csc\left(\frac{m}{2}\right)}{2m} \begin{cases} \cos\left(m\left(\frac{1}{2}+s-t\right)\right) + \frac{\delta_2 \cos\left(m\left(\frac{1}{2}-t\right)\right)+m\delta_1 \sin\left(m\left(\frac{1}{2}-t\right)\right)}{m^2-\delta_2}, & 0 \le s \le t \le 1, \\ \cos\left(m\left(\frac{1}{2}+t-s\right)\right) + \frac{\delta_2 \cos\left(m\left(\frac{1}{2}-t\right)\right)+m\delta_1 \sin\left(m\left(\frac{1}{2}-t\right)\right)}{m^2-\delta_2}, & 0 \le t < s \le 1, \end{cases}$$

and for $M = -m^2$, $m > 0$, the expression of $G_{\delta_1,\delta_2,M}$ is given by:

$$G_{M,\delta_1,\delta_2}(t,s) = \frac{\operatorname{csch}\left(\frac{m}{2}\right)}{2m} \begin{cases} -\cosh\left(m\left(\frac{1}{2}+s-t\right)\right) + \frac{\delta_2 \cosh\left(m\left(\frac{1}{2}-t\right)\right)-m\,\delta_1 \sinh\left(m\left(\frac{1}{2}-t\right)\right)}{m^2+\delta_2}, & 0 \le s \le t \le 1 \\ -\cosh\left(m\left(\frac{1}{2}+t-s\right)\right) + \frac{\delta_2 \cosh\left(m\left(\frac{1}{2}-t\right)\right)-m\,\delta_1 \sinh\left(m\left(\frac{1}{2}-t\right)\right)}{m^2+\delta_2}, & 0 \le t < s \le 1. \end{cases}$$

5.1.2. $M = 0$

For the case $M = 0$, we cannot apply Formula (10), and we need to compute G_{0,δ_1,δ_2} directly.

It is clear that the solutions of the equation $u''(t) = \sigma(t)$, $t \in I$ are given by the expression:

$$u(t) = c_1 + c_2 t + \int_0^t (t-s)\sigma(s)ds. \tag{17}$$

So, $u(0) - u(1) = -c_2 + \int_0^1 (s-1)\sigma(s)ds$.

On the other hand,

$$\int_0^1 u(t)dt = \int_0^1 \left(c_1 + c_2 t + \int_0^t (t-s)\sigma(s)ds\right)dt = c_1 + \frac{c_2}{2} + \int_0^1 \int_0^t (t-s)\sigma(s)\,ds\,dt.$$

Applying Fubini's theorem, we have that:

$$\int_0^1 \int_0^t (t-s)\sigma(s)ds = \int_0^1 \int_s^1 (t-s)\sigma(s)\,dt\,ds = \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)ds.$$

Imposing the boundary conditions in (1), we arrive at the following system of equations:

$$\begin{aligned} \delta_1 c_1 + \left(\frac{\delta_1}{2}+1\right)c_2 &= \int_0^1 (s-1)\sigma(s)ds - \delta_1 \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)ds, \\ \delta_2 c_1 + \frac{\delta_2}{2}c_2 &= -\int_0^1 \sigma(s)ds - \delta_2 \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)ds, \end{aligned}$$

whose solutions are:

$$\begin{aligned} c_1 &= -\frac{1}{2}\int_0^1 (s-1)\sigma(s)ds - \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)ds - \frac{\delta_1+2}{2\,\delta_2}\int_0^1 \sigma(s)ds, \\ c_2 &= \int_0^1 (s-1)\sigma(s)ds + \frac{\delta_1}{\delta_2}\int_0^1 \sigma(s)ds. \end{aligned}$$

Substituting c_1 and c_2 in (17), we have that:

$$\begin{aligned} u(t) = &-\frac{1}{2}\int_0^1 (s-1)\sigma(s)ds - \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)ds - \frac{\delta_1+2}{2\,\delta_2}\int_0^1 \sigma(s)ds \\ &+ \int_0^1 t(s-1)\sigma(s)ds + \frac{\delta_1}{\delta_2}\int_0^1 t\,\sigma(s)ds + \int_0^1 (t-s)\sigma(s)ds \\ = &\int_0^1 G_{0,\delta_1,\delta_2}(t,s)\,\sigma(s)ds, \end{aligned}$$

being:

$$G_{0,\delta_1,\delta_2}(t,s) = \frac{1}{2\,\delta_2}\begin{cases} -2+\delta_1\,(-1+2t)-s\,\delta_2\,(1+s-2t), & 0\le s\le t\le 1,\\ \\ -2+\delta_1\,(-1+2t)-\delta_2\,(s-1)\,(s-2t), & 0\le t< s\le 1. \end{cases} \tag{18}$$

5.2. Regions of Constant Sign of Green's Function

Now, we are in a position to obtain the regions of constant sign of Green's function as a function of the parameters M, δ_1, and δ_2.

To this end, we notice that, by direct differentiation on (16) and (18), the following identities hold:

$$\frac{\partial}{\partial \delta_1} G_{M,\delta_1,\delta_2}(t,s) = \frac{\omega_1(t)}{M-\delta_2} \quad \text{for } M\neq 0,\ M\neq \delta_2$$

and:

$$\frac{\partial}{\partial \delta_1} G_{0,\delta_1,\delta_2}(t,s) = \frac{1}{\delta_2}\left(t-\frac{1}{2}\right),$$

which implies that $\frac{\partial}{\partial\delta_1} G_{M,\delta_1,\delta_2}$ will change sign depending on t. As a consequence, there will be some values of t for which G_{M,δ_1,δ_2} will increase with respect to δ_1 and some other values of t for which G_{M,δ_1,δ_2} will decrease with respect to δ_1. As an immediate consequence, we deduce the following result.

Corollary 1. *The two following properties hold:*

- *If M and δ_2 are such that $G_{M,0,\delta_2} > 0$ on $I\times I$, then G_{M,δ_1,δ_2} is either positive or changes its sign on $I\times I$;*
- *If M and δ_2 are such that $G_{M,0,\delta_2} < 0$ on $I\times I$, then G_{M,δ_1,δ_2} is either negative or changes its sign on $I\times I$.*

Furthermore, the following result can be easily verified.

Lemma 8. *If M and δ_2 are such that $G_{M,0,\delta_2}$ changes sign on $I\times I$, then G_{M,δ_1,δ_2} also changes its sign on $I\times I$ for every $\delta_1\in\mathbb{R}$.*

Proof. It is immediately verified using arguments similar to Theorems 2, 3, 6, and 7. In particular, it is obtained that:

1. If $M=m^2\in(0,\pi^2)$ and $\delta_2 < -\frac{m^2\cos\left(\frac{m}{2}\right)}{1-\cos\left(\frac{m}{2}\right)}$, then $G_{M,\delta_1,\delta_2}\left(\frac{1}{2},\frac{1}{2}\right)<0$ and $G_{M,\delta_1,\delta_2}(0,0)>0$;
2. If $M=m^2\in(0,\pi^2)$ and $\delta_2 > \dfrac{m^2}{1-\cos\left(\frac{m}{2}\right)}$, then $G_{M,\delta_1,\delta_2}\left(\frac{1}{2},\frac{1}{2}\right)>0$ and $G_{M,\delta_1,\delta_2}(0,0)<0$;
3. If $M=-m^2$, with $m\in(0,\infty)$, and $\delta_2 > \dfrac{2m^2e^{\frac{m}{2}}}{1+e^m-2e^{\frac{m}{2}}}$, then $G_{M,\delta_1,\delta_2}\left(\frac{1}{2},\frac{1}{2}\right)>0$ and $G_{M,\delta_1,\delta_2}(0,0)<0$;
4. If $M=-m^2$, with $m\in(0,\infty)$, and $\delta_2 < -\dfrac{m^2(1+e^m)}{1+e^m-2e^{\frac{m}{2}}}$ then $G_{M,\delta_1,\delta_2}\left(\frac{1}{2},\frac{1}{2}\right)<0$ and $G_{M,\delta_1,\delta_2}(0,0)>0$;
5. If $M>\pi^2$ then $G_{M,\delta_1,\delta_2}(t,0)$ is sign-changing on I.

□

Moreover, since for any fixed $t\in I$, $G_{M,\delta_1,\delta_2}(t,s)$ is either increasing or decreasing with respect to δ_1, we deduce the following facts:

- If M and δ_2 are such that $G_{M,0,\delta_2}>0$ on $I\times I$, then G_{M,δ_1,δ_2} will be positive on $I\times I$ for some values (both positive and negative) of δ_1. In particular, by Lemma 2, we know that G_{M,δ_1,δ_2} will be positive on $I\times I$ for $\delta_1\in(-\delta_1(\delta_2,M),\delta_1(\delta_2,M))$, where the optimal value $\delta_1(\delta_2, M)$ will be either $+\infty$ or the smallest positive real value for which $G_{M,\delta_1(\delta_2,M),\delta_2}$ attains the value of zero at some point;

- If M and δ_2 are such that $G_{M,0,\delta_2} < 0$ on $I \times I$, then G_{M,δ_1,δ_2} will be negative on $I \times I$ for some values (both positive and negative) of δ_1. In particular, by Lemma 2, we know that G_{M,δ_1,δ_2} will be negative on $I \times I$ for $\delta_1 \in (-\delta_1(\delta_2, M), \delta_1(\delta_2, M))$, where the optimal value $\delta_1(\delta_2, M)$ will be either $+\infty$ or the smallest positive real value for which $G_{M,\delta_1(\delta_2,M),\delta_2}$ attains the value of zero at some point.

Similar to Lemmas 4 and 5, we can make precise the points where a constant-sign Green's function G_{M,δ_1,δ_2} may vanish.

Lemma 9. *Let $M < \pi^2$ and $\delta_1 > 0$. If G_{M,δ_1,δ_2} has a constant sign on $I \times I$ and vanishes at some point (t_0, s_0), then either $t_0 = 1$ or $t_0 = s_0$.*

Proof. Let us suppose that $G_{M,\delta_1,\delta_2} \geq 0$ (the case $G_{M,\delta_1,\delta_2} \leq 0$ would be analogous). From Lemma 3, we only need to discard the case $t_0 = 0$. Suppose then that $G_{M,\delta_1,\delta_2}(0, s_0) = 0$ for some $s_0 \in (0, 1)$. In such a case, from the equality:

$$G_{M,\delta_1,\delta_2}(0, s_0) - G_{M,\delta_1,\delta_2}(1, s_0) = \delta_1 \int_0^1 G_{M,\delta_1,\delta_2}(t, s_0)\, dt > 0,$$

we deduce that $G_{M,\delta_1,\delta_2}(1, s_0) < 0$, which is a contradiction. Therefore, G_{M,δ_1,δ_2} cannot vanish at $(0, s_0)$. $\square$

5.3. Negativeness of G_{M,δ_1,δ_2}

Now, we study the region where Green's function is negative on the square of definition. We distinguish two situations.

5.3.1. $M \neq 0$

We analyze the region where G_{M,δ_1,δ_2} is negative on $I \times I$. To do this, taking into account Corollary 1, we fix $M \neq 0$ and δ_2 for which $G_{M,0,\delta_2}$ is negative on $I \times I$, that is $\delta_2 \in (M, f(M))$, with:

$$f(M) = \begin{cases} \frac{m^2}{1-\cos\left(\frac{m}{2}\right)}, & M = m^2,\ m \in (0, \pi), \\ \frac{2m^2 e^{\frac{m}{2}}}{1+e^m-2e^{\frac{m}{2}}}, & M = -m^2 < 0,\ m \in (0, \infty). \end{cases}$$

Taking into account Lemma 2, we only need to perform the calculations for $\delta_1 > 0$ (since the case $\delta_1 < 0$ follows by symmetry). On the other hand, it is immediate to verify that function ω_1 is strictly decreasing on I, $\omega_1(0) = \frac{1}{2}$, and $\omega_1(1) = -\frac{1}{2}$.

The characterization of the set is given in the following result.

Theorem 8. *Let $M < \pi^2, M \neq 0$, and $\delta_2 \in (M, f(M))$, then G_{M,δ_1,δ_2} is strictly negative on $I \times I$ if and only if:*

$$|\delta_1| < 2\,(M - \delta_2)\, G_{M,0,\delta_2}\left(1, \frac{1}{2}\right).$$

Proof. For $\delta_1 > 0$, it is immediately deduced from expression:

$$G_{M,\delta_1,\delta_2}(t, s) = G_{M,0,\delta_2}(t, s) + \frac{\delta_1}{M - \delta_2}\, \omega_1(t)$$

and the fact that ω_1 attains its minimum at $t = 1$ and $G_{M,0,\delta_2}$ attains its maximum at $(t,s) = \left(1, \frac{1}{2}\right)$. As a consequence:

$$\max_{t,s\in I} G_{M,\delta_1,\delta_2}(t,s) = G_{M,\delta_1,\delta_2}\left(1, \frac{1}{2}\right) = G_{M,0,\delta_2}\left(1, \frac{1}{2}\right) - \frac{\delta_1}{2(M-\delta_2)}.$$

Thus, G_{M,δ_1,δ_2} is negative if and only if $\delta_1 < 2\,(M-\delta_2)\,G_{M,0,\delta_2}\left(1, \frac{1}{2}\right)$.

The case $\delta_1 < 0$ follows by symmetry. □

5.3.2. $M = 0$

For the negative case, we set $\delta_2 \in (0,8)$ where $G_{0,0,\delta_2}$ is negative.

Theorem 9. *If $M = 0$ and $\delta_2 \in (0,8)$, then G_{0,δ_1,δ_2} is negative on $I \times I$ if and only if:*

$$|\delta_1| < 2 - \tfrac{\delta_2}{4}.$$

Proof. Suppose that $\delta_1 > 0$, and let us calculate the maximum of G_{0,δ_1,δ_2} (whose expression is given by (18)). From Lemma 9, we know that such a maximum is either at $t = 1$ or at $t = s$.

At points of the form $(1,s)$, we have that $r(s) = G_{0,\delta_1,\delta_2}(1,s)$ has an absolute maximum at $s = \frac{1}{2}$. Therefore, $r(s) < 0$ for all $s \in I$ if and only if $r\left(\frac{1}{2}\right) < 0$, that is $0 < \delta_1 < 2 - \frac{\delta_2}{4}$.

Let us consider now the restriction to the diagonal, that is $h(s) = G_{0,\delta_1,\delta_2}(s,s)$. Given $c = \frac{1}{2} - \frac{\delta_1}{\delta_2} < \frac{1}{2}$, it holds that $h'(s) < 0$ for $s < c$ and $h'(c) = 0$ and $h'(s) > 0$ for $s > c$. Thus, c is a minimum of h. If $c \in (0, \frac{1}{2})$, h attains its maximum either at $s = 1$ or at $s = 0$, while if $c \leq 0$, then $h' > 0$ on $(0,1]$, and the maximum is attained at $s = 1$. In any case, $h(0) = \frac{-2-\delta_1}{2\delta_2} < 0$ and $h(1) = \frac{-2+\delta_1}{2\delta_2} > h(0)$. Therefore, $h(s) < 0$ if and only if $h(1) < 0$, that is $0 < \delta_1 < 2$.

Therefore, for $\delta_1 > 0$, $G_{M,\delta_1,\delta_2} < 0$ on $I \times I$ if and only if:

$$\delta_1 < \min\left\{2 - \frac{\delta_2}{4}, 2\right\} = 2 - \frac{\delta_2}{4}.$$

Using the symmetry of G_{0,δ_1,δ_2} with respect to δ_1, we conclude the result. □

5.4. Positiveness of G_{M,δ_1,δ_2}

Let us calculate now the regions where G_{M,δ_1,δ_2} is positive. As usual, we distinguish two cases.

5.4.1. $M \neq 0$

Taking into account Corollary 1, let us fix $M \neq 0$ and δ_2 such that $G_{M,0,\delta_2}$ is positive on $I \times I$, that is $\delta_2 \in (g(M), M)$, with:

$$g(M) := \begin{cases} g_1\left(\sqrt{M}\right), & M \in (0,\pi^2), \\ g_2\left(\sqrt{-M}\right), & M < 0, \end{cases} \tag{19}$$

where $g_1(m) = -\dfrac{m^2\cos\left(\frac{m}{2}\right)}{1-\cos\left(\frac{m}{2}\right)}$ and $g_2(m) = -\frac{m^2\cosh\left(\frac{m}{2}\right)}{\cosh\left(\frac{m}{2}\right)-1}$.

Now, we define the function:

$$k(M) := \begin{cases} k_1\left(\sqrt{M}\right), & M \in (0,\pi^2), \\ k_2\left(\sqrt{-M}\right), & M < 0, \end{cases} \tag{20}$$

where $k_1(m) = -m^2\cot^2\left(\frac{m}{2}\right)$ and $k_2(m) = -m^2\coth^2\left(\frac{m}{2}\right)$.

It is easy to verify that:

$$g(M) < k(M) < M, \quad \text{for all} \quad M \neq 0.$$

We shall consider now two different subcases, depending on the sign of the parameter M. We start with $M > 0$.

Theorem 10. *Let functions g and k be defined in (19) and (20), respectively. Assume that $M = m^2$, $m \in (0, \pi)$ and $\delta_2 \in (g(M), M)$, then the two following properties are fulfilled:*

1. *If $\delta_2 \in (k(M), M)$, then G_{M,δ_1,δ_2} is strictly positive on $I \times I$ if and only if:*

$$|\delta_1| < m \cot\left(\frac{m}{2}\right);$$

2. *If $\delta_2 \in (g(M), k(M)]$, then G_{M,δ_1,δ_2} is strictly positive on $I \times I$ if and only if:*

$$|\delta_1| < \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2 - \delta_2)^2}}{m}.$$

Proof. Let us assume that $\delta_1 > 0$ and calculate the minimum of G_{M,δ_1,δ_2}. From Lemma 9, we know that such a minimum is either at $t = 1$ or $t = s$.

Let us distinguish several cases:

1. $\delta_2 \geq 0$ (that is, $\delta_2 \in [0, M)$):
At the points of the form $(1, s)$, the function to be studied is:

$$r(s) = G_{M,\delta_1,\delta_2}(1, s) = \frac{\cos\left(\frac{m}{2}(1-2s)\right)\csc\left(\frac{m}{2}\right)}{2m} + \frac{\delta_2 \cot\left(\frac{m}{2}\right)}{2m(m^2 - \delta_2)} - \frac{\delta_1}{2(m^2 - \delta_2)},$$

whose minimum is attained at $s = 0$ and $s = 1$ (indeed, $r(0) = r(1)$). Thus, $r(s) > 0$ for all $s \in I$ if and only if $r(0) = r(1) > 0$, that is $0 < \delta_1 < m \cot\left(\frac{m}{2}\right)$.
At the diagonal $t = s$, we obtain the following function:

$$h(s) = G_{M,\delta_1,\delta_2}(s, s) = \frac{\csc\left(\frac{m}{2}\right)}{2m}\left(\cos\left(\frac{m}{2}\right) + \frac{\delta_1\, m \sin\left(m\left(\frac{1}{2} - s\right)\right) + \delta_2 \cos\left(m\left(\frac{1}{2} - s\right)\right)}{m^2 - \delta_2}\right),$$

which attains its minimum at $s = 1$, and so, $h(s) > 0$ on I if and only if $h(1) = r(1) > 0$. Thus, from Lemma 9, we have that $G_{M,\delta_1,\delta_2} > 0$ on $I \times I$ if and only if $0 < \delta_1 < m \cot\left(\frac{m}{2}\right)$;

2. $\delta_2 < 0$ (that is, $\delta_2 \in (g(M), 0)$):
At the points of the form $(1, s)$, analogous to the previous case, we obtain that $r(s) > 0$ on I if and only if $r(1) > 0$, that is $0 < \delta_1 < m \cot\left(\frac{m}{2}\right)$.
At the diagonal $t = s$, we have that $h'(c) = 0$, $h'(s) > 0$ for $s > c$, and $h'(s) < 0$ for $s < c$, with $c = \frac{1}{2} - \frac{1}{m}\arctan\left(\frac{m\,\delta_1}{\delta_2}\right)$. Therefore, c is a minimum of h. Moreover, we note that $c \in I$ if and only if $\frac{1}{m}\arctan\left(\frac{m\delta_1}{\delta_2}\right) \in \left[-\frac{1}{2}, 0\right)$, that is $0 < \delta_1 \leq -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right)$.
Therefore, we subdivide the case $\delta_2 < 0$ into two cases:
   (a) If $\delta_1 \geq -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right)$, then $h'(s) < 0$ for all $s \in [0, 1)$ and the minimum of h is attained at $s = 1$. Thus, $h(s) > 0$ on I if and only if $h(1) = r(1) > 0$, that is $0 < \delta_1 < m \cot\left(\frac{m}{2}\right)$. As a consequence, $G_{M,\delta_1,\delta_2} > 0$ on $I \times I$ for $0 < \delta_1 < m \cot\left(\frac{m}{2}\right)$.
   We note that the two previous conditions, $\delta_1 \geq -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right)$ and $0 < \delta_1 < m \cot\left(\frac{m}{2}\right)$, are compatible if and only if $\delta_2 > -m^2 \cot^2\left(\frac{m}{2}\right) \equiv k(M)$;

(b) If $0 < \delta_1 < -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right)$, then h attains an absolute minimum at $c \in (0,1)$. In this case, $h(c) > 0$ if and only if:

$$\delta_1 < \frac{\sqrt{m^4 - 2m^2\delta_2 - \delta_2^2 + (m^2-\delta_2)^2\cos(m)}}{\sqrt{2}\,m} = \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}.$$

We note that $-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2 > 0$ for $m \in (0,\pi)$ and $\delta_2 \in (g(M),0)$. Indeed, $-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2 > 0$ if and only if $\left|\cos\left(\frac{m}{2}\right)\right| |m^2-\delta_2| \geq |\delta_2|$. Since $m \in (0,\pi)$ and $\delta_2 < 0$, the previous inequality is equivalent to:

$$\delta_2 \geq -\frac{m^2\cos\left(\frac{m}{2}\right)}{1-\cos\left(\frac{m}{2}\right)} \equiv g(M).$$

Moreover, we note that:

$$\min\left\{-\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right), \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}\right\}$$

$$= \begin{cases} \frac{\sqrt{-\delta_2^2+\cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}, & \delta_2 \in (g(M),k(M)), \\ -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right), & \delta_2 \in (k(M),M). \end{cases}$$

As a consequence, we conclude that:

- If $\delta_2 \in (g(M),k(M)]$ then, from (b), $G_{M,\delta_1,\delta_2} > 0$ for:

$$0 < \delta_1 < \min\left\{-\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right), \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}\right\}$$

$$= \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}.$$

- If $\delta_2 \in (k(M),M)$, then, from (a), $G_{M,\delta_1,\delta_2} > 0$ for:

$$-\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right) < \delta_1 < m\cot\left(\frac{m}{2}\right)$$

and, from (b), $G_{M,\delta_1,\delta_2} > 0$ for:

$$0 < \delta_1 \leq \min\left\{-\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right), \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}\right\}$$

$$= \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}.$$

Thus, $G_{M,\delta_1,\delta_2} > 0$ on $I \times I$ for $0 < \delta_1 < m\cot\left(\frac{m}{2}\right)$.

The fact that the obtained bounds are optimal follows from Lemma 9.
Using the symmetry with respect to δ_1, we conclude the result. □

In the sequel, we shall consider the case $M < 0$.

Theorem 11. *Let functions g and k be defined in (19) and (20), respectively. For any $M = -m^2$, with $m > 0$ and $\delta_2 \in (g(M),M)$, it holds that:*

1. *If* $\delta_2 \in (g(M), k(M)]$, *then* G_{M,δ_1,δ_2} *is strictly positive on* $I \times I$ *if and only if:*
$$|\delta_1| < \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2 \cosh^2\left(\frac{m}{2}\right)}}{m}.$$
2. *If* $\delta_2 \in (k(M), M)$, *then* G_{M,δ_1,δ_2} *is strictly positive on* $I \times I$ *if and only if:*
$$|\delta_1| < m \coth\left(\frac{m}{2}\right).$$

Proof. Let us assume that $\delta_1 > 0$ and calculate the minimum of G_{M,δ_1,δ_2}. From Lemma 9, we know that such a minimum is either at $t = 1$ or $t = s$.

At the points of the form $(1, s)$, we obtain the function:
$$r(s) = G_{M,\delta_1,\delta_2}(1,s) = \frac{\operatorname{csch}\left(\frac{m}{2}\right)}{2m}\left(\frac{\delta_1 m \sinh\left(\frac{m}{2}\right) + \delta_2 \cosh\left(\frac{m}{2}\right)}{m^2+\delta_2} - \cosh\left(m\left(s - \frac{1}{2}\right)\right)\right),$$
whose minimum is attained at $s = 0$ and $s = 1$ (indeed, $r(0) = r(1)$). Thus, $r(s) > 0$ for all $s \in I$ if and only if $r(0) = r(1) > 0$, that is $0 < \delta_1 < m \coth\left(\frac{m}{2}\right)$.

At the diagonal $t = s$, we obtain the following function:
$$h(s) = G_{M,\delta_1,\delta_2}(s,s) = \frac{\operatorname{csch}\left(\frac{m}{2}\right)}{2m}\left(\frac{\delta_2 \cosh\left(m\left(\frac{1}{2} - s\right)\right) - \delta_1 m \sinh\left(m\left(\frac{1}{2} - s\right)\right)}{m^2+\delta_2} - \cosh\left(\frac{m}{2}\right)\right).$$

It occurs that $h'(s) = 0$ if and only if $\tanh\left(\frac{m}{2}(1-2s)\right) = \frac{m\delta_1}{\delta_2}$. Since $\delta_2 < 0$ and $\tanh^{-1}(x)$ exists for $x \in [-1, 1]$, we have that h has a critical point $c = \frac{1}{2} - \frac{1}{m}\tanh^{-1}\left(\frac{m\delta_1}{\delta_2}\right)$ if and only if $-1 \le \frac{m\delta_1}{\delta_2} < 0$, that is $\delta_1 \le -\frac{\delta_2}{m}$. In such a case, it occurs that $h'(s) < 0$ for $s < c$ and $h'(s) > 0$ for $s > c$. Moreover, we can see that $c \in I$ if and only if $0 < \delta_1 \le -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right)\left(< -\frac{\delta_2}{m}\right)$. Therefore, we distinguish two cases:

(a) If $\delta_1 > -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right)$, then $h'(s) < 0$ for all $s \in I$ and h has a minimum at $s = 1$. Then, $h(s) > 0$ for all $s \in I$ if and only if $h(1) = r(1) > 0$, that is if and only if $0 < \delta_1 < m\coth\left(\frac{m}{2}\right)$. As a consequence, $G_{M,\delta_1,\delta_2} > 0$ on $I \times I$ for $0 < \delta_1 < m\coth\left(\frac{m}{2}\right)$. We note that the two previous conditions, $\delta_1 > -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right)$ and $0 < \delta_1 < m\coth\left(\frac{m}{2}\right)$, are compatible if and only if $\delta_2 > -m^2\coth^2\left(\frac{m}{2}\right) \equiv k(M)$;

(b) If $0 < \delta_1 \le -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right)$, then h attains an absolute minimum $c \in (0,1)$. In this case, $h(c) > 0$ (and, consequently, $h(s) > 0$ for $s \in I$) if and only if:
$$0 < \delta_1 < \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m}.$$
Note that, analogous to what was done in Theorem 10, it can be proven that $\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right) > 0$ for $M = -m^2 < 0$ and $\delta_2 \in (g(M), M)$.
Therefore, since $h(c) < h(1) = r(1)$, $G_{M,\delta_1,\delta_2} > 0$ on $I \times I$ for:
$$0 < \delta_1 < \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m}.$$
Moreover, we note that:
$$\min\left\{-\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right), \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m}\right\}$$

$$= \begin{cases} \frac{\sqrt{\delta_2^2-(m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m}, & \delta_2 \in (g(M), k(M)), \\ -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right), & \delta_2 \in (k(M), M). \end{cases}$$

As a consequence, reasoning analogously to the previous theorem and using the symmetry with respect to δ_1, we conclude that the attained bounds are optimal, and so, the result holds. □

5.4.2. $M = 0$

As in the previous case, we shall compute the positive sign of G_{0,δ_1,δ_2} fixing the value of δ_2.

Theorem 12. *Let $M = 0$ and $\delta_2 \in (-8, 0)$, then $G_{0,\delta_1,\delta_2} > 0$ on $I \times I$ if and only if:*

$$|\delta_1| < \begin{cases} \frac{1}{2}\sqrt{-8\delta_2-\delta_2^2}, & \delta_2 \in (-8,-4), \\ 2, & \delta_2 \in [-4, 0). \end{cases}$$

Proof. Let us assume that $\delta_1 > 0$ and calculate the minimum of G_{0,δ_1,δ_2}. From Lemma 9, we know that such a minimum is either at $t = 1$ or $t = s$.

As we saw in Theorem 9, $r(s) = G_{0,\delta_1,\delta_2}(1, s)$ has its maximum at $s = \frac{1}{2}$ and the minimum at $s = 0$ and $s = 1$. Hence, $r > 0$ if and only if $r(0) = r(1) > 0$, that is $0 < \delta_1 < 2$.

On the other hand, as we saw in Theorem 9, $h(s) = G_{0,\delta_1,\delta_2}(s, s)$ has an absolute minimum at $c = \frac{1}{2} - \frac{\delta_1}{\delta_2}$, $h'(s) < 0$ for $s < c$, and $h'(s) > 0$ for $s > c$.

We distinguish the following cases:

- If $\delta_1 \geq -\frac{\delta_2}{2}$, then $c \geq 1$, and the minimum of h is attained at $s = 1$ and $h(1) > 0$ if and only if $\delta_1 < 2$. Since $h(1) = r(1)$, we deduce that if $-\frac{\delta_2}{2} \leq \delta_1 < 2$, then $G_{0,\delta_1,\delta_2} > 0$ on $I \times I$. We note that this is only possible when $\delta_2 > -4$;
- If $0 < \delta_1 < -\frac{\delta_2}{2}$, then $c \in (0, 1)$ and $h(c) > 0$ if and only if $\delta_1 < \frac{1}{2}\sqrt{-8\delta_2-\delta_2^2}$. Since $r(1) = h(1) > h(c)$, we deduce that $G_{0,\delta_1,\delta_2} > 0$ for:

$$0 < \delta_1 < \min\left\{-\frac{\delta_2}{2}, \frac{1}{2}\sqrt{-8\delta_2-\delta_2^2}\right\} = \begin{cases} \frac{1}{2}\sqrt{-8\delta_2-\delta_2^2}, & \delta_2 \in (-8,-4), \\ -\frac{\delta_2}{2}, & \delta_2 \in [-4, 0). \end{cases}$$

In conclusion, $G_{0,\delta_1,\delta_2}(t, s) > 0$ on $I \times I$ for all $t, s \in I$ for:

$$0 < \delta_1 < \begin{cases} \frac{1}{2}\sqrt{-8\delta_2-\delta_2^2}, & \delta_2 \in (-8,-4), \\ 2, & \delta_2 \in [-4, 0). \end{cases}$$

Using the symmetry with respect to δ_1, we conclude the result. □

5.5. *A Particular Case: $\delta_2 = 0$*

Finally, as a consequence of the previous results, we arrive at the following corollary.

Corollary 2. *Let us consider the perturbed periodic problem:*

$$\begin{cases} u''(t) + Mu(t) = \sigma(t), \quad t \in I, \\ u(0) - u(1) = \delta_1 \int_0^1 u(s)ds, \\ u'(0) - u'(1) = 0, \end{cases} \tag{21}$$

for $M \in \mathbb{R} \setminus \{0\}$. The following statements holds:

1. *If* $M = m^2 > 0$*, then* $G_{M,\delta_1,0} > 0$ *on* $I \times I$ *if and only if* $m \in (0, \pi)$ *and* $|\delta_1| < m \cot\left(\frac{m}{2}\right)$*;*
2. *If* $M = -m^2 < 0$*, then* $G_{M,\delta_1,0} < 0$ *on* $I \times I$ *if and only if* $|\delta_1| < \frac{2me^{\frac{m}{2}}}{1-e^m}$*.*

In this case, the graph showing the sign of Green's function on the (M, δ_1) plane can be seen in Figure 2.

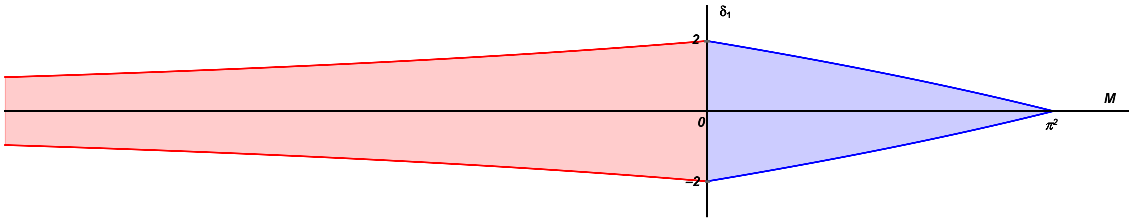

Figure 2. The blue and red areas represent the regions of the positive and negative sign of Green's function, respectively.

6. Conclusions

We studied and characterized the regions of constant sign of Green's function related to the problem:

$$\begin{cases} u''(t) + Mu(t) = \sigma(t), \quad t \in I, \\ u(0) - u(1) = \delta_1 \displaystyle\int_0^1 u(s)ds, \\ u'(0) - u'(1) = \delta_2 \displaystyle\int_0^1 u(s)ds. \end{cases}$$

In the case $M \neq 0$, we obtained the expression of Green's function as a linear combination of the Green's function of the related homogeneous problem (with $\delta_1 = \delta_2 = 0$) and some particular solutions of the equation. However, for the case $M = 0$, we obtained the expression of Green's function by direct integration.

We started our study by analyzing the regions of constant sign of Green's function in the particular case of $\delta_1 = 0$, and later on, we extended this study to any real value of parameter δ_1.

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