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Quasi-Density of Sets, Quasi-Statistical Convergence and the Matrix Summability Method

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Abstract: In this paper, we define the quasi-density of subsets of the set of natural numbers and show several of the properties of this density. The quasi-density $d_p(A)$ of the set $A \subseteq N$ is dependent on the sequence $p = (p_n)$. Different sequences (p_n), for the same set A, will yield new and distinct densities. If the sequence (p_n) does not differ from the sequence (n) in its order of magnitude, i.e., $\lim_{n\to\infty} \frac{p_n}{n} = 1$, then the resulting quasi-density is very close to the asymptotic density. The results for sequences that do not satisfy this condition are more interesting. In the next part, we deal with the necessary and sufficient conditions so that the quasi-statistical convergence will be equivalent to the matrix summability method for a special class of triangular matrices with real coefficients.

Keywords: statistical convergence; quasi-statistical convergence; asymptotic density; quasi-density; the matrix summability method

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1. Introduction

The notion of asymptotic density for a subset of the set of natural numbers is known. It determines the size of the given subset compared to the set N.

Let $A \subseteq N$. We define $A(n) = |k \in A,\ k \le n|$, i.e., as the number of elements of set A smaller than n.

Then

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}$$

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$$

is the lower and upper asymptotic density of the set $A \subseteq N$, respectively.

If $\underline{d}(A) = \overline{d}(A)$, then there exists $\lim_{n\to\infty} \frac{A(n)}{n} = d(A)$ that is called the asymptotic density of set A. It is evident that if for some set A there exists $d(A)$, then $0 \le d(A) \le 1$ (see [1]).

A different method for defining the density is based on the matrix method of limiting sequences of ones and zeros (see [2,3]).

Let

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/n & 1/n & 1/n & \cdots & 1/n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

be a regular (Cesáro) matrix (see [4]) defined as follows:

$$C = (c_{nk}),$$

where

$$\begin{array}{lll} c_{nk} = 1/n & \text{for } k \leq n & \\ c_{nk} = 0 & \text{for } k > n & n,k = 1,\ 2,\ \ldots \end{array}$$

Then, we define the asymptotic density of the set $A \subseteq N$ by the relation

$$d(A) = \lim_{n\to\infty} \sum_{k=1}^{\infty} c_{nk} \cdot \chi_A(k),$$

where $\chi_A(k)$ is the characteristic function of set $A \subseteq N$ (see [5,6]),

$$\chi_A(k) = \begin{cases} 1, & \text{if } k \in A \\ 0, & \text{if } k \notin A. \end{cases}$$

Agnew in [2] defined the sufficient condition for a matrix so that at least one sequence of ones and zeros to be limitable (summable) by the matrix.

Let $A = (a_{nk})$ be an infinite matrix with real elements. The requirements of summability are:

(a) $\sum_{k=1}^{\infty} |a_{nk}| \leq M < \infty$, $\forall n = 1,\ 2,\ \ldots$
(b) $\lim\limits_{n\to\infty} \max\limits_{1\leq k\leq n} |a_{nk}| = 0.$

Let $A = (a_{nk})$ be an infinite matrix with real elements. We say that the sequence $x = (x_k)$ is $A-$limitable to the number $s \epsilon R$ ($A - \lim x_k = s$), if $\lim\limits_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} x_k = s$.

If the implication $\lim x_k = s \implies A - \lim x_k = s$ holds true, we say that the matrix A is regular [2].

The necessary and sufficient condition for the matrix $A = (a_{nk})$ to be regular is

(a) $\exists K > 0\ \forall n = 1, 2, \ldots\ \sum_{k=1}^{\infty} |a_{nk}| \leq K$
(b) $\forall k = 1, 2, \ldots\ \lim\limits_{n\to\infty} a_{nk} = 0$
(c) $\lim\limits_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$ (see [6]).

Example 1. *If $T = (t_{nk})$ is a regular matrix, then we can use it to define the density $d_T(A.)$ (see [5,6]). Let the matrix $T = (t_{nk})$, where*

$$\begin{array}{lll} t_{nk} = \frac{\frac{1}{k}}{s_n} & \text{for } \ k \leq n & \\ t_{nk} = 0 & \text{for } \ k > n & ,\ n,k = 1,\ 2,\ \ldots \end{array}$$

and $s_n = \sum_{j=1}^{n} \frac{1}{j}$, then

$$\delta(A) = \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} \cdot \chi_A(k)$$

is the logarithmic density of the set A.

Let the matrix $T = (t_{nk})$, where

$$\begin{array}{lll} t_{nk} = \frac{\phi(k)}{n} & \text{for } \ k \leq n,\ k \mid n & \\ t_{nk} = 0 & \text{for } \ k \leq n,\ k \nmid n & ,\ n,k = 1,2,\ldots \\ t_{nk} = 0 & \text{for } \ k > n & \end{array}$$

and ϕ is the Euler function, then

$$d_T(A) = \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} \cdot \chi_A(k)$$

is the Schoenberg density of the set A.

In this paper we define the quasi-density using a matrix, whose members satisfy special conditions.

In the next section, we will present the connection between statistical convergence and the matrix method of summability of the sequence of real numbers.

We say that $x = (x_n)$ converges statistically to the number $L \in R$, if

$$\forall \varepsilon > 0 : d(N_\varepsilon) = 0, \text{ where } d(N_\varepsilon) = \{k \epsilon N : \ |x_k - L| \geq \varepsilon\}.$$

Numerous writers extended this convergence by substituting a different density for the asymptotic density (or by a function with suitable properties, respectively) (see [7–11]).

We will endeavor to characterize the quasi-statistical convergence by using the matrix method.

In the paper [12] the authors defined the quasi-statistical convergence as:

Let $p = (p_n)$ be a sequence of positive real numbers with the properties:

(a) $\lim\limits_{n\to\infty} p_n = +\infty$

(b) $\limsup\limits_{n\to\infty} \frac{p_n}{n} < +\infty$.

The quasi-density of the set $A \subseteq N$ for the sequence $p = (p_n)$ is

$$d_p(A) = \lim_{n\to\infty} \frac{1}{p_n} |\{k \epsilon A, \ k \leq n \ \}|,$$

if such a limit exists.

If $p_n = n$, then $d_p(A)$ is the asymptotic density of set A.

We say that the sequence $x = (x_k)$ converges quasi-statistically (given the sequence $p = (p_n)$) to the number $L \in R$ ($\text{stq}_p - \lim x_k = L$), if $\forall \varepsilon > 0$ the set E_ε has a quasi-density equal to zero (t.j. $d_p(E_\varepsilon) = 0$), where $E_\varepsilon = \{k \epsilon N, \ |x_k - L| \geq \varepsilon\}$.

If we define $p_n = n, \ n = 1, 2, \ldots$, then the quasi-statistical convergence is identical to the statistical convergence.

If the sequence $x = (x_n)$ quasi-statistically converges to the number L, then it converges statistically as well. However, the reverse does not hold [12].

If $\liminf\limits_{n\to\infty} \frac{p_n}{n} > 0$, then if the sequence $x = (x_n)$ statistically converges to the number L, then it converges quasi-statistically as well (see [12]).

2. The Quasi-Density

Let $p = (p_n)$ be a sequence of positive real numbers that satisfies the following properties:

(a) $\lim\limits_{n\to\infty} p_n = +\infty$

(b) $\limsup\limits_{n\to\infty} \frac{p_n}{n} < +\infty$.

We will call such a sequence permissible.

The lower quasi-density of the set $A \subseteq N$ for a permissible sequence $p = (p_n)$ is

$$\underline{d_p}(A) = \liminf_{n\to\infty} \frac{1}{p_n} |\{k \epsilon A, \ k \leq n\}|,$$

if such a limit exists.

The upper quasi-density of the set $A \subseteq N$ for a permissible sequence $p = (p_n)$ is

$$\overline{d_p}(A) = \limsup_{n\to\infty} \frac{1}{p_n} |\{k \epsilon A, \ k \leq n\}|,$$

if such a limit exists.

In case $\underline{d_p}(A) = \overline{d_p}(A)$, then there exists a quasi-density of set A and we denote it as $d_p(A) = \lim\limits_{n\to\infty} \frac{1}{p_n} |\{k \epsilon A, \ k \leq n\}|$.

Example 2. *Sequences that satisfy these properties (a permissible sequences) are, for example,*

$$(p_n) = (\log n)_{n=1}^{\infty},\ (p_n) = (n\cdot\alpha + d)_{n=1}^{\infty},\ \alpha \in R^+,\ d \in R,\ (p_n) = (n^{\alpha})_{n=1}^{\infty},\ \alpha \in (0,1).$$

If the permissible sequence satisfies the following property, we can define the quasi-density of the set A using a matrix.

Let $p = (p_n)$ be a permissible sequence, let in addition $\liminf\limits_{n\to\infty} \frac{p_n}{n} = h,\ h \in R^+,\ h \neq 0$. We will create a matrix $B = (b_{nk})$ as follows:

$$b_{nk} = \begin{cases} 1/p_n, & k \le n \\ 0, & k > n \end{cases},\ \text{i.e.,}$$

$$B = \begin{pmatrix} 1/p_1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1/p_2 & 1/p_2 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/p_n & 1/p_n & 1/p_n & \cdots & 1/p_n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The matrix defined in this way does meet the Angew's conditions.

It is true that

$$\sum_{k=1}^{\infty} b_{nk} = 1/p_n + 1/p_n + \ldots + 1/p_n = n\cdot 1/p_n = 1/{p_n/n} \le 1/h$$

and

$$\lim_{n\to\infty} \max_{1\le k\le n} |b_{nk}| = \lim_{n\to\infty} 1/p_n = 0,\ \text{because}\ \lim_{n\to\infty} p_n = \infty.$$

Then, we can define the quasi-density of the set $A \subseteq N$ as follows:

Let $\chi_A(k)$be the characteristic function of set A.

Then,

$$\underline{d_p}(A) = \liminf_{n\to\infty} \sum_{k=1}^{\infty} b_{nk}\cdot\chi_A(k)$$

and

$$\overline{d_p}(A) = \limsup_{n\to\infty} \sum_{k=1}^{\infty} b_{nk}\cdot\chi_A(k)$$

are the lower and upper quasi-density of set A, respectively.

In case $\underline{d_p}(A) = \overline{d_p}(A)$, then there exists a quasi-density of set A and we denote it as $d_p(A) = \underline{d_p}(A) = \overline{d_p}(A) = \lim\limits_{n\to\infty} \sum_{k=1}^{\infty} b_{nk}\cdot\chi_A(k)$.

We will now state several properties of a quasi-density.

Proposition 1. *Let$A \subseteq N$ be a finite set. Then, $d_p(A) = 0$ for every permissible sequence $p = (p_n)$.*

Proof of Proposition 1. If A is a finite set, then

$$d_p(A) = \lim_{n\to\infty} \frac{1}{p_n} |\{k\epsilon A,\ k \le n\}| \le \lim_{n\to\infty} \frac{|A|}{p_n} = 0.$$

The quasi-density of the set of all natural numbers N is dependent on the sequence (p_n). □

Proposition 2. *Let (p_n) be a permissible sequence.*

(a) *If $\limsup\limits_{n\to\infty} \frac{p_n}{n} = T \neq 0$, then $\overline{d_p}(N) = \frac{1}{T}$ (if $\limsup\limits_{n\to\infty} \frac{p_n}{n} = 1$, then $\overline{d_p}(N) = 1$).*

(b) *If* $\limsup\limits_{n\to\infty}\frac{p_n}{n} = 0$, *then* $\overline{d_p}(N) = \infty$.

Proof of Proposition 2. (a) Suppose that $\limsup\limits_{n\to\infty}\frac{p_n}{n} = T \neq 0$. Then

$$\overline{d_p}(N) = \limsup_{n\to\infty}\sum_{k=1}^{\infty} b_{nk}\cdot\chi_k(N) = \limsup_{n\to\infty}\frac{n}{p_n} = \frac{1}{T}.$$

(b) Similarly

$$\overline{d_p}(N) = \limsup_{n\to\infty}\sum_{k=1}^{\infty} b_{nk}\cdot\chi_k(N) = \limsup_{n\to\infty}\frac{n}{p_n} = \limsup_{n\to\infty}\frac{1}{\frac{p_n}{n}} = \infty.$$

□

Note: Let exists a finite $\lim\limits_{n\to\infty}\frac{p_n}{n}$

(a) In the case of $\lim\limits_{n\to\infty}\frac{p_n}{n} = L \neq 0$, then $d_p(N) = \frac{1}{L}$.

(b) In the case of $\lim\limits_{n\to\infty}\frac{p_n}{n} = 0$, then $d_p(N) = \infty$.

(c) In the case of $\lim\limits_{n\to\infty}\frac{p_n}{n} = 1$, then $d_p(N) = 1$.

We see that, generally, for any $A \subseteq N$: $0 \leq \underline{d_p}(A) \leq \overline{d_p}(A) \leq +\infty$, i.e., quasi-density does not behave like any of the densities studied up to now.

If the sequence $p = (p_n)$ is such a permissible sequence, for which there exists a finite and non-zero limit $\lim\limits_{n\to\infty}\frac{p_n}{n}$, we can determine the relation between the asymptotic density and the quasi-density of a set.

Proposition 3. *Let* $A \subseteq N$ *be such a set, for which its asymptotic density is* $d(A) = m,\ m\epsilon 0,1$. *Let there exists a non-zero* $\lim\limits_{n\to\infty}\frac{p_n}{n} = L$. *Then, there also exists a quasi-density of set* A *and* $d_p(A) = \frac{1}{L}\cdot m$ *holds true.*

Proof of Proposition 3. When we use the definition of quasi-density we get the following.

$$d_p(A) = \lim_{n\to\infty}\frac{1}{p_n}|\{k\epsilon A,\ k \leq n\}| = \lim_{n\to\infty}\frac{n}{p_n}\cdot\frac{1}{n}|\{k\epsilon A,\ k \leq n\}| = \frac{1}{L}\cdot m.$$

□

Corollary 1. *Let* $p = (p_n)$ *be any arithmetic sequence of the type*

$$p_n = n\cdot\alpha + d,\ n = 1, 2, \ldots,\ \alpha\epsilon R^+,\ d\epsilon R.$$

Let $A \subseteq N$ be such a set that its asymptotic density $d(A) = m,\ m\epsilon R$.
Then, $d_p(A) = \frac{m}{\alpha}$.

Proof of Corollary 1. For an arithmetic sequence the following applies:

$$\lim_{n\to\infty}\frac{p_n}{n} = \lim_{n\to\infty}\frac{n\cdot\alpha + d}{n} = \alpha.$$

From the previous theorem we obtain $d_p(A) = \frac{1}{\alpha}\cdot m$. □

Example 3. *Let* $A = \{1^2,\ 2^2,\ \ldots, n^2, \ldots\}$. *It is evident that* $d(A) = 0$.

We define the sequence $p = (p_n)$ by:

$$p_n = \begin{cases} 2n, & n = 2k \\ \frac{n}{2}, & n = 2k-1 \end{cases} \quad k = 1, 2, \ldots$$

This sequence satisfies the requirements of the definition. The quasi-density of set $A = \{1^2, 2^2, \ldots, n^2, \ldots\}$ given the sequence p is

$$d_p(A) = \lim_{n\to\infty} \frac{|A \cap \langle 1, n\rangle|}{p_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{p_n} = 0.$$

Quasi-density of set $B = \{1, 2, \ldots, n, \ldots\} = N$ given the sequence p does not exist, because

$$\underline{d_p}(B) = \liminf_{n\to\infty} \frac{|N \cap \langle 1, n\rangle|}{p_n} = \liminf_{n\to\infty} \frac{n}{p_n} = 2,$$

$$\overline{d_p}(B) = \limsup_{n\to\infty} \frac{|N \cap \langle 1, n\rangle|}{p_n} = \limsup_{n\to\infty} \frac{n}{p_n} = \frac{1}{2}.$$

Example 4. *Let $p_n = \log n,\ n = 2, 3, \ldots$. It is evident that a sequence (p_n) defined as such is permisible, because $\lim_{n\to\infty} \log n = \infty$ and $\lim_{n\to\infty} \frac{\log n}{n} = 0$.*

Quasi-densities of the sets $A = \{1^2, 2^2, \ldots, n^2, \ldots\}$ and $B = \{1, 2, \ldots, n, \ldots\}$ given a sequence defined as preceding (p_n) exists and is identical: $d_p(A) = \infty$ a $d_p(B) = \infty$.

The asymptotic densities of these sets are not the same, because $d(A) = 0$ and $d(B) = 1$. We can say the following corollary:

Corollary 2. *If $\limsup_{n\to\infty} \frac{p_n}{n} = 0$, then there is a set $C \subseteq N$ such that it exists $d(C)$, but it does not exist $d_p(C)$.*

Proposition 4. *Let the following hold true for sequences $p = (p_n)$*

$$0 < \liminf_{n\to\infty} \frac{p_n}{n} \le \limsup_{n\to\infty} \frac{p_n}{n} = T < \infty$$

Then, for any sequence $A \subseteq N$, $0 \le \underline{d_p}(A) \le \overline{d_p}(A)$ and $\overline{d_p}(A) \le \frac{1}{T}$ is valid.

Proof of Proposition 4.

$$0 \le \underline{d_p}(A) = \liminf_{n\to\infty} \frac{|k \in A, k \le n|}{p_n} = \liminf_{n\to\infty} \frac{n}{p_n} \cdot \frac{|k \in A, k \le n|}{n} \le$$

$$\le \limsup_{n\to\infty} \frac{n}{p_n} \cdot \frac{|k \in A, k \le n|}{n} = \limsup_{n\to\infty} \frac{|k \in A, k \le n|}{p_n} = \overline{d_p}(A).$$

In addition to that $\overline{d_p}(A) = \limsup_{n\to\infty} \frac{n}{p_n} \cdot \frac{|k \in A, k \le n|}{n} \le \frac{1}{T} \cdot \overline{d}(A) \le \frac{1}{T}$. □

It is sufficient to realize that for every set $A \subseteq N$ there exists a $\underline{d}(A)$ and $\overline{d}(A)$ (an asymptotic density $d(A)$ does not have to exist).

Corollary 3. *If there exists an asymptotic density $d(A)$ of set $A \subseteq N$, the the quasi-density $d_p(A)$ of this set exists if and only if $\lim_{n\to\infty} \frac{p_n}{n} = L \neq 0$ and $d_p(A) \epsilon {<} 0, \frac{1}{L} {>}$ holds true.*

In the next example, we will assume that $\lim_{n\to\infty} \frac{p_n}{n} = L \neq 0$.

Proposition 5. *Let $A, B \subseteq N$ be a non-empty set for which their quasi-densities are $d_p(A)$ and $d_p(B)$. Let $d_p : \mathcal{P}_N \to \langle 0, \infty)$ be a function. Then*

(a) *If $A \subseteq B$ then $d_p(A) \leq d_p(B)$.*

(b) $\underline{d_p}(A \cap B) \leq d_p(A) + d_p(B), \overline{d_p}(A \cap B) \leq d_p(A) + d_p(B)$.

(c) *If $A \cap B = \varnothing$ then $d_p(A \cup B) = d_p(A) + d_p(B)$.*

Proof of Proposition 5. (a) Let $A \subseteq B$. Then, for every $n \in N$ the following holds true

$$|\{k\epsilon A,\ k \leq n\}| \leq |\{k\epsilon B,\ k \leq n\}|.$$

Then

$$\frac{1}{p_n}|\{k\epsilon A,\ k \leq n\}| \leq \frac{1}{p_n}|\{k\epsilon B,\ k \leq n\}|.$$

Transitioning to the limit, we obtain

$$\lim_{n\to\infty}\frac{1}{p_n}|\{k\epsilon A,\ k \leq n\}|\ \leq \lim_{n\to\infty}\frac{1}{p_n}|\{k\epsilon B,\ k \leq n\}|, \text{ i.e., } d_p(A) \leq d_p(B).$$

(b) It is evident that $|\{k\epsilon A \cup B,\ k \leq n\}| \leq |\{k\epsilon A,\ k \leq n\}| + |\{k\epsilon B,\ k \leq n\}|$.
From that we obtain the following:

$$\underline{d_p}(A \cup B) = \liminf_{n\to\infty}\frac{1}{p_n}|\{k\epsilon(A \cup B),\ k \leq n\}| \leq \limsup_{n\to\infty}\frac{1}{p_n}|\{k\epsilon(A \cup B),\ k \leq n\}| \leq$$

$$\leq \limsup_{n\to\infty}\frac{1}{p_n}(|\{k\epsilon A,\ k \leq n\}\ | + |\{k\epsilon B,\ k \leq n\}\ |) \leq$$

$$\leq \limsup_{n\to\infty}\frac{1}{p_n}|\{k\epsilon A,\ k \leq n\}| + \limsup_{n\to\infty}\frac{1}{p_n}|\{k\epsilon B,\ k \leq n\}| = \overline{d_p}(A) + \overline{d_p}(B) =$$

$$= d_p(A) + d_p(B).$$

(c) $A \cap B = \varnothing \Rightarrow |\{k\epsilon A \cup B, k \leq n\}| = |\{k\epsilon A,\ k \leq n\}| + |\{k\epsilon B,\ k \leq n\}|$

$$\overline{d_p}(A \cup B) \leq d_p(A) + d_p(B) = \liminf_{n\to\infty}\frac{1}{p_n}|\{k\epsilon A,\ k \leq n\}| + \liminf_{n\to\infty}\frac{1}{p_n}|\{k\epsilon B,\ k \leq n\}| \leq$$

$$\leq \liminf_{n\to\infty}\frac{1}{p_n}|\{k\epsilon(A \cup B),\ k \leq n\}| = \underline{d_p}(A \cup B).$$

□

Now, we will show that quasi-densities have the almost Darboux property.

Definition 1. *We say that the density $d(A)$ has the almost Darboux property, if for every real number $t \in < 0, d(N))$ the exists such a set $A \subseteq N$, for which its density is $d(A) = t$.*

Theorem 1. *For every real number $t \in 0, \infty)$ there exists such a set $A \subseteq N$ and a permissible sequence $p = (p_n)$, that $d_p(A) = t$.*

Proof of Theorem 1. If $t = 0$, then we can choose A to be any finite set (Proposition 1).
Let $t \in (0, \infty)$, and let us choose any $m \in (0, 1]$.
For these chosen immutable numbers, we define a sequence $p = (p_n) = \left(\frac{m}{t}\cdot n\right)_{n=1}^{\infty}$.
This sequence is permissible, because

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty}\frac{m}{t}\cdot n = +\infty \text{ a } \limsup_{n\to\infty}\frac{p_n}{n} = \lim_{n\to\infty}\frac{\frac{m}{t}\cdot n}{n} = \frac{m}{t} < +\infty.$$

An asymptotic density has the almost Darboux property:

For every $m \in \langle 0, 1)$ exists such a set $A \subseteq N$ for which its asymptotic density is $d(A) = m$.

Let A be such a subset of natural numbers, such that its asymptotic density is m and the sequence $(p_n) = \left(\frac{m}{t} \cdot n\right)_{n=1}^{\infty}$. Then,

$$d_p(A) = \lim_{n\to\infty} \frac{1}{p_n} |\{k\epsilon A,\ k \le n\}| = \lim_{n\to\infty} \frac{t}{m \cdot n} |\{k\epsilon A,\ k \le n\}| = \frac{t}{m} \lim_{n\to\infty} \frac{|\{k\epsilon A,\ k \le n\ \}|}{n} = \frac{t}{m} \cdot m = t.$$

□

Theorem 2. *Let $A \subseteq N$ be such a subset of natural numbers, for which its asymptotic density $d(A) = 0$. Let $p = (p_n)$ be a permissible sequence that satisfies the condition $\limsup\limits_{n\to\infty} \frac{p_n}{n} = T \neq 0$. Then, $\overline{d_p}(A) = 0$.*

Proof of Theorem 2. Let $d(A) = 0$ and $\limsup\limits_{n\to\infty} \frac{p_n}{n} = T \neq 0$. The upper quasi-density of this set in regard to the sequence $p = (p_n)$ is

$$0 \le \overline{d_p}(A) = \limsup_{n\to\infty} \frac{|k \in A,\ k \le n|}{p_n} = \limsup_{n\to\infty} \frac{n}{p_n} \cdot \frac{|k \in A,\ k \le n|}{n} \le \limsup_{n\to\infty} \frac{n}{p_n} \cdot \limsup_{n\to\infty} \frac{|k \in A,\ k \le n|}{n} = \frac{1}{T} \cdot 0 = 0.$$

□

3. The Quasi-Statistical Convergence and the Matrix Transformation

In the final part of this paper, we will focus on the quasi-statistical convergence of sequences of real numbers.

We will show the equivalence between this convergence and a matrix transformation of the same sequence.

Let $p = (p_n)$ be a permissible sequence. By $\mathcal{T}_p$ we will denote the class of matrices with non-negative real members

$$B = (b_{nk}) \quad n, k = 1, 2, \ldots$$

for which the following conditions are true:

(a) $\sum_{k=1}^{n} b_{nk} = 1$

(b) If D is a subset of natural numbers for which $d_p(D) = 0$, then $\lim\limits_{n\to\infty} \sum_{k\in D} b_{nk} = 0$.

It is evident that if a matrix belong to the class $\mathcal{T}_p$, then it is regular. However, the reverse does not hold.

Example 5. *Let $p = (p_n) = (2n+1)_{n=1}^{\infty}$. Let the set $D = \{1^2, 2^2, 3^2, \ldots\}$. According to Proposition 3, the quasi-density of this set in regard to sequence p is*

$$d_p(D) = \frac{1}{L} \cdot m = \frac{1}{2} \cdot 0 = 0,$$

where $L = \lim\limits_{n\to\infty} \frac{p_n}{n} = \lim\limits_{n\to\infty} \frac{2n+1}{n} = 2$ a $m = d(D) = 0$.

Let us define the matrix $C = (c_{nk})$ as follows:

$$\begin{aligned} c_{11} &= 1,\ c_{1k} = 0 \text{ for } k > 1 \\ c_{nk} &= \tfrac{1}{2k \log n} \text{ for } k \notin D,\ k \le n \\ c_{nk} &= \tfrac{1}{k \log n} \text{ for } k \in D,\ k \le n \\ c_{nk} &= 0 \text{ for } k > n. \end{aligned}$$

This matrix is the lower triangular regular, but do not belong to the class $\mathcal{T}_p$, because

$$\sum_{\substack{k < n^2 \\ k \in D}} b_{nk} = \frac{1}{\log n^2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) \geq \frac{\log n}{2 \log n} = \frac{1}{2} \nrightarrow 0 \text{ for } n \to \infty.$$

That is, if the matrix belongs to the class $\mathcal{T}_p$, so it is regular, the reverse is not true.

Lemma 1. *If the bounded sequence $x = (x_k)$ is not quasi-statistically convergent, then there exist real numbers $\lambda < \mu$ such that neither of the sets $\{n \in N,\ x_n < \lambda\}$ and $\{n \in N,\ x_n > \mu\}$ has quasi-density zero.*

Proof of Lemma 1. The proof is the same as the proof of the Lemma in [6].

We will now utter a theorem that connects quasi-statistically convergent the sequences of real number and a matrix transformation of the same sequence using matrices from the class $\mathcal{T}_p$. □

Theorem 3. *The bounded sequence $x = (x_k)$ of real numbers is quasi-statistically convergent to $L \in R$ $\left(\mathrm{st_q} - \lim x_k = L\right)$ if and only if it is summable to $L \in R$ for each matrix $B = (b_{nk}) \in \mathcal{T}_p$.*

Proof of Theorem 3. Let $\mathrm{st_q} - \lim x_k = L,\ L \in R$ and $B = (b_{nk}) \in \mathcal{T}_p$.

As B is regular there exist a $K \in R$ such that $\forall n = 1, 2, \ldots\ \sum_{k=1}^{\infty} |b_{nk}| \leq K$.

It is sufficient to show that $\lim\limits_{n \to \infty} a_n = 0$ where $a_n = \sum_{k=1}^{\infty} b_{nk} \cdot (x_k - L)$.

For $\varepsilon > 0$ put $B(\varepsilon) = \{k \in N,\ |x_k - L| \geq \varepsilon\}$.

By the assumption we have $d_p(B(\varepsilon)) = 0$ we have $\lim\limits_{k \in B(\varepsilon)} \sum_{k \in B(\varepsilon)} |b_{nk}| = 0$.

As the sequence $x = (x_k)$ is bounded, there exist $M > 0$ such that $\forall k = 1, 2, \ldots :$ $|x_k - L| \leq M$.

Let $\varepsilon > 0$. Then

$$|a_n| \leq \sum_{k \in B(\frac{\varepsilon}{2K})} |b_{nk}| \cdot |x_k - L| + \sum_{k \notin B(\frac{\varepsilon}{2K})} |b_{nk}| \cdot |x_k - L| \leq$$

$$\leq M \sum_{k \in B(\frac{\varepsilon}{2K})} |b_{nk}| + \frac{\varepsilon}{2K} \sum_{k \notin B(\frac{\varepsilon}{2K})} |b_{nk}| \leq M \sum_{k \in B(\frac{\varepsilon}{2K})} |b_{nk}| + \frac{\varepsilon}{2}.$$

By the condition (b) there exists an integer n_0 such that for all $n > n_0 :$

$$\sum_{k \in B(\frac{\varepsilon}{2K})} |b_{nk}| < \frac{\varepsilon}{2M}.$$

Together we obtain $\lim\limits_{n \to \infty} a_n = 0$.

Conversely, suppose that $\mathrm{st_q} - \lim x_k = L$ does not apply. We show that it exists a matrix $B^* = (b_{nk}{}^*) \in \mathcal{T}_p$ such that $B - \lim x_k = L$ does not apply too.

If $\mathrm{st_q} - \lim x_k = L' \neq L$ then from the first part of proof it follows that $B - \lim x_k = L'$ for any $B \in \mathcal{T}_p$. Thus we way assume that $x = (x_k)$ is not quasi-statistically convergent and by the above Lema there exist λ and μ $(\lambda < \mu)$ such that either the set $U = \{k \in N :\ x_k < \lambda\}$ nor $V = \{k \in N :\ x_k > \mu\}$ has quasi-density zero.

It is clear that $U \cap V = \varnothing$. Let $U_n = U \cap \{1,\ 2,\ \ldots,\ n\}$ and $V_n = V \cap \{1,\ 2,\ \ldots,\ n\}$. Therefore, there exists an $\varepsilon > 0$ and subsets $U' = \{u_k\}_{k=1}^{\infty} \subset U$ and $V' = \{v_k\}_{k=1}^{\infty} \subset V$ such for each $k \in N$: $\frac{1}{u_k} \cdot |\{n \in N : n \leq u_k\}| > \varepsilon$ and $\frac{1}{v_k} \cdot |\{n \in N : n \leq v_k\}| > \varepsilon$.

Now define the matrix $B^* = (b_{nk}{}^*)$ in the following way

$$b_{nk}{}^* = \begin{cases} 0 & k > n \\ \frac{1}{n} & k \le n \ \wedge \ n \notin U' \cap V' \\ \frac{1}{|U_n|} & k \in U_n \ \wedge \ n \in U' \\ \frac{1}{|V_n|} & k \in V_n \ \wedge \ n \in V' \end{cases}$$

Check that $B^* = (b_{nk}{}^*) \in \mathcal{T}_p$. Obviously, B^* is a lower triangular nonnegative matrix. Condition (a) is clear from the definition of B^*, $\sum_{k=1}^{n} b_{nk}{}^* = 1$ for each n.

Condition (b) for this matrix:

Let for the set $C \subseteq N: \ d_p(C) = 0$.

Then,

$$\sum_{\substack{k \in C \\ n \notin U' \cup V'}} b_{nk}{}^* = \sum_{\substack{k \in C \\ n \notin U' \cup V'}} \frac{1}{n}\chi_k(C) = \sum_{\substack{k \in C \\ n \notin U' \cup V'}} \frac{p_n}{n} \cdot \frac{1}{p_n} \cdot \chi_k(C).$$

For $n \to \infty$ we have

$$\lim_{n\to\infty} \sum_{\substack{k \in C \\ n \notin U' \cup V'}} \frac{p_n}{n} \cdot \frac{\chi_k(C)}{p_n} \le \limsup_{n\to\infty} \frac{p_n}{n} \cdot d_p(C) = 0$$

.

We proved that $B^* = (b_{nk}{}^*)$ belongs to $\mathcal{T}_p$.

Next, we will show $B^* - \lim x_k$ does not exist.

For $n \in U' : \ \sum_{k=1}^{\infty} b_{nk}{}^* \cdot x_k = \sum_{k \in U_n} \frac{1}{|U_n|} \cdot x_k < \lambda \cdot 1 = \lambda$ and for $n \in V' : \ \sum_{k=1}^{\infty} b_{nk}{}^* \cdot x_k = \sum_{k \in V_n} \frac{1}{|V_n|} \cdot x_k > \mu \cdot 1 = \mu$. □

4. Conclusions

In this paper we define the lower quasi-density $\underline{d_p}(A)$, the upper quasi-density $\overline{d_p}(A)$ and the quasi-density $d_p(A)$ of subsets of natural numbers, which we use to define the quasi-statistical convergence of sequences.

We proved some of the properties of quasi-densities (e.g., the quasi-density of a finite subset of natural numbers is zero and has the almost Darboux property). Given a permissible sequence, for which $\lim_{n\to\infty} \frac{p_n}{n} = L \neq 0$ there is a relation between the asymptotic and quasi-densities of set A, we have $d_p(A) = \frac{d(A)}{L}$.

The final section pertains to the quasi-statistical converge. We showed that the bounded sequence of real numbers is quasi-statistically convergent to $L \in R$ if and only if it is summable to $L \in R$ for each matrix $\mathrm{B} = (\mathrm{b_{nk}}) \in \mathcal{T}_\mathrm{p}$.

One of the most important applications of quasi-densities is connecting the quasi-statistical convergence with the summability method and by doing so generalize the term convergence.

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