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QM-BZ-Algebras and Quasi-Hyper BZ-Algebras

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Abstract: BZ-algebra, as the common generalization of BCI-algebra and BCC-algebra, is a kind of important logic algebra. Herein, the new concepts of QM-BZ-algebra and quasi-hyper BZ-algebra are proposed and their structures and constructions are studied. First, the definition of QM-BZ-algebra is presented, and the structure of QM-BZ-algebra is obtained: Each QM-BZ-algebra is KG-union of quasi-alter BCK-algebra and anti-grouped BZ-algebra. Second, the new concepts of generalized quasi-left alter (hyper) BZ-algebras and QM-hyper BZ-algebra are introduced, and some characterizations of them are investigated. Third, the definition of quasi-hyper BZ-algebra is proposed, and the relationships among BZ-algebra, hyper BZ-algebra, quasi-hyper BCI-algebra, and quasi-hyper BZ-algebra are discussed. Finally, several special classes of quasi-hyper BZ-algebras are studied in depth and the following important results are proved: (1) an anti-grouped quasi-hyper BZ-algebra is an anti-grouped BZ-algebra; (2) every generalized anti-grouped quasi-hyper BZ-algebra corresponds to a semihypergroup.

Keywords: BCI-algebra; BZ-algebra; QM-BZ-algebra; quasi-hyper BZ-algebra; anti-grouped BZ-algebra

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1. Introduction

BCI-algebra as well as BCK-algebra are two kinds of algebraic structures closely related to combinatorial logic and fuzzy logic, which have been studied extensively and deeply (see [1–10]). As a extension of BCI-algebra, pseudo-BCI-algebra was proposed in 2008 by Wieslaw A. Dudek and Young Bae Jun (see [11]). Then, X.Y. Wu and X.H. Zhang introduced a quasi-maximal element into pseudo-BCI algebra and studied the structure of QM-pseudo-BCI algebra (see [12]). Dually, X.H. Zhang and Y.D. Du proposed introducing a quasi-minimal element into BCI-algebra and proved the adjoint semigroup of QM-BCI-algebra is a commutative Clifford semigroup in [13]. As another generalization of BCI-algebra, BZ-algebra was first proposed by Ye (see [14]). Naturally, we introduce a quasi-minimal element into BZ-algebra and study the structure theorem of QM-BZ-algebra.

In 1934, the definition of hyperstructure (also called multialgebra) was proposed by F. Marty (see [15]), and hyperstructures have been used widely in pure and applied sciences (see [16–19]). Naturally, the idea of hyperstructures is also applied to the study of non-classical logic algebras. In 2000, Young Bae Jun et al. proposed the concept of hyper BCK-algebra, and investigated hyper BCK-ideals and some related hyper algebras, such as hyper K-algebra and hyper MV-algebra (see [20–25]). In 2006, Jun and Borzooei et al. independently proposed the new concept of hyper BCC-algebra; also in 2006, Xin introduced hyper BCI-algebra, and since then, many research papers on hyper logical algebras have emerged (see [26–33]). In 2021, Y.D. Du and X.H. Zhang introduced the hyper structure into BZ-algebra and discussed the relationships between hyper BZ-algebra and semihypergroups (see [34]). In this paper, they gave the concepts of anti-grouped hyper BZ-algberas as well as generalized anti-grouped hyper BZ-algebras and discussed the connection between them and BZ-algebras. Additionally, in [13], X.H. Zhang and Y.D. Du proposed quasi-hyper BCI-algebra and discussed relationships among quasi-hyper BCI-algebra, H_v-groups and hyper groups. Similarly, we present the definition of quasi-hyper BZ-algebra, which is an extension of hyper BZ-algebra and quasi-hyper BCI-algebra.

The arrangement of the whole paper is as below. In Section 2, we give the definitions and properties of logical algebras and some related hyper structures. In Section 3, we firstly introduce a method to construct a BZ-algebra by using BCC-algebra and anti-grouped BZ-algebra. Then, QM-BZ-algebra is defined and the relationships among QM-BZ-algebra, QM-hyper BZ-algebra, and generalized quasi-left alter (hyper) BZ-algebra are discussed. In Section 4, we introduce quasi-hyper BZ-algebra and study its properties, discussing the relationships among BZ-algebra, hyper BZ-algebra, quasi-hyper BCI-algebra, and quasi-hyper BZ-algebra. Moreover, we investigate some kinds of quasi-hyper BZ-subalgebra and propose the definition of anti-grouped quasi-hyper BZ-algebra, generalized anti-grouped quasi-hyper BZ-algebra, and associative quasi-hyper BZ-algebra, and discuss their relationships. Finally, we introduce QM-quasi-hyper BZ-algebra and generalized quasi-left alter quasi-hyper BZ-algebra.

2. Preliminaries

Firstly, we give some concepts of some logical algebras and connections between BZ-algebra and BCI-algebra that play an important role in discussions of special BZ-algebra. In $BCK/BCI/BCC/BZ$-algebra, define $\leq$: $x \leq y$ iff $x * y = 0$.

Definition 1 ([1,2])**.** *Assume that* $<X; *, 0>$ *is an algebraic structure; then, it is a BCI-algebra if it meets:* $\forall x, y, z \in X$,
(1) $((x * z) * (y * z)) * (x * y) = 0$;
(2) $(x * (x * y)) * y = 0$;
(3) $x * 0 = x$;
(4) $x * y = 0$ *and* $y * x = 0 \Rightarrow x = y$.
If a BCI-algebra meets the below condition: $\forall x \in X$,
(5) $0 * x = 0$,
we call it a BCK-algebra.

Definition 2 ([10])**.** *Assume that* $<X; *, 0>$ *is an algebraic structure; then, it is a BCC-algebra if it meets:* $\forall x, y, z \in X$,
(1) $((x * z) * (y * z)) * (x * y) = 0$;
(2) $x * 0 = x$;
(3) $x * x = 0$;
(4) $0 * x = 0$;
(5) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

Definition 3 ([14])**.** *Assume that* $<X; *, 0>$ *is an algebraic structure; then, it is a BZ-algebra if it meets:* $\forall x, y, z \in X$,
(1) $((x * z) * (y * z)) * (x * y) = 0$;
(2) $x * 0 = x$;
(3) $x * y = 0$, $y * x = 0$ *imply* $x = y$.

An algebra of type (2,0) means that this algebra is composed of a non-empty set and an operation as well as a constant. Clearly, all the above algebras are algebras of type (2,0).

Definition 4 ([9])**.** *An algebraic structure* $<X; *, 0>$ *is a quasi-alter BCK-algebra iff it meets:* $\forall x, y \in X$, *if* $x = y$, $x * y = 0$, *otherwise*, $x * y = x$.

Theorem 1 ([35])**.** *Assume that* $<X; *, 0>$ *is a* BZ*-algebra; it is a* BCI*-algebra iff* $\forall x, y \in X$,

$$x * (x * y) \leq y.$$

In paper [36], Xiaohong Zhang and Ruifen Ye introduced a kind of BZ-algebra and revealed the connection between BZ-algebra and a general group.

Definition 5 ([36])**.** *A BZ-algebra* $<X;*,0>$ *is called an anti-grouped BZ-algebra if, for any* $x \in X$*, it satisfies* $0*(0*x) = x$.

Theorem 2 ([36])**.** *A BZ-algebra* $<X;*,0>$ *is anti-grouped iff it meets:*

$$(x*y)*(z*y) = x*z, \forall x,y,z \in X.$$

Theorem 3 ([36])**.** *Let* $<X;*,0>$ *be an anti-grouped BZ-algebra. Define* $"\circledast"$:

$$x \circledast y = x*(0*y), \forall x,y \in X.$$

Then, $<X;\circledast,0>$ *is a group.*

Theorem 4 ([36])**.** *Assume that* $<G;\circ,e>$ *is a group. Define* $"\cdot"$:

$$x \cdot y = x \circ y^{-1}, \forall x,y \in G.$$

Then, $<G;\cdot,e>$ *is an anti-grouped BZ-algebra.*

In the following, we give the definitions of semihypergroup and some hyper logical algebras.

Definition 6 ([16])**.** *Assume that* $(H,\circ)$ *is a hypergroupoid. If* $\forall x,y,z \in H$*, we have* $(x \circ y) \circ z = x \circ (y \circ z)$*, so* $(H,\circ)$ *is a semihypergroup. Then, there is*

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

Note that, if $(H,\circ)$ *is a semihypergroup, there is* $(A \circ B) \circ C = A \circ (B \circ C)$ *for all* $A,B,C \in P^*(H)$*, where* $P^*(H)$ *represents nonempty subset of* H.

Definition 7 ([16])**.** *Assume that* $(H,\circ)$ *is a semihypergroup.* $(H,\circ)$ *is called a hypergroup if* $(\forall a \in H)\ a \circ H = H \circ a = H$.

In a study of hyperstructures, $a \ll b$ represents $0 \in a \circ b$. For each $S,B \subseteq H$, $S \ll B$ represents that, for all $s \in S$, there is $b \in B$, s.t. $s \ll b$.

Definition 8 ([20])**.** *Assume that* $(H,\circ)$ *is a hypergroupoid containing 0. If it meets these axioms:* $\forall x,y,z \in H$,
$(HBK1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$;
$(HBK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$;
$(HBK3)$ $x \circ H \ll x$;
$(HBK4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$,
then it is a hyper BCK-algebra.

Definition 9 ([26])**.** *Assume that* $(H,\circ)$ *is a hypergroupoid containing 0. If it meets these axioms:* $\forall x,y,z \in H$,
$(HBC1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$;
$(HBC2)$ $x \ll x$;
$(HBC3)$ $x \circ y \ll x$;
$(HBC4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$,
then it is a hyper BCC-algebra (Jun's definition).

Definition 10 ([27]). *Assume that* $(H, \circ)$ *is a hypergroupoid containing 0. If it meets these axioms:* $\forall x, y, z \in H$,
$(HC1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$;
$(HBC2)$ $0 \circ x = \{0\}$;
$(HBC3)$ $x \circ 0 = \{x\}$;
$(HBC4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$,
then it is a hyper BCC-algebra (Borzooei's definition).

Jun and Borzooei, in [26,27], gave different definitions of hyper BCC-algebra, respectively. However, in this paper, we mainly use Definition 10 for hyper BCC-algebra.

Definition 11 ([28]). *Assume that* $(H, \circ)$ *is a hypergroupoid containing 0. If it meets these axioms:* $\forall x, y, z \in H$,
$(HBK1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$;
$(HBK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$;
$(HBI3)$ $x \ll x$;
$(HBK4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$;
$(HBI5)$ $0 \circ (0 \circ x) \ll x$,
then it is a hyper BCI*-algebra.*

Definition 12 ([13]). *Assume that* $(H, \circ)$ *is a hypergroupoid containing 0. If it meets these axioms:* $\forall x, y, z \in H$,
$(QHCI1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$;
$(QHCI2)$ $(x \circ y) \circ z = (x \circ z) \circ y$;
$(QHCI3)$ $x \ll x$;
$(QHCI4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$;
$(QHCI5)$ $x \ll x \circ 0$;
$(QHCI6)$ $x \ll 0$ *implies* $x = 0$,
then it is a quasi-hyper BCI*-algebra.*

Definition 13 ([34]). *Assume that* $(H, \circ)$ *is a hypergroupoid containing 0. If it meets these axioms:* $\forall x, y, z \in H$,
$(HZ1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$;
$(HZ2)$ $x \ll x$;
$(HZ3)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$;
$(HZ4)$ $0 \circ (0 \circ x) \ll x$;
$(HZ5)$ $x \ll x \circ 0$,
then it is a hyper BZ-algebra.

Proposition 1 ([34]). *In any hyper BZ-algebra* $(H, \circ)$*, the following holds: For all* $x, y, z \in H$ *and for all non-empty subsets* A *and* B *of* H,
(1) $x \ll 0 \Rightarrow x = 0$;
(2) $0 \circ (x \circ y) \ll y \circ x$;
(3) $A \ll A$;
(4) $A \subseteq B \Rightarrow A \ll B$;
(5) $A \ll 0 \Rightarrow A = 0$;
(6) $0 \circ 0 = 0$;
(7) $(0 \circ x) \circ (0 \circ x) = 0$;
(8) $0 \circ x$ *is a singleton set;*
(9) $x \circ y = 0 \Rightarrow (x \circ z) \circ (y \circ z) = 0$ *and* $x \circ z \ll y \circ z$;
(10) $A \circ 0 = 0 \Rightarrow A = 0$;
(11) $x \ll y \Rightarrow 0 \ll y \circ x$;
(12) $0 \circ (0 \circ (0 \circ x)) \ll 0 \circ x$;
(13) $x \circ x = 0 \Rightarrow |y \circ z| = 1$.

Definition 14 ([34])**.** *A hyper BZ-algebra* $(H, \circ)$ *is called a generalized anti-grouped hyper BZ-algebra if* $\forall x, y, z \in H$*, it satisfies* $(x \circ (0 \circ y)) \circ (0 \circ z) = x \circ (0 \circ (y \circ (0 \circ z)))$*.*

3. QM-*BZ*-Algebra

Firstly, we introduce a method to construct BZ-algebra by using BCC-algebra and anti-grouped BZ-algbera.

Proposition 2. *Assume that* $(K, *, 0)$ *is a BCC-algebra and* $(G, \cdot, 0)$ *is an anti-grouped BZ-algebra. Denote* $A = K \cup G$*, and* $K \cap G = \{0\}$*. An operation on* A *is as below:*

$$x \circ y = \begin{cases} x * y, & x, y \in K \\ x \cdot y, & x, y \in G \\ 0 \cdot y, & x \in K, y \in G - \{0\} \\ x, & x \in G - \{0\}, y \in K \end{cases}$$

Therefore, $(A, \circ, 0)$ *is BZ-algebra.*

Proof. (1) $\forall x, y, z \in A$,

Case 1: $\forall x, y, z \in K$, $((x \circ z) \circ (y \circ z)) \circ (x \circ y) = ((x * z) * (y * z)) * (x * y) = 0$;

Case 2: $\forall x, y, z \in G$, $((x \circ z) \circ (y \circ z)) \circ (x \circ y) = ((x \cdot z) \cdot (y \cdot z)) \cdot (x \cdot y) = 0$;

Case 3: $\forall x, y \in K, \forall z \in G - \{0\}$, $((x \circ z) \circ (y \circ z)) \circ (x \circ y) = ((0 \cdot z) \circ (0 \cdot z)) \circ (x \circ y) = ((0 \cdot z) \cdot (0 \cdot z)) \circ (x \circ y) = 0 \circ (x * y) = 0$;

Case 4: $\forall x \in K, \forall y \in G - \{0\}, \forall z \in K$, $((x \circ z) \circ (y \circ z)) \circ (x \circ y) = ((x * z) \circ y) \circ (0 \cdot y) = (0 \cdot y) \circ (0 \cdot y) = (0 \cdot y) \cdot (0 \cdot y) = 0$;

Case 5: $\forall x \in G - \{0\}, \forall y \in K, \forall z \in K$, $((x \circ z) \circ (y \circ z)) \circ (x \circ y) = (x \circ (y * z)) \circ x = x \circ x = x \cdot x = 0$;

Case 6: $\forall x, y \in G - \{0\}, \forall z \in K$, $((x \circ z) \circ (y \circ z)) \circ (x \circ y) = (x \circ y) \circ (x \circ y) = (x \cdot y) \cdot (x \cdot y) = 0$;

Case 7: $\forall x \in G - \{0\}, \forall y \in K, \forall z \in G - \{0\}$, $((x \circ z) \circ (y \circ z)) \circ (x \circ y) = ((x \cdot z) \circ (0 \cdot z)) \circ x = ((x \cdot z) \cdot (0 \cdot z)) \cdot x = (x \cdot 0) \cdot x = x \cdot x = 0$;

Case 8: $\forall x \in K, \forall y \in G - \{0\}, \forall z \in G - \{0\}$, $((x \circ z) \circ (y \circ z)) \circ (x \circ y) = ((0 \cdot z) \circ (y \cdot z)) \circ (0 \cdot y) = ((0 \cdot z) \cdot (y \cdot z)) \cdot (0 \cdot y) = (0 \cdot y) \cdot (0 \cdot y) = 0$.

Above all, (BZ1) holds.

(2) Obviously, $\forall x \in A$, $x \circ x = 0$, that is, (BZ2) holds.

(3) $\forall x, y \in A$, (i) $\forall x, y \in K$, $x \circ y = 0$, and $y \circ x = 0$ imply $x = y$; (ii) $\forall x, y \in G$, $x \circ y = 0$, and $y \circ x = 0$ imply $x = y$; (iii) $\forall x \in G - \{0\}, \forall y \in K$, if $x \circ y = 0$, there is $x = 0$. Because $x \in G - \{0\}$, there does not exist $x \circ y = 0$. Then, (BZ3) holds.

To sum up, $(A, \circ, 0)$ is a BZ-algebra. □

Definition 15. *Assume that K is a BCC-algebra and G is an anti-grouped BZ-algebra,* $K \cap G = \{0\}$*. Denote* $A = K \cup G$ *according to Proposition 2; then, A is BZ-algebra. Then, A is called the KG-union of K and G, and is written as* $A = K \oplus_{KG} G$*.*

Assume that $<X, *, 0>$ is a BZ-algebra. $\forall a, x \in X$, denote a map ρ_a:

$$\rho_a : X \to X : X \mapsto x * a.$$

$\forall a, b \in X, \forall x \in X$, denote $\rho_a * \rho_b$:

$$(\rho_a * \rho_b)(x) = \rho_a(y) \text{ and } y = \rho_b(x),$$

where $*$ means the composition operation of mappings.

Theorem 5. *Denote* $M(X)$ *as a set which is all compositional results of finite mappings which are for all* $a \in H$*; thus, we have* ρ_a*. Then,* $M(X)$ *is a monoid.*

Proof. $\forall a, b, c \in X$, and $\forall x \in X$; therefore:

$$(\rho_a * \rho_b) * \rho_c(x) = \rho_a * \rho_b(x * c) = ((x * c) * b) * a,$$

$$\rho_a * (\rho_b * \rho_c)(x) = \rho_a * (\rho_b * \rho_c(x)) = \rho_a((x * c) * b) = ((x * c) * b) * a.$$

Obviously, $(\rho_a * \rho_b) * \rho_c(x) = \rho_a * (\rho_b * \rho_c)(x)$. So, $(M(X), *)$ satisfies associative law, that is, $(M(X), *)$ is a semigroup.

$\forall x \in X$, $\rho_a \in M(X)$, there is,

$$\rho_0 * \rho_a(x) = \rho_0(x * a) = (x * a) * 0 = x * a = \rho_a(x),$$

$$\rho_a * \rho_0(x) = \rho_a(x * 0) = (x * 0) * a = x * a = \rho_a(x).$$

Then, ρ_0 is the identity element in $M(X)$. Thus, $M(X)$ is a monoid. □

Example 1. *Let $X = \{0, 1, 2, 3, 4, 5\}$. The operation $*$ on X is shown in Table 1.*

Table 1. BZ-algebra.

*	**0**	**1**	**2**	**3**	**4**	**5**
0	0	0	0	0	5	4
1	1	0	3	3	5	4
2	2	2	0	2	5	4
3	3	0	3	0	5	4
4	4	4	4	4	0	5
5	5	5	5	5	4	0

Then, $<X, *, 0>$ is a BZ-algebra and $M(X) = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_{12}\}$, where $\rho_{12} = \rho_1 * \rho_2$.

We can verify the following:

$\rho_0 * \rho_0 = \rho_0, \rho_0 * \rho_1 = \rho_1, \rho_0 * \rho_2 = \rho_2, \rho_0 * \rho_3 = \rho_3, \rho_0 * \rho_4 = \rho_4, \rho_0 * \rho_5 = \rho_5, \rho_0 * \rho_{12} = \rho_{12}$;

$\rho_1 * \rho_0 = \rho_1, \rho_1 * \rho_1 = \rho_1, \rho_1 * \rho_2 = \rho_{12}, \rho_1 * \rho_3 = \rho_1, \rho_1 * \rho_4 = \rho_4, \rho_1 * \rho_5 = \rho_5, \rho_1 * \rho_{12} = \rho_{12}$;

$\rho_2 * \rho_0 = \rho_2, \rho_2 * \rho_1 = \rho_{12}, \rho_2 * \rho_2 = \rho_2, \rho_2 * \rho_3 = \rho_1, \rho_2 * \rho_4 = \rho_4, \rho_2 * \rho_5 = \rho_5, \rho_2 * \rho_{12} = \rho_{12}$;

$\rho_3 * \rho_0 = \rho_3, \rho_3 * \rho_1 = \rho_1, \rho_3 * \rho_2 = \rho_{12}, \rho_3 * \rho_3 = \rho_2, \rho_3 * \rho_4 = \rho_4, \rho_3 * \rho_5 = \rho_5, \rho_3 * \rho_{12} = \rho_{12}$;

$\rho_4 * \rho_0 = \rho_4, \rho_4 * \rho_1 = \rho_4, \rho_4 * \rho_2 = \rho_4, \rho_4 * \rho_3 = \rho_4, \rho_4 * \rho_4 = \rho_5, \rho_4 * \rho_5 = \rho_{12}, \rho_4 * \rho_{12} = \rho_4$;

$\rho_5 * \rho_0 = \rho_5, \rho_5 * \rho_1 = \rho_5, \rho_5 * \rho_2 = \rho_5, \rho_5 * \rho_3 = \rho_5, \rho_5 * \rho_4 = \rho_{12}, \rho_5 * \rho_5 = \rho_4, \rho_5 * \rho_{12} = \rho_5$;

$\rho_{12} * \rho_0 = \rho_{12}, \rho_{12} * \rho_1 = \rho_{12}, \rho_{12} * \rho_2 = \rho_{12}, \rho_{12} * \rho_3 = \rho_{12}, \rho_{12} * \rho_4 = \rho_4, \rho_{12} * \rho_5 = \rho_5, \rho_{12} * \rho_{12} = \rho_{12}$.

Then, $M(X)$ is a monoid, and the operation $*$ on it is shown in Table 2. However, it is not commutative, since $\rho_3 * \rho_2 = \rho_{12} \neq \rho_1 = \rho_2 * \rho_3$.

Table 2. The adjoint semigroup of BZ-algebra.

$*$	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_{12}
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_{12}
ρ_1	ρ_1	ρ_1	ρ_{12}	ρ_1	ρ_4	ρ_5	ρ_{12}
ρ_2	ρ_2	ρ_{12}	ρ_2	ρ_1	ρ_4	ρ_5	ρ_{12}
ρ_3	ρ_3	ρ_1	ρ_{12}	ρ_2	ρ_4	ρ_5	ρ_{12}
ρ_4	ρ_4	ρ_4	ρ_4	ρ_4	ρ_5	ρ_{12}	ρ_4
ρ_5	ρ_5	ρ_5	ρ_5	ρ_5	ρ_{12}	ρ_4	ρ_5
ρ_{12}	ρ_{12}	ρ_{12}	ρ_{12}	ρ_{12}	ρ_4	ρ_5	ρ_{12}

Definition 16. *Assume that $(X, \leq)$ is a partial order containing a constant 0. x is called a quasi-minimal element in X, if $\forall a \in X$, $a \leq x$ implies $x = a$ or $a = 0$.*

Definition 17. *A BZ-algebra $(X, \leq, *, 0)$ is called QM-BZ-algebra if all elements of X are quasi-minimal elements.*

Theorem 6. *Assume that $(X, \leq, *, 0)$ is a BZ-algebra. Then, X is a QM-BZ-algebra iff it meets: $\forall x, y \in X - \{0\}$,*

$$x \leq y \text{ implies } x = y.$$

Proof. $(\Rightarrow)$ $\forall x, y \in X - \{0\}$, assume that $x \leq y$, according to Definition 16, $y = x$ or $x = 0$. However, $x \neq 0$. So, $x = y$.

$(\Leftarrow)$ Assume that $x, y \in X$, $x \leq y$. If $y = 0$, then $x \leq y = 0$, and we can obtain $x = y = 0$. If $x \neq 0$, $y \neq 0$, there is $x = y$ by condition. So, y is a quasi-minimal element. Thus, X is a QM-BZ-algebra. □

Theorem 7. *Assume that $(X, \leq, *, 0)$ is a BZ-algebra, $K(X)$ is a BCC-part of X, and AG(X) is an anti-grouped part of X. Then, the below conditions are equivalent:*
(1) X is QM-BZ-algebra;
(2) $K(X)$ is quasi-alter BCK-algebra and $AG(X) = (X - K(X)) \cup \{0\}$.

Proof. (1)⇒(2) Assume that X is a QM-BZ-algebra. Then, $\forall x, y \in K(X)$, if $x = y$, $x * y = 0$. If $x \neq y$, it can be divided into the below three cases:

Case 1: $x = 0$, $y \neq 0$, $x * (x * y) = 0 * (0 * y) = 0 * 0 = 0$, that is, $x \leq x * y$; $(x * y) * x = (0 * y) * 0 = 0 * 0 = 0$, that is, $x * y \leq x$. According to Definition 3, $x = x * y$;

Case 2: $x \neq 0$, $y = 0$, $x * (x * y) = x * (x * 0) = x * x = 0$, that is, $x \leq x * y$; $(x * y) * x = (x * 0) * x = x * x = 0$, that is, $x * y \leq x$. According to Definition 3, $x = x * y$;

Case 3: $x \neq 0$, $y \neq 0$, $(x * y) * x = ((x * y) * 0) * x = ((x * y) * (0 * y)) * (x * 0) = 0$, that is, $x * y \leq x$. Because $x * y \neq 0$ and $x \neq 0$, according to Theorem 6, $x * y = x$.

According to Definition 4, $K(X)$ is a quasi-alter BCK-algebra. If $x \in X - K(X)$, then $0 * x \neq 0$ and $0 * (0 * x) \neq 0$. Because $(0 * (0 * x)) * x = (0 * x) * (0 * x) = 0$, that is, $0 * (0 * x) \leq x$. According to Theorem 6, $0 * (0 * x) = x$. Thus, $(X - K(X)) \cup \{0\} \subseteq AG(X)$. Additionally, $AG(X) \subseteq (X - K(X)) \cup \{0\}$; then, $AG(X) = (X - K(X)) \cup \{0\}$.

(2)⇒(1) Let (2) hold; then, $\forall x, y \in X$, $x \neq y$:

Case 1: $x, y \in K(X)$, assume that $x \leq y$, $x = x * y = 0$, and it does not hold. So, $x = y$;

Case 2: $x, y \in AG(X)$, assume that $x \leq y$, that is, $x * y = 0$. Then,

$$0 * (y * x) = (x * x) * (y * x) \leq x * y = 0.$$

So, $0 * (y * x) = 0$. Then, $y * x = 0 * (0 * (y * x)) = 0 * 0 = 0$, and $y * x = 0$. That is, $y \leq x$. So, $x = y$.

Case 3: $x \in K(X)$, $y \in AG(X)$, assume that $x \leq y$, that is, $x * y = 0$. Then,

$$0 = 0 * (0 * (x * y)) = (0 * (0 * x)) * (0 * (0 * y)) = 0 * y.$$

Clearly, $0 * y = 0$ does not hold. Then, $x = y$.
Case 4: $x \in AG(X), y \in K(X)$, assume that $x \leq y$, that is, $x * y = 0$. Then,

$$x = x * 0 = (0 * (0 * x)) * (0 * (0 * y)) = 0 * (0 * (x * y)) = 0 * 0 = 0.$$

Clearly, $x = 0$ does not hold. Then, $x = y$.
According to Theorem 6, X is a QM-BZ-algebra. □

According to Theorem 7, the KG-union of quasi-alter BCK-algebra and anti-grouped BZ-algebra is QM-BZ-algebra.

Theorem 8. *Assume that $(X, \leq, *, 0)$ is QM-BZ-algebra, that $K(X)$ is the BCC-part, and that $G(X)$ is the BCC-remainder. Then:*
(1) $x * y = x$, $x \in G(X), y \in K(X)$;
(2) $x * y = 0 * y$, $x \in K(X), y \in G(X)$.

Proof. (1) $\forall x \in G(X), y \in K(X)$, assume that $x * y \in K(X)$. Then, $0 = 0 * 0 = 0 * (0 * (x * y)) = (0 * (0 * x)) * (0 * (0 * y)) = x * 0 = x$. Clearly, $x = 0$ does not hold. So, $x * y \in G(X)$, that is, $x * y = 0 * (0 * (x * y)) = (0 * (0 * x)) * (0 * (0 * y)) = x * 0 = x$.

(2) $\forall x \in K(X), y \in G(X)$, assume that $x * y \in K(X)$. Then, $0 = 0 * 0 = 0 * (0 * (x * y)) = (0 * (0 * x)) * (0 * (0 * y)) = 0 * y$. Clearly, $0 * y = 0$ does not hold. So, $x * y \in G(X)$, that is, $x * y = 0 * (0 * (x * y)) = (0 * (0 * x)) * (0 * (0 * y)) = 0 * y$. □

Definition 18. *BZ-algebra <X; $*$, 0> is called generalized quasi-left alter BZ-algebra if it meets: $\forall x, y \in X$,*

$$x * (x * y) = 0 * (0 * y), x \neq y.$$

Theorem 9. *Let <X, $*$, 0> be a generalized quasi-left alter BZ-algebra. $\forall x \in X$, either $0 * x \neq 0$, or $0 * (0 * x) = x$.*

Proof. Let $0 * x \neq 0$. Assume that $0 * (0 * x) = x$. Because $0 * (0 * (0 * x)) = 0 * x$, then:

$$x * (x * (0 * (0 * x))) = 0 * (0 * (0 * (0 * x))) = 0 * (0 * x).$$

There is $x * (0 * (0 * x)) \neq x$, because if $x * (0 * (0 * x)) = x$, $0 * (0 * x) = x * (x * (0 * (0 * x))) = x * x = 0$. Then, $0 * x = 0 * (0 * (0 * x)) = 0 * 0 = 0$, and this conflicts with assumption $0 * x \neq 0$. So, $x * (0 * (0 * x)) \neq x$.
According to Definition 18:

$$\begin{aligned} x * (x * (x * (0 * (0 * x)))) &= 0 * (0 * (x * (0 * (0 * x)))) \\ &= (0 * (0 * x)) * (0 * (0 * (0 * (0 * x)))) \\ &= (0 * (0 * x)) * (0 * (0 * x)) = 0. \end{aligned}$$

Therefore, $x * (x * (x * (0 * (0 * x)))) = 0$. In addition, $x * (x * (x * (0 * (0 * x)))) = x * (0 * (0 * x)) = 0$, $0 * (0 * x) = x * (x * (0 * (0 * x))) = x * 0 = x$. This conflicts with assumption $0 * (0 * x) \neq x$, so $0 * (0 * x) = x$. □

According to Theorem 9, let $K(X)$ be the BCC-part of a generalized quasi-left alter BZ-algebra, $G(X)$ be BCC-remainder of a generalized quasi-left alter BZ-algebra. Then, $G(X) \cup \{0\}$ is an anti-grouped BZ-subalgebra.

Theorem 10. *Assume that <X, $*$, 0> is a generalized quasi-left alter BZ-algebra. Therefore, X is BCI-algebra.*

Proof. Let $K(X)$ be the BCC-part of X and $G(X)$ be the BCC-remainder of X. $\forall x, y \in X$:

If $x = y$, $(x*(x*y))*y = (x*(x*x))*x = (x*0)*x = x*x = 0$, that is, $x*(x*y) \leq y$.

If $x \neq y$, there are two cases according to Theorem 9:

Case 1: $y \in K(X)$, $(x*(x*y))*y = (0*(0*y))*y = 0*y = 0$, that is, $x*(x*y) \leq y$;

Case 2: $y \in G(X)$, $(x*(x*y))*y = (0*(0*y))*y = y*y = 0$, that is $x*(x*y) \leq y$.

To sum up, $\forall x, y \in X$, $x*(x*y) \leq y$. According to Theorem 1, X is BCI-algebra. □

Theorem 11. *Assume that <X, ∗, 0> is a generalized quasi-left alter BZ-algebra. Then, X is a QM-BZ-algebra.*

Proof. Let $K(X)$ be the BCC-part of X, and $G(X)$ be the BCC-remainder of X. Then, $\forall x, y \in X$, assume that $x \leq y$ and $x \neq y$. Therefore,

$$x = x*0 = x*(x*y) = 0*(0*y).$$

(1) When $y \in K(X)$, $x = 0*(0*y) = 0*0 = 0$.

(2) When $y \in G(X)$, $x = 0*(0*y) = y$, but $x \neq y$. So, y is a quasi-minimal element of X. Because y is arbitrary, X is a QM-BZ-algebra. □

In the following, the concepts of QM-hyper BZ-algebra and generalized quasi-left alter hyper BZ-algebra are shown.

Definition 19. *Let $(H, \ll)$ be a partial order that contains a constant 0 in a hyper structure. x is said to be a quasi-minimal element in H, if for any element a in H, $a \ll x$ implies $x = a$ or $a = 0$.*

Definition 20. *A hyper BZ-algebra $(H, \ll, \circ, 0)$ is said to be QM-hyper BZ-algebra if all elements of H are quasi-minimal elements.*

Theorem 12. *Assume that $(H, \ll, \circ, 0)$ is a hyper BZ-algebra. Then, H is a QM-hyper BZ-algebra iff it meets: for all $x, y \in H - \{0\}$,*

$$x \ll y \text{ implies } x = y.$$

Proof. ($\Rightarrow$) For all $x, y \in H - \{0\}$, let $x \ll y$, according to Definition 19, $y = x$ or $x = 0$. However, $x \neq 0$. So, $x = y$.

($\Leftarrow$) Assume that $x, y \in H$, $x \ll y$. If $y = 0$, then $x \ll y = 0$, and we can obtain $x = y = 0$. If $x \neq 0$, $y \neq 0$; therefore, $x = y$ by condition. So, y is a quasi-minimal element of H. Thus, H is a QM-hyper BZ-algebra. □

Definition 21. *Hyper BZ-algebra $(H, \circ)$ is called a generalized quasi-left alter hyper BZ-algebra if, $\forall x, y \in H$,*

$$x \circ (x \circ y) = 0 \circ (0 \circ y), x \neq y.$$

Theorem 13. *Assume that $(H, \circ)$ is a generalized quasi-left alter hyper BZ-algebra. Thus, H is BCI-algebra.*

Proof. Assume that $(H, \circ)$ is a generalized quasi-left alter hyper BZ-algebra. Let $B(H)$ be hyper BCC-part of H. $\forall x \in B(H)$ and $x \neq 0$, $0 \circ (x \circ 0) = (0 \circ 0) \circ (x \circ 0) \ll 0 \circ x = 0$, according to Proposition 1, $0 \circ (x \circ 0) = 0$. According to Definition 21, $x \circ (x \circ 0) = 0 \circ (0 \circ 0) = 0$. Assume that $x \neq x \circ 0$, $x \circ 0 = x \circ (x \circ (x \circ 0)) = 0 \circ (0 \circ (x \circ 0)) = 0 \circ 0 = 0$; clearly, $x \neq x \circ 0$ is not true. So, $x = x \circ 0$.

If $x \notin B(H)$, and $0 \circ (x \circ 0) \neq 0$, there is $x \circ (x \circ 0) = 0 \circ (0 \circ 0) = 0$. Assume that $x \neq x \circ 0$, $x \circ 0 = x \circ (x \circ (x \circ 0)) = 0 \circ (0 \circ (x \circ 0))$. So, $(x \circ 0) \circ (x \circ 0) = (0 \circ (0 \circ (x \circ 0))) \circ (0 \circ (0 \circ (x \circ 0))) \ll 0 \circ 0 = 0$, according to Proposition 1, $(x \circ 0) \circ (x \circ 0) = 0$. Therefore, $0 \circ ((x \circ 0) \circ x) \subset (x \circ x) \circ ((x \circ 0) \circ x) \ll x \circ (x \circ 0) = 0$. That is, $0 \circ ((x \circ 0) \circ x) = 0$.

Beasuse $0 \circ x \neq 0, 0 \circ (x \circ 0) \neq 0$, so $0 \circ ((x \circ 0) \circ x) = 0$ is not true. That is, $x \neq x \circ 0$. So, $x = x \circ 0$.

Above all, $\forall x \in H, x = x \circ 0$. Moreover, $x \circ x = x \circ (x \circ 0) = 0 \circ (0 \circ 0) = 0$, according to Proposition 1, $(H, \circ)$ is BZ-algebra. According to Theorem 10, $(H, \circ)$ is BCI-algebra. □

Theorem 14. *Let $(H, \circ)$ be generalized quasi-left alter hyper BZ-algebra. Then, H is a QM-hyper BZ algebra.*

Proof. Assume that $(H, \circ)$ be a generalized quasi-left alter hyper BZ-algebra. According to Theorem 13, H is BCI-algebra. Let $K(H)$ be the BCC-part of H, and $G(H)$ be the BCC-remainder of H. Then, $\forall x, y \in H$, let $x \ll y$ and $x \neq y$. Therefore,

$$x = x \circ 0 = x \circ (x \circ y) = 0 \circ (0 \circ y).$$

(1) When $y \in K(H)$, $x = 0 \circ (0 \circ y) = 0 \circ 0 = 0$.

(2) When $y \in G(H)$, $x = 0 \circ (0 \circ y) = y$, but $x \neq y$. So, y is a quasi-minimal element of H. Because y is arbitrary, H is QM-hyper BZ-algebra. □

However, not every QM-hyper BZ-algebra is generalized quasi-left alter hyper BZ-algebra; see Example 2.

Example 2. *Let $H = \{0, 1, 2, 3, 4\}$. The operation $\circ$ on H is shown in Table 3.*

Table 3. QM-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**	**4**
0	0	1	0	0	0
1	1	0	1	1	1
2	2	1	$\{0, 2\}$	2	2
3	3	1	3	$\{0, 3\}$	3
4	4	1	4	4	0

Clearly, $(H, \circ)$ is QM-hyper BZ-algebra, but it is not a generalized quasi-left alter hyper BZ-algebra, since $2 \circ (2 \circ 0) = \{0, 2\}, 0 \circ (0 \circ 0) = 0, 2 \neq 0$.

Additionally, Example 2 shows that not every QM-hyper BZ-algebra is QM-BZ-algebra.

4. Quasi-Hyper *BZ*-Algebras

In this part, we propose the definition of quasi-hyper BZ-algebras. In the following, we replace the singleton set $\{x\}$ with x.

Definition 22. *Assume that $(H, \circ)$ is a hypergroupoid containing 0. If it meets these axioms:*
$\forall x, y, z \in H$,
$(QHZ1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$;
$(QHZ2)$ $x \ll x$;
$(QHZ3)$ $x \ll y$ *and* $y \ll x \Rightarrow x = y$;
$(QHZ4)$ $x \ll x \circ 0$;
$(QHZ5)$ $x \ll 0 \Rightarrow x = 0$,
then it is a quasi-hyper BZ-algebra.

Remark 1. (1) *Every BZ-algebra is a quasi-hyper BZ-algebra;*
(2) *Every hyper BZ-algebra is a quasi-hyper BZ-algebra;*
(3) *Every quasi-hyper BCI-algebra is a quasi-hyper BZ-algebra.*

Now, we give some examples about quasi-hyper BZ-algebras and some examples which explain that not every quasi-hyper BZ-algebra is a hyper BZ-algebra.

Example 3. (1) *Assume that* $H = \{0, 1, 2, 3, 4\}$. *Define an operation* $*$ *on* H *in Table 4,*

Table 4. BZ-algebra.

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	2	2	0
1	1	0	3	2	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	4	3	3	0

Then, $(H, *, 0)$ *is a BZ-algebra, and it is a quasi-hyper BZ-algebra.*

(2) *Assume that* $H = \{0, 1, 2, 3, 4\}$. *Define an operation* $\circ$ *on* H *in Table 5,*

Table 5. Quasi-hyper BZ-algebra.

$*$	**0**	**1**	**2**	**3**	**4**
0	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
1	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
2	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
3	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
4	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$

Then, $(H, \circ, 0)$ *is a quasi-hyper BZ-algebra, but it is not a hyper BZ-algebra since* $0 \circ (0 \circ 0) = \{0, 1\}$ *and* $0 \notin 1 \circ 0$(*i.e.*, $1 \ll 0$ *is not true); this means that the condition (HZ4) in Definition 13 does not hold.*

(3) *Assume that* $H = \{0, 1, 2, 3, 4\}$. *Define an operation* $\circ$ *on* H *in Table 6,*

Table 6. Quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**	**4**
0	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
1	$\{2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
2	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
3	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
4	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$

Then $(H, \circ, 0)$ *is a quasi-hyper BZ-algebra, but it is not a quasi-hyper BCI-algebra, since* $(1 \circ 0) \circ 1 = \{2, 3, 4\}$, $(1 \circ 1) \circ 0 = \{0, 1, 2, 3, 4\}$ *and* $0 \neq 1$.

Proposition 3. *In any quasi-hyper BZ-algebra* $(H, \circ)$, *the followings hold: for all* $x, y, z \in H$ *and for all non-empty subsets* A *and* B *of* H:
(1) $0 \circ (x \circ y) \ll y \circ x$;
(2) $A \ll A$;
(3) $A \subseteq B \Rightarrow A \ll B$;
(4) $A \ll 0 \Rightarrow A = 0$;
(5) $x \circ y = 0 \Rightarrow (x \circ z) \circ (y \circ z) = 0$ *and* $x \circ z \ll y \circ z$;
(6) $A \circ 0 = 0 \Rightarrow A = 0$;
(7) $x \ll y \Rightarrow 0 \ll y \circ x$;
(8) $x \circ x = 0 \Rightarrow |x \circ y| = 1$.

Proof. (1) By (QHZ1) and (QHZ2), $0 \circ (x \circ y) \subseteq (y \circ y) \circ (x \circ y) \ll y \circ x$. Then, $0 \circ (x \circ y) \ll y \circ x$.

(2) By (QHZ2), for any $x \in A$, there is $x \ll x$, that is $0 \in x \circ x$. Then, $A \ll A$.
(3) Let $a \in A$. Then, $a \in B$. By (QHZ2), $x \ll x$ and $0 \in x \circ x$. Then, $A \ll B$.
(4) Let $a \in A$. Then, $a \ll 0$, and so $a = 0$. Then, $A = \{0\}$.
(5) By (QHZ1), $(x \circ z) \circ (y \circ z) \ll x \circ y = \{0\}$. By (4), $(x \circ z) \circ (y \circ z) = \{0\}$. So, $x \circ z \ll y \circ z$.
(6) Assume that $A \circ 0 = 0$; then, $A \ll 0$. So, $A = 0$.
(7) Assume that $x \ll y$. Then, $0 \in x \circ y$, and so $0 \in 0 \circ 0 \subseteq (y \circ y) \circ (x \circ y) \ll y \circ x$. Hence, $0 \ll y \circ x$.
(8) For any $x \in H$, let $x \circ x = \{0\}$. $\forall y \in H$, assume that $|x \circ y| > 1$, let $a, b \in x \circ y$, and $a \neq b$. Then,

$$a \circ b \subseteq (x \circ y) \circ (x \circ y) \ll x \circ x = 0 \text{ and } b \circ a \subseteq (x \circ y) \circ (x \circ y) \ll x \circ x = 0;$$

thus, $a \circ b \ll 0$, $b \circ a \ll 0$, and $a \ll b, b \ll a$. Thus, $a = b$, and so $|x \circ y| = 1$. □

Proposition 4. *In any quasi-hyper BZ-algebra $(H, \circ)$ satisfying $0 \circ 0 = 0$, the followings hold: $\forall x \in H$,*
(1) $(0 \circ x) \circ (0 \circ x) = 0$;
(2) $0 \circ x$ is a singleton set;
(3) $(0 \circ x) \circ 0 = 0 \circ x$.

Proof. (1) $\forall x \in H$, $(0 \circ x) \circ (0 \circ x) \ll 0 \circ 0 = 0$. By Proposition 3 (4), $(0 \circ x) \circ (0 \circ x) = 0$.
(2) For any $a, b \in 0 \circ x$, and $a \neq b$. $a \circ b \subseteq (0 \circ x) \circ (0 \circ x) = 0$, $b \circ a \subseteq (0 \circ x) \circ (0 \circ x) = 0$, by (QHZ3), $a \ll b, b \ll a$, so $a = b$. Thus, $0 \circ x$ is a singleton set.
(3) By (2), let $0 \circ x = m$. By(QHZ4), $m \ll m \circ 0$ and $m \circ m = 0$. Assume that $|m \circ 0| > 1$, let $a, b \in m \circ 0$. $a \circ b \subseteq (m \circ 0) \circ (m \circ 0) \ll m \circ m = 0$, $b \circ a \subseteq (m \circ 0) \circ (m \circ 0) \ll m \circ m = 0$, so $a \ll b$, $b \ll a$. In addition, $a = b$. So, $|m \circ 0| = 1$. Because $m \circ 0 = (0 \circ x) \circ 0 \subseteq (0 \circ x) \circ (x \circ x) \ll 0 \circ x = m$, and $|m \circ 0| = 1$, $m = m \circ 0$. That is, $(0 \circ x) \circ 0 = 0 \circ x$. □

In the following, we give the concepts of standard quasi-hyper BZ-algebra and transitive quasi-hyper BZ-algebra.

Definition 23. *A quasi-hyper BZ algebra $(H, \circ)$ is called a standard quasi-hyper BZ algebra if, $\forall x \in H$, it satisfies $x \circ 0 = x$.*

Proposition 5. *Every standard quasi-hyper BZ-algebra is a hyper BZ-algebra.*

Proof. Assume that $(H, \circ)$ is a standard quasi-hyper BZ algebra. For all $x \in H$, $0 \circ (0 \circ x) \subseteq (x \circ x) \circ (0 \circ x) \ll x \circ 0 = x$. That is, $0 \circ (0 \circ x) \ll x$. So, $(H, \circ)$ is a hyper BZ-algebra. □

Definition 24. *A quasi-hyper BZ-algebra $(H, \circ)$ is called a transitive quasi-hyper BZ algebra if, $\forall x, y, z \in H$, it satisfies $x \ll y$ and $y \ll z \Rightarrow x \ll z$.*

Proposition 6. *In any transitive quasi-hyper BZ-algebra $(H, \circ)$, the following conditions hold: for all $x, y, z, u \in H$, and for all non-empty subsets A, B, and C of H:*
$(TQHZ1)$ $A \ll B$ and $B \ll C$ imply $A \ll C$;
$(TQHZ2)$ $x \circ y \ll z$ implies $(x \circ u) \circ (y \circ u) \ll z$.

Proof. (TQHZ1) Let $a \in A$. $\exists b \in B$ s.t. $a \ll b$. Additionally, for any $b \in B$, $\exists c \in C$ such that $b \ll c$. So, $a \ll c$. Then, for any $a \in A$, $\exists c \in C$ s.t. $a \ll c$, that is $A \ll C$.
(TQHZ2) By (QHZ1), $(x \circ u) \circ (y \circ u) \ll x \circ y$, and $x \circ y \ll z$. So, $(x \circ u) \circ (y \circ u) \ll z$. □

Example 4. (1) *Assume that $H = \{0, 1, 2, 3, 4\}$. Define an operation $\circ$ on H in Table 7,*

Table 7. Transitive quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	$\{0,1\}$	0	0	0
2	2	2	$\{0,2\}$	$\{0,2\}$	0
3	3	2	$\{1,2\}$	$\{0,1,2\}$	0
4	4	4	4	4	0

Then, $(H,\circ)$ is a transitive quasi-hyper BZ-algebra and a transitive hyper BZ-algebra.
(2) Assume that $H = \{0,1,2,3,4\}$. Define an operation $\circ$ on H in Table 8,

Table 8. Transitive quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**	**4**
0	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
1	4	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
2	4	4	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
3	4	4	4	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
4	4	4	4	4	$\{0,1,2,3,4\}$

Then, $(H,\circ)$ is a transitive quasi-hyper BZ-algebra, but it is not standard.

Proposition 7. *Every transitive standard quasi-hyper BZ-algebra is a hyper BZ-algebra.*

Proof. According to Proposition 5. □

In the following, we investigate some kinds of quasi-hyper BZ-subalgebra.

Definition 25. *Assume that $(H,\circ)$ is a quasi-hyper BZ-algebra, that S is a subset of H, and that S contains 0. If S is a quasi-hyper BZ-algebra under the hyper operation "$\circ$" on H, then S is a hyper subalgebra of H.*

Proposition 8. *Assume that S is a nonempty subset of a quasi-hyper BZ-algebra $(H,\circ)$. If $\forall x,y \in S$ $x\circ y \subseteq S$, there is $0 \in S$.*

Proof. Let $\forall x,y \in S$, $x\circ y \subseteq S$ and $a \in S$. Because $a \ll a$, there is $0 \in a\circ a \subseteq S$. □

Theorem 15. *Assume that S is a nonempty subset of a quasi-hyper BZ-algebra $(H,\circ)$. S is a hyper subalgebra of H iff $\forall x,y \in S$, $x\circ y \subseteq S$.*

Proof. ($\Rightarrow$) This is clear.

($\Leftarrow$) $\forall x,y \in S$, assume that $x\circ y \subseteq S$. By Proposition 8, $0 \in S$. $\forall x,y,z \in S$, there is $x\circ z \subseteq S$, $y\circ z \subseteq S$, and $x\circ y \subseteq S$. Thus,

$$(x\circ z)\circ(y\circ z) = \bigcup_{a\in x\circ z, b\in y\circ z} a\circ b \subseteq S.$$

Because $S \subseteq H$, $(x\circ z)\circ(y\circ z) \ll x\circ y \subseteq S$. Therefore, in S, (QHZ1) holds. In a similar way, (QHZ2), (QHZ3), (QHZ4), and (QHZ5) hold in S. So, S is a hyper subalgebra of H. □

Theorem 16. *Assume that $(H,\circ)$ is a quasi-hyper BZ-algebra. Then,*

$$S_I := \{m \in H | m\circ m = 0\}$$

is a hyper subalgebra of H and $\forall x,y \in S_I$, $x\circ y$ is a singleton set whenever $S_I \neq \varnothing$.

Proof. Assume that $x, y \in S_I$ and $a \in x \circ y$. Then, there is $(x \circ y) \circ (x \circ y) \ll x \circ x = 0$ and $(x \circ y) \circ (x \circ y) = 0$ and $a \circ a \subseteq (x \circ y) \circ (x \circ y) = 0$. Therefore, $x \circ y \subseteq S_I$. By Theorem 15, S_I is a hyper subalgebra of H.

According to Proposition 3 (8), $\forall x, y \in S_I$, $x \circ y$ is a singleton set. □

However, S_I is not necessarily a BZ-algebra; see Example 5.

Example 5. *Assume that $H = \{0, 1, 2, 3\}$. Define an operation $\circ$ on H in Table 9,*

Table 9. Quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	$\{1,2\}$	$\{0,2\}$	0	0
2	2	2	0	2
3	2	2	0	0

Then, $(H, \circ)$ is a quasi-hyper BZ-algebra, and $S_I = \{0, 2, 3\}$ is a quasi-hyper BZ-subalgebra. But it is not a BZ-algebra, since $3 \circ 0 = 2 \neq 3$.

Theorem 17. *Assume that $(H, \circ)$ is a standard quasi-hyper BZ-algebra. Then,*

$$BCC(H) := \{m \in H | 0 \circ m = 0\},$$

is a hyper BCC-subalgebra of H.

Theorem 18. *Assume that $(H, \circ)$ is a transitive standard quasi-hyper BZ-algebra. Then,*

$$AG(H) := \{m \in H | 0 \circ (0 \circ m) = m\},$$

is a quasi-hyper BZ-subalgebra of H and an anti-grouped BZ-algebra.

Proof. Obviously, $0 \in AG(H)$. Let $x, y \in AG(H)$. According to Proposition 7 and (TSHZ4), $0 \circ (0 \circ (x \circ y)) = (0 \circ (0 \circ x)) \circ (0 \circ (0 \circ y))$, then $0 \circ (0 \circ (x \circ y)) = x \circ y$. So, $x \circ y \subseteq AG(H)$. By Theorem 15, $AG(H)$ is a hyper subalgebra of H. $\forall x \in AG(H)$, by Proposition 4 (2), $0 \circ x$ is a singleton set, $x \circ x = (0 \circ (0 \circ x)) \circ (0 \circ (0 \circ x)) \ll 0 \circ 0 = 0$. So, for any $x, y \in AG(H)$, $|x \circ y| = 1$. Because $AG(H)$ is a standard quasi-hyper BZ-algebra, it is a BZ-algebra. According to Definition 5, we know $(AG(H), \circ)$ is an anti-grouped BZ-algebra. □

In the following, we study the connection between quasi-hyper BZ-algebra and semigroups.

Assume that $(H, \circ)$ is a quasi-hyper BZ-algebra. $\forall a, x \in H$, denote a map:

$$\rho_a : H \to P^*(H); x \mapsto x \circ a,$$

where $P^*(H)$ represents a nonempty subset of H.

$\forall a, b \in H$, $\forall x \in H$, denote $\rho_a \circ \rho_b$:

$$(\rho_a \circ \rho_b)(x) = \bigcup_{\forall y \in \rho_b(x)} \rho_a(y),$$

where $\circ$ means the composition operation of mappings.

Theorem 19. *Denote $M(H)$ as a set which is all compositional results of finite mappings which are for all $a \in H$; therefore, ρ_a. Then, $M(H)$ is a semigroup.*

Proof. $\forall x \in H$, $a, b, c \in H$, for any $s \in ((\rho_a \circ \rho_b) \circ \rho_c)(x)$, there exists $y \in \rho_c(x)$ s.t. $s \in (\rho_a \circ \rho_b)(y)$. Then, $\exists u \in \rho_b(y)$ s.t. $u \in \rho_b(\rho_c(x)) = \rho_b \circ \rho_c(x)$ and $s \in \rho_a(u)$. Then, $s \in (\rho_a \circ (\rho_b \circ \rho_c))(x)$ and $((\rho_a \circ \rho_b) \circ \rho_c)(x) \subseteq (\rho_a \circ (\rho_b \circ \rho_c))(x)$.

For any $t \in (\rho_a \circ (\rho_b \circ \rho_c))(x)$, there exists $m \in \rho_b \circ \rho_c(x)$ such that $t \in \rho_a(m)$. Then, $\exists n \in \rho_c(x)$ s.t. $m \in \rho_b(n)$ and $t \in \rho_a(\rho_b(n)) = \rho_a \circ \rho_b(n)$. Then, $t \in ((\rho_a \circ \rho_b) \circ \rho_c)(x)$ and $(\rho_a \circ (\rho_b \circ \rho_c))(x) \subseteq ((\rho_a \circ \rho_b) \circ \rho_c)(x)$.

So, $(\rho_a \circ (\rho_b \circ \rho_c))(x) = ((\rho_a \circ \rho_b) \circ \rho_c)(x)$. Then, $M(H)$ satisfies associative law. □

Example 6. *Assume that* $H = \{0, 1, 2, 3\}$*. The operation* $\circ$ *on* H *is shown in Table 10,*

Table 10. Quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**
0	$\{0,1\}$	$\{0,1\}$	2	2
1	1	$\{0,1\}$	2	2
2	2	2	$\{0,1\}$	$\{0,1\}$
3	3	3	1	$\{0,1\}$

Then, $(H, \circ)$ *is a quasi-hyper BZ-algebra, and* $M(H) = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_2^2\}$*, where* $\rho_2^2 = \rho_2 \circ \rho_2$*.*

We can verify the following:

$\rho_0 \circ \rho_0 = \rho_0, \rho_0 \circ \rho_1 = \rho_1, \rho_0 \circ \rho_2 = \rho_2, \rho_0 \circ \rho_3 = \rho_3, \rho_0 \circ \rho_2^2 = \rho_2^2;$
$\rho_1 \circ \rho_0 = \rho_1, \rho_1 \circ \rho_1 = \rho_1, \rho_1 \circ \rho_2 = \rho_3, \rho_1 \circ \rho_3 = \rho_3, \rho_1 \circ \rho_2^2 = \rho_2^2;$
$\rho_2 \circ \rho_0 = \rho_2, \rho_2 \circ \rho_1 = \rho_2, \rho_2 \circ \rho_2 = \rho_2^2, \rho_2 \circ \rho_3 = \rho_2^2, \rho_2 \circ \rho_2^2 = \rho_3;$
$\rho_3 \circ \rho_0 = \rho_3, \rho_3 \circ \rho_1 = \rho_3, \rho_3 \circ \rho_2 = \rho_2^2, \rho_3 \circ \rho_3 = \rho_2^2, \rho_3 \circ \rho_2^2 = \rho_3;$
$\rho_2^2 \circ \rho_0 = \rho_2^2, \rho_2^2 \circ \rho_1 = \rho_2^2, \rho_2^2 \circ \rho_2 = \rho_3, \rho_2^2 \circ \rho_3 = \rho_3, \rho_2^2 \circ \rho_2^2 = \rho_2^2.$

Then $(M(H), \circ)$ *is a semigroup, but it is not commutative, since* $\rho_1 \circ \rho_2 = \rho_3 \neq \rho_2 = \rho_2 \circ \rho_1$*. The operation on* $M(H)$ *is shown in Table 11.*

Table 11. The adjoint semigroup of quasi-hyper BZ-algebra.

$\circ$	ρ_0	ρ_1	ρ_2	ρ_3	ρ_2^2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	ρ_2^2
ρ_1	ρ_1	ρ_1	ρ_3	ρ_3	ρ_2^2
ρ_2	ρ_2	ρ_2	ρ_2^2	ρ_2^2	ρ_3
ρ_3	ρ_3	ρ_3	ρ_2^2	ρ_2^2	ρ_3
ρ_2^2	ρ_2^2	ρ_2^2	ρ_3	ρ_3	ρ_2^2

In the following, we give the concepts of anti-grouped quasi-hyper BZ-algebra and generalized anti-grouped quasi-hyper BZ-algebra. Moreover, we study the relations between quasi-hyper BZ-algebra and semihypergroups by generalized anti-grouped quasi-hyper BZ-algebra and associative quasi-hyper BZ-algebra.

Definition 26. *A quasi-hyper BZ-algebra* $(H, \circ)$ *is called an anti-grouped quasi-hyper BZ-algebra if,* $\forall x, y, z \in H$*, it satisfies* $(x \circ z) \circ (y \circ z) = x \circ y$*.*

Proposition 9. *Assume that* $(H, \circ)$ *is an anti-grouped quasi-hyper BZ-algebra. Therefore, it is an anti-grouped BZ-algebra.*

Proof. Assume that $|0 \circ 0| > 1$, let $0 \circ 0 = \{0, m\}$, and $m \neq 0$. According to Definition 26, $(0 \circ 0) \circ (0 \circ 0) = 0 \circ 0$, and $(0 \circ 0) \circ (0 \circ 0) = \{0 \circ 0, 0 \circ m, m \circ 0, m \circ m\}$, $0 \circ 0 = \{0, m\}$. Because $m \circ 0 \neq 0$, $m \circ 0 = m$. Then, $(m \circ 0) \circ (0 \circ 0) = m \circ (0 \circ 0) = \{m \circ 0, m \circ m\}$, $m \circ 0 = m$, according to Definition 26, $\{m \circ 0, m \circ m\} = m$. So, $0 \in m \circ m = m$, but $m \neq 0$. So, $|0 \circ 0| = 1$ and $0 \circ 0 = 0$. According to Proposition 4, $\forall x \in H$, $0 \circ x$ is singleton set.

$\forall x, y \in H$, assume that $x \ll y$, that is, $0 \in x \circ y$. Then, $0 = 0 \circ 0 = (y \circ y) \circ (x \circ y) = y \circ x$, and $y \ll x$. According to Definition 22, $x = y$. Then, $x \ll y$ implies $x = y$. By Definition 22, $x \ll x \circ 0$, then $x \circ 0 = x$. Then, $0 \circ (0 \circ x) \subset (x \circ x) \circ (0 \circ x) = x \circ 0 = x$. So, $0 \circ (0 \circ x) = x$, because $0 \circ x$ is singleton set. Therefore, $x \circ x = (0 \circ (0 \circ x)) \circ (0 \circ (0 \circ x)) = 0$. According to Proposition 3, $\forall x, y, z \in H$, $((x \circ z) \circ (y \circ z)) \circ (x \circ y) = 0$, and $x = x \circ 0$. So, H is anti-grouped BZ-algebra. □

Definition 27. *A quasi-hyper BZ-algebra $(H, \circ)$ is called a generalized anti-grouped quasi-hyper BZ-algebra if, $\forall x, y, z \in H$, it satisfies $(x \circ (0 \circ y)) \circ (0 \circ z) = x \circ (0 \circ (y \circ (0 \circ z)))$.*

Remark 2. *Every generalized anti-grouped hyper BZ-algebra is a generalized anti-grouped quasi-hyper BZ-algebra.*

The following example shows that not every generalized anti-grouped quasi-hyper BZ-algebra is a generalized anti-grouped hyper BZ-algebra.

Example 7. *Assume that $H = \{0, 1, 2, 3\}$. The operation $\circ$ on H is shown in Table 12,*

Table 12. Generalized anti-grouped quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**
0	$\{0,1\}$	$\{0,1\}$	3	3
1	1	$\{0,1\}$	3	3
2	3	3	$\{0,1\}$	$\{0,1\}$
3	3	3	1	$\{0,1\}$

Then, $(H, \circ)$ is a generalized anti-grouped quasi-hyper BZ-algebra. However, it is not a hyper BZ-algebra, since $0 \circ (0 \circ 0) = \{0, 1\}$ and $1 \ll 0$ is not true.

Proposition 10. *In any generalized anti-grouped quasi-hyper BZ-algebra $(H, \circ)$, $\forall x, y \in H$, define "$\oplus$" :*

$$x \oplus y = x \circ (0 \circ y).$$

Then, $(H, \oplus, 0)$ is a semihypergroup.

Proof. $\forall x, y, z \in H$, $(x \oplus y) \oplus z = (x \circ (0 \circ y)) \circ (0 \circ z) = x \circ (0 \circ (y \circ (0 \circ z))) = x \oplus (y \oplus z)$. Then, $(H, \oplus, 0)$ is a semihypergroup. □

Example 8. (1) *Assume that $H = \{0, 1, 2, 3, 4\}$. An operation $\circ$ on H is shown in Table 13,*

Table 13. Generalized anti-grouped quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**	**4**
0	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
1	4	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
2	4	4	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
3	4	4	4	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
4	$\{1,2,3,4\}$	$\{2,3,4\}$	$\{3,4\}$	4	$\{0,1,2,3,4\}$

Then, $(H, \circ)$ is a generalized anti-grouped quasi-hyper BZ-algebra. According to Proposition 10, we get a semihypergroup $(H, \oplus)$ and the operation $\oplus$ on it is shown in Table 14.

Table 14. Semihypergroup derived from Table 13.

$\oplus$	**0**	**1**	**2**	**3**	**4**
0	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
1	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
2	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
3	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
4	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$

(2) *Assume that* $H = \{0,1,2,3\}$. *An operation* $\circ$ *on* H *in shown in Table 15.*

Table 15. Generalized anti-grouped quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	$\{1,2\}$	$\{0,2\}$	0	2
2	2	2	0	2
3	3	3	0	0

Then $(H,\circ)$ *is a generalized anti-grouped quasi-hyper BZ-algebra. According to Proposition 10, we get a semihypergroup* $(H,\oplus)$ *and the operation* $\oplus$ *on it is shown in Table 16.*

Table 16. Semihypergroup derived from Table 15.

$\oplus$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
2	2	2	2	2
3	3	3	3	3

Definition 28. *A quasi-hyper BZ-algebra* $(H,\circ)$ *is called associative if,* $\forall x,y,z \in H$*, it satisfies* $x\circ(y\circ z) = (x\circ y)\circ z$.

Remark 3. *Every associative quasi-hyper BZ algebra is a generalized anti-grouped quasi-hyper BZ algebra.*

Example 7 shows that not every generalized anti-grouped quasi-hyper BZ algebra is an associative quasi-hyper BZ algebra, since $1\circ(0\circ 0) = \{0,1\} \neq 1 = (1\circ 0)\circ 0$.

Proposition 11. *Assume that* $(H,\circ)$ *is an associative quasi-hyper BZ-algebra. Then, the below conditions hold:* $\forall x \in H$,
(1) $0\circ 0 = 0$;
(2) $(0\circ x)\circ(0\circ x) = 0$;
(3) $0\circ x$ *is a singleton set;*
(4) $(0\circ x)\circ 0 = 0\circ x$;
(5) $0\circ x = x$;
(6) $x\circ x = 0$;
(7) $x\circ 0 = x$.

Proof. (1) Assume that $0\circ 0 \neq 0$, and let $0\circ 0 = \{0,m\}$ and $m \neq 0$. Therefore, $0 \in m\circ m \subseteq m\circ(0\circ 0) = (m\circ 0)\circ 0$. So, $\exists p \in m\circ 0$ s.t. $0 \in p\circ 0$. That is, $p \ll 0$, and by (QHZ5), $p = 0$. Therefore, $0 \in m\circ 0$. That is, $m \ll 0$ and $m = 0$. So, $m \neq 0$ is not true, and $0\circ 0 = 0$.

(2) This follows from Proposition 4.

(3) This follows from Proposition 4.

(4) This follows from Proposition 4.

(5) By (2) and (4), for any $x \in H$, $0 \in 0 \circ 0 \subseteq 0 \circ (x \circ x) = (0 \circ x) \circ x = ((0 \circ x) \circ 0) \circ x = (0 \circ x) \circ (0 \circ x) = 0$. Because $0 \circ x$ is a singleton set, $0 \circ x \ll x$. For any $x \in H$, $0 \in (x \circ 0) \circ (x \circ 0) = x \circ ((0 \circ x) \circ 0) = x \circ (0 \circ x)$, so $x \ll 0 \circ x$. By (QHZ3), $x = 0 \circ x$.

(6) $x \circ x = (0 \circ x) \circ (0 \circ x) = 0$.

(7) According to (6) and Proposition 3 (8), $\forall x, y \in H$, $|x \circ y| = 1$. Then, $x \circ 0 = (0 \circ x) \circ (x \circ x) \ll 0 \circ x = x$. By (QHZ4), $x \ll x \circ 0$, and by (QHZ3), $x = x \circ 0$. □

Theorem 20. *Assume that $(H, \circ)$ is an associative quasi-hyper BZ-algebra. Then, it is an associative BCI-algebra.*

Proof. By Propositions 11 (6) and 3 (8), $\forall x, y, z \in H$, $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = (x \circ y) \circ (x \circ (z \circ z) \circ y) = (x \circ y) \circ ((x \circ 0) \circ y) = (x \circ y) \circ (x \circ y) = 0$, and Definition 1 (1) holds. $(x \circ (x \circ y)) \circ y = (x \circ x) \circ (y \circ y) = 0 \circ 0 = 0$, and Definition 1 (2) holds. Obviously, $x \circ x = 0$ and Definition 1 (3) holds. Clearly, Definition 1 (4) holds. So, $(H, \circ)$ is an associative BCI-algebra. □

In the following, we give the concepts of generalized quasi-left alter quasi-hyper BZ-algebra and QM-quasi-hyper BZ-algebra.

Definition 29. *A quasi-hyper BZ-algebra $(H, \ll, \circ, 0)$ is called QM-quasi-hyper BZ-algebra if every element of H is a quasi-minimal element.*

Theorem 21. *Assume that $(H, \ll, \circ, 0)$ is a quasi-hyper BZ-algebra. Then, H is a QM-quasi-hyper BZ-algebra iff it meets: $\forall x, y \in H - \{0\}$,*

$$x \ll y \text{ implies } x = y.$$

Proof. The proof is similar to Theorem 12. □

According to Remark 1, we know that both BZ-algebra and hyper BZ-algbera are quasi-hyper BZ-algebra. So, both QM-BZ-algebra and QM-hyper BZ-algebra are QM-quasi-hyper BZ-algebra, but not every QM-quasi-hyper BZ-algebra is QM-BZ-algebra and QM-hyper BZ-algebra (see Example 9).

Example 9. *Assume that $H = \{0, 1, 2\}$. An operation on H is shown in Table 17.*

Table 17. QM-quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**
0	$\{0,2\}$	$\{1,2\}$	$\{0,2\}$
1	$\{1,2\}$	$\{0,2\}$	2
2	2	2	$\{0,2\}$

Then, $(H, \circ)$ is a QM-quasi-hyper BZ-algebra, but it is not a QM-hyper BZ-algebra since $0 \circ (0 \circ 0) = \{0, 2\}$,$2 \ll 0$ is not true. Moreover, it is not a QM-BZ-algebra.

Definition 30. *Quasi-hyper BZ-algebra $(H, \circ)$ is called a generalized quasi-left alter quasi-hyper BZ algebra if, $\forall x, y \in H$,*

$$x \circ (x \circ y) = 0 \circ (0 \circ y), x \neq y.$$

Example 10. *Assume that $H = \{0, 1, 2, 3, 4\}$. Define an operation $\circ$ on H in Table 18,*

Table 18. Generalized quasi-left alter quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**	**4**
0	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
1	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
2	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
3	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
4	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{2,3,4\}$	$\{0,1,2,3,4\}$

Then, $(H,\circ)$ is generalized quasi-left alter quasi-hyper BZ-algebra.

Proposition 12. *Let $(H,\circ)$ be a generalized quasi-left alter quasi-hyper BZ-algebra satisfying $0\circ 0=0$. Therefore, H is a BCI-algebra.*

Proof. The proof is similar to Theorem 13. □

According to Theorems 11 and 14, we know generalized quasi-left alter BZ-algebra is QM-BZ-algebra and generalized quasi-left alter hyper BZ-algebra is QM-hyper BZ-algebra. However, not every generalized quasi-left alter quasi-hyper BZ-algebra is QM-quasi-hyper BZ-algebra; see Example 10.

In addition, not every QM-quasi-hyper BZ algebra is a generalized quasi-left alter quasi-hyper BZ-algbera; see Example 11.

Example 11. *Let $H=\{0,1,2,3,4\}$. The operation on H is shown in Table 19.*

Table 19. Generalized quasi-left alter quasi-hyper BZ-algebra.

$\circ$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
2	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
3	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
4	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{0,1,2,3,4\}$

Then, $(H,\circ)$ is QM-quasi-hyper BZ-algebra, but it is not a generalized quasi-left alter quasi-hyper BZ-algebra since $1\circ(1\circ 0)=\{0,1,2,3,4\}$, $0\circ(0\circ 0)=0$ and $1\neq 0$.

5. Discussion

In this paper, we firstly proposed a construction theorem of BZ-algebra through BCC-algebra and anti-grouped BZ-algebra. Secondly, the definition of QM-BZ-algebra was proposed and their properties were studied. Moreover, the structure of QM-BZ-algebra was obtained: each QM-BZ-algebra is a KG-union of quasi-alter BCK-algebra and anti-grouped BZ-algebra. Thirdly, we introduced generalized quasi-left alter BZ-algebra and proved that every generalized quasi-left alter BZ-algebra is QM-BZ-algebra. Forthly, we introduced the notion of quasi-minimal elements into hyper BZ-algebra, and QM-hyper BZ-algebra was obtained. Additionally, the relationship between QM-hyper BZ-algebra and generalized quasi-left alter hyper BZ-algebra was discussed. Next, quasi-hyper BZ-algebra, which is an extension of hyper BZ-algebra and quasi-hyper BCI-algebra, was proposed. We also gave the concepts of anti-grouped quasi-hyper BZ-algebra, generalized anti-grouped quasi-hyper BZ-algebra, and associative quasi-hyper BZ-algebra, and proved that: (1) an anti-grouped quasi-hyper BZ-algebra is an anti-grouped BZ-algebra; (2) every generalized anti-grouped quasi-hyper BZ-algebra corresponds to a semihypergroup; (3) every associative quasi-hyper BZ-algebra is an associative BCI-algebra.

The above research results are helpful for revealing the connections among the relevant logic algebras (and their hyper structures), and can be used for reference for other non-classical logic algebras. As a further research topic, we will consider the relationship among (quasi-hyper) BZ-algebras, semihypergroups (see [37]) and BI-algebras (basic implication algebras, see [38,39]) as well as their ideal (filter) theories .

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