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Existence of Mild Solutions for Hilfer Fractional Evolution Equations with Almost Sectorial Operators

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Abstract: In this paper, we obtain new sufficient conditions of the existence of mild solutions for Hilfer fractional evolution equations in the cases that the semigroup associated with an almost sectorial operator is compact as well as noncompact. Our results improve and extend some recent results in references.

Keywords: fractional evolution equations; Hilfer derivative; almost sectorial operator; mild solutions

MSC: 26A33; 34A08; 34K37

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1. Introduction

In the past two decades, fractional differential equations are widely used in the mathematical modeling of real-world phenomena. These applications have motivated many researchers in the field of differential equations to investigate fractional differential equations with different fractional derivatives, see the monographs [1–4] and the recent references.

The main motivation of studying fractional evolution equation comes from two aspects. Firstly, many mathematical models in physics and fluid mechanics are characterized by fractional partial differential equations. Secondly, many types of fractional partial differential equations, such as fractional diffusion equations, wave equations, Navier–Stokes equations, Rayleigh–Stokes equations, Fokker–Planck equations, Schrödinger equations, and so on, can be abstracted as fractional evolution equations, for example, see [5–7]. Therefore, the study of fractional evolution equations is very valuable in both theory and application. Indeed, the well-posedness of fractional evolution equations has become an important research topic of evolution equations (see [8–18]).

In this paper, we consider the Cauchy problem of fractional evolution equations with an almost sectorial operator

$$\begin{cases} {}^{H}D_{0+}^{\lambda,\nu}y(t) = Ay(t) + g(t, y(t)), \quad t \in (0, T], \\ I_{0+}^{(1-\lambda)(1-\nu)}y(0) = y_0, \end{cases} \tag{1}$$

where ${}^{H}D_{0+}^{\lambda,\nu}$ is the Hilfer fractional derivative of order $0 < \lambda < 1$ and type $0 \le \nu \le 1$, $I_{0+}^{(1-\lambda)(1-\nu)}$ is Riemann–Liouville fractional integral of order $(1-\lambda)(1-\nu)$, A is an almost sectorial operator in Banach space X, $g : [0, T] \times X \to X$ is a function to be defined later, $y_0 \in X, T \in (0, \infty)$.

The Hilfer fractional derivative is a natural generalization of Riemann–Liouville derivative and Caputo derivative, see [1]. It is obvious that fractional differential equations with Hilfer derivatives include fractional differential equations with Riemann–Liouville derivative or Caputo derivative as special cases. In the past few years, fractional differential equations with Hilfer fractional derivative received great attention from many researchers (see [8–18]).

In this paper, we will prove new existence theorems of mild solutions for (1) in the cases that the semigroup associated with the almost sectorial operator is compact as well as noncompact. In particular, our results obtained in this paper essentially improve and extend the known results in [4,9,10]. The rest of this paper is organized as follows: in Section 2, we will introduce almost sectorial operators, fractional calculus and the measure of noncompactness which will be used in this paper. In Section 3, we will give some useful lemmas before proving the main results. In Section 4, we will show some new existence results of mild solutions for Cauchy problem (1). In Section 5, we will point out that the definitions of the operators in [10,16–18] are inappropriate.

2. Preliminaries

We first introduce some notations and definitions about almost sectorial operators, fractional calculus and the Kuratowski's measure of noncompactness. For more details, we refer to [1,2,19,20].

Assume that X is a Banach space with the norm $|\cdot|$. Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = (0, \infty)$ and J be a finite interval of $\mathbb{R}$. By $C(J, X)$ we denote the Banach space of all continuous functions from J to X with the norm $\|u\| = \sup_{t \in J} |u(t)| < \infty$. We denote by $\mathcal{L}(X)$ the space of all bounded linear operators from X to X with the usual operator norm $\|\cdot\|_{\mathcal{L}(X)}$.

Let A be a linear operator from X to itself. Denote by $D(A)$ the domain of A, by $\sigma(A)$ its spectrum, while $\rho(A) := \mathbb{C} - \sigma(A)$ is the resolvent set of A. Let $S_\mu^0 = \{z \in \mathbb{C} \backslash \{0\} : |\arg z| < \mu\}$ be the open sector for $0 < \mu < \pi$, and S_μ be its closure, i.e., $S_\mu = \{z \in \mathbb{C} \backslash \{0\} : |\arg z| \leq \mu\} \cup \{0\}$.

Definition 1. *Let $0 < k < 1$ and $0 < \omega < \frac{\pi}{2}$. We denote $\Theta_\omega^{-k}(X)$ as a family of all closed linear operators $A : D(A) \subset X \to X$ such that*

(i) $\sigma(A) \subset S_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \omega\} \cup \{0\}$ and

(ii) for any $\mu \in (\omega, \pi)$, there exists C_μ such that

$$\|R(z; A)\|_{\mathcal{L}(X)} \leq C_\mu |z|^{-k}, \text{ for all } z \in \mathbb{C} \setminus S_\mu,$$

where $R(z; A) = (zI - A)^{-1}, z \in \rho(A)$ is the resolvent operator of A. The linear operator A will be called an almost sectorial operator on X if $A \in \Theta_\omega^{-k}(X)$.

Define the power of A as

$$A^\beta = \frac{1}{2\pi i} \int_{\Gamma_\rho} z^\beta R(z; A) dz, \quad \beta > 1 - k,$$

where $\Gamma_\rho = \{\mathbb{R}^+ e^{i\rho}\} \bigcup \{\mathbb{R}^+ e^{-i\rho}\}$ is an appropriate path oriented counterclockwise and $\omega < \rho < \mu$. Then, the linear power space $X_\beta := D(A^\beta)$ can be defined and X_β is a Banach space with the graph norm $\|y\|_\beta = |A^\beta y|, y \in D(A^\beta)$.

Next, let us introduce the semigroup associated with A. We denote the semigroup associated with A by $\{Q(t)\}_{t \geq 0}$. For $t \in S^0_{\frac{\pi}{2} - \omega}$

$$Q(t) = e^{-tz}(A) = \frac{1}{2\pi i} \int_{\Gamma_\rho} e^{-tz} R(z; A) dz,$$

where the integral contour $\Gamma_\rho = \{\mathbb{R}^+ e^{i\rho}\} \bigcup \{\mathbb{R}^+ e^{-i\rho}\}$ is oriented counter-clockwise and $\omega < \rho < \mu < \frac{\pi}{2} - |\arg t|$, forms an analytic semigroup of growth order $1 - k$.

Lemma 1 (see [19]). *Assume that $0 < k < 1$ and $0 < \omega < \frac{\pi}{2}$. Set $A \in \Theta_\omega^{-k}(X)$. Then*

(i) *$Q(s + t) = Q(s)Q(t)$, for any $s, t \in S^0_{\frac{\pi}{2} - \omega}$;*

(ii) *there exists a constant $C_0 > 0$ such that $\|Q(t)\|_{\mathcal{L}(X)} \leq C_0 t^{k-1}$, for any $t > 0$;*

(iii) *The range $R(Q(t))$ of $Q(t)$, $t \in S^0_{\frac{\pi}{2}-\omega}$ is contained in $D(A^\infty)$. Particularly, $R(Q(t)) \subset D(A^\beta)$ for all $\beta \in \mathbb{C}$ with $Re(\beta) > 0$,*

$$A^\beta Q(t)y = \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz} R(z;A)y dz, \text{ for all } y \in X,$$

and hence there exists a constant $C' = C'(\gamma,\beta) > 0$ such that

$$\|A^\beta Q(t)\|_{B(X)} \le C' t^{-\gamma - Re(\beta) - 1}, \text{ for all } t > 0;$$

(iv) *If $\beta > 1-k$, then $D(A^\beta) \subset \Sigma_Q = \{y \in X : \lim_{t\to 0+} Q(t)y = y\}$;*
(v) *$R(\lambda, A) = \int_0^\infty e^{-\lambda t} Q(t)dt$, for every $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$.*

Definition 2 (Riemann-Liouville fractional integral, see [2])**.** *The fractional integral of order λ for a function $y : [0,\infty) \to \mathbb{R}$ is defined as*

$$I^\lambda_{0+} y(t) = \frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1} y(s)ds, \quad \lambda > 0,\ t > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 3 (Hilfer fractional derivative, see [1])**.** *Let $0 < \lambda < 1$ and $0 \le \nu \le 1$. The Hilfer fractional derivative of order λ and type ν for a function $y : [0,\infty) \to \mathbb{R}$ is defined as*

$${}^H D^{\lambda,\nu}_{0+} y(t) = I^{\nu(1-\lambda)}_{0+} \frac{d}{dt} I^{(1-\lambda)(1-\nu)}_{0+} y(t).$$

In particular, when $\nu = 0$, $0 < \lambda < 1$, then

$${}^H D^{\lambda,0}_{0+} y(t) = \frac{d}{dt} I^{1-\lambda}_{0+} y(t) =: {}^L D^\lambda_{0+} y(t),$$

where ${}^L D^\lambda_{0+}$ is Riemann–Liouville derivative.

If $\nu = 1$, $0 < \lambda < 1$, then

$${}^H D^{\lambda,1}_{0+} y(t) = I^{1-\lambda}_{0+} \frac{d}{dt} y(t) =: {}^C D^\lambda_{0+} y(t),$$

where ${}^C D^\lambda_{0+}$ is Caputo derivative.

Let D be a nonempty subset of X. The Kuratowski's measure of noncompactness α is defined as follows:

$$\alpha(D) = \inf\left\{ d > 0 :\ D \subset \bigcup_{j=1}^n M_j \text{ and } \mathrm{diam}(M_j) \le d \right\},$$

where the diameter of M_j is given by $\mathrm{diam}(M_j) = \sup\{|x-y| :\ x, y \in M_j\}$, $j = 1, \ldots, n$.

Lemma 2 ([21])**.** *Let X be a Banach space, and let $\{u_n(t)\}_{n=1}^\infty : [0,T] \to X$ be a continuous function family. If there exists $\xi \in L[0,T]$ such that*

$$|u_n(t)| \le \xi(t), \quad t \in [0,T],\ n = 1, 2, \ldots.$$

Then $\alpha(\{u_n(t)\}_{n=1}^\infty)$ is integrable on $[0,T]$, and

$$\alpha\left(\left\{ \int_0^t u_n(s)ds \right\}_{n=1}^\infty\right) \le 2\int_0^t \alpha(\{u_n(s)\}_{n=1}^\infty)ds.$$

Definition 4 ([22])**.** *Define the wright function* $M_\lambda(\theta)$ *by*

$$M_\lambda(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\lambda n)}, \quad 0 < \lambda < 1,\, \theta \in \mathbb{C},$$

with the following property

$$\int_0^\infty \theta^\delta M_\lambda(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\lambda\delta)}, \quad \text{for } \delta \geq 0.$$

Lemma 3 ([9])**.** *The problem* (1) *is equivalent to the integral equation*

$$\begin{aligned} y(t) =& \frac{y_0}{\Gamma(\nu(1-\lambda)+\lambda)} t^{-(1-\lambda)(1-\nu)} \\ &+ \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1}[Ay(s) + g(s, y(s))] ds, \quad t \in (0, T]. \end{aligned} \tag{2}$$

Lemma 4. *Assume that* $y(t)$ *satisfies integral Equation* (2)*. Then*

$$y(t) = \mathcal{S}_{\lambda,\nu}(t) y_0 + \int_0^t \mathcal{K}_\lambda(t-s) g(s, y(s)) ds, \quad t \in (0, T],$$

where

$$\mathcal{S}_{\lambda,\nu}(t) = I_{0+}^{\nu(1-\lambda)} \mathcal{K}_\lambda(t),\ \mathcal{K}_\lambda(t) = t^{\lambda-1}\mathcal{Q}_\lambda(t), \text{ and } \mathcal{Q}_\lambda(t) = \int_0^\infty \lambda\theta M_\lambda(\theta) Q(t^\lambda\theta) d\theta.$$

Proof. This proof is similar to [9], so we omit it. □

In view of Lemma 4, we have the following definition.

Definition 5. *If* $y \in C((0, T], X)$ *satisfies*

$$y(t) = \mathcal{S}_{\lambda,\nu}(t) y_0 + \int_0^t \mathcal{K}_\lambda(t-s) g(s, y(s)) ds, \quad t \in (0, T],$$

then $y(t)$ *is called a mild solution of the Cauchy problem* (1)*.*

Lemma 5 ([10])**.** *If* $\{Q(t)\}_{t>0}$ *is a compact operator, then* $\{\mathcal{S}_{\lambda,\nu}(t)\}_{t>0}$ *and* $\{\mathcal{Q}_\lambda(t)\}_{t>0}$ *are also compact operators.*

Lemma 6 ([4])**.** *Let* $\beta > 1-k$*. For all* $y \in D(A^\beta)$*, we have* $\lim_{t\to 0+} \mathcal{Q}_\lambda(t) y = \frac{y}{\Gamma(\lambda)}$*.*

Lemma 7. *Assume that* $\{Q(t)\}_{t>0}$ *is a compact operator. Then* $\{Q(t)\}_{t>0}$ *is equicontinuous.*

Lemma 8 (See also [10])**.** *For any fixed* $t > 0$*,* $\mathcal{Q}_\lambda(t)$*,* $\mathcal{K}_\lambda(t)$ *and* $\mathcal{S}_{\lambda,\nu}(t)$ *are linear operators, and for any* $y \in X$*,*

$$|\mathcal{Q}_\lambda(t) y| \leq L_1 t^{\lambda(k-1)}|y|,\ |\mathcal{K}_\lambda(t) y| \leq L_1 t^{\lambda k-1}|y|, \text{ and } |\mathcal{S}_{\lambda,\nu}(t) y| \leq L_2 t^{-1+\nu-\lambda\nu+\lambda k}|y|,$$

where

$$L_1 = \frac{C_0\Gamma(k)}{\Gamma(\lambda k)}, \quad L_2 = \frac{C_0\Gamma(k)}{\Gamma(\nu(1-\lambda)+\lambda k)}.$$

Proof. By

$$\int_0^\infty \theta^\delta M_\lambda(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\lambda\delta)}, \text{ for } \delta \geq 0,$$

we have

$$
\begin{aligned}
|\mathcal{Q}_\lambda(t)y| =& \left| \int_0^\infty \lambda\theta M_\lambda(\theta)Q(t^\lambda\theta)yd\theta \right| \\
\leq& \lambda C_0 \int_0^\infty M_\lambda(\theta)\theta^k t^{\lambda(k-1)}|y|d\theta \\
\leq& L_1 t^{\lambda(k-1)}|y|, \text{ for } t \in (0,T] \text{ and } y \in X.
\end{aligned}
$$

Moreover, for $t \in (0,T]$ and $y \in X$,

$$
|\mathcal{K}_\lambda(t)y| = |t^{\lambda-1}\mathcal{Q}_\lambda(t)y| \leq L_1 t^{\lambda k-1}|y|,
$$

and

$$
\begin{aligned}
|\mathcal{S}_{\lambda,\nu}(t)y| = |I_{0+}^{\nu(1-\lambda)}\mathcal{K}_\lambda(t)y| =& \left| \frac{1}{\Gamma(\nu(1-\lambda))} \int_0^t (t-s)^{\nu(1-\lambda)-1}\mathcal{K}_\lambda(s)yds \right| \\
\leq& \frac{C_0\Gamma(k)}{\Gamma(\lambda k)\Gamma(\nu(1-\lambda))} \int_0^t (t-s)^{\nu(1-\lambda)-1}s^{\lambda k-1}|y|ds \\
\leq& L_2 t^{-1+\nu-\lambda\nu+\lambda k}|y|.
\end{aligned}
$$

This completes the proof. □

Lemma 9 ([10]). *Assume that $\{Q(t)\}_{t>0}$ is equicontinuous. Then $\{\mathcal{Q}_\lambda(t)\}_{t>0}$, $\{\mathcal{K}_\lambda(t)\}_{t>0}$ and $\{\mathcal{S}_{\lambda,\nu}(t)\}_{t>0}$ are strongly continuous, that is, for any $y \in X$ and $t'' > t' > 0$,*

$$
\begin{aligned}
&|\mathcal{Q}_\lambda(t')y - \mathcal{Q}_\lambda(t'')y| \to 0,\ |\mathcal{K}_\lambda(t')y - \mathcal{K}_\lambda(t'')y| \to 0, \\
&|\mathcal{S}_{\lambda,\nu}(t')y - \mathcal{S}_{\lambda,\nu}(t'')y| \to 0, \quad \text{as } t'' \to t'.
\end{aligned}
$$

3. Some Lemmas

Throughout this paper, we assume that $A \in \Theta_\omega^{-k}(X)$, $0 < k < 1$ and $0 < \omega < \frac{\pi}{2}$. Furthermore, we suppose that $y_0 \in D(A^\beta)$ with $\beta > 1-k$.

We introduce the following hypotheses:

(H1) $Q(t)$ is continuous in the uniform operator topology for $t > 0$, i.e., $\{Q(t)\}_{t>0}$ is equicontinuous.

(H2) the map $t \to g(t,y)$ is measurable for all $y \in X$ and the map $y \to g(t,y)$ is continuous for a.e. $t \in [0,T]$.

(H3) there exists a function $m \in L((0,T],\mathbb{R}^+)$ satisfying

$$
I_{0+}^{\lambda k}m \in C((0,T],\mathbb{R}^+), \quad \lim_{t\to 0+} t^{1-\nu+\lambda\nu-\lambda k}I_{0+}^{\lambda k}m(t) = 0
$$

and $|g(t,y)| \leq m(t)$, for a.e. $t \in (0,T]$ and any $y \in X$.

(H4) there exists a constant $r > 0$ such that

$$
L_2|y_0| + L_1 \sup_{t\in[0,T]} \left\{ t^{1-\nu+\lambda\nu-\lambda k} \int_0^t (t-s)^{\lambda k-1}m(s)ds \right\} \leq r,
$$

where

$$
L_1 = \frac{C_0\Gamma(k)}{\Gamma(\lambda k)}, \quad L_2 = \frac{C_0\Gamma(k)}{\Gamma(\nu(1-\lambda)+\lambda k)}.
$$

Let

$$
C_\lambda((0,T],X) = \left\{ y \in C((0,T],X) : \lim_{t\to 0+} t^{1-\nu+\lambda\nu-\lambda k}|y(t)| \text{ exists and is finite} \right\},
$$

with the norm

$$\|y\|_\lambda = \sup_{t\in(0,T]} \{t^{1-\nu+\lambda\nu-\lambda k}|y(t)|\}.$$

Then $(C_\lambda((0,T],X), \|\cdot\|_\lambda)$ is a Banach space (see Lemma 3.2 of [23]).

For any $y \in C_\lambda((0,T],X)$, define an operator $\mathcal{T}$ as follows

$$(\mathcal{T}y)(t) = (\mathcal{T}_1 y)(t) + (\mathcal{T}_2 y)(t),$$

where

$$(\mathcal{T}_1 y)(t) = \mathcal{S}_{\lambda,\nu}(t)y_0, \quad (\mathcal{T}_2 y)(t) = \int_0^t \mathcal{K}_\lambda(t-s)g(s,y(s))ds, \quad \text{for } t\in(0,T].$$

Clearly, the problem (1) has a mild solution $y^* \in C_\lambda((0,T],X)$ if and only if $\mathcal{T}$ has a fixed point $y^* \in C_\lambda((0,T],X)$.

It is easy to show that

$$\lim_{t\to 0+} t^{1-\nu+\lambda\nu-\lambda k}\mathcal{S}_{\lambda,\nu}(t)y_0 = 0. \tag{3}$$

In fact,

$$\begin{aligned} t^{1-\nu+\lambda\nu-\lambda k}\mathcal{S}_{\lambda,\nu}(t)y_0 &= \frac{t^{1-\nu+\lambda\nu-\lambda k}}{\Gamma(\nu(1-\lambda))}\int_0^t (t-s)^{\nu(1-\lambda)-1}s^{\lambda-1}\mathcal{Q}_\lambda(s)y_0 ds \\ &= \frac{1}{\Gamma(\nu(1-\lambda))}\int_0^1 (1-z)^{\nu(1-\lambda)-1}z^{\lambda-1}t^{\lambda(1-k)}\mathcal{Q}_\lambda(tz)y_0 dz. \end{aligned}$$

By lemma 6, $\lim_{t\to 0+} t^{\lambda(1-k)}\mathcal{Q}_\lambda(tz)y_0 = 0$ and $\int_0^1 (1-z)^{\nu(1-\lambda)-1}z^{\lambda-1}dz$ exists, so (3) holds.

In addition, from Lemma 8 and (H3), we have

$$\begin{aligned} \left|t^{1-\nu+\lambda\nu-\lambda k}\int_0^t \mathcal{K}_\lambda(t-s)g(s,y(s))ds\right| \le& L_1 t^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}m(s)ds \\ \to& 0, \quad \text{as } t\to 0. \end{aligned} \tag{4}$$

For any $u \in C([0,T],X)$, set

$$y(t) = t^{-(1-\nu+\lambda\nu-\lambda k)}u(t), \quad t\in(0,T].$$

Clearly, $y \in C_\lambda((0,T],X)$. Define an operator $\mathcal{F}$ as follows

$$(\mathcal{F}u)(t) = (\mathcal{F}_1 u)(t) + (\mathcal{F}_2 u)(t),$$

where

$$(\mathcal{F}_1 u)(t) = \begin{cases} t^{1-\nu+\lambda\nu-\lambda k}(\mathcal{T}_1 y)(t), & \text{for } t\in(0,T], \\ 0, & \text{for } t=0, \end{cases}$$
$$(\mathcal{F}_2 u)(t) = \begin{cases} t^{1-\nu+\lambda\nu-\lambda k}(\mathcal{T}_2 y)(t), & \text{for } t\in(0,T], \\ 0, & \text{for } t=0. \end{cases}$$

Let

$$\Omega_r = \{u \in C([0,T],X) : \ \|u\| \le r\}.$$

and

$$\widetilde{\Omega}_r = \{y \in C_\lambda((0,T],X) : \ \|y\|_\lambda \le r\}.$$

Clearly, Ω_r and $\widetilde{\Omega}_r$ are nonempty, convex and closed subsets of $C([0,T],X)$ and $C_\lambda((0,T],X)$, respectively.

Before giving the main results, we first prove the following lemmas.

Lemma 10. *Assume that (H1)–(H4) hold. Then, the set* $\{\mathcal{F}u : u \in \Omega_r\}$ *is equicontinuous.*

Proof. Step I. We first prove that $\{\mathcal{F}_1 u : u \in \Omega_r\}$ is equicontinuous.

For $t_1 = 0$, $t_2 \in (0,T]$, by (3), we obtain

$$\Big|(\mathcal{F}_1 u)(t_2) - (\mathcal{F}_1 u)(0)\Big| \le \Big|{t_2}^{1-\nu+\lambda\nu-\lambda k}\mathcal{S}_{\lambda,\nu}(t_2)y_0 - 0\Big| \to 0, \quad \text{as } t_2 \to 0.$$

For any $t_1, t_2 \in (0,T]$ and $t_1 < t_2$, we have

$$\begin{aligned}\Big|(\mathcal{F}_1 u)(t_2) - (\mathcal{F}_1 u)(t_1)\Big| \le&\Big|{t_2}^{1-\nu+\lambda\nu-\lambda k}\mathcal{S}_{\lambda,\nu}(t_2)y_0 - {t_1}^{1-\nu+\lambda\nu-\lambda k}\mathcal{S}_{\lambda,\nu}(t_1)y_0\Big| \\ \le&|{t_2}^{1-\nu+\lambda\nu-\lambda k}||\mathcal{S}_{\lambda,\nu}(t_2)y_0 - \mathcal{S}_{\lambda,\nu}(t_1)y_0| \\ &+ |{t_2}^{1-\nu+\lambda\nu-\lambda k} - {t_1}^{1-\nu+\lambda\nu-\lambda k}||\mathcal{S}_{\lambda,\nu}(t_1)y_0| \\ \to&0, \quad \text{as } t_2 \to t_1.\end{aligned}$$

Hence, $\{\mathcal{F}_1 u : u \in \Omega_r\}$ is equicontinuous.

Step II. We prove that $\{\mathcal{F}_2 u : u \in \Omega_r\}$ is equicontinuous.

Let $y(t) = t^{-(1-\nu+\lambda\nu-\lambda k)}u(t)$, for any $u \in \Omega_r$, $t \in (0,T]$. Then $y \in \widetilde{\Omega}_r$.

For $t_1 = 0$, $0 < t_2 < T$, by (4), we have

$$\begin{aligned}\Big|(\mathcal{F}_2 u)(t_2) - (\mathcal{F}_2 u)(0)\Big| =&\Big|{t_2}^{1-\nu+\lambda\nu-\lambda k}\int_0^{t_2}\mathcal{K}_\lambda(t_2 - s)g(s,y(s))ds\Big| \\ \to&0, \quad \text{as } t_2 \to 0.\end{aligned}$$

For $0 < t_1 < t_2 \le T$, we get

$$\begin{aligned}&|(\mathcal{F}_2 u)(t_2) - (\mathcal{F}_2 u)(t_1)| \\ \le&\Big|{t_1}^{1-\nu+\lambda\nu-\lambda k}\int_{t_1}^{t_2}(t_2 - s)^{\lambda-1}\mathcal{Q}_\lambda(t_2 - s)g(s,y(s))ds\Big| \\ &+ \Big|{t_1}^{1-\nu+\lambda\nu-\lambda k}\int_0^{t_1}\left((t_2 - s)^{\lambda-1} - (t_1 - s)^{\lambda-1}\right)\mathcal{Q}_\lambda(t_2 - s)g(s,y(s))ds\Big| \\ &+ \Big|{t_1}^{1-\nu+\lambda\nu-\lambda k}\int_0^{t_1}(t_1 - s)^{\lambda-1}\big(\mathcal{Q}_\lambda(t_2 - s) - \mathcal{Q}_\lambda(t_1 - s)\big)g(s,y(s))ds\Big| \\ &+ \Big|{t_2}^{1-\nu+\lambda\nu-\lambda k} - {t_1}^{1-\nu+\lambda\nu-\lambda k}\Big|\Big|\int_0^{t_2}(t_2 - s)^{\lambda-1}\mathcal{Q}_\lambda(t_2 - s)g(s,y(s))ds\Big| \\ \le&I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where

$$\begin{aligned}I_1 =&L_1{t_1}^{1-\nu+\lambda\nu-\lambda k}\Big|\int_0^{t_2}(t_2 - s)^{\lambda k-1}m(s)ds - \int_0^{t_1}(t_1 - s)^{\lambda k-1}m(s)ds\Big|, \\ I_2 =&2L_1{t_1}^{1-\nu+\lambda\nu-\lambda k}\int_0^{t_1}\left((t_1 - s)^{\lambda-1} - (t_2 - s)^{\lambda-1}\right)(t_2 - s)^{\lambda(k-1)}m(s)ds, \\ I_3 =&{t_1}^{1-\nu+\lambda\nu-\lambda k}\Big|\int_0^{t_1}(t_1 - s)^{\lambda-1}\big(\mathcal{Q}_\lambda(t_2 - s) - \mathcal{Q}_\lambda(t_1 - s)\big)g(s,y(s))ds\Big|, \\ I_4 =&\Big|{t_2}^{1-\nu+\lambda\nu-\lambda k} - {t_1}^{1-\nu+\lambda\nu-\lambda k}\Big|\Big|L_1\int_0^{t_2}(t_2 - s)^{\lambda k-1}m(s)ds\Big|.\end{aligned}$$

One can deduce that $\lim_{t_2 \to t_1} I_1 = 0$, since $I_{0^+}^{\lambda k} m \in C((0,T], \mathbb{R}^+)$. Noting that

$$((t_1-s)^{\lambda-1} - (t_2-s)^{\lambda-1})(t_2-s)^{\lambda(k-1)} m(s) \le (t_1-s)^{\lambda k-1} m(s), \quad \text{for } s \in [0, t_1),$$

then by Lebesgue dominated convergence theorem, we have

$$\int_0^{t_1} \big((t_1-s)^{\lambda-1} - (t_2-s)^{\lambda-1}\big)(t_2-s)^{\lambda(k-1)} m(s) ds \to 0, \quad \text{as } t_2 \to t_1,$$

which implies $I_2 \to 0$ as $t_2 \to t_1$.

By (H3), for $\varepsilon > 0$, we have

$$\begin{aligned} I_3 \le & {t_1}^{1-\nu+\lambda\nu-\lambda k} \int_0^{t_1-\varepsilon} (t_1-s)^{\lambda-1} \| \mathcal{Q}_\lambda(t_2-s) - \mathcal{Q}_\lambda(t_1-s) \|_{\mathcal{L}(X)} |g(s,y(s))| ds \\ & + {t_1}^{1-\nu+\lambda\nu-\lambda k} \left| \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\lambda-1} \big(\mathcal{Q}_\lambda(t_2-s) - \mathcal{Q}_\lambda(t_1-s)\big) g(s,y(s)) ds \right| \\ \le & {t_1}^{1-\nu+\lambda\nu-\lambda k} \int_0^{t_1} (t_1-s)^{\lambda-1} m(s) ds \sup_{s\in[0,t_1-\varepsilon]} \| \mathcal{Q}_\lambda(t_2-s) - \mathcal{Q}_\lambda(t_1-s) \|_{\mathcal{L}(X)} \\ & + 2L_1 {t_1}^{1-\nu+\lambda\nu-\lambda k} \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\lambda k-1} m(s) ds \\ \le & I_{31} + I_{32} + I_{33}, \end{aligned}$$

where

$$\begin{aligned} I_{31} =& {t_1}^{1-\nu+\lambda\nu-\lambda k} \int_0^{t_1} (t_1-s)^{\lambda-1} m(s) ds \sup_{s\in[0,t_1-\varepsilon]} \| \mathcal{Q}_\lambda(t_2-s) - \mathcal{Q}_\lambda(t_1-s) \|_{\mathcal{L}(X)}, \\ I_{32} =& 2L_1 {t_1}^{1-\nu+\lambda\nu-\lambda k} \left| \int_0^{t_1} (t_1-s)^{\lambda k-1} m(s) ds - \int_0^{t_1-\varepsilon} (t_1-\varepsilon-s)^{\lambda k-1} m(s) ds \right|, \\ I_{33} =& 2L_1 {t_1}^{1-\nu+\lambda\nu-\lambda k} \int_0^{t_1-\varepsilon} ((t_1-\varepsilon-s)^{\lambda k-1} - (t_1-s)^{\lambda k-1}) m(s) ds. \end{aligned}$$

By (H1) and Lemma 9, it is easy to see that $I_{31} \to 0$ as $t_2 \to t_1$. Similar to the proof that I_1, I_2 tend to zero, we get $I_{32} \to 0$ and $I_{33} \to 0$ as $\varepsilon \to 0$. Thus, I_3 tends to zero as $t_2 \to t_1$. Clearly, $I_4 \to 0$ as $t_2 \to t_1$.

Therefore, $\{\mathcal{F}_2 u : u \in \Omega_r\}$ is equicontinuous. Furthermore, $\{\mathcal{F} u : u \in \Omega_r\}$ is equicontinuous. □

Lemma 11. *Assume that (H2)–(H4) hold. Then $\mathcal{F}\Omega_r \subset \Omega_r$.*

Proof. Let $y(t) = t^{-(1-\nu+\lambda\nu-\lambda k)} u(t)$, for $u \in \Omega_r$, $t \in (0,T]$. Then $y \in \widetilde{\Omega}_r$.

From Lemmas 10, we know that $\mathcal{F}\Omega_r \subset C([0,T], X)$. For $t > 0$ and any $u \in \Omega_r$, by (H4), we have

$$\begin{aligned} |(\mathcal{F}u)(t)| \le & \left| t^{1-\nu+\lambda\nu-\lambda k} \mathcal{S}_{\lambda,\nu}(t) y_0 \right| + \left| t^{1-\nu+\lambda\nu-\lambda k} \int_0^t \mathcal{K}_\lambda(t-s) g(s,y(s)) ds \right| \\ \le & L_2 |y_0| + L_1 t^{1-\nu+\lambda\nu-\lambda k} \int_0^t (t-s)^{\lambda k-1} m(s) ds \le r. \end{aligned}$$

For $t = 0$, we have $|(\mathcal{F}u)(0)| = 0 < r$. Therefore, $\mathcal{F}\Omega_r \subset \Omega_r$. □

Lemma 12. *Assume that (H2)–(H4) hold. Then $\mathcal{F}$ is continuous.*

Proof. Let $\{u_n\}_{n=1}^\infty$ be a sequence in Ω_r which is convergent to $u \in \Omega_r$. Consequently,

$$\lim_{n\to\infty} u_n(t) = u(t), \text{ and } \lim_{n\to\infty} t^{-(1-\nu+\lambda\nu-\lambda k)} u_n(t) = t^{-(1-\nu+\lambda\nu-\lambda k)} u(t), \text{ for } t \in (0,T].$$

Let $y(t) = t^{-(1-\nu+\lambda\nu-\lambda k)}u(t)$, $y_n(t) = t^{-(1-\nu+\lambda\nu-\lambda k)}u_n(t)$, $t \in (0,T]$. Then $y, y_n \in \widetilde{\Omega}_r$. In view of (H2), we have

$$\lim_{n\to\infty} g(t, y_n(t)) = \lim_{n\to\infty} g(t, t^{-(1-\nu+\lambda\nu-\lambda k)}u_n(t)) = g(t, t^{-(1-\nu+\lambda\nu-\lambda k)}u(t)) = g(t, y(t)).$$

For each $t \in (0,T]$, $(t-s)^{\lambda k-1}|g(s,y_n(s)) - g(s,y(s))| \leq 2(t-s)^{\lambda k-1}m(s)$. By Lebesgue dominated convergence theorem, we obtain

$$\int_0^t (t-s)^{\lambda k-1}|g(s,y_n(s)) - g(s,y(s))|ds \to 0, \quad \text{as } n \to \infty.$$

Thus, for $t \in [0,T]$,

$$\begin{aligned}
&\Big|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\Big| \\
\leq& t^{1-\nu+\lambda\nu-\lambda k}\int_0^t |\mathcal{K}_\lambda(t-s)(g(s,y_n(s)) - g(s,y(s)))|ds \\
\leq& L_1 t^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}|g(s,y_n(s)) - g(s,y(s))|ds \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, $\|\mathcal{F}u_n - \mathcal{F}u\| \to 0$ as $n \to \infty$. Hence, $\mathcal{F}$ is continuous. The proof is completed. □

4. Main Results

Theorem 1. *Assume that $Q(t)(t > 0)$ is compact. Furthermore suppose that (H2)–(H4) hold. Then the Cauchy problem (1) has at least one mild solution in $\widetilde{\Omega}_r$.*

Proof. Clearly, the problem (1) exists a mild solution $y \in \widetilde{\Omega}_r$ if and only if the operator $\mathcal{F}$ has a fixed point $u \in \Omega_r$, where $u(t) = t^{1-\nu+\lambda\nu-\lambda k}y(t)$. Hence, we only need to prove that the operator $\mathcal{F}$ has a fixed point in Ω_r. From Lemmas 11 and 12, we know that $\mathcal{F}\Omega_r \subset \Omega_r$ and $\mathcal{F}$ is continuous. In view of Lemma 10, the set $\{\mathcal{F}u : u \in \Omega_r\}$ is equicontinuous. It remains to prove that for $t \in [0,T]$, $\{(\mathcal{F}u)(t) : u \in \Omega_r\}$ is relatively compact in X. Clearly, $\{(\mathcal{F}u)(0) : u \in \Omega_r\}$ is relatively compact in X. We only consider the case $t > 0$. For any $\varepsilon \in (0,t)$ and $\delta > 0$, define $\mathcal{F}_{\varepsilon,\delta}$ on Ω_r as follows

$$\begin{aligned}
(\mathcal{F}_{\varepsilon,\delta}u)(t) :=& t^{1-\nu+\lambda\nu-\lambda k}(\mathcal{T}_{\varepsilon,\delta}y)(t) \\
:=& t^{1-\nu+\lambda\nu-\lambda k}\Bigg(\mathcal{S}_{\lambda,\nu}(t)y_0 + \int_0^{t-\varepsilon}\int_\delta^\infty \lambda\theta(t-s)^{\lambda-1}M_\lambda(\theta) \\
&\times Q((t-s)^\lambda\theta)g(s,y(s))d\theta ds\Bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
(\mathcal{F}_{\varepsilon,\delta}u)(t) =& t^{1-\nu+\lambda\nu-\lambda k}\Bigg(\mathcal{S}_{\lambda,\nu}(t)y_0 + Q(\varepsilon^\lambda\delta)\int_0^{t-\varepsilon}\int_\delta^\infty \lambda\theta(t-s)^{\lambda-1}M_\lambda(\theta) \\
&\times Q((t-s)^\lambda\theta - \varepsilon^\lambda\delta)g(s,y(s))d\theta ds\Bigg).
\end{aligned}$$

By Lemma 5, we know that $\mathcal{S}_{\lambda,\nu}(t)$ is compact because $Q(t)$ is compact for $t > 0$. Furthermore, $Q(\varepsilon^\lambda\delta)$ is compact, then the set $\{(\mathcal{F}_{\varepsilon,\delta}u)(t),\ u \in \Omega_r\}$ is relatively compact in X for any $\varepsilon \in (0,t)$ and for any $\delta > 0$. Moreover, for every $u \in \Omega_r$, we find

$$\begin{aligned}
&\left|(\mathcal{F}u)(t) - (\mathcal{F}_{\varepsilon,\delta}u)(t)\right| \\
\leq & t^{1-\nu+\lambda\nu-\lambda k}\left|\int_0^t \int_0^\delta \lambda\theta(t-s)^{\lambda-1}M_\lambda(\theta)Q((t-s)^\lambda\theta)g(s,y(s))d\theta ds\right| \\
&+ t^{1-\nu+\lambda\nu-\lambda k}\left|\int_{t-\varepsilon}^t \int_\delta^\infty \lambda\theta(t-s)^{\lambda-1}M_\lambda(\theta)Q((t-s)^\lambda\theta)g(s,y(s))d\theta ds\right| \\
\leq & \lambda C_0 t^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}|g(s,y(s))|ds \int_0^\delta \theta^k M_\lambda(\theta)d\theta \\
&+ \lambda C_0 t^{1-\nu+\lambda\nu-\lambda k}\int_{t-\varepsilon}^t (t-s)^{\lambda k-1}|g(s,y(s))|ds \int_0^\infty \theta^k M_\lambda(\theta)d\theta \\
\leq & \lambda C_0 t^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}m(s)ds \int_0^\delta \theta^k M_\lambda(\theta)d\theta \\
&+ \lambda C_0 t^{1-\nu+\lambda\nu-\lambda k}\int_{t-\varepsilon}^t (t-s)^{\lambda k-1}m(s)ds \int_0^\infty \theta^k M_\lambda(\theta)d\theta \\
\to & 0, \quad \text{as } \varepsilon \to 0,\ \delta \to 0.
\end{aligned}$$

Therefore, $\{(\mathcal{F}u)(t) : u \in \Omega_r\}$ is also a relatively compact set in X for $t \in [0,T]$. Thus, $\{\mathcal{F}u : u \in \Omega_r\}$ is relatively compact by Ascoli–Arzela Theorem. Hence, $\mathcal{F}$ is a completely continuous operator. Schauder's fixed point theorem shows that $\mathcal{F}$ has at least a fixed point $u^* \in \Omega_r$. Let $y^*(t) = t^{-(1-\nu+\lambda\nu-\lambda k)}u^*(t)$. Thus,

$$y^*(t) = \mathcal{S}_{\lambda,\nu}(t)y_0 + \int_0^t \mathcal{K}_\lambda(t-s)g(s,y^*(s))ds, \quad t \in (0,T],$$

which implies that y^* is a mild solution of (1) in $\widetilde{\Omega}_r$. The proof is completed. □

In the case that $Q(t)$ is noncompact for $t > 0$, we give an assumption as follows:

(H5) there exists a constant $K > 0$ such that for any bounded $D \subseteq X$,

$$\alpha(g(t,D)) \leq Kt^{1-\nu+\lambda\nu-\lambda k}\alpha(D), \quad \text{for a.e. } t \in [0,T],$$

where α is the Kuratowski's measure of noncompactness.

Theorem 2. *Assume that (H1)–(H5) hold. Then the Cauchy problem (1) has at least one mild solution in $\widetilde{\Omega}_r$.*

Proof. Let $u_0(t) = t^{1-\nu+\lambda\nu-\lambda k}\mathcal{S}_{\lambda,\nu}(t)y_0$ for all $t \in [0,T]$ and $u_{n+1} = \mathcal{F}u_n$, $n = 0,1,2,\cdots$. By Lemma 11, $\mathcal{F}u_n \in \Omega_r$, for $u_n \in \Omega_r$. Consider set $\mathcal{V} = \{\mathcal{F}u_n) : u_n \in \Omega_r\}_{n=0}^\infty$, and we will prove set $\mathcal{V}$ is relatively compact. In view of Lemmas 10, the set $\mathcal{V}$ is equicontinuous. We only need to prove $\mathcal{V}(t) = \{(\mathcal{F}u_n)(t),\ u_n \in \Omega_r\}_{n=0}^\infty$ is relatively compact in X for $t \in [0,T]$.

By the properties of measure of noncompactness, for any $t \in [0,T]$ we have

$$\alpha\Big(\Big\{u_n(t)\Big\}_{n=0}^\infty\Big) = \alpha\Big(\Big\{u_0(t)\Big\} \cup \Big\{u_n(t)\Big\}_{n=1}^\infty\Big) = \alpha\Big(\Big\{u_n(t)\Big\}_{n=1}^\infty\Big) = \alpha(\mathcal{V}(t)). \tag{5}$$

Let $y_n(t) = t^{-1+\nu-\lambda\nu+\lambda k}u_n(t)$, $t \in (0,T]$, $n = 0,1,2,\cdots$. By the condition (H5) and Lemma 2, we have

$$\begin{aligned}
\alpha(\mathcal{V}(t)) =&\alpha\Big(\Big\{(\mathcal{F}u_n)(t)\Big\}_{n=0}^{\infty}\Big)\\
=&\alpha\Big(\Big\{t^{1-\nu+\lambda\nu-\lambda k}\mathcal{S}_{\lambda,\nu}(t)y_0 + t^{1-\nu+\lambda\nu-\lambda k}\int_0^t \mathcal{K}_\lambda(t-s)g(s,y_n(s))ds\Big\}_{n=0}^{\infty}\Big)\\
=&\alpha\Big(\Big\{t^{1-\nu+\lambda\nu-\lambda k}\int_0^t \mathcal{K}_\lambda(t-s)g(s,y_n(s))ds\Big\}_{n=0}^{\infty}\Big)\\
\le& 2L_1 t^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}\alpha\Big(g(s,\{s^{-1+\nu-\lambda\nu+\lambda k}u_n(s)\}_{n=0}^{\infty})\Big)ds\\
\le& 2L_1 K T^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}s^{1-\nu+\lambda\nu-\lambda k}\alpha\Big(\{s^{-1+\nu-\lambda\nu+\lambda k}u_n(s)\}_{n=0}^{\infty}\Big)ds\\
\le& 2L_1 K T^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}\alpha\Big(\Big\{u_n(s)\Big\}_{n=0}^{\infty}\Big)ds.
\end{aligned}$$

In view of (5), we obtain

$$\alpha(\mathcal{V}(t)) \le 2L_1 K T^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}\alpha(\mathcal{V}(s))ds.$$

Therefore, by the inequality in ([24], p.188), we obtain that $\alpha(\mathcal{V}(t)) = 0$, then $\mathcal{V}(t)$ is relatively compact. Consequently, it follows from Ascoli–Arzela Theorem that set $\mathcal{V}$ is relatively compact, i.e., there exists a convergent subsequence of $\{u_n\}_{n=0}^{\infty}$. With no confusion, let $\lim_{n\to\infty} u_n = u^* \in \Omega_r$.

Thus, by continuity of the operator $\mathcal{F}$, we have

$$u^* = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \mathcal{F}u_{n-1} = \mathcal{F}\Big(\lim_{n\to\infty} u_{n-1}\Big) = \mathcal{F}u^*.$$

Let $y^*(t) = t^{-1+\nu-\lambda\nu+\lambda k}u^*(t)$. Thus, y^* is a mild solution of (1) in $\widetilde{\Omega}_r$. The proof is completed. □

In the following, we prove the existence and uniqueness of a mild solution of the Cauchy problem (1).

(H6) There exists a function $L \in C([0,T],\mathbb{R}^+)$ such that $I_{0+}^{\lambda k}L \in C([0,T],\mathbb{R}^+)$,

$$|g(t,y_1(t)) - g(t,y_2(t))| \le L(t)\|y_1 - y_2\|_\lambda, \text{ for any } y_1, y_2 \in \widetilde{\Omega}_r,$$

and

$$\sup_{t\in[0,T]}\Big\{L_1 T^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}L(s)ds\Big\} \le l_0 < 1.$$

Theorem 3. *Assume that the conditions (H2)–(H4) and (H6) hold. Then the Cauchy problem (1) has a unique mild solution in* $\widetilde{\Omega}_r$.

Proof. From Lemmas 11, we know that $\mathcal{F}\Omega_r \subset \Omega_r$. For any $u_1, u_2 \in \Omega_r$, $t \in [0,T]$, we have

$$\begin{aligned}
&\Big|(\mathcal{F}u_1)(t) - (\mathcal{F}u_2)(t)\Big|\\
\le& T^{1-\nu+\lambda\nu-\lambda k}\int_0^t |\mathcal{K}_\lambda(t-s)(g(s,y_1(s)) - g(s,y_2(s)))|ds\\
\le& L_1 T^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}|g(s,y_1(s)) - g(s,y_2(s))|ds\\
\le& L_1 T^{1-\nu+\lambda\nu-\lambda k}\int_0^t (t-s)^{\lambda k-1}L(s)\|y_1 - y_2\|_\lambda ds\\
\le& l_0\|u_1 - u_2\|.
\end{aligned}$$

Thus
$$\|(\mathcal{F}u_1) - (\mathcal{F}u_2)\| \leq l_0\|u_1 - u_2\|,$$
which implies that $\mathcal{F}$ is a contraction mapping. In view of the contraction mapping principle, $\mathcal{F}$ has the unique fixed point $u^* \in \Omega_r$. Let $y^*(t) = t^{-(1-\nu+\lambda\nu-\lambda k)}u^*(t)$. Thus, y^* is a unique mild solution of (1) in $\widetilde{\Omega}_r$. The proof is completed. □

5. Remarks

In recent paper [10], the authors studied the problem (1) and obtained the following result by Schauder's fixed point theorem.

Theorem 4 (see Theorem 3 in [10])**.** *Let $0 < k < 1$, $0 < \omega < \frac{\pi}{2}$ and $A \in \Theta_\omega^{-k}(X)$. If we assume, $Q(t)(t > 0)$ is compact and the following hypotheses hold:*

(h$_1$) *for each fixed $t \in (0, T]$, $g(t, \cdot) : X \to X$ is continuous function and for each $y \in C((0, T], X)$, $g(\cdot, y) : (0, T] \to X$ is strongly measurable.*

(h$_2$) *there exists a function $l \in L^1((0, T], \mathbb{R}^+)$ satisfying*
$$I_{0+}^{\lambda k} l \in C((0, T], \mathbb{R}^+), \quad \lim_{t\to 0+} t^{(1-\lambda k)(1-\nu)} I_{0+}^{\lambda k} l(t) = 0$$
and $|g(t, u)| \leq l(t)$ for all $u \in \mathcal{B}_r^{\mathcal{Y}}((0, T])$ and almost all $t \in [0, T]$.

(h$_3$)
$$\sup_{t\in[0,T]} \left(t^{(1-\lambda k)(1-\nu)} |\mathcal{S}_{\lambda,\nu}(t)y_0| + t^{(1-\lambda k)(1-\nu)} \int_0^t (t - s)^{\lambda k-1} l(s)ds \right) \leq r,$$
for a constant $r > 0$ and $y_0 \in D(A^\theta)$, $\theta > 1 - k$, where $\mathcal{S}_{\lambda,\nu}(t) = I_{0+}^{\nu(1-\lambda)} t^{\lambda-1}\mathcal{Q}_\lambda(t)$.

Then there exist a mild solution of the Cauchy problem (1) in $\mathcal{B}_r^{\mathcal{Y}}((0, T])$ for every $y_0 \in D(A^\beta)$ with $\beta > 1 - k$.

Remark 1. *In [10], the authors claimed that $\lim_{t\to 0+} t^{(1-\lambda k)(1-\nu)}\mathcal{S}_{\lambda,\nu}(t)y_0 = 0$ (see, (12) in [10]). However, this claim is incorrect.*

In fact, when $\nu = 1$ and $y_0 \neq 0$, from Lemma 6, we know that $\lim_{t\to 0+} \mathcal{Q}_\lambda(t)y_0 = y_0/\Gamma(\lambda)$. Furthermore, we have
$$\begin{aligned}\lim_{t\to 0+} \mathcal{S}_{\lambda,1}(t)y_0 =& \frac{1}{\Gamma(1-\lambda)} \lim_{t\to 0+} \int_0^t (t - s)^{-\lambda} s^{\lambda-1}\mathcal{Q}_\lambda(s)y_0 ds \\ =& \frac{1}{\Gamma(1-\lambda)} \lim_{t\to 0+} \int_0^1 (1 - z)^{-\lambda} z^{\lambda-1}\mathcal{Q}_\lambda(tz)y_0 dz \\ =& y_0 \neq 0.\end{aligned}$$

Therefore, the definition of the operator $\mathcal{E}$ in (14) of [10] is incorrect. Because there is the same shortcoming in the papers [16–18], the definitions of the operator $\mathcal{P}$ in [16], the operator Φ in the proof of Theorem 3.1 in [17] and the operator $\mathfrak{F}$ in the proof of Theorem 3 in [18] are inappropriate.

Remark 2. *The condition (h$_3$) contains the abstract operator $\mathcal{S}_{\lambda,\nu}(t)$. It is difficult to verify whether the condition (h$_3$) is satisfied for one fractional evolution equation.*

Remark 3. *The results obtained in this paper essentially improve and correct Theorem 3 in [10], and extend Theorem 2.1 in [4] and the known results in [9]. It is worth mentioning that all conditions of our theorems do not contain the abstract operator $\mathcal{S}_{\lambda,\nu}(t)$.*

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