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(ω, c)-Periodic Solutions to Fractional Differential Equations with Impulses

Lulu Ren and JinRong Wang *

Department of Mathematics, Guizhou University, Guiyang 550025, China; llrenmath@126.com
* Correspondence: jrwang@gzu.edu.cn

Abstract: This paper deals with the (ω, c)-periodic solutions to impulsive fractional differential equations with Caputo fractional derivative with a fixed lower limit. Firstly, a necessary and sufficient condition of the existence of (ω, c)-periodic solutions to linear problem is given. Secondly, the existence and uniqueness of (ω, c)-periodic solutions to semilinear problem are proven. Lastly, two examples are given to demonstrate our results.

Keywords: (ω, c)-periodic solutions; fractional differential equation; impulse

MSC: 26A33

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Citation: Ren, L.; Wang, J. (ω, c)-Periodic Solutions to Fractional Differential Equations with Impulses. *Axioms* **2022**, *11*, 83. https://doi.org/10.3390/axioms11030083

Academic Editors: Viktor N. Orlov, Michal Feckan and Chris Goodrich

Received: 19 January 2022
Accepted: 21 February 2022
Published: 22 February 2022

1. Introduction

Alvarez et al. [1] introduced a new concept of (ω, c)-periodic functions: a continuous function $f : \mathbb{R} \to X$, where X is a complex Banach space, is (ω, c)-periodic if $f(t + \omega) = cf(t)$ holds for all $t \in \mathbb{R}$, where $\omega > 0, c \in \mathbb{C} \setminus \{0\}$. Then, Alvarez et al. [2] proved the existence and uniqueness of (N, λ)-periodic solutions to a a class of Volterra difference equations. For more research on (ω, c)-period systems, we refer the readers to [3–6].

In recent years, impulsive fractional differential equations have attracted more and more scholars' attentions. For the existence of solutions and control problems, we refer to [7–11]. Recently, Fečkan et al. [12] proved the existence of the periodic solutions of impulsive fractional differential equations. However, to our knowledge, the existence of (ω, c)-periodic solutions of impulsive fractional differential equations has not been studied. Motivated by [1,7,12–14], we study the following impulsive fractional differential equations with fixed lower limits

$$\begin{cases} {}^cD^q_{t_0}u(t) = f(t, u(t)),\ q \in (0,1),\ t \neq t_k,\ t \in [t_0, \infty), \\ u(t^+_k) = u(t^-_k) + \Delta_k,\ k \in \mathbb{N}, \end{cases}$$

where ${}^cD^q_{t_0}u(t)$ is the Caputo fractional derivative with the lower time at t_0, and for any $k \in \mathbb{N}$, $t_k < t_{k+1}$, $\lim_{k\to\infty} t_k = \infty$.

In this paper, we deal with the existence of (ω, c)-periodic solutions impulsive fractional differential equations with fixed lower limit. We first study the existence of (ω, c)-periodic solutions to the linear problem, i.e., $f(t, u) = \rho u$. Then, we prove the existence of (ω, c)-periodic solutions to the semilinear problem. Finally, we give two examples to illustrate our results.

2. Preliminaries

We introduce a Banach space $PC(\mathbb{R}, \mathbb{R}^n) = \{x : \mathbb{R} \to \mathbb{R}^n : x \in C((t_k, t_{k+1}], \mathbb{R}^n)$, and $x(t^-_k) = x(t_k),\ x(t^+_k)$ exists $\forall k \in \mathbb{N}\}$ endowed with the norm $\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|$.

Definition 1. *(see [15]) Let $n \in \mathbb{N}^+$ and u be a n time differentiable function. The Caputo fractional derivative of order $\alpha > 0$ with the lower limit zero for u is given by*

$$ {}^c D_0^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, \;\; n-1 < \alpha \leq n. $$

Lemma 1. *Assume that $f : \mathbb{R} \times \mathbb{R}^n$ is continuous. A solution $u \in PC(\mathbb{R}, \mathbb{R}^n)$ of the following impulsive fractional differential equations with fixed lower limit*

$$ \begin{cases} {}^c D_{t_0}^q u(t) = f(t, u(t)), \; q \in (0,1), \; t \neq t_k, \; t \in [t_0, \infty), \\ u(t_k^+) = u(t_k^-) + \Delta_k, \; k \in \mathbb{N}, \\ u(t_0) = u_{t_0}, \end{cases} \tag{1} $$

is given by

$$ u(t) = u(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-\tau)^{q-1} f(\tau, u(\tau)) d\tau + \sum_{t_0 < t_i < t} \Delta_i, \quad t \geq t_0. \tag{2} $$

Proof. From Lemma 3.2 in [7], a solution u of Equation (1) is given by

$$ u(t) = u(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-\tau)^{q-1} f(\tau, u(\tau)) d\tau + \sum_{i=1}^k \Delta_i, \quad t \in (t_k, t_{k+1}]. \tag{3} $$

Using

$$ \sum_{i=1}^k \Delta_i = \sum_{t_0 < t_i < t} \Delta_i, \;\; \forall t \in (t_k, t_{k+1}], $$

we get that (3) is equivalent to

$$ u(t) = u(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-\tau)^{q-1} f(\tau, u(\tau)) d\tau + \sum_{t_0 < t_i < t} \Delta_i \tag{4} $$

on $(t_k, t_{k+1}]$. Using the arbitrariness of k, we obtain that (4) holds on $\bigcup_{k=1}^\infty (t_k, t_{k+1}]$. Since (4) is independent of k, we obtain that (2) holds on $[t_0, \infty)$. □

Definition 2. *(see [16], Theorem 2.4) A solution $u \in PC(\mathbb{R}, \mathbb{R}^n)$ of following linear impulsive fractional differential equations with fixed lower limit*

$$ \begin{cases} {}^c D_{t_0}^q u(t) = \rho u(t), \; \rho \in \mathbb{R}, \; q \in (0,1), \; t \neq t_k, \; t \in [t_0, \infty), \\ u(t_k^+) = (1 + \alpha_k) u(t_k^-), \; k \in \mathbb{N}, \\ u(t_0) = u_{t_0}, \end{cases} $$

is given by

$$ u(t) = \begin{cases} u_{t_0} E_q\big(\rho(t-t_0)^q\big), \;\; t \in [t_0, t_1] \\ u_{t_0} \prod_{i=1}^k \Big(1 + \alpha_i E_q\big(\rho(t_i - t_0)^q\big)\Big) E_q\big(\rho(t-t_0)^q\big), \;\; t \in (t_k, t_{k+1}], \;\; k \in \mathbb{N}, \end{cases} $$

where $E_q(\cdot)$ is the Mittag–Leffler function.

Definition 3. *(see [1]) Let $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$, X denote a complex Banach space with norm $\| \cdot \|$. A continuous function $f : \mathbb{R} \to X$ is said to be (ω, c)-periodic if $f(t+\omega) = cf(t)$ for all $t \in \mathbb{R}$.*

Lemma 2. *(see [3], Lemma 2.2) Set* $\Phi_{\omega,c} := \{u : u \in PC(\mathbb{R}, \mathbb{R}^n)\}$ *and* $u(\cdot + \omega) = cu(\cdot)\}$. *Then,* $u \in \Phi_{\omega,c}$ *if, and only if, it holds*

$$u(\omega) = cu(0). \tag{5}$$

3. $((\omega, c))$-Periodic Solutions to Linear Problem

Set $t_0 = 0$, we consider the following linear impulsive fractional differential equation with fixed lower limit

$$\begin{cases} {}^cD_0^q u(t) = \rho u(t), \ \ \rho \in \mathbb{R}, \ q \in (0,1), \ \ t \neq t_k, \ t \in [0, \infty), \\ u(t_k^+) = (1 + \alpha_k) u(t_k^-), \ k \in \mathbb{N}, \\ u(0) = u_0. \end{cases} \tag{6}$$

Theorem 1. *Assume that there exists a constant* $N \in \mathbb{N}$ *such that*

$$\omega = t_N, \ t_{k+N} = t_k + \omega, \ \forall k \in \mathbb{N}, \ \text{ and } \ \alpha_{i+N} = \alpha_i, \ \ \forall i \in \mathbb{N}.$$

Then, the linear impulsive fractional differential Equation (6) has a (ω, c)*-periodic solution* $u \in \Phi_{\omega,c}$ *if, and only if*

$$u_0 \left(c - \prod_{i=1}^{N} \left(1 + \alpha_i E_q\left(\rho t_i^q\right) \right) E_q\left(\rho \omega^q\right) \right) = 0. \tag{7}$$

Proof. "$\Rightarrow$" If (6) has a (ω, c)-periodic solution $u \in \Phi_{\omega,c}$, i.e., $u(\cdot + \omega) = cu(\cdot)$, then $u(\omega) = cu(0)$, i.e.,

$$u_0 \prod_{i=1}^{N} \left(1 + \alpha_i E_q\left(\rho t_i^q\right) \right) E_q\left(\rho \omega^q\right) = cu_0$$

which implies that (7) holds.

"$\Leftarrow$" It follows from Definition 2 that Equation (7) has a solution u given by

$$u(t) = \begin{cases} u_0 E_q\left(\rho t^q\right), \ \ t \in [0, t_1] \\ u_0 \displaystyle\prod_{i=1}^{k} \left(1 + \alpha_i E_q\left(\rho t_i^q\right) \right) E_q\left(\rho t^q\right), \ \ t \in (t_k, t_{k+1}], \ \ k \in \mathbb{N}. \end{cases}$$

If (7) holds, we obtain $u(t_N) = u(\omega) = cu_0$. Now, we prove that the solution $u \in \Phi_{\omega,c}$.

Case 1: For $t \in (0, t_1]$, we have $t + \omega \in (t_N, t_{N+1}]$, then

$$\begin{aligned} u(t + \omega) &= u_{t_N} E_q\left(\rho (t + \omega - t_N)^q\right) \\ &= u_{t_N} E_q\left(\rho t^q\right) = cu_0 E_q\left(\rho t^q\right) = cu(t). \end{aligned}$$

Case 2: For $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, we have $t + \omega \in (t_{k+N}, t_{k+N+1}]$, then

$$\begin{aligned} u(t + \omega) &= u_{t_N} \prod_{i=1}^{k} \left(1 + \alpha_{i+N} E_q\left(\rho (t_{i+N} - t_N)^q\right) \right) E_q\left(\rho (t + \omega - t_N)^q\right) \\ &= u_{t_N} \prod_{i=1}^{k} \left(1 + \alpha_i E_q\left(\rho t_i^q\right) \right) E_q\left(\rho t^q\right) \\ &= cu_0 \prod_{i=1}^{k} \left(1 + \alpha_i E_q\left(\rho t_i^q\right) \right) E_q\left(\rho t^q\right) \\ &= cu(t). \end{aligned}$$

So, we obtain that (6) has a (ω, c)-periodic solution $u \in \Phi_{\omega,c}$. □

4. (ω, c)-Periodic Solutions to Semilinear Problem

Set $t_0 = 0$, we consider the (ω, c)-periodic solutions of following impulsive fractional differential equations with fixed lower limit

$$\begin{cases} {}^cD_0^q u(t) = f(t, u(t)),\ q \in (0,1),\ \ t \neq t_k,\ t \in [0, \infty), \\ u(t_k^+) = u(t_k^-) + \Delta_k,\ k \in \mathbb{N}, \\ u(0) = u_0. \end{cases} \tag{8}$$

We assume the following conditions:

(I) $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and

$$f(t + \omega, cu) = cf(t, u), \quad \forall t \in \mathbb{R},\ \forall u \in \mathbb{R}^n.$$

(II) There exists a constant $A > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq A\|u - v\|, \quad \forall t \in \mathbb{R},\ \forall u, v \in \mathbb{R}^n.$$

(III) There exist constant $B > 0$, $P > 0$ such that

$$\|f(t, u)\| \leq B\|u\| + P, \quad \forall t \in \mathbb{R},\ \forall u \in \mathbb{R}^n.$$

(IV) $\Delta_k \in \mathbb{R}^n$ and there exists a constant $M \in \mathbb{N}$ such that $\omega = t_M$, $t_{k+M} = t_k + \omega$ and $\Delta_{k+M} = \Delta_k$ hold for any $k \in \mathbb{N}$.

Lemma 3. *Suppose that conditions (I), (IV) hold and $c \neq 1$. Then, the solution $u \in \Psi := PC([0, \omega], \mathbb{R}^n)$ of Equation (8) satisfying (5) is given by*

$$\begin{aligned} u(t) \ &= \ (c-1)^{-1}\frac{1}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}f(\tau, u(\tau))d\tau + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}f(\tau, u(\tau))d\tau \\ &\quad + (c-1)^{-1}\sum_{k=1}^{M}\Delta_k + \sum_{0<t_k<t}\Delta_k \quad t \in [0, \omega]. \end{aligned}$$

Proof. It follows from (2) that the solution $u \in PC([0, \omega], \mathbb{R}^n)$ is given by

$$u(t) = u(0) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}f(\tau, u(\tau))d\tau + \sum_{t_0<t_k<t}\Delta_k, \quad t \in [0, \omega]. \tag{9}$$

So we get

$$u(\omega) = u(0) + \frac{1}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}f(\tau, u(\tau))d\tau + \sum_{t_0<t_k<\omega}\Delta_k = cu_0$$

which is equivalent to

$$u_0 = (c-1)^{-1}\Big(\frac{1}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}f(\tau, u(\tau))d\tau + \sum_{t_0<t_k<\omega}\Delta_k\Big). \tag{10}$$

By (9) and (10), we obtain

$$\begin{aligned} u(t) \ &= \ (c-1)^{-1}\frac{1}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}f(\tau, u(\tau))d\tau + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}f(\tau, u(\tau))d\tau \\ &\quad + (c-1)^{-1}\sum_{k=1}^{M}\Delta_k + \sum_{0<t_k<t}\Delta_k. \end{aligned}$$

The proof is finished. □

Theorem 2. *Suppose that conditions (I), (II), (IV) hold and $c \neq 1$. If $0 < \frac{A\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)} < 1$, then the impulsive fractional differential Equation (8) has a unique (ω, c)-periodic solution $u \in \Phi_{\omega,c}$. Furthermore, we have*

$$\|u\|_\infty \leq \frac{\mu\omega^q(|c-1|^{-1}+1) + \Gamma(q+1)(|c-1|^{-1}+1)\sum_{k=1}^{M}\|\Delta_k\|}{\Gamma(q+1) - A\omega^q(|c-1|^{-1}+1)},$$

where $\mu = \sup_{t\in[0,\omega]}\|f(t,0)\|$.

Proof. It follows from (I) that for any $u \in \Phi_{\omega,c}$, we have

$$f(t+\omega, u(t+\omega)) = f(t+\omega, cu(t)) = cf(t, u(t)), \quad \forall t \in \mathbb{R}$$

which implies that $f(\cdot, u(\cdot)) \in \Phi_{\omega,c}$.

Define the operator $F : \Psi \to \Psi$ by

$$\begin{aligned}(Fu)(t) \quad = \quad & (c-1)^{-1}\frac{1}{\Gamma(q)}\int_0^\omega(\omega-\tau)^{q-1}f(\tau,u(\tau))d\tau + \frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}f(\tau,u(\tau))d\tau \\ & +(c-1)^{-1}\sum_{k=1}^{M}\Delta_k + \sum_{0<t_k<t}\Delta_k. \end{aligned} \tag{11}$$

From Lemmas 2 and 3, we obtain that the fixed points of F determine the (ω, c)-periodic solutions of Equation (8). It is easy to see that $F(\Psi) \subseteq \Psi$. For any $u, v \in \Psi$, we have

$$\begin{aligned} & \big\|(Fu)(t) - (Fv)(t)\big\| \\ = \; & \Big\|(c-1)^{-1}\frac{1}{\Gamma(q)}\int_0^\omega(\omega-\tau)^{q-1}f(\tau,u(\tau))d\tau + \frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}f(\tau,u(\tau))d\tau \\ & -(c-1)^{-1}\frac{1}{\Gamma(q)}\int_0^\omega(\omega-\tau)^{q-1}f(\tau,v(\tau))d\tau - \frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}f(\tau,v(\tau))d\tau\Big\| \\ \leq \; & |c-1|^{-1}\frac{1}{\Gamma(q)}\int_0^\omega(t-\tau)^{q-1}\big\|f(\tau,u(\tau)) - f(\tau,v(\tau))\big\|d\tau \\ & +\frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}\big\|f(\tau,u(\tau)) - f(\tau,v(\tau))\big\|d\tau \\ \leq \; & |c-1|^{-1}\frac{A}{\Gamma(q)}\int_0^\omega(\omega-\tau)^{q-1}\|u(\tau)-v(\tau)\|d\tau \\ & +\frac{A}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}\|u(\tau)-v(\tau)\|d\tau \\ \leq \; & \frac{A}{\Gamma(q)}\|u-v\|_\infty\left(|c-1|^{-1}\int_0^\omega(\omega-\tau)^{q-1}d\tau + \int_0^t(t-\tau)^{q-1}d\tau\right) \\ \leq \; & \frac{A\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)}\|u-v\|_\infty \end{aligned}$$

which implies that

$$\|Fu - Fv\|_\infty \leq \frac{A\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)}\|u-v\|_\infty.$$

From the condition $0 < \frac{A\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)} < 1$, we obtain that F is a contraction mapping. So, there exists a unique fixed point u of (11) satisfying $u(\omega) = cu(0)$. It follows from Lemma 2 that $u \in \Phi_{\omega,c}$. Then, we obtain that Equation (8) has a unique (ω, c)-periodic solution $u \in \Phi_{\omega,c}$.

Furthermore, we have

$$\begin{aligned}
\|u(t)\| \leq\ & |c-1|^{-1}\frac{1}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}\|f(\tau,u(\tau))-f(\tau,0)\|d\tau \\
&+|c-1|^{-1}\frac{1}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}\|f(\tau,0)\|d\tau \\
&+\frac{1}{\Gamma(q)}\int_0^{t}(t-\tau)^{q-1}\|f(\tau,u(\tau))-f(\tau,0)\|d\tau \\
&+\frac{1}{\Gamma(q)}\int_0^{t}(t-\tau)^{q-1}\|f(\tau,0)\|d\tau \\
&+|c-1|^{-1}\sum_{k=1}^{M}\|\Delta_k\|+\sum_{t_0<t_k<t}\|\Delta_k\| \\
\leq\ & |c-1|^{-1}\frac{A}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}\|u(\tau)\|d\tau+|c-1|^{-1}\frac{\mu}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}d\tau \\
&+\frac{A}{\Gamma(q)}\int_0^{t}(t-\tau)^{q-1}\|u(\tau)\|d\tau+\frac{\mu}{\Gamma(q)}\int_0^{t}(t-\tau)^{q-1}d\tau \\
&+\left(|c-1|^{-1}+1\right)\sum_{k=1}^{M}\|\Delta_k\| \\
\leq\ & \frac{A\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)}\|u\|_\infty+\frac{\mu\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)}+\left(|c-1|^{-1}+1\right)\sum_{k=1}^{M}\|\Delta_k\|,
\end{aligned}$$

which implies that

$$\|u\|_\infty \leq \frac{\mu\omega^q(|c-1|^{-1}+1)+\Gamma(q+1)(|c-1|^{-1}+1)\sum_{k=1}^{M}\|\Delta_k\|}{\Gamma(q+1)-A\omega^q(|c-1|^{-1}+1)}.$$

The proof is completed. □

Theorem 3. *Suppose that conditions (I), (III), (IV) hold and $c \neq 1$. If $B\omega^q(|c-1|^{-1}+1) < \Gamma(q+1)$, then the impulsive fractional differential Equation (8) has at least one (ω, c)-periodic solution $u \in \Phi_{\omega,c}$.*

Proof. Let $\mathbb{B}_r = \{u \in \Psi : \|u\|_\infty \leq r\}$, where

$$r \geq \frac{P\omega^q(|c-1|^{-1}+1)+\Gamma(q+1)(|c-1|^{-1}+1)\sum_{k=1}^{M}\|\Delta_k\|}{\Gamma(q+1)-B\omega^q(|c-1|^{-1}+1)}.$$

We consider F defined in (11) on $\mathbb{B}_r$. For any $t \in [0,\omega]$ and any $u \in \mathbb{B}_r$

$$\begin{aligned}
\|F(u)(t)\| \leq\ & |c-1|^{-1}\frac{B}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}\|u(\tau)\|d\tau+|c-1|^{-1}\frac{P}{\Gamma(q)}\int_0^{\omega}(\omega-\tau)^{q-1}d\tau \\
&+\frac{B}{\Gamma(q)}\int_0^{t}(t-\tau)^{q-1}\|u(\tau)\|d\tau+\frac{P}{\Gamma(q)}\int_0^{t}(t-\tau)^{q-1}d\tau \\
&+|c-1|^{-1}\sum_{k=1}^{M}\|\Delta_k\|+\sum_{0<t_k<t}\|\Delta_k\| \qquad (12)\\
\leq\ & \frac{B\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)}\|u\|_\infty+\frac{P\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)}+\left(|c-1|^{-1}+1\right)\sum_{k=1}^{M}\|\Delta_k\| \\
\leq\ & r,
\end{aligned}$$

which implies $\|Fu\|_\infty \leq r$. So, $F(\mathbb{B}_r) \subseteq \mathbb{B}_r$.

We prove that F is continuous on $\mathbb{B}_r$.

Let $\{u_i\}_{i\geq 1} \subseteq \mathbb{B}_r$ and $u_i \to u$ on $\mathbb{B}_r$ as $i \to \infty$. By the continuity of f, we get $f(\tau, u_i(\tau)) \to f(\tau, u(\tau))$ as $i \to \infty$. Thus, we have

$$\begin{aligned}(\omega-\tau)^{q-1}f(\tau,u_i(\tau)) &\to (\omega-\tau)^{q-1}f(\tau,u(\tau)) \quad \text{as } i\to\infty,\\ (t-\tau)^{q-1}f(\tau,u_i(\tau)) &\to (t-\tau)^{q-1}f(\tau,u(\tau)) \quad \text{as } i\to\infty.\end{aligned}$$

Using condition (III), we obtain that for any $0 \leq \tau \leq t \leq \omega$,

$$\begin{aligned}&\int_0^{\omega} \left\|(\omega-\tau)^{q-1}f(\tau,u_i(\tau)) - (\omega-\tau)^{q-1}f(\tau,u(\tau))\right\| d\tau\\ &\leq 2(Br+P)\int_0^{\omega}(\omega-\tau)^{q-1}d\tau \leq 2(Br+P)q^{-1}\omega^q < \infty,\end{aligned}$$

and

$$\begin{aligned}&\int_0^{t} \left\|(t-\tau)^{q-1}f(\tau,u_i(\tau)) - (t-\tau)^{q-1}f(\tau,u(\tau))\right\| d\tau\\ &\leq 2(Br+P)\int_0^{t}(t-\tau)^{q-1}d\tau \leq 2(Br+P)q^{-1}\omega^q < \infty.\end{aligned}$$

Then, by Lebesgue dominated convergence theorem, we get

$$\int_0^{\omega} \left\|(\omega-\tau)^{q-1}f(\tau,u_i(\tau)) - (\omega-\tau)^{q-1}f(\tau,u(\tau))\right\| d\tau \to 0 \quad \text{as } i\to\infty,$$

and

$$\int_0^{t} \left\|(t-\tau)^{q-1}f(\tau,u_i(\tau)) - (t-\tau)^{q-1}f(\tau,u(\tau))\right\| d\tau \to 0 \quad \text{as } i\to\infty.$$

So, for any $t \in [0,\omega]$, it holds

$$\begin{aligned}&\|(Fu_i)(t) - (Fu)(t)\|\\ \leq\ & (c-1)^{-1}\frac{1}{\Gamma(q)}\int_0^{\omega}\left\|(\omega-\tau)^{q-1}f(\tau,u_i(\tau)) - (\omega-\tau)^{q-1}f(\tau,u(\tau))\right\| d\tau\\ &+\frac{1}{\Gamma(q)}\int_0^{t}\left\|(t-\tau)^{q-1}f(\tau,u_i(\tau)) - (t-\tau)^{q-1}f(\tau,u(\tau))\right\| d\tau \to 0 \quad \text{as } i\to\infty.\end{aligned}$$

Then, F is continuous on $\mathbb{B}_r$.

We prove that F is pre-compact.

For any $t_i < t \leq s \leq t_{i+1}$, $i \in \mathbb{N}_0$, we have

$$\left\|\sum_{0<t_k<t}\Delta_k - \sum_{0<t_k<s}\Delta_k\right\| = \left\|\sum_{k=1}^{i}\Delta_k - \sum_{k=1}^{i}\Delta_k\right\| = 0$$

which implies that

$$\left\|\sum_{0<t_k<t}\Delta_k - \sum_{0<t_k<s}\Delta_k\right\| \to 0, \quad \text{as } t\to s.$$

So, for any $0 \le s_1 < s_2 \le \omega$, and any $u \in \mathbb{B}_r$, it holds

$$
\begin{aligned}
& \|(Fu)(s_1) - (Fu)(s_2)\| \\
\le\ & \left\| \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1-\tau)^{q-1} f(\tau, u(\tau)) d\tau - \frac{1}{\Gamma(q)} \int_0^{s_2} (s_2-\tau)^{q-1} f(\tau, u(\tau)) d\tau \right\| \\
& + \left\| \sum_{0<t_k<s_1} \Delta_k - \sum_{0<t_k<s_2} \Delta_k \right\| \\
\le\ & \frac{1}{\Gamma(q)} \int_0^{s_1} \left((s_1-\tau)^{q-1} - (s_2-\tau)^{q-1}\right) \|f(\tau, u(\tau))\| d\tau \\
& + \frac{1}{\Gamma(q)} \int_{s_1}^{s_2} (s_2-\tau)^{q-1} \|f(\tau, u(\tau))\| d\tau + \left\| \sum_{0<t_k<s_1} \Delta_k - \sum_{0<t_k<s_2} \Delta_k \right\| \\
\le\ & \frac{Br+P}{\Gamma(q)} \int_0^{s_1} \left((s_1-\tau)^{q-1} - (s_2-\tau)^{q-1}\right) d\tau + \frac{Br+P}{\Gamma(q)} \int_{s_1}^{s_2} (s_2-\tau)^{q-1} d\tau \\
& + \left\| \sum_{0<t_k<s_1} \Delta_k - \sum_{0<t_k<s_2} \Delta_k \right\| \\
\le\ & \frac{Br+P}{\Gamma(q+1)} \left((s_2^q - s_1^q) + 2(s_2-s_1)^q \right) + \left\| \sum_{0<t_k<s_1} \Delta_k - \sum_{0<t_k<s_2} \Delta_k \right\| \to 0 \quad \text{as } s_1 \to s_2.
\end{aligned}
$$

So, $F(\mathbb{B}_r)$ is equicontinuous. By (12), we obtain that $F(\mathbb{B}_r)$ is uniformly bounded. Using Arzelà-Ascoli theorem, we obtain that $F(\mathbb{B}_r)$ is pre-compact.

It follows from Schauder's fixed point theorem that the impulsive fractional differential Equation (8) has at least one (ω, c) periodic solution $u \in \Phi_{\omega,c}$. The proof is finished. □

Remark 1. *If $c = 1$, (ω, c)-periodic solution is standard ω-periodic solution. If $c = -1$, (ω, c)-periodic solution is ω-antiperiodic solution. Moreover, all results obtained in this paper are based on the fixed lower limit of Caputo fractional derivative.*

5. Examples

Example 1. *We consider the following impulsive fractional differential equation:*

$$
\begin{cases}
{}^cD_0^{\frac{1}{2}} u(t) = \lambda \cos 2t \sin u(t), \ t \neq t_k, \ t \in [0, \infty), \\
u(t_k^+) = u(t_k^-) + \cos k\pi, \ k = 1, 2, 3, \cdots,
\end{cases}
\tag{13}
$$

where $\lambda \in \mathbb{R}$, $t_k = \frac{k\pi}{2}$, $\Delta_k = \cos k\pi$, $f(t, u) = \lambda \cos 2t \sin u(t)$. Set $\omega = \pi$, $c = -1$. It is easy to see that for any $k \in \mathbb{N}$, $t_{k+2} = t_k + \pi$, $\Delta_{k+2} = \Delta_k$. So, we obtain $M = 2$, and (IV) holds. For any $t \in \mathbb{R}$ and any $u \in \mathbb{R}$, we have

$$
f(t+\omega, cu) = f(t+\pi, -u) = -\lambda \cos 2t \sin u(t) = -f(t, u) = cf(t, u)
$$

which implies that (I) holds. For any $t \in \mathbb{R}$ and any $u, v \in \mathbb{R}$, we have $|f(t, u) - f(t, v)| \le |\lambda||u - v|$ which implies that $A = |\lambda|$ and (II) holds. Note that $\frac{A\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)} = \frac{3|\lambda|\sqrt{\pi}}{\Gamma(\frac{1}{2})}$. Letting $0 < |\lambda| < \frac{\Gamma(\frac{1}{2})}{3\sqrt{\pi}}$, we obtain $0 < \frac{A\omega^q(|c-1|^{-1}+1)}{\Gamma(q+1)} < 1$. Then, all assumptions in Theorem 2 hold for Equation (13).

Hence, if $0 < |\lambda| < \frac{\Gamma(\frac{1}{2})}{3\sqrt{\pi}}$, (13) has a unique $(\pi, -1)$-periodic solution $u \in \Phi_{\pi,-1}$. Furthermore, we have

$$
\|u\|_\infty \le \frac{\mu\omega^q(|c-1|^{-1}+1) + \Gamma(q+1)(|c-1|^{-1}+1)\sum_{k=1}^M \|\Delta_k\|}{\Gamma(q+1) - A\omega^q(|c-1|^{-1}+1)} = \frac{3\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}) - 3|\lambda|\sqrt{\pi}}.
$$

Example 2. *We consider the following impulsive fractional differential equation:*

$$\begin{cases} {}^cD_0^{\frac{1}{2}}u(t) = \lambda u(t)\sin\left(3^{-t}u(t)\right),\ t \neq t_k,\ t \in [0,\infty), \\ u(t_k^+) = u(t_k^-) + 2,\ k = 1,2,3,\cdots, \end{cases} \tag{14}$$

where $\lambda \in \mathbb{R}$, $t_k = \frac{k}{2}$, $\Delta_k = 2$, $f(t,u) = \lambda u \sin(3^{-t}u)$. *Set* $\omega = 1$, $c = 3$. *Obviously,* $t_{k+2} = t_k + 1$, $\Delta_{k+2} = \Delta_k$ *hold for all* $k \in \mathbb{N}$. *So we obtain* $M = 2$, *and* (IV) *holds. For any* $t \in \mathbb{R}$ *and any* $u \in \mathbb{R}$, *we have*

$$f(t+\omega, cu) = f(t+1, 3u) = 3\lambda u \sin(3^{-t}u) = 3f(t,u) = cf(t,u)$$

which implies that (I) *holds. For any* $t \in \mathbb{R}$ *and any* $u \in \mathbb{R}$, *we have* $|f(t,u)| \leq |\lambda||u|$ *which implies that* $B = |\lambda|$, $P = 0$ *and* (III) *holds. Note that* $B\omega^q(|c-1|^{-1}+1) = \frac{3}{2}|\lambda|$. *Letting* $|\lambda| < \frac{1}{\Gamma(\frac{5}{2})}$, *we get* $B\omega^q(|c-1|^{-1}+1) < \Gamma(q+1)$. *Then, all assumptions in Theorem 3 hold for Equation (13).*

Therefore, if $|\lambda| < \frac{1}{\Gamma(\frac{5}{2})}$, *Equation (14) has at least one* $(1,3)$*-periodic solution* $u \in \Phi_{1,3}$.

6. Conclusions

In this paper, we mainly study the existence of (ω,c)-periodic solutions for impulsive fractional differential equations with fixed lower limits. In future work, we shall study the (ω,c)-periodic solutions for impulsive fractional differential equations with varying lower limits.

Author Contributions: The contributions of all authors (L.R. and J.W.) are equal. All the main results were developed together. All authors have read and agreed to the published version of the manuscript.

Funding: This work is partially supported by Foundation of Postgraduate of Guizhou Province (YJSCXJH[2019]031), Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006), Major Research Project of Innovative Group in Guizhou Education Department ([2018]012), and Guizhou Data Driven Modeling Learning and Optimization Innovation Team ([2020]5016).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the referees for their careful reading of the manuscript and valuable comments. The authors thank the editor too.

Conflicts of Interest: The authors declare no conflict of interest.

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