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Subclasses of Yamakawa-Type Bi-Starlike Functions Associated with Gegenbauer Polynomials

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Abstract: In this paper, we introduce and investigate new subclasses (Yamakawa-type bi-starlike functions and another class of Lashin, both mentioned in the reference list) of bi-univalent functions defined in the open unit disk, which are associated with the Gegenbauer polynomials and satisfy subordination conditions. Furthermore, we find estimates for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Several known or new consequences of the results are also pointed out.

Keywords: starlike and convex functions; hadamard product; subordination; bi-univalent functions; Fekete–Szegő problem; Gegenbauer polynomials; Yamakawa-type bi-starlike functions

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1. Introduction and Preliminaries

In geometric function theory, there have been numerous interesting and fruitful usages of a wide variety of special functions, q-calculus and special polynomials; for example, the Fibonacci polynomials, the Faber polynomials, the Lucas polynomials, the Pell polynomials, the Pell–Lucas polynomials, and the Chebyshev polynomials of the second kind. The Horadam polynomials are potentially important in a variety of disciplines in the mathematical, physical, statistical, and engineering sciences. Gegenbauer polynomials or ultra spherical polynomials $\mathfrak{G}_n^{\lambda}$ can be obtained using the Gram–Schmidt orthogonalization process for polynomials in the domain $(-1,1)$ with the weight factor $\left(1-\ell^2\right)^{\lambda-\frac{1}{2}}$, $\lambda>-\dfrac{1}{2}$. Also, $\mathfrak{G}_n^{0}(\ell)$ is defined as $\lim\limits_{\lambda\to 0}\dfrac{\mathfrak{G}_n^{\lambda}(\ell)}{\lambda}$, and for $\lambda\neq 0$ the resulting polynomial $R_n(\ell)$ is multiplied by a number which makes the value at $\ell=1$ equal to $(2\lambda)_n/n! = 2\lambda(2\lambda+1)(2\lambda+2)\ldots(2\lambda+n-1)/n!$. For $\lambda=0$ and $n\neq 0$, the value at $\ell=1$ is $\dfrac{2}{n}$, while $\mathfrak{G}_0^{0}(\ell)=1$.

The Gegenbauer polynomials (for details, see Kim et al. [1] and references cited therein) are given in terms of the Jacobi polynomials $P_n^{(\nu,\upsilon)}$, with $\nu=\upsilon=\lambda-\dfrac{1}{2}$, $\left(\lambda>-\dfrac{1}{2},\ \lambda\neq 0\right)$, defined by

$$\begin{aligned}\mathfrak{G}_n^{\lambda}(\ell) &= \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma\left(n+\lambda+\frac{1}{2}\right)}P_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(\ell)\\ &= \binom{n+2\lambda-1}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}(2\lambda+n)_k}{\left(\lambda+\frac{1}{2}\right)_k}\left(\frac{\ell-1}{2}\right)^k, \end{aligned}\qquad (1)$$

where $(a)_n := a(a+1)(a+2)\ldots(a+n-1)$, and $(a)_0 := 1$.

From (1), it follows that $\mathfrak{G}_n^\lambda(\ell)$ is a polynomial of degree n with real coefficients, and $\mathfrak{G}_n^\lambda(1) = \binom{n+2\lambda-1}{n}$, while the leading coefficient of $\mathfrak{G}_n^\lambda(\ell)$ is $2^n\binom{n+\lambda-1}{n}$. By the theory of Jacobi polynomials, for $\mu = v = \lambda - \frac{1}{2}$, with $\lambda > -\frac{1}{2}$, and $\lambda \neq 0$, we get

$$\mathfrak{G}_n^\lambda(-\ell) = (-1)^n \mathfrak{G}_n^\lambda(\ell).$$

It is easy to show that $\mathfrak{G}_n^\lambda(\ell)$ is a solution of the Gegenbauer differential equation

$$(1-\ell^2)y'' - (2\lambda)\ell y' + n(n+2\lambda)y = 0,$$

with $\ell = 0$ an ordinary point; this means that we can express the solution in the form of a power series $y = \sum_{n=0}^{\infty} a_n \ell^n$, and the Rodrigues formula for the Gegenbauer polynomials is (see [2,3]) as follows:

$$\left(1-\ell^2\right)^{\lambda-\frac{1}{2}} \mathfrak{G}_n^\lambda(\ell) = \frac{(-2)^n(\lambda)_n}{n!(n+2\lambda)_n}\left(\frac{d}{d\ell}\right)^n \left(1-\ell^2\right)^{n+\lambda-\frac{1}{2}},$$

and the above relation can be easily derived from the properties of Jacobi polynomials.

The generating function of Gegenbauer polynomials is given by (see [1,4])

$$\frac{2^{\lambda-\frac{1}{2}}}{(1-2\ell t+t^2)^{\frac{1}{2}}\left(1-\ell t+\sqrt{1-2\ell t+t^2}\right)^{\lambda-\frac{1}{2}}} = \frac{\left(\lambda-\frac{1}{2}\right)_n}{(2\lambda)_n}\mathfrak{G}_n^\lambda(\ell)t^n, \tag{2}$$

and this equality can be derived from the generating function of Jacobi polynomials.

From the above relation (2), we note that

$$\frac{1}{(1-2\ell t+t^2)^\lambda} = \sum_{n=0}^{\infty} \mathfrak{G}_n^\lambda(\ell)t^n,\ t\in\mathbb{C},\ |t|<1,\ \ell\in[-1,1],\ \lambda\in\left(-\frac{1}{2},+\infty\right)\setminus\{0\}, \tag{3}$$

and the proof is given in [4] and Kim et al. [1] (also, see [5]) where the authors extensively studied many results from different perspectives. For $\lambda = 1$, the relation (3) gives the ordinary generating function for the Chebyshev polynomials, and for $\lambda = \frac{1}{2}$, we obtain the ordinary generating function for the Legendre polynomials (see also [6]).

In 1935, Robertson [7] proved an integral representation for the typically real-valued function class T_R having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z\in\Delta := \{z\in\mathbb{C} : |z|<1\}, \tag{4}$$

which is holomorphic in the open unit disc Δ, real for $z \in (-1,1)$, and satisfies the condition

$$\operatorname{Im} f(z)\operatorname{Im} z > 0,\ z\in\Delta\backslash(-1,1).$$

Namely, $f \in T_R$ if and only if it has the representation

$$f(z) = \int_{-1}^{1}\frac{z}{1-2\ell z+z^2}\,d\mu,\ z\in\Delta,$$

where μ is a probability measure on $[-1,1]$. The class T_R has been extended in [8] to the class $T_R(\lambda)$, $\lambda > 0$, which was defined by

$$f(z) = \int_{-1}^{1} \Phi_\ell^\lambda(z)\, d\mu(\ell),\ z \in \Delta,\ -1 \leq \ell \leq 1, \tag{5}$$

where

$$\Phi_\ell^\lambda(z) := \frac{z}{(1 - 2\ell z + z^2)^\lambda},\ z \in \Delta,\ -1 \leq \ell \leq 1, \tag{6}$$

and μ is a probability measure on $[-1, 1]$. The function $\Phi_\ell^\lambda(z)$ has the following Taylor–Maclaurin series expansion:

$$\Phi_\ell^\lambda(z) = z + \mathfrak{G}_1^\lambda(\ell) z^2 + \mathfrak{G}_2^\lambda(\ell) z^3 + \mathfrak{G}_3^\lambda(\ell) z^4 + \cdots + \mathfrak{G}_{n-1}^\lambda(\ell) z^n + \ldots, \tag{7}$$

where $\mathfrak{G}_n^\lambda(\ell)$ denotes the Gegenbauer (or ultra spherical) polynomials of order λ and degree n in ℓ, which are generated by

$$\Phi_\ell^\lambda(z) = \sum_{n=0}^{\infty} \mathfrak{G}_n^\lambda(\ell) z^n = z\left(1 - 2\ell z + z^2\right)^{-\lambda}.$$

In particular,

$$\mathfrak{G}_0^\lambda(\ell) = 1, \quad \mathfrak{G}_1^\lambda(\ell) = 2\lambda\ell, \quad \mathfrak{G}_2^\lambda(\ell) = 2\lambda(\lambda+1)\ell^2 - \lambda = 2(\lambda)_2\ell^2 - \lambda. \tag{8}$$

Of course, we have $T_R(1) \equiv T_R$, and if f given by (5) is written in the power expansion series (4), then we have

$$a_n = \int_{-1}^{1} \mathfrak{G}_{n-1}^\lambda(\ell)\, d\mu(\ell).$$

One can easily see that the class $T_R(\lambda)$, $\lambda > 0$, is a compact and convex set in the linear space of holomorphic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are holomorphic in Δ, endowed with the topology of local uniform convergence on compact subsets of Δ. The importance of the class $T_R(\lambda)$, $\lambda > 0$, follows as well from the paper of Hallenbeck [9], who studied the extreme points of some families of univalent functions and proved that

$$\operatorname{co} \mathcal{S}_R^*(1-\lambda) = T_R(\lambda), \quad \text{and} \quad \operatorname{ext} \operatorname{co} \mathcal{S}_R^*(1-\lambda) = \left\{ \frac{z}{(1 - 2\ell z + z^2)^\lambda} : \ell \in [-1;1] \right\},$$

where "co A" denotes the closed convex hull of A, "ext A" represents the set of the extremal points of A, while $\mathcal{S}_R^*(\vartheta)$ denotes the class of holomorphic functions given by (5), which are univalent and starlike of order ϑ, $\vartheta \in [0, 1)$, in Δ, and have real coefficients.

Let $\mathcal{A}$ represents the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \Delta, \tag{9}$$

which are analytic in Δ, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in Δ. The Koebe one quarter theorem [10] ensures that the image of Δ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z,\ (z \in \Delta) \quad \text{and} \quad f\left(f^{-1}(w)\right) = w,\ \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ, and let Σ denote the class of bi-univalent functions defined in the unit disk Δ. Since $f \in \Sigma$

has the Maclaurin series given by (9), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 + \dots . \tag{10}$$

We notice that the class Σ is not empty. For instance, the functions

$$f_1(z) = \frac{z}{1-z}, \quad f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad f_3(z) = -\log(1-z)$$

with their corresponding inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \quad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}, \quad f_3^{-1}(w) = \frac{e^{w}-1}{e^{w}}$$

are elements of Σ. However, the Koebe function is not a member of Σ. Lately, Srivastava et al. [11] have essentially revived the study of analytic and bi-univalent functions; this was followed by such works as those of [12–17]. Several authors have introduced and examined subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [11–13,15]), bi-close-to-convex functions [18,19], and bi-prestarlike functions by Jahangiri and Hamidi [20].

Orthogonal polynomials have been broadly considered in recent years from various perceptions due to their importance in mathematical physics, mathematical statistics, engineering, and probability theory. Orthogonal polynomials that appear most often in applications are the classical orthogonal polynomials (Hermite polynomials, Laguerre polynomials, and Jacobi polynomials). The previously mentioned Fibonacci polynomials, Faber polynomials, the Lucas polynomials, the Pell polynomials, the Pell–Lucas polynomials, the Chebyshev polynomials of the second kind, and Horadam polynomials have been studied in several papers from a theoretical point of view and recently in the case of bi-univalent functions (see [21–28] also the references cited therein).

Here, in this article, we associate certain bi-univalent functions with Gegenbauer polynomials and then explore some properties of the class of bi-starlike functions based on earlier work of Srivastava et al. (also, see [11]). In addition, motivated by recent works by Murugusundaramoorthy et al. [29], Wannas [30] and Amourah et al. [31], we introduce a new subclass of the Yamakawa-type bi-starlike function class (see [32]) associated with Gegenbauer polynomials, obtain upper bounds of the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the functions $f \in \mathcal{GY}_\Sigma\left(\Phi_\ell^\lambda\right)$ defined by subordination, and consider the remarkable Fekete–Szegő problem. We also provide relevant connections of our results with those of some earlier investigations.

First, we define a new subclass Yamakawa-type bi-starlike in the open unit disk, associated with Gegenbauer polynomials as below.

Unless otherwise stated, we let $0 \le \vartheta \le 1$, $\lambda > \frac{1}{2}$ and $\ell \in \left(\frac{1}{2}, 1\right]$.

Definition 1. *For $0 \le \vartheta \le 1$ and $\ell \in \left(\frac{1}{2}, 1\right]$, a function $f \in \Sigma$ of the form (9) is said to be in the class $\mathcal{GY}_\Sigma\left(\vartheta, \Phi_\ell^\lambda\right)$ if the following subordinations hold:*

$$\frac{f(z)}{(1-\vartheta)z + \vartheta z f'(z)} \prec \Phi_\ell^\lambda(z), \tag{11}$$

and

$$\frac{g(w)}{(1-\vartheta)w + \vartheta w g'(w)} \prec \Phi_\ell^\lambda(w) \tag{12}$$

where $z, w \in \Delta$, Φ_ℓ^λ is given by (6), and $g = f^{-1}$ is given by (10).

By specializing the parameter ϑ, we state a new subclass of Yamakawa-type bi-starlike in the open unit disk, associated with Gegenbauer polynomials as below:

Remark 1. *For $\vartheta = 1$, we get $\mathcal{YS}^*_\Sigma(\Phi^\lambda_\ell) := \mathcal{GY}_\Sigma(1, \Phi^\lambda_\ell)$, thus $f \in \mathcal{YS}^*_\Sigma(\Phi^\lambda_\ell)$ if $f \in \Sigma$ and the following subordinations hold:*

$$\frac{f(z)}{zf'(z)} \prec \Phi^\lambda_\ell(z) \quad \text{and} \quad \frac{g(w)}{wg'(w)} \prec \Phi^\lambda_\ell(w)$$

where $z, w \in \Delta$, and $g = f^{-1}$ is given by (10).

Remark 2. *For $\vartheta = 0$, we get $\mathcal{N}_\Sigma(\Phi^\lambda_\ell) := \mathcal{GY}_\Sigma(0, \Phi^\lambda_\ell)$, thus $f \in \mathcal{N}_\Sigma(\Phi^\lambda_\ell)$ if $f \in \Sigma$ and the following subordinations hold:*

$$\frac{f(z)}{z} \prec \Phi^\lambda_\ell(z) \quad \text{and} \quad \frac{g'(w)}{w} \prec \Phi^\lambda_\ell(w)$$

where $z, w \in \Delta$ and $g = f^{-1}$ is given by (10).

Note that if in the above Remarks 1 and 2, we choose $\lambda = 1$ or $\lambda = \frac{1}{2}$, then we can state the new subclasses of $\mathcal{YS}^*_\Sigma(\Phi^\lambda_\ell)$ and $\mathcal{N}_\Sigma(\Phi^\lambda_\ell)$ related with Chebyshev polynomials and Legendre polynomials, respectively.

2. Initial Taylor Coefficients Estimates for the Functions of $\mathcal{GY}_\Sigma\left(\vartheta, \Phi^\lambda_\ell\right)$

To obtain our first results, we need the following lemma:

Lemma 1 ([33], p. 172)**.** *Assume that $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$, $z \in \mathbb{U}$, is an analytic function in $\mathbb{U}$ such that $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. Then,*

$$|\omega_1| \le 1, \quad |\omega_n| \le 1 - |\omega_1|^2, \ n = 2, 3, \dots.$$

In the next result, we obtain the upper bounds for the modules of the first two coefficients for the functions that belong to the class $\mathcal{GY}_\Sigma(\vartheta, \Phi^\lambda_\ell)$.

Theorem 1. *Let f given by (9) be in the class $\mathcal{GY}_\Sigma(\vartheta, \Phi^\lambda_\ell)$. Then,*

$$|a_2| \le \frac{2\lambda\ell\sqrt{2\lambda\ell}}{\sqrt{\left|(1 - 6\vartheta + 6\vartheta^2)4\lambda^2\ell^2 - 2\big(2(\lambda)_2\ell^2 - \lambda\big)(1 - 2\vartheta)^2\right|}}, \tag{13}$$

and

$$|a_3| \le \frac{2(\lambda\ell)^2(1 - 2\vartheta - 2\vartheta^2)}{|(1 - 3\vartheta)(1 - 2\vartheta)^2|} + \frac{2\lambda\ell}{|1 - 3\vartheta|}, \tag{14}$$

where $\vartheta \neq \frac{1}{3}$.

Proof. Let $f \in \mathcal{GY}_\Sigma(\vartheta, \Phi^\lambda_\ell)$ and $g = f^{-1}$. From the definition in Formulas (11) and (12), we have

$$\frac{f(z)}{(1 - \vartheta)z + \vartheta z f'(z)} = \Phi^\lambda_\ell(u(z)) \tag{15}$$

and

$$\frac{g(w)}{(1 - \vartheta)w + \vartheta w g'(w)} = \Phi^\lambda_\ell(v(w)), \tag{16}$$

where the functions u and v are of the form

$$u(z) = c_1 z + c_2 z^2 + \ldots, \tag{17}$$

and

$$v(w) = d_1 w + d_2 w^2 + \ldots, \tag{18}$$

are analytic in Δ with $u(0) = 0 = v(0)$, and $|u(z)| < 1$, $|v(w)| < 1$, for all $z, w \in \Delta$. From Lemma 1 it follows that

$$|c_j| \leq 1 \quad \text{and} \quad |d_j| \leq 1, \text{ for all } j \in \mathbb{N}. \tag{19}$$

Replacing (17) and (18) in (15) and (16), respectively, we have

$$\frac{f(z)}{(1-\vartheta)z + \vartheta z f'(z)} = 1 + \mathfrak{G}_1^{\lambda}(\ell)u(z) + \mathfrak{G}_2^{\lambda}(\ell)u^2(z) + \ldots, \tag{20}$$

and

$$\frac{g(w)}{(1-\vartheta)w + \vartheta w g'(w)} = 1 + \mathfrak{G}_1^{\lambda}(\ell)v(w) + \mathfrak{G}_2^{\lambda}(\ell)v^2(w) + \ldots. \tag{21}$$

In view of (9) and (10), from (20) and (21), we obtain

$$\begin{aligned} 1 + (1-2\vartheta)a_2 z + \Big[(1-3\vartheta)a_3 - 2\vartheta(1-2\vartheta)a_2^2\Big]z^2 + \ldots \\ = 1 + \mathfrak{G}_1^{\lambda}(\ell)c_1 z + \Big[\mathfrak{G}_1^{\lambda}(\ell)c_2 + \mathfrak{G}_2^{\lambda}(\ell)c_1^2\Big]z^2 + \ldots, \end{aligned}$$

and

$$\begin{aligned} 1 - (1-2\vartheta)(\alpha)a_2 w + \Big\{\Big(1 - 4\vartheta + 2\vartheta^2\Big)a_2^2 - (1-3\lambda)a_3\Big\}w^2 + \ldots \\ = 1 + \mathfrak{G}_1^{\lambda}(\ell)d_1 w + \Big[\mathfrak{G}_1^{\lambda}(\ell)d_2 + \mathfrak{G}_2^{\lambda}(\ell)d_1^2\Big]w^2 + \ldots, \end{aligned}$$

which yields the following relations:

$$(1-2\vartheta)a_2 = \mathfrak{G}_1^{\lambda}(\ell)c_1, \tag{22}$$

$$(1-3\vartheta)a_3 - 2\vartheta(1-2\vartheta)a_2^2 = \mathfrak{G}_1^{\lambda}(\ell)c_2 + \mathfrak{G}_2^{\lambda}(\ell)c_1^2, \tag{23}$$

and

$$-(1-2\vartheta)a_2 = \mathfrak{G}_1^{\lambda}(\ell)d_1, \tag{24}$$

$$-(1-3\vartheta)a_3 + \Big(1 - 4\vartheta + 2\vartheta^2\Big)a_2^2 = \mathfrak{G}_1^{\lambda}(\ell)d_2 + \mathfrak{G}_2^{\lambda}(\ell)d_1^2. \tag{25}$$

From (22) and (24), it follows that

$$c_1 = -d_1, \tag{26}$$

and

$$2(1-2\vartheta)^2 a_2^2 = [\mathfrak{G}_1^{\lambda}(\ell)]^2(c_1^2 + d_1^2),$$

$$a_2^2 = \frac{[\mathfrak{G}_1^{\lambda}(\ell)]^2}{2(1-2\vartheta)^2}(c_1^2 + d_1^2) \tag{27}$$

Adding (23) and (25), using (27), we obtain

$$a_2^2 = \frac{[\mathfrak{G}_1^{\lambda}(\ell)]^3(c_2 + d_2)}{(1 - 6\vartheta + 6\vartheta^2)[\mathfrak{G}_1^{\lambda}(\ell)]^2 - 2(1-2\vartheta)^2\mathfrak{G}_2^{\lambda}(\ell)}. \tag{28}$$

Applying (19) for the coefficients c_2 and d_2 and using (8), we obtain the Inequality (13).

By subtracting (25) from (23), using (26) and (27), we get

$$a_3 = \frac{\mathfrak{G}_1^{\lambda}(\ell)(c_2 - d_2)}{2(1-3\vartheta)} + \frac{(1-2\vartheta-2\vartheta^2)[\mathfrak{G}_1^{\lambda}(\ell)]^2}{2(1-3\vartheta)} a_2^2 \quad (29)$$
$$= \frac{(1-2\vartheta-2\vartheta^2)[\mathfrak{G}_1^{\lambda}(\ell)]^2(c_1^2+d_1^2)}{4(1-3\vartheta)(1-2\vartheta)^2} + \frac{\mathfrak{G}_1^{\lambda}(\ell)(c_2 - d_2)}{2(1-3\vartheta)}.$$

Using (8) and once again applying (19) for the coefficients c_1, c_2, d_1, and d_2, we deduce the required Inequality (14). □

By taking $\vartheta = 0$ or $\vartheta = 1$ and $\ell \in (0,1)$, one can easily state the upper bounds for $|a_2|$ and $|a_3|$ for the function classes $\mathcal{GY}_{\Sigma}(0,\Phi) =: \mathcal{N}_{\Sigma}(\Phi_{\ell}^{\lambda})$ and $\mathcal{GY}_{\Sigma}(1,\Phi) =: \mathcal{YS}_{\Sigma}^{*}(\Phi_{\ell}^{\lambda})$, respectively, as follows:

Remark 3. *Let f given by (9) be in the class $\mathcal{N}_{\Sigma}(\Phi_{\ell}^{\lambda})$. Then,*

$$|a_2| \leq \frac{2\lambda\ell\sqrt{2\lambda\ell}}{\sqrt{|4\lambda^2\ell^2 - 2(2(\lambda)_2\ell^2 - \lambda)|}},$$

and

$$|a_3| \leq 2(\lambda\ell)^2 + 2\lambda\ell.$$

Remark 4. *Let f given by (9) be in the class $\mathcal{YS}_{\Sigma}^{*}(\Phi_{\ell}^{\lambda})$. Then,*

$$|a_2| \leq \frac{2\lambda\ell\sqrt{2\lambda\ell}}{\sqrt{|4\lambda^2\ell^2 - 2(2(\lambda)_2\ell^2 - \lambda)|}},$$

and

$$|a_3| \leq 3(\lambda\ell)^2 + \lambda\ell.$$

Remark 5. *Let f given by (9) be in the class $\mathcal{GY}_{\Sigma}^{*}(\vartheta,\Phi_{\ell}^{1})$. Then,*

$$|a_2| \leq \frac{2\ell\sqrt{2\ell}}{\sqrt{|(1-6\vartheta+6\vartheta^2)4\ell^2 - 2(4\ell^2-1)(1-2\vartheta)^2|}},$$

and

$$|a_3| \leq \frac{2\ell^2(1-2\vartheta-2\vartheta^2)}{|(1-3\vartheta)(1-2\vartheta)^2|} + \frac{2\ell}{|1-3\vartheta|},$$

where $\vartheta \neq \frac{1}{3}$.

Remark 6. *Let f given by (9) be in the class $\mathcal{GY}_{\Sigma}^{*}\left(\vartheta,\Phi_{\ell}^{1/2}\right)$. Then, for $\ell \neq \frac{1}{\sqrt{2}}$,*

$$|a_2| \leq \frac{\ell\sqrt{\ell}}{\sqrt{|(1-6\vartheta+6\vartheta^2)\ell^2 - (3\ell^2-1)(1-2\vartheta)^2|}},$$

and

$$|a_3| \leq \frac{\ell^2(1-2\vartheta-2\vartheta^2)}{2|(1-3\vartheta)(1-2\vartheta)^2|} + \frac{\ell}{|1-3\vartheta|},$$

where $\vartheta \neq \frac{1}{3}$.

In the above Remarks 3 and 4, by fixing $\lambda = 1$ and $\lambda = \frac{1}{2}$, we obtain the new estimates of $|a_2|$ and $|a_3|$ for the function classes $\mathcal{YS}^*_\Sigma(\Phi^\lambda_\ell)$ and $\mathcal{N}_\Sigma(\Phi^\lambda_\ell)$ related with Chebyshev polynomials and Legendre polynomials, respectively.

3. Fekete–Szegő Inequality for the Function Class $\mathcal{GY}_\Sigma\left(\vartheta, \Phi^\lambda_\ell\right)$

Due to the result of Zaprawa [34], in this section, we obtain the Fekete–Szegő inequality for the function classes $\mathcal{GY}_\Sigma(\vartheta, \Phi^\lambda_\ell)$.

Theorem 2. *Let f given by (9) be in the class $\mathcal{GY}_\Sigma(\vartheta, \Phi^\lambda_\ell)$, and $\mu \in \mathbb{R}$. Then, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{2\lambda\ell}{|1-3\vartheta|}, & \textit{if} \quad |h(\mu)| \leq \dfrac{1}{2|1-3\vartheta|}, \\ 4\lambda\ell|h(\mu)|, & \textit{if} \quad |h(\mu)| \geq \dfrac{1}{2|1-3\vartheta|}, \end{cases}$$

where

$$h(\mu) := \frac{2\lambda\ell^2\left[2\lambda^2\ell^2(1-2\vartheta-2\vartheta^2) - \mu(1-3\vartheta)\right]}{(1-3\vartheta)\left\{2\lambda\ell^2(1-6\vartheta+6\vartheta^2) - (1-2\vartheta)^2[2(\lambda+1)\ell^2-1]\right\}},$$

and $\vartheta \neq \frac{1}{3}$.

Proof. If $f \in \mathcal{GY}_\Sigma(\vartheta, \Phi^\lambda_\ell)$ is given by (9), from (28) and (29), we have

$$\begin{aligned} a_3 - \mu a_2^2 =& \frac{\mathfrak{G}^\lambda_1(\ell)(c_2-d_2)}{2(1-3\vartheta)} + \left(\frac{(1-2\vartheta-2\vartheta^2)[\mathfrak{G}^\lambda_1(\ell)]^2}{2(1-3\vartheta)} - \mu\right)a_2^2 \\ =& \frac{\mathfrak{G}^\lambda_1(\ell)(c_2-d_2)}{2(1-3\vartheta)} + \left(\frac{(1-2\vartheta-2\vartheta^2)[\mathfrak{G}^\lambda_1(\ell)]^2}{2(1-3\vartheta)} - \mu\right) \\ &\times \frac{[\mathfrak{G}^\lambda_1(\ell)]^3(c_2+d_2)}{(1-6\vartheta+6\vartheta^2)[\mathfrak{G}^\lambda_1(\ell)]^2 - 2(1-2\vartheta)^2\mathfrak{G}^\lambda_2(\ell)} \\ =& \mathfrak{G}^\lambda_1(\ell)\left[\left(h(\mu) + \frac{1}{2(1-3\vartheta)}\right)c_2 + \left(h(\mu) - \frac{1}{2(1-3\vartheta)}\right)d_2\right], \end{aligned}$$

where

$$h(\mu) = \frac{((1-2\vartheta-2\vartheta^2)[\mathfrak{G}^\lambda_1(\ell)]^2 - 2\mu(1-3\vartheta))[\mathfrak{G}^\lambda_1(\ell)]^3}{2(1-3\vartheta)\{(1-6\vartheta+6\vartheta^2)[\mathfrak{G}^\lambda_1(\ell)]^2 - 2(1-2\vartheta)^2\mathfrak{G}^\lambda_2(\ell)\}}.$$

Now, by using (8)

$$a_3 - \mu a_2^2 = 2\lambda\ell\left[\left(h(\mu) + \frac{1}{2(1-3\vartheta)}\right)c_2 + \left(h(\mu) - \frac{1}{2(1-3\vartheta)}\right)d_2\right],$$

where

$$\begin{aligned} h(\mu) &= \frac{2\lambda^2\ell^2\left[2\lambda^2\ell^2(1-2\vartheta-2\vartheta^2) - \mu(1-3\vartheta)\right]}{(1-3\vartheta)\left\{2\lambda^2\ell^2(1-2\vartheta+2\vartheta^2) - \lambda(1-2\vartheta)^2[2(\lambda+1)\ell^2-1]\right\}} \\ &= \frac{2\lambda\ell^2\left[2\lambda^2\ell^2(1-2\vartheta-2\vartheta^2) - \mu(1-3\vartheta)\right]}{(1-3\vartheta)\left\{2\lambda\ell^2(1-6\vartheta+6\vartheta^2) - (1-2\vartheta)^2[2(\lambda+1)\ell^2-1]\right\}} \end{aligned}$$

Therefore, in view of (8) and (19), we conclude that the required inequality holds. □

4. The Subclass $\mathfrak{M}_{\Sigma}\left(\tau, \Phi_{\ell}^{\lambda}\right)$ of Bi-Univalent Functions

In [35] Obradović et al. gave some criteria for univalence expressed by $\operatorname{Re} f'(z) > 0$ for the linear combination

$$\tau\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\tau)\frac{1}{f'(z)},\ \tau \geq 1,\ z \in \Delta.$$

Based on the above definitions, recently, Lashin [36] introduced and studied new subclasses of the bi-univalent function. In our further discussions, unless otherwise stated, we let $\tau \geq 1$, $\lambda > \frac{1}{2}$, and $\ell \in \left(\frac{1}{2}, 1\right]$.

Definition 2. *A function $f \in \Sigma$ given by (9) is said to be in the class $\mathfrak{M}_{\Sigma}(\tau, \Phi_{\ell}^{\lambda})$ if it satisfies the conditions*

$$\tau\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\tau)\frac{1}{f'(z)} \prec \Phi_{\ell}^{\lambda}(z) \tag{30}$$

and

$$\tau\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\tau)\frac{1}{g'(w)} \prec \Phi_{\ell}^{\lambda}(w) \tag{31}$$

where $\tau \geq 1$, $z, w \in \Delta$, Φ_{ℓ}^{λ} is given by (6), and the function $g = f^{-1}$ is given by (10).

Remark 7. *For the particular case $\tau = 1$, a function $f \in \Sigma$ given by (9) is said to be in the class $\mathfrak{M}_{\Sigma}(\Phi_{\ell}^{\lambda}) =: \mathfrak{K}_{\Sigma}(\Phi_{\ell}^{\lambda})$ if it satisfies the subordination relations*

$$1+\frac{zf''(z)}{f'(z)} \prec \Phi_{\ell}^{\lambda}(z) \quad \textit{and} \quad 1+\frac{wg''(w)}{g'(w)} \prec \Phi_{\ell}^{\lambda}(w),$$

$z, w \in \Delta$, Φ_{ℓ}^{λ} is given by (6), and $g = f^{-1}$ is given by (10).

Theorem 3. *Let f be given by (9) and $f \in \mathfrak{M}_{\Sigma}(\tau, \Phi_{\ell}^{\lambda})$, with $\tau \geq 1$. Then,*

$$|a_2| \leq \min\left\{\frac{\lambda\ell}{2(2\tau-1)};\ \frac{\lambda\ell\sqrt{2\lambda\ell}}{2\sqrt{\left|(1+\tau)\lambda^2\ell^2-4(2\tau-1)^2[2\ell^2(\lambda)_2-\lambda]\right|}}\right\}, \tag{32}$$

and

$$|a_3| \leq \min\left\{\frac{2\lambda\ell}{3(3\tau-1)}+\frac{\lambda^2\ell^2}{4(2\tau-1)^2};\ \frac{2\lambda\ell}{3(3\tau-1)}+\frac{2\lambda^3\ell^3}{\left|(1+\tau)\lambda^2\ell)_1^2-(2\tau-1)^2[2\ell^2(\lambda)_2-\lambda]\right|}\right\}.$$

Proof. $f \in \mathfrak{M}_{\Sigma}(\tau, \Phi_{\ell}^{\lambda})$, from (30) and (31) it follows that

$$\tau\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\tau)\frac{1}{f'(z)} = \Phi_{\ell}^{\lambda}(u(z)), \tag{33}$$

and

$$\tau\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\tau)\frac{1}{g'(w)} = \Phi_{\ell}^{\lambda}(v(w)), \tag{34}$$

where the functions u and v are analytic in Δ with $u(0) = 0 = v(0)$, such that $|u(z)| < 1$, $|v(w)| < 1$, for all $z, w \in \Delta$, and are of the form (17) and (18), respectively.

From (33) and (34), we have

$$1+2(2\tau-1)a_2 z+\Big[3(3\tau-1)a_3+4(1-2\tau)_2 a_2^2\Big]z^2+\dots$$
$$=1+\mathfrak{G}_1^{\lambda}(\ell)c_1 z+\Big[\mathfrak{G}_1^{\lambda}(\ell)c_2+\mathfrak{G}_2^{\lambda}(\ell)c_1^2\Big]z^2+\dots,$$

and

$$1-2(2\tau-1)a_2 w+\Big[2(5\tau-1)a_2^2-3(3\tau-1)a_3\Big]w^2-\dots$$
$$=1+\mathfrak{G}_1^{\lambda}(\ell)d_1 w+\Big[\mathfrak{G}_1^{\lambda}(\ell)d_2+\mathfrak{G}_2^{\lambda}(\ell)d_1^2\Big]w^2+\dots,$$

and equating the coefficients of the above two relations, we get

$$2(2\tau-1)a_2=\mathfrak{G}_1^{\lambda}(\ell)c_1, \tag{35}$$
$$3(3\tau-1)a_3+4(1-2\tau)a_2^2=\mathfrak{G}_1^{\lambda}(\ell)c_2+\mathfrak{G}_2^{\lambda}(\ell)c_1^2, \tag{36}$$

and

$$-2(2\tau-1)a_2=\mathfrak{G}_1^{\lambda}(\ell)d_1, \tag{37}$$
$$2(5\tau-1)a_2^2-3(3\tau-1)a_3=\mathfrak{G}_1^{\lambda}(\ell)d_2+\mathfrak{G}_2^{\lambda}(\ell)d_1^2. \tag{38}$$

From (35) and (37), we get

$$p_1=-q_1 \tag{39}$$

From (35), by using the Inequality (19) for the coefficients c_j and d_j, from (8), we have

$$|a_2|\le\frac{\mathfrak{G}_1^{\lambda}(\ell)}{2(2\tau-1)}=\frac{\lambda\ell}{(2\tau-1)}.$$

Furthermore,

$$8(2\tau-1)^2a_2^2=\Big(\mathfrak{G}_1^{\lambda}(\ell)\Big)^2\Big(c_1^2+d_1^2\Big),$$

that is,

$$a_2^2=\frac{\big(\mathfrak{G}_1^{\lambda}(\ell)\big)^2\big(c_1^2+d_1^2\big)}{8(2\tau-1)^2}. \tag{40}$$

Thus, from the Inequality (19) and using (8), we obtain

$$|a_2|\le\frac{\mathfrak{G}_1^{\lambda}(\ell)}{4(2\tau-1)}=\frac{\lambda\ell}{2(2\tau-1)}. \tag{41}$$

Now, from (36), (38) and using (40), we get

$$\big[2(1+\tau)\Big(\mathfrak{G}_1^{\lambda}(\ell)\Big)^2-8(2\tau-1)^2\mathfrak{G}_2^{\lambda}(\ell)\big]a_2^2=\Big(\mathfrak{G}_1^{\lambda}(\ell)\Big)^3(c_2+d_2). \tag{42}$$

Thus, according to (42), we obtain

$$a_2^2=\frac{\big(\mathfrak{G}_1^{\lambda}(\ell)\big)^3(c_2+d_2)}{2(1+\tau)\big(\mathfrak{G}_1^{\lambda}(\ell)\big)^2-8(2\tau-1)^2\mathfrak{G}_2^{\lambda}(\ell)},$$

hence,

$$|a_2|\le\frac{\lambda\ell\sqrt{2\lambda\ell}}{2\sqrt{\big|(1+\tau)\lambda^2\ell^2-4(2\tau-1)^2[2\ell^2(\lambda)_2-\lambda]\big|}}, \tag{43}$$

and the Inequality (32) is proved.

From (36), (38) and using (39), we get

$$a_3 = \frac{\mathfrak{G}_1^{\lambda}(\ell)(c_2 - d_2)}{6(3\tau - 1)} + a_2^2, \tag{44}$$

which implies

$$|a_3| \leq \frac{2\lambda\ell}{3(3\tau - 1)} + |a_2^2|. \tag{45}$$

From this inequality, using (41), we obtain

$$|a_3| \leq \frac{2\lambda\ell}{3(3\tau - 1)} + \frac{\lambda^2\ell^2}{4(2\tau - 1)^2}.$$

Combining (45) and (43), it follows that

$$|a_3| \leq \frac{2\lambda\ell}{3(3\tau - 1)} + \frac{2\lambda^3\ell^3}{\left|(1+\tau)\lambda^2\ell)_1^2 - (2\tau - 1)^2[2\ell^2(\lambda)_2 - \lambda]\right|}.$$

□

Motivated by the result of Zaprawa [34], we discuss the Fekete–Szegő inequality [37] for the functions $f \in \mathfrak{M}_{\Sigma}\left(\tau, \Phi_{\ell}^{\lambda}\right)$.

Theorem 4. *For $\nu \in \mathbb{R}$, let $f \in \mathfrak{M}_{\Sigma}\left(\tau, \Phi_{\ell}^{\lambda}\right)$ be given by (9). Then,*

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{2\lambda\ell}{3(3\tau - 1)}, & \text{if} \quad |h(\nu)| \leq \dfrac{1}{6(3\tau - 1)}, \\ 4|h(\nu)|, & \text{if} \quad |h(\nu)| \geq \dfrac{1}{6(3\tau - 1)}, \end{cases}$$

where

$$h(\nu) = \frac{(1-\nu)\lambda\ell^2}{4\left\{(1+\tau)\lambda\ell^2 - (2\tau - 1)^2[2\ell^2(\lambda + 1) - 1]\right\}}. \tag{46}$$

Proof. If $f \in \mathfrak{M}_{\Sigma}\left(\tau, \Phi_{\ell}^{\lambda}\right)$ be given by (9), from (44) we have

$$a_3 - \nu a_2^2 = \frac{\mathfrak{G}_1^{\lambda}(\ell)(c_2 - d_2)}{6(3\tau - 1)} + (1-\nu)a_2^2. \tag{47}$$

By substituting (42) in (47), we obtain

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{\mathfrak{G}_1^{\lambda}(\ell)(c_2 - d_2)}{6(3\tau - 1)} + \frac{(1-\nu)\left(\mathfrak{G}_1^{\lambda}(\ell)\right)^3(c_2 + d_2)}{2(1+\tau)\left(\mathfrak{G}_1^{\lambda}(\ell)\right)^2 - 8(2\tau - 1)^2\mathfrak{G}_2^{\lambda}(\ell)} \\ &= \mathfrak{G}_1^{\lambda}(\ell)\left[\left(h(\nu) + \frac{1}{6(3\tau - 1)}\right)c_2 + \left(h(\nu) - \frac{1}{6(3\tau - 1)}\right)d_2\right], \end{aligned}$$

where

$$h(\nu) = \frac{(1-\nu)\left(\mathfrak{G}_1^{\lambda}(\ell)\right)^2}{2(1+\tau)\left(\mathfrak{G}_1^{\lambda}(\ell)\right)^2 - 8(2\tau - 1)^2\mathfrak{G}_2^{\lambda}(\ell)}.$$

From (8), it follows

$$a_3 - \nu a_2^2 = 2\lambda\ell\left[\left(h(\nu) + \frac{1}{6(3\tau - 1)}\right)c_2 + \left(h(\nu) - \frac{1}{6(3\tau - 1)}\right)d_2\right], \tag{48}$$

where the function h is given by (46). Hence, by using the triangle inequality for the modulus of (48) together with (19), we get our result. □

For $\nu = 1$ the above theorem reduces to the following special case:

Remark 8. *If* $f \in \mathfrak{M}_{\Sigma}\left(\tau, \Phi_{\ell}^{\lambda}\right)$ *is given by* (9), *then*

$$\left|a_3 - a_2^2\right| \leq \frac{2\lambda\ell}{3(3\tau - 1)}.$$

5. Conclusions

Yamakawa-type bi-starlike functions related with the Gegenbauer polynomials are defined for the first time, and initial Taylor coefficients and Fekete–Szegő inequality are obtained. Further, by fixing $\lambda = 1$ or $\lambda = \frac{1}{2}$, the Gegenbauer polynomials lead to the Chebyshev polynomials and the Legendre polynomials, respectively. Hence, our results represent a new study of the Yamakawa family of bi-starlike functions associated with Chebyshev and Legendre polynomials, which are also not considered in the literature. We have left this as an exercise to interested readers.

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