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The Local Antimagic Total Chromatic Number of Some Wheel-Related Graphs

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Abstract: Let $G = (V, E)$ be a connected graph with $|V| = n$ and $|E| = m$. A bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \cdots, n+m\}$ is called local antimagic total labeling if, for any two adjacent vertices u and v, $\omega_t(u) \neq \omega_t(v)$, where $\omega_t(u) = f(u) + \sum_{e \in E(u)} f(e)$, and $E(u)$ is the set of edges incident to u. Thus, any local antimagic total labeling induces a proper coloring of G, where the vertex x in G is assigned the color $\omega_t(x)$. The local antimagic total chromatic number, denoted by $\chi_{lat}(G)$, is the minimum number of colors taken over all colorings induced by local antimagic total labelings of G. In this paper, we present the local antimagic total chromatic numbers of some wheel-related graphs, such as the fan graph F_n, the bowknot graph $B_{n,n}$, the Dutch windmill graph D_4^n, the analogous Dutch graph AD_4^n and the flower graph $\mathscr{F}_n$.

Keywords: local antimagic total labeling; local antimagic total chromatic number; wheel-related graphs

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1. Introduction

Let $G = (V, E)$ be a connected simple graph with n vertices and m edges. In 2017, Arumugam et al. [1] and Bensmail et al. [2] independently introduced the notation of local antimagic labeling of graphs. A bijection $f : E(G) \rightarrow \{1, 2, \cdots, m\}$ is called a *local antimagic labeling* of G if any two adjacent vertices u and v in G satisfy $\omega(u) \neq \omega(v)$, where $\omega(u) = \sum_{e \in E(u)} f(e)$, and $E(u)$ is the set of edges incident to u. It is clear that assigning $\omega(x)$ to x for each $x \in V(G)$ naturally induces a proper vertex coloring of G, which is called a local antimagic coloring of G. A graph G is called local antimagic if G has a local antimagic labeling. The *local antimagic chromatic number* [1] of G, denoted by $\chi_{la}(G)$, is the minimum number of colors taken over all colorings of G induced by local antimagic labelings of G.

Arumugam et al. [1] presented the local antimagic chromatic numbers of some families of graphs, such as star S_n, path P_n, cycle C_n, wheel W_n, friendship graph $\mathcal{F}_n$, complete graph K_n, complete bipartite graph $K_{2,n}$ and the join graph $G \vee \overline{K_2}$, where G is a graph of order $n \geq 4$ and $\overline{K_2}$ is the complement graph of the complete graph K_2. Meanwhile, many researchers have studied the local antimagic chromatic numbers of classes of many graphs. In [3], Lau et al. gave counterexamples to the lower bound of $\chi_{la}(G \vee \overline{K_2})$ that was obtained in [1]. Another counterexample was independently found by Shaebani [4]. Lau et al. gave affirmative solutions on Problem 3.3 of [1] and settled Theorem 2.15 of [1]. Moreover, they also completely determined the local antimagic chromatic number of a complete bipartite graph. In [5], Lau et al. presented some sufficient conditions for $\chi_{la}(H) \leq \chi_{la}(G)$, where H is obtained from G by deleting or adding a certain edge. They then determined the exact values of the local antimagic chromatic numbers of many cycle-related join graphs. Nazula et al. [6] determined the local antimagic chromatic number of unicyclic graphs. Premalatha et al. [7] determined the local antimagic chromatic number of the corona product of two graphs, such as paths with null graphs. In [8], Bača et al. estimated that for the bounds of the local antimagic chromatic number for disjoint union of multiple copies of a graph, there are trees and graphs with vertices of even degrees and with chromatic index 3. From the results proved by Haslegrave [9], Bača et al. obtained that the local antimagic chromatic numbers of disjoint union of arbitrary graphs are finite if and only if none of these graphs contain an isolated edge as a subgraph.

Recently, Putri et al. [10] extended this notion by labeling the vertices and edges of a graph G to establish a vertex coloring. The *local antimagic total labeling* on a graph G is defined to be an assignment $f : V(G) \cup E(G) \to \{1, 2, \cdots, |V(G)| + |E(G)|\}$ so that the weights of any two adjacent vertices u and v are distinct, that is, $\omega_t(u) \neq \omega_t(v)$, where $\omega_t(u) = f(u) + \sum_{e \in E(u)} f(e)$. Analogous to the local antimagic labeling, any local antimagic total labeling induces a proper vertex coloring of G, where the vertex x in G is assigned the color $\omega_t(x)$. The *local antimagic total chromatic number* of G, denoted by $\chi_{lat}(G)$, is the minimum number of colors taken over all colorings induced by local antimagic total labelings of G. Clearly, $\chi_{la}(G) \geq \chi_{lat}(G) \geq \chi(G)$ for any graph G. Putri et al. [10] presented the local antimagic total chromatic numbers of some families tree, such as star, double star, banana tree graph, centipede graph, and the amalgamation of the star graph. In [11], Kurniawati et al. determined the exact values of local antimagic total chromatic numbers of graphs $G \odot K_2$, when G is the star, path, cycle and friendship graphs. Moreover, Kurniawati et al. [12] determined the exact value of graphs $G \odot mK_1$, when G is the star, path, cycle and friendship graphs.

In this paper, we present the local antimagic total chromatic numbers of some wheel-related graphs, such as the fan graph F_n, the bowknot graph $B_{n,n}$, the Dutch windmill graph D_4^n, the analogous Dutch windmill graph AD_4^n and the flower graph $\mathscr{F}_n$.

2. Main Results

In this section, we compute the local antimagic total chromatic numbers of some wheel-related graphs. In [1], Arumugam et al. presented the exact value of $\chi_{la}(W_n)$ with three cases: (i) $\chi_{la}(W_n) = 4$ for $n \equiv 1 \pmod 4$, (ii) $\chi_{la}(W_n) = 4$ for $n \equiv 3 \pmod 4$ and (iii) $\chi_{la}(W_n) = 3$ for $n \equiv 2 \pmod 4$. However, they only presented the range of $\chi_{la}(W_n)$ for $n \equiv 0 \pmod 4$. Then Lau et al. [3] gave the exact value of $\chi_{la}(W_n)$ that $\chi_{la}(W_n) = 3$ for $n \equiv 0 \pmod 4$. In [5], Lau et al. corrected three errors of the local antimagic labeling for $n \equiv 1 \pmod 4$ and one error of the local antimagic labeling for $n \equiv 3 \pmod 4$. Arumugam et al. and Lau et al. completely determined the exact value of the local antimagic chromatic number of the wheel in the following lemma.

Lemma 1 ([1,3])**.** *For the wheel W_n of order $n+1$, we have*

$$\chi_{la}(W_n) = \begin{cases} 3, & \text{for even } n, \\ 4, & \text{for odd } n. \end{cases}$$

Lemma 2 ([10])**.** *For any graph G, we have $\chi_{lat}(G) \geq \chi(G)$.*

The fan graph F_n of order $n+1$ is obtained by deleting a rim edge of the wheel W_n, where the central vertex of W_n is also the central vertex of F_n. The fan graph F_5 is shown in Figure 1. In fact, Slamin et al. [13] in 2018 and Amalia et al. [14] in 2021 presented the exact value of the local antimagic total chromatic number of the fan graph. However, the following local antimagic total labeling of the fan graph in Theorem 1 is different from that of these authors.

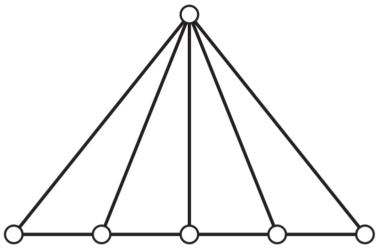

Figure 1. The fan graph F_5.

Theorem 1. *For the fan graph F_n and $n \geq 4$, then $\chi_{lat}(F_n) = 3$.*

Proof. Let $V(F_n) = \{c\} \cup \{u_i | 1 \leq i \leq n\}$ and $E(F_n) = \{cu_i | 1 \leq i \leq n\} \cup \{u_i u_{i+1} | 1 \leq i \leq n-1\}$ be vertex set and edge set of the fan graph, respectively. Then we obtain $|V(F_n)| = n+1$ and $|E(F_n)| = 2n-1$. It is clear that $\chi_{lat}(F_n) \geq \chi(F_n) = 3$. In order to prove that $\chi_{lat}(F_n) = 3$, it suffices to prove that $\chi_{lat}(F_n) \leq 3$, which means that we should obtain a local antimagic total labeling using three distinct colors. Define $f : V(F_n) \cup E(F_n) \to \{1, 2, \cdots, 3n\}$. Let $f(c) = 3n$, and label the edges $u_i u_{i+1}$ for i such that $1 \leq i \leq n-1$ as follows:

$$f(u_i u_{i+1}) = \begin{cases} n - \frac{i+1}{2}, & \text{for odd } i, \\ \frac{i}{2}, & \text{for even } i. \end{cases}$$

Then label the remaining edges and vertices of graph F_n. Let us discuss two cases for n.

Case 1. For odd n.

$$f(cu_i) = \begin{cases} 2n + \frac{i-1}{2}, & \text{for odd } i \text{ and } 1 \leq i < n, \\ \frac{5n-3}{2} + \frac{i}{2}, & \text{for even } i, \text{ and } 1 \leq i \leq n, \\ 3n - 1 & \text{for } i = n. \end{cases}$$
$$f(u_i) = \begin{cases} \frac{3n}{2} - \frac{i}{2}, & \text{for odd } i \text{ and } 1 \leq i \leq n, \\ 2n - \frac{i}{2}, & \text{for even } i \text{ and } 1 \leq i \leq n. \end{cases}$$

From the above labelings, we have

$$\omega_t(u_i) = \begin{cases} \frac{9n-3}{2}, & \text{for odd } i \text{ and } 1 \leq i \leq n, \\ \frac{11n-3}{2}, & \text{for even } i \text{ and } 1 \leq i \leq n. \end{cases}$$
$$\omega_t(c) = \frac{5n^2+5n}{2}.$$

Thus f is a local antimagic total labeling using three colors, and we obtain $\chi_{lat}(F_n) \leq 3$.

Case 2. For even n.

$$f(cu_i) = \begin{cases} \frac{5n-1}{2} - \frac{i}{2}, & \text{for odd } i \text{ and } 1 \leq i \leq n, \\ 3n - 1 - \frac{i}{2}, & \text{for even } i \text{ and } 1 \leq i < n, \\ 3n - 1, & \text{for } i = n. \end{cases}$$
$$f(u_i) = \begin{cases} n + \frac{i-1}{2}, & \text{for odd } i \text{ and } 1 \leq i \leq n, \\ \frac{3n-2}{2} + \frac{i}{2}, & \text{for even } i \text{ and } 1 \leq i \leq n. \end{cases}$$

From the above labelings, we have

$$\omega_t(u_i) = \begin{cases} \frac{9n-4}{2}, & \text{for odd } i \text{ and } 1 \leq i \leq n, \\ \frac{11n-4}{2}, & \text{for even } i \text{ and } 1 \leq i \leq n. \end{cases}$$
$$\omega_t(c) = \frac{5n^2+5n}{2}.$$

Thus f is a local antimagic total labeling using three colors, and we obtain $\chi_{lat}(F_n) \leq 3$. The proof is complete. □

All figures in this paper, the red front represents the local antimagic total labeling of edges and vertices of graph; other colors represent the sum of weights of vertex and the edges incident with the vertex in the local antimagic total labeling of graphs. Different colors are selected to clearly see the number of different colors of vertices in the figure.

Example 1. *The local antimagic total labelings of the fan graph F_9 and F_8 are shown in Figures 2 and 3.*

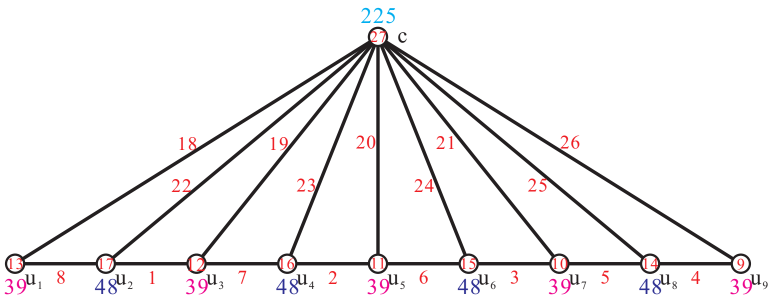

Figure 2. The local antimagic total labeling of F_9.

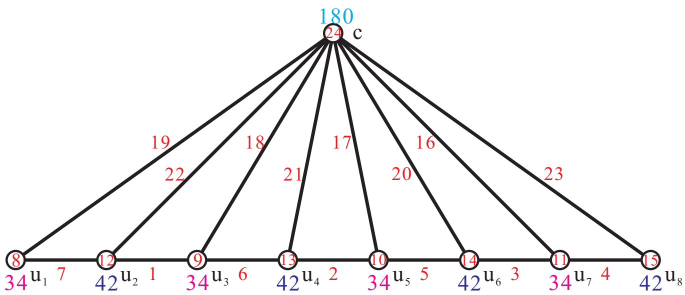

Figure 3. The local antimagic total labeling of F_8.

The bowknot graph, denoted by $B_{n,n}$, is the graph by gluing two central vertices of double fan graphs F_n. Obviously, the bowknot graph $B_{n,n}$ is obtained from the wheel W_{2n} by deleting two edges every $n-1$ edges on the rim of the wheel, and shown in Figure 4. It has $2n+1$ vertices and $4n-2$ edges.

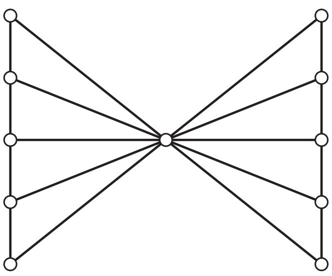

Figure 4. The bowknot graph $B_{5,5}$.

Theorem 2. *For the bowknot graph $B_{n,n}$ $(n \geq 5)$, we have $\chi_{lat}(B_{n,n}) = 3$.*

Proof. Let $V(B_{n,n}) = \{c\} \cup \{u_i, v_i | 1 \leq i \leq n\}$ be the vertex set of graph $B_{n,n}$, and let $E(B_{n,n}) = \{cu_i, cv_i | 1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1} | 1 \leq i \leq n-1\}$ be the edge set of $B_{n,n}$. Since K_3 is an induced subgraph of $B_{n,n}$, we have $\chi_{lat}(B_{n,n}) \geq \chi(B_{n,n}) = 3$. Define $f : V(B_{n,n}) \cup E(B_{n,n}) \rightarrow \{1, 2, \cdots, 6n-1\}$ and consider the following two cases.

Case 1. If n is odd.

Firstly, label the edges of $B_{n,n}$ as follows:

$$f(u_iu_{i+1}) = \begin{cases} i, & \text{if } i \text{ is odd and } 1 \le i \le n-1, \\ 2n-i, & \text{if } i \text{ is even and } 1 \le i \le n-1. \end{cases}$$
$$f(v_iv_{i+1}) = \begin{cases} i+1, & \text{if } i \text{ is odd and } 1 \le i \le n-1, \\ 2n-1-i, & \text{if } i \text{ is even and } 1 \le i \le n-1. \end{cases}$$
$$f(cu_i) = \begin{cases} 4n-3, & \text{if } i = 1, \\ 2n-4+i & \text{if } 3 \le i \le n-1 \text{ and } i \text{ is odd}, \\ 3n-5+i, & \text{if } 1 \le i \le n-1 \text{ and } i \text{ is even}. \\ 4n-4, & \text{if } i = n. \end{cases}$$
$$f(cv_i) = \begin{cases} 3n-4, & \text{if } i = 1, \\ 2n-3+i & \text{if } 3 \le i \le n-1 \text{ and } i \text{ is odd}, \\ 3n-4+i, & \text{if } 1 \le i \le n-1 \text{ and } i \text{ is even}. \\ 4n-2, & \text{if } i = n. \end{cases}$$

Secondly, label the vertices of $B_{n,n}$ by the following way:

$$f(u_i) = \begin{cases} 5n-i & \text{if } 1 \le i \le n \text{ and } i \text{ is odd}, \\ 6n-i, & \text{if } 1 \le i \le n \text{ and } i \text{ is even}. \end{cases}$$
$$f(v_i) = \begin{cases} 6n-1, & \text{if } i = 1, \\ 5n-1-i & \text{if } 3 \le i \le n \text{ and } i \text{ is odd}, \\ 6n-1-i, & \text{if } 1 \le i \le n \text{ and } i \text{ is even}. \end{cases}$$
$$f(c) = 5n-2.$$

Accordingly, we have

$$\begin{aligned} &\omega_t(u_i) = \omega_t(v_i) = 9n-3, && \text{if } i \text{ is odd and } 1 \le i \le n, \\ &\omega_t(u_i) = \omega_t(v_i) = 11n-6, && \text{if } i \text{ is even and } 1 \le i \le n, \\ &\omega_t(c) = 6n^2+2n-2. \end{aligned}$$

It is clear that f is a local antimagic total labeling of $B_{n,n}$ using three distinct colors and $\chi_{lat}(B_{n,n}) \le 3$ for odd n.

Case 2. If n is even.

Label the edges of $B_{n,n}$ as follows:

$$f(u_iu_{i+1}) = \begin{cases} i, & \text{if } i \text{ is odd and } 1 \le i \le n-1, \\ 2n-i, & \text{if } i \text{ is even and } 1 \le i \le n-1. \end{cases}$$
$$f(v_iv_{i+1}) = \begin{cases} i+1, & \text{if } i \text{ is odd and } 1 \le i \le n-1, \\ 2n-1-i, & \text{if } i \text{ is even and } 1 \le i \le n-1. \end{cases}$$
$$f(cu_i) = \begin{cases} 4n-3, & \text{if } i = 1, \\ 3n-5+i, & \text{if } 3 \le i \le 5 \text{ and } i \text{ is odd}, \\ 3n-4+i, & \text{if } 7 \le i \le n \text{ and } i \text{ is odd}, \\ 2n-3+i, & \text{if } 1 \le i \le n \text{ and } i \text{ is even}. \end{cases}$$
$$f(cv_i) = \begin{cases} 4n-2, & \text{if } i = 1, \\ 3n-1, & \text{if } i = 3, \\ 3n-3+i, & \text{if } 5 \le i \le n \text{ and } i \text{ is odd}, \\ 2n-2+i, & \text{if } 1 \le i < n \text{ and } i \text{ is even}. \\ 4n-1, & \text{if } i = n. \end{cases}$$

Then, give the exact values of the vertices of $B_{n,n}$:

$$f(u_i) = \begin{cases} 6n-1, & \text{if } i = 1, \\ 5n+1-i & \text{if } 3 \le i \le 5 \text{ and } i \text{ is odd}, \\ 5n-i & \text{if } 7 \le i \le n \text{ and } i \text{ is odd}, \\ 6n-2-i, & \text{if } 1 \le i < n \text{ and } i \text{ is even}, \\ 6n-2, & \text{if } i = n. \end{cases}$$
$$f(v_i) = \begin{cases} 6n-3, & \text{if } i = 1, \\ 5n-3, & \text{if } i = 3, \\ 5n-1-i & \text{if } 5 \le i \le n \text{ and } i \text{ is odd}, \\ 6n-3-i, & \text{if } 1 \le i < n \text{ and } i \text{ is even}, \\ 5n-5, & \text{if } i = n. \end{cases}$$
$$f(c) = 3n+1.$$

From above labelings under f, we obtain

$$\omega_t(u_i) = \omega_t(v_i) = 10n-3, \quad \text{if } i \text{ is odd and } 1 \le i \le n,$$
$$\omega_t(u_i) = \omega_t(v_i) = 10n-6, \quad \text{if } i \text{ is even and } 1 \le i \le n,$$
$$\omega_t(c) = 6n^2+n-1.$$

Clearly, f is a local antimagic total labeling of $B_{n,n}$ using three distinct colors and $\chi_{lat}(B_{n,n}) \le 3$. Hence, we obtain $\chi_{lat}(B_{n,n}) = 3$ for $n \ge 2$, and the proof is completed. □

Example 2. *Let $n = 5$ and $n = 6$. We have the local antimagic total labelings of the bowknot graph $B_{5,5}$ and $B_{6,6}$ in Figures 5 and 6.*

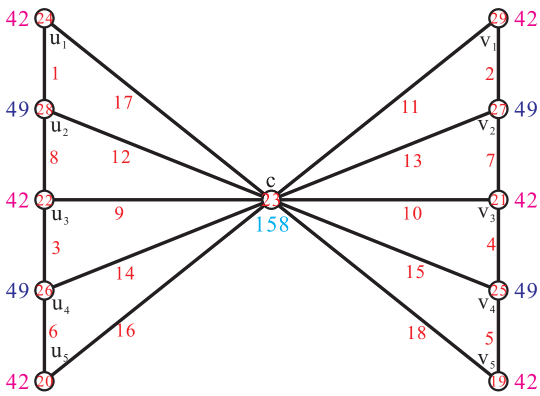

Figure 5. The local antimagic total labeling of $B_{5,5}$.

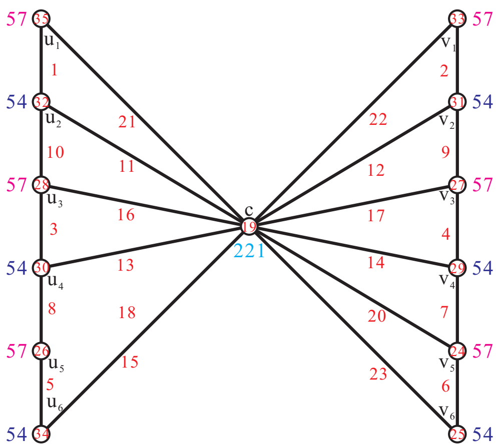

Figure 6. The local antimagic total labeling of $B_{6,6}$.

The Dutch windmill graph, denoted by $D_4^n (n \geq 2)$*, is a graph of order* $3n + 1$ *and size* $4n$ *by gluing a common vertex of n cycles* C_4*. A example of the Dutch windmill graph is shown in Figure 7. Note that the Dutch windmill graph* D_4^n *is obtained from the wheel* W_{3n} *by deleting one edge continuously every 2 edges on the rim of the wheel* W_{3n}*, and then delete the middle spoke edge of each vane in the resulting graph. Let* $V(D_4^n) = \{c\} \cup \{u_i, v_i, w_i | 1 \leq i \leq n\}$ *and* $E(D_4^n) = \{cu_i, cv_i, w_iu_i, w_iv_i | 1 \leq i \leq n\}$ *be the vertex and edge sets of graph* D_4^n*, respectively.*

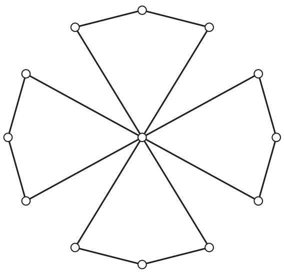

Figure 7. The Dutch graph D_4^4.

Theorem 3. *For the Dutch windmill graph* D_4^n*, we have*

$$\chi_{lat}(D_4^n) = \begin{cases} 2, & \text{if } n \leq 7, \\ 3, & \text{otherwises.} \end{cases}$$

Proof. Obviously, the lower bound of the local antiamgic total chromatic number for graph D_4^n is two since $\chi_{lat}(D_4^n) \geq \chi(D_4^n) = 2$. Then we consider the upper bound of $\chi_{lat}(D_4^n)$.

Define $f : V(D_4^n) \cup E(D_4^n) \to \{1, 2, \cdots, 7n + 1\}$. Let σ be the sum of the weights of the vertex c and the edges incident with the vertex c and let ξ be the sum of the weights of the vertices w_i for $1 \leq i \leq n$. If we use the minimum weights, label the vertex c and the edges incident with the vertex c, then $\sigma \geq 1 + 2 + \cdots + 2n + 1$, and if we use the maximum weights, label the edges w_iu_i, w_iv_i and the vertices w_i for $1 \leq i \leq n$, then $\xi \leq (4n + 2) + (4n + 3) + \cdots + 7n + 1$. Suppose that there is a local antiamgic total labeling using two distinct colors labeling the graph D_4^n; thus, the color of the vertex c is same as the vertices w_i for $1 \leq i \leq n$. It means $\sigma \leq \frac{\xi}{n}$ and so $n \leq 7$. When $\sigma \leq \frac{\xi}{n}$, there possibly is obtained a local antimagic total labeling f using two distinct colors such that the vertex c and the vertices w_i have the same color for $n \leq 7$. However, when $\sigma > \frac{\xi}{n}$, there must exist three different colors. Consider two cases as follows:

Case 1. For $n > 7$.

According to the parity of n, there are two subcases to confirm the exact values of the local antimagic total labeling.

Subcase 1. If n is odd.

Label the edges and vertices of the graph D_4^n by the following way:

$$f(cu_i) = i, \quad \text{for } 1 \leq i \leq n,$$

$$f(cv_i) = \begin{cases} \frac{3n-1}{2} - 2i, & \text{for } 1 \leq i \leq \lfloor \frac{n-3}{4} \rfloor, \\ \frac{5n-1}{2} - 2i, & \text{for } \lfloor \frac{n+1}{4} \rfloor \leq i \leq 2\lceil \frac{n-1}{4} \rceil - \lfloor \frac{n-3}{4} \rfloor, \\ \frac{7n-1}{2} - 2i. & \text{for } 2\lceil \frac{n-1}{4} \rceil - \lfloor \frac{n-3}{4} \rfloor < i \leq n. \end{cases}$$

$$f(w_iu_i) = \begin{cases} \frac{11n+3}{2} + i, & \text{for } 1 \leq i \leq \frac{n-1}{2}, \\ \frac{9n+3}{2} + i, & \text{for } \frac{n+1}{2} \leq i \leq n. \end{cases}$$

$$f(w_iv_i) = \begin{cases} 7n + 1 - 2i, & \text{for } 1 \leq i \leq \frac{n-1}{2}, \\ 8n + 1 - 2i, & \text{for } \frac{n+1}{2} \leq i < n, \\ 7n + 1, & \text{for } i = n. \end{cases}$$

Then

$$f(c) = 5n+1,$$
$$f(u_i) = \begin{cases} 5n+1-2i, & \text{for } 1 \le i \le \frac{n-1}{2}, \\ 6n+1-2i, & \text{for } \frac{n+1}{2} \le i \le n. \end{cases}$$
$$f(v_i) = \begin{cases} 2n+2+4i, & \text{for } 1 \le i \le \frac{n-1}{2} - \lceil \frac{n+1}{4} \rceil, \\ n+2+4i, & \text{for } \frac{n+1}{2} - \lceil \frac{n+1}{4} \rceil \le i \le \frac{n-1}{2}, \\ 2+4i, & \text{for } \frac{n+1}{2} \le i \le \frac{n-3}{2} + \lceil \frac{n+1}{4} \rceil, \\ 2-n+4i, & \text{for } \frac{n-1}{2} + \lceil \frac{n+1}{4} \rceil \le i < n, \\ 2n+2, & \text{for } i = n. \end{cases}$$
$$f(w_i) = \begin{cases} 3n+1+i, & \text{for } 1 \le i \le n-1, \\ 3n+1, & \text{for } i = n, \end{cases}$$

From the above vertex weights, we have

$$\begin{aligned} \omega_t(u_i) = \omega_t(v_i) &= \tfrac{21n+5}{2}, && \text{for } 1 \le i \le n, \\ \omega_t(w_i) &= \tfrac{31n+7}{2}, && \text{for } 1 \le i \le n, \\ \omega_t(c) &= 2n^2+6n+1. \end{aligned}$$

Therefore, f is a local antimagic total labeling of the graph D_4^n using three distinct colors and so $\chi_{lat}(D_4^n) = 3$ for odd n.

Subcase 2. If n is even.

Label the edges and vertices of the graph D_4^n by the following way:

$$\begin{aligned} f(cu_i) &= i, && \text{for } 1 \le i \le n, \\ f(cv_i) &= n+i, && \text{for } 1 \le i \le n, \\ f(w_iv_i) &= 4n+1-2i, && \text{for } 1 \le i \le n, \end{aligned}$$
$$f(w_iu_i) = \begin{cases} 5n+2, & \text{for } i = 1, \\ 6n+1+i, & \text{for } 2 \le i \le n. \end{cases}$$

Then

$$f(c) = 5n+1,$$
$$f(v_i) = 5n+2+i, \quad \text{for } 1 \le i \le n,$$
$$f(u_i) = \begin{cases} 5n, & \text{for } i = 1, \\ 4n+2-2i, & \text{for } 2 \le i \le n, \end{cases}$$
$$f(w_i) = \begin{cases} 5n-1, & \text{for } i = 1, \\ 4n-2+i, & \text{for } 2 \le i \le n. \end{cases}$$

For the vertex weights under the labeling f, we have

$$\begin{aligned} \omega_t(u_i) = \omega_t(v_i) &= 10n+3, && \text{for } 1 \le i \le n, \\ \omega_t(w_i) &= 14n, && \text{for } 1 \le i \le n, \\ \omega_t(c) &= 2n^2+6n+1. \end{aligned}$$

The above arguments indicate that f is a local antimagic total labeling of D_4^n with three colors, and so $\chi_{lat}(D_4^n) \le 3$ for even n.

Case 2. For $n \le 7$.

Present the detailed local antimagic total labeling for each n when $n \le 7$. The following figure is shown the local antimagic total labeling for $n = 7$ in Figure 8.

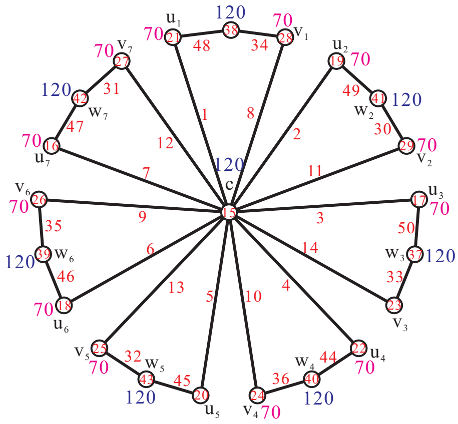

Figure 8. The local antimagic total labeling of D_4^7.

The exact value of each edge and vertex of the local antimagic total labelings for $n \leq 6$ are given in Tables 1–5.

When $n = 2$, we obtain that $\omega_t(u_i) = \omega_t(v_i) = 22$ and $\omega_t(w_i) = \omega_t(c) = 25$ for $i = 1, 2$. Similarly, when $n = 3$, we obtain that $\omega_t(u_i) = \omega_t(v_i) = 34$ and $\omega_t(w_i) = \omega_t(c) = 37$ for $i = 1, 2, 3$. When $n = 4$, we obtain that $\omega_t(u_i) = \omega_t(v_i) = 39$ and $\omega_t(w_i) = \omega_t(c) = 56$ for $1 \leq i \leq 4$. Therefore $\chi_{lat}(D_4^2) = \chi_{lat}(D_4^3) = \chi_{lat}(D_4^4) = 2$.

Table 1. D_4^2.

The Weights of Local Antimagic Total Labeling of D_4^2		
i	1	2
$f(cu_i)$	1	3
$f(cv_i)$	2	4
$f(w_iu_i)$	10	12
$f(w_iv_i)$	6	5
$f(u_i)$	11	7
$f(v_i)$	14	13
$f(w_i)$	9	8
$f(c)$	15	

Table 2. D_4^3.

The Weights of Local Antimagic Total Labeling of D_4^3			
i	1	2	3
$f(cu_i)$	1	2	3
$f(cv_i)$	6	4	5
$f(w_iu_i)$	19	17	18
$f(w_iv_i)$	7	8	9
$f(u_i)$	14	15	13
$f(v_i)$	21	22	20
$f(w_i)$	11	12	10
$f(c)$		16	

Table 3. D_4^4.

The Weights of Local Antimagic Total Labeling of D_4^4				
i	1	2	3	4
$f(cu_i)$	2	4	14	6
$f(cv_i)$	5	1	8	3
$f(w_iu_i)$	12	11	7	10
$f(w_iv_i)$	15	17	22	20
$f(u_i)$	25	24	18	23
$f(v_i)$	19	21	9	16
$f(w_i)$	29	28	27	26
$f(c)$	13			

Table 4. D_4^5.

The Weights of Local Antimagic Total Labeling of D_4^5					
i	1	2	3	4	5
$f(cu_i)$	1	2	3	4	5
$f(cv_i)$	10	8	6	9	7
$f(w_iu_i)$	30	31	27	28	29
$f(w_iv_i)$	34	32	35	33	36
$f(u_i)$	24	22	25	23	21
$f(v_i)$	11	15	14	13	12
$f(w_i)$	17	18	19	20	16
$f(c)$	26				

Table 5. D_4^6.

The Weights of Local Antimagic Total Labeling of D_4^6						
i	1	2	3	4	5	6
$f(cu_i)$	3	6	2	5	4	1
$f(cv_i)$	12	8	11	9	10	7
$f(w_iu_i)$	26	27	28	29	31	36
$f(w_iv_i)$	41	42	40	43	39	35
$f(u_i)$	38	34	37	33	32	30
$f(v_i)$	14	17	16	15	18	25
$f(w_i)$	24	22	23	19	21	20
$f(c)$	13					

When $n = 5$, $\omega_t(u_i) = \omega_t(v_i) = 55$ and $\omega_t(w_i) = \omega_t(c) = 81$ for $1 \leq i \leq 5$. Similarly, when $n = 6$, $\omega_t(u_i) = \omega_t(v_i) = 67$ and $\omega_t(w_i) = \omega_t(c) = 91$ for $1 \leq i \leq 6$. Therefore $\chi_{lat}(D_4^5) = \chi_{lat}(D_4^6) = 2$.

In conclusion, the local antimagic total chromatic number of the graph D_4^n is $\chi_{lat}(D_4^n) = 2$ for $n \leq 7$, and $\chi_{lat}(D_4^n) = 3$ for $n > 7$, respectively. The proof is done. □

Example 3. *The local antimagic total labelings of the graph D_4^9 and D_4^8 are shown in Figures 9 and 10.*

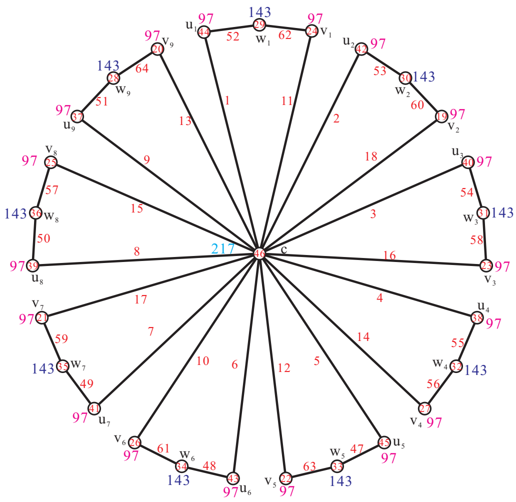

Figure 9. The local antimagic total labeling of D_4^9.

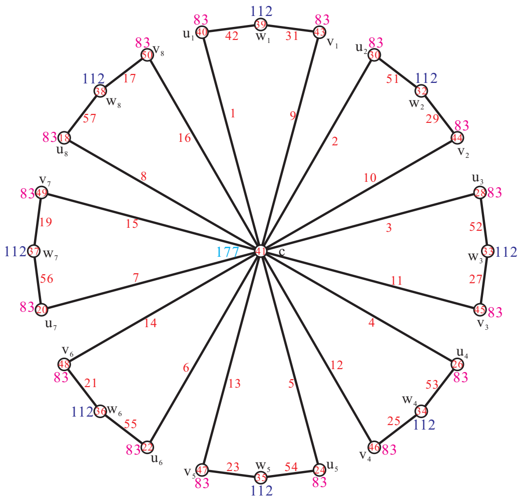

Figure 10. The local antimagic total labeling of D_4^8.

The analogous Dutch windmill graph, denoted by AD_4^n, is obtained from the Dutch windmill graph D_4^n by adding edges cw_i for each $i \in 1, 2, \ldots, n$. The graph AD_4^n has $3n + 1$ vertices and $5n$ edges. It can be seen that the analogous Dutch windmill graph AD_4^n can be viewed as from the wheel W_{3n} by deleting one edge continuously every 2 edges on the rim of the wheel. The analogous Dutch windmill graph AD_4^4 is shown in Figure 11.

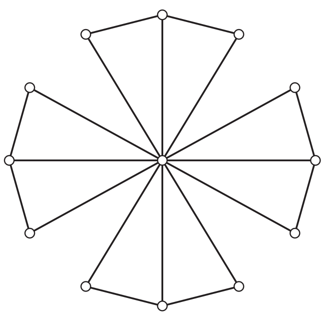

Figure 11. The analogous Dutch windmill graph AD_4^4.

Theorem 4. *For the analogous Dutch windmill graph, AD_4^n, we have $\chi_{lat}(AD_4^n) = 3$.*

Proof. The lower bound of the local antimagic total chromatic number of graph AD_4^n is 3 since K_3 is an induced subgraph of graph AD_4^n and $\chi_{lat}(AD_4^n) \geq \chi(AD_4^n)$. Define $f : V(AD_4^n) \cup E(AD_4^n) \to \{1, 2, \cdots, 8n+1\}$. We discuss two cases for the exact value of each vertex and edge as follows.

Case 1. If n is odd.

For $n = 1$, the local antimagic total labeling of the graph AD_4^1 is shown in Figure 12.

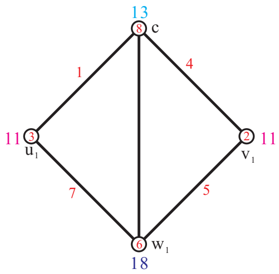

Figure 12. The local antimagic total labeling of AD_4^1.

Label the edges of AD_4^n by the following way:

$$\begin{aligned}
&f(cu_i) = \begin{cases} \frac{n-1}{2} + i, & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\ i - \frac{n+1}{2}, & \text{for } \frac{n+3}{2} \leq i \leq n. \end{cases} \\
&f(cv_i) = \begin{cases} \frac{3n+1}{2} - i, & \text{for } 1 \leq i \leq \frac{n-1}{2}, \\ \frac{5n+1}{2} - i, & \text{for } \frac{n+1}{2} \leq i \leq n. \end{cases} \\
&f(cw_i) = 8n + 1 - i, \quad \text{for } 1 \leq i \leq n. \\
&f(w_iu_i) = 5n + i, \quad \text{for } 1 \leq i \leq n. \\
&f(w_iv_i) = 7n + 1 - i, \quad \text{for } 1 \leq i \leq n.
\end{aligned}$$

Then label the vertices of AD_4^n as follows:

$$\begin{aligned}
&f(c) = 8n + 1, \\
&f(w_i) = 3n + i, \quad \text{for } 1 \leq i \leq n, \\
&f(u_i) = \begin{cases} 5n + 2 - 2i, & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\ 6n + 2 - 2i, & \text{for } \frac{n+3}{2} \leq i \leq n. \end{cases} \\
&f(v_i) = \begin{cases} 2n + 2i, & \text{for } 1 \leq i \leq \frac{n-1}{2}, \\ n + 2i, & \text{for } \frac{n+1}{2} \leq i \leq n. \end{cases}
\end{aligned}$$

For the vertex weights under the labeling f, we have

$$\begin{aligned}
&\omega_t(u_i) = \omega_t(v_i) = \tfrac{21n+3}{2}, \quad \text{for } 1 \leq i \leq n, \\
&\omega_t(w_i) = 23n + 2, \quad \text{for } 1 \leq i \leq n, \\
&\omega_t(c) = \tfrac{19n^2+19n+2}{2}.
\end{aligned}$$

The above arguments indicate that f is a local antimagic labeling of AD_4^n with three colors, and so $\chi_{lat}(AD_4^n) \leq 3$ for odd n.

Case 2. If n is even.

We give the following exact values of edges of AD_4^n:

$$\begin{aligned}
&f(cu_i) = \begin{cases} i, & \text{for } 1 \leq i < n, \\ 2n, & \text{for } i = n. \end{cases} \\
&f(cv_i) = 2n - i, \quad \text{for } 1 \leq i \leq n, \\
&f(cw_i) = 8n + 1 - i, \quad \text{for } 1 \leq i \leq n, \\
&f(w_iu_i) = 6n + i, \quad \text{for } 1 \leq i \leq n, \\
&f(w_iv_i) = 6n + 1 - i, \quad \text{for } 1 \leq i \leq n,
\end{aligned}$$

Then label the vertices as follows:

$$
\begin{aligned}
f(c) &= 8n+1, \\
f(w_i) &= 2n+1+i, && \text{for } 1 \le i \le n, \\
f(u_i) &= \begin{cases} 5n+1-2i, & \text{for } 1 \le i < n, \\ 2n+1, & \text{for } i = n. \end{cases} \\
f(v_i) &= 3n+2i, && \text{for } 1 \le i \le n.
\end{aligned}
$$

We obtain that,

$$
\begin{aligned}
\omega_t(u_i) &= \omega_t(v_i) = 11n+1, && \text{for } 1 \le i \le n, \\
\omega_t(w_i) &= 22n+3, && \text{for } 1 \le i \le n, \\
\omega_t(c) &= \frac{19n^2+19n+2}{2}.
\end{aligned}
$$

It is clear that f is a local antimagic total labeling of AD_4^n using three colors and so $\chi_{lat}(AD_4^n) \le 3$ for even n. The proof is complete. □

Example 4. *The local antimagic total labelings of the graph AD_4^5 and AD_4^6 are shown in Figures 13 and 14.*

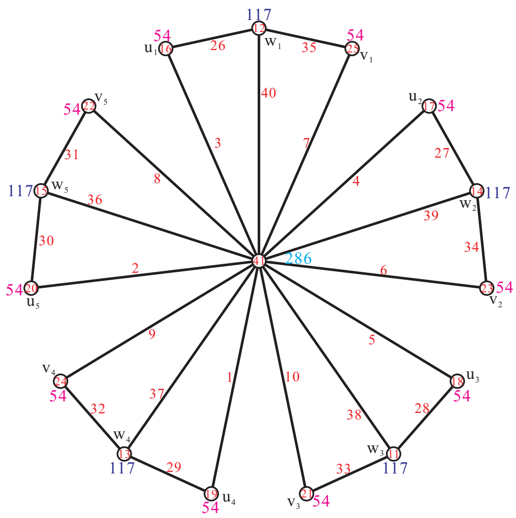

Figure 13. The local antimagic total labeling of AD_4^5.

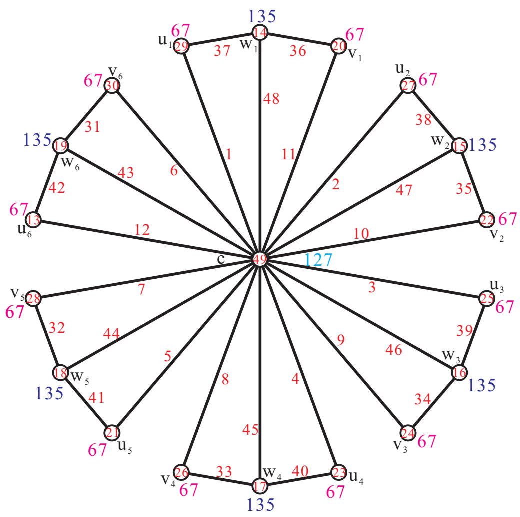

Figure 14. The local antimagic total labeling of AD_4^6.

The flower graph $\mathscr{F}_n$ of order $2n+1$ is a graph obtained by adjoining firstly a pendant edge to each vertex on the rim of the wheel graph W_n, then joining every pendant vertex to the central vertex of W_n by an edge. The flower graph $\mathscr{F}_5$ is shown in Figure 15. Then we respectively obtain the lower and upper bounds of the local antimagic vertex total chromatic number of the flower graph $\mathscr{F}_n$.

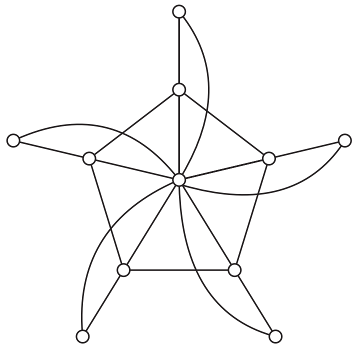

Figure 15. The flower graph $\mathscr{F}_5$.

Theorem 5. *For the flower graph $\mathscr{F}_n$, we have*

$$\begin{cases} 4 \le \chi_{lat}(\mathscr{F}_n) \le 5, & \text{for odd } n, \\ 3 \le \chi_{lat}(\mathscr{F}_n) \le 4, & \text{for even } n. \end{cases}$$

Proof. $\chi_{lavt}(\mathscr{F}_n) \ge \chi(\mathscr{F}_n) = 4$ for odd n and $\chi_{lavt}(\mathscr{F}_n) \ge \chi(\mathscr{F}_n) = 3$ for even n since W_n is the induced subgraph of graph $\mathscr{F}_n$. Then give the upper bound of the local antimigic total chromatic number of the flower graph.

The flower graph $\mathscr{F}_n$ has $2n+1$ vertices and $4n$ edges. The vertex set of $\mathscr{F}_n$ is $V(\mathscr{F}_n) = \{u_i | 1 \le i \le n\} \cup \{v_i | 1 \le i \le n\} \cup \{c\}$ and the edge set of $\mathscr{F}_n$ is $E(\mathscr{F}_n) = \{u_i u_{i+1} | 1 \le i \le n\} \cup \{cu_i | 1 \le i \le n\} \cup \{u_i v_i | 1 \le i \le n\} \cup \{cv_i | 1 \le i \le n\}$, where the edge $u_n u_{n+1}$ is the edge $u_n u_1$. Let f be the local antimagic labeling of the graph W_n defined in Lemma 1 such that the vertices are assigned four distinct colors for odd n and three colors for even n.

Define a bijection $g : V(\mathscr{F}_n) \cup E(\mathscr{F}_n) \to \{1, 2, \cdots, 6n+1\}$. Firstly, label the edges of the subgraph W_n of the $\mathscr{F}_n$ such that $g(e) = f(e)$, where $e \in W_n$. Secondly, label the remaining edges and vertices of $\mathscr{F}_n$ using $\{2n+1, \cdots, 6n+1\}$ since the wheel has $2n$ edges. Let us discuss two cases for n.

Case 1. If n is odd.

By Lemma 1, for $n \equiv 1 \pmod 4$, the vertex weights of graph W_n are, respectively, $\omega(u_1) = \frac{5n+11}{4}$, $\omega(u_i) = \frac{11n+13}{4}$ for odd i and $i \ne 1$, $\omega(u_i) = \frac{9n+11}{4}$ for even i and $\omega(c) = \frac{6n^2+n+1}{4}$. For $n \equiv 3 \pmod 4$, the vertex weights of graph W_n are, respectively, $\omega(u_1) = 2n+2$, $\omega(u_i) = \frac{9n+9}{4}$ for odd i and $i \ne 1$, $\omega(u_i) = \frac{11n+7}{4}$ for even i and $\omega(c) = \frac{(3n+1)n}{2}$. Then

$$\begin{aligned} & g(c) = 6n+1, \\ & g(cv_i) = 4n+1-2i, \quad \text{for } 1 \le i \le n, \\ & g(u_i v_i) = \begin{cases} 5n-1+i, & \text{for odd } i \text{ and } 1 \le i \le n, \\ 4n-1+i, & \text{for even } i \text{ and } 1 \le i \le n. \end{cases} \end{aligned}$$

$$\begin{aligned} & g(v_i) = \begin{cases} 2n+1+i, & \text{for odd } i \text{ and } 1 \le i \le n, \\ 3n+1+i, & \text{for even } i \text{ and } 1 \le i \le n. \end{cases} \\ & g(u_i) = \begin{cases} 5n+2-i, & \text{for odd } i \text{ and } 1 \le i \le n, \\ 6n+2-i, & \text{for even } i \text{ and } 1 \le i \le n. \end{cases} \end{aligned}$$

Conclude the vertex weights under labeling g for each vertex of the graph $\mathscr{F}_n$ as follows:

For $n \equiv 1 \pmod 4$,

$$\begin{aligned} \omega_t(c) &= \tfrac{18n^2+25n+5}{4}, \\ \omega_t(v_i) &= 11n+1, \quad \text{for } 1 \le i \le n. \\ \omega_t(u_i) &= \begin{cases} \frac{51n+17}{4}, & \text{for odd } i \text{ and } 2 \le i \le n, \\ \frac{49n+11}{4}, & \text{for even } i \text{ and } 1 \le i \le n, \\ \frac{45n+15}{4}, & \text{for } i = 1. \end{cases} \end{aligned}$$

For $n \equiv 3 \pmod 4$,

$$\begin{aligned} \omega_t(c) &= \tfrac{9n^2+13n+2}{2}, \\ \omega_t(v_i) &= 11n+1, \quad \text{for } 1 \le i \le n. \\ \omega_t(u_i) &= \begin{cases} \frac{49n+13}{4}, & \text{for odd } i \text{ and } 2 \le i \le n, \\ \frac{51n+11}{4}, & \text{for even } i \text{ and } 1 \le i \le n, \\ 12n+3, & \text{for } i = 1. \end{cases} \end{aligned}$$

It is clear that g is a local antimagic total labeling of the graph $\mathscr{F}_n$ using five colors, and so $\chi_{lat}(\mathscr{F}_n) \le 5$.

Case 2. If n is even.

By Lemma 1, for $n \equiv 2 \pmod 4$, the vertex weights of graph W_n are, respectively, $\omega(u_i) = \frac{9n+6}{4}$ for odd i and $\omega(u_i) = \frac{11n+6}{4}$ for even i and $\omega(c) = \frac{(3n+1)n}{2}$. For $n \equiv 4 \pmod 4$, the vertex weights of graph W_n are, respectively, $\omega(u_i) = \frac{9n+8}{4}$ for odd i and $i \ne 1$, $\omega(u_i) = \frac{11n+4}{4}$ for even i and $\omega(c) = \frac{(3n+1)n}{2}$. Then

$$\begin{aligned} g(c) &= 5n+1, \\ g(u_i) &= 2n+2i, \quad \text{for } 1 \le i \le n, \\ g(u_iv_i) &= 4n+1-2i, \quad \text{for } 1 \le i \le n, \\ g(v_i) &= \begin{cases} 4n+i, & \text{for odd } i \text{ and } 1 \le i \le n, \\ 5n+1+i, & \text{for even } i \text{ and } 1 \le i \le n. \end{cases} \\ g(cv_i) &= \begin{cases} 5n+1+i, & \text{for odd } i \text{ and } 1 \le i \le n, \\ 4n+i, & \text{for even } i \text{ and } 1 \le i \le n. \end{cases} \end{aligned}$$

Accordingly, we obtain the vertex weights under labeling g for each vertex of the graph $\mathscr{F}_n$.

For $n \equiv 2 \pmod 4$,

$$\begin{aligned} \omega_t(x) &= \tfrac{13n^2+13n+2}{2}, \\ \omega_t(v_i) &= 13n+2, \quad \text{for } 1 \le i \le n. \\ \omega_t(u_i) &= \begin{cases} \frac{33n+10}{4}, & \text{for odd } i \text{ and } 1 \le i \le n, \\ \frac{35n+10}{4}, & \text{for even } i \text{ and } 1 \le i \le n. \end{cases} \end{aligned}$$

For $n \equiv 4 \pmod 4$,

$$\begin{aligned} \omega_t(x) &= \tfrac{13n^2+13n+2}{2}, \\ \omega_t(v_i) &= 13n+2, \quad \text{for } 1 \le i \le n. \\ \omega_t(u_i) &= \begin{cases} \frac{33n+12}{4}, & \text{for odd } i \text{ and } 1 \le i \le n, \\ \frac{35n+8}{4}, & \text{for even } i \text{ and } 1 \le i \le n. \end{cases} \end{aligned}$$

Therefore g is a local antimagic total labeling of graph $\mathscr{F}_n$ with four colors and $\chi_{lat}(\mathscr{F}_n) \le 4$. The proof is done. □

Example 5. *The local antimagic total labelings of the graph $\mathscr{F}_9$ and $\mathscr{F}_{11}$ are shown in Figures 16 and 17.*

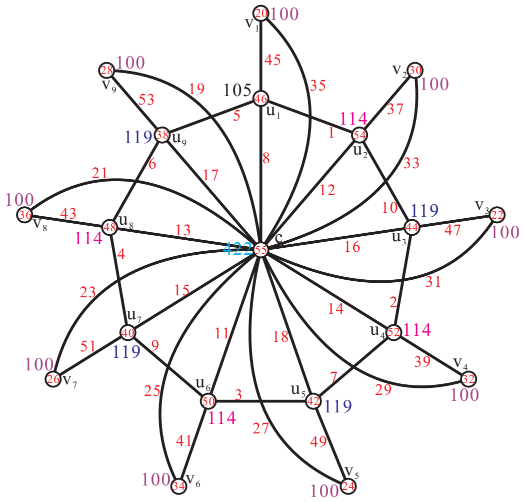

Figure 16. The local antimagic total labeling of $\mathscr{F}_9$.

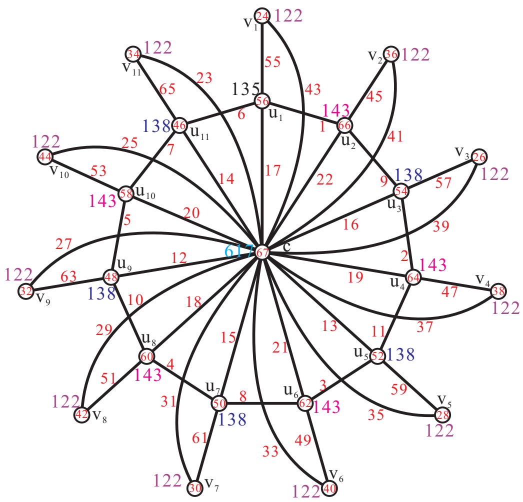

Figure 17. The local antimagic total labeling of $\mathscr{F}_{11}$.

Example 6. *The local antimagic total labelings of the graph $\mathscr{F}_{10}$ and $\mathscr{F}_{12}$ are shown in Figures 18 and 19.*

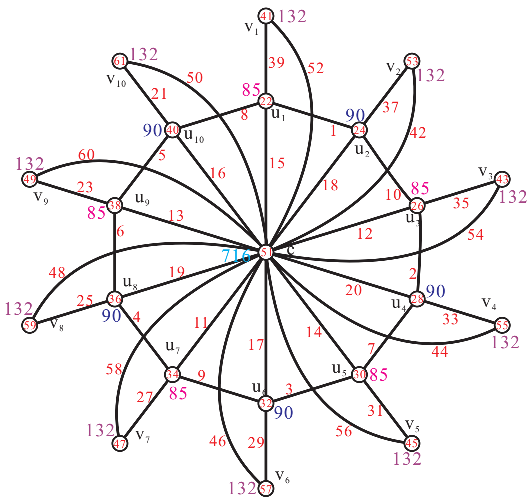

Figure 18. The local antimagic total labeling of $\mathscr{F}_{10}$.

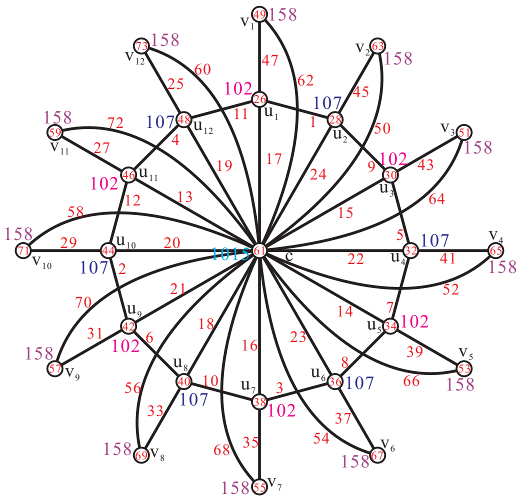

Figure 19. The local antimagic total labeling of $\mathscr{F}_{12}$.

3. Conclusions

Note that the difference of the local antimagic chromatic coloring and the local antimagic total chromatic coloring is that the former is assigning weighted values for the edge set of graph G, which induces a proper vertex coloring of G; the latter is assigning the vertex and edge set of graph G simultaneously, which also induces a proper vertex coloring of G.

For a graph G with independent vertices, the determination of the local antimagic chromatic number is easier than that of the local antimagic total chromatic number. In fact, the weighted values of independent edges in the local antimagic coloring of G must be different, but may be the same in the local antimagic total chromatic coloring of G. In this paper, we consider the local antimagic total chromatic colorings of wheel-related graphs which have no independent edges, and present the local antimagic total chromatic numbers of the fan graph F_n, the bowknot graph $B_{n,n}$, the Dutch windmill graph D_4^n, the analogous Dutch graph AD_4^n and the flower graph $\mathscr{F}_n$.

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