axioms

MDPI

Article

Hadamard Product Properties for Certain Subclasses of p-Valent Meromorphic Functions

Alaa H. El-Qadeem [1,*] and Ibrahim S. Elshazly [2]

1 Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt
2 Department of Basic Sciences, Common First Year, King Saud University, Alriyad 11451, Saudi Arabia; iali2.c@ksu.edu.sa
* Correspondence: ahhassan@science.zu.edu.eg

Abstract: We study the Hadamard product features of certain subclasses of p-valent meromorphic functions defined in the punctured open-unit disc using the q-difference operator. For functions belonging to these subclasses, we obtained certain coefficient estimates and inclusion characteristics. Furthermore, linkages between the results given here and those found in previous publications are highlighted.

Keywords: analytic function; univalent function; starlike function; convex function; meromorphic function; q-difference operator

MSC: 30C45; 30D30

Citation: El-Qadeem, A.H.; Elshazly, I.S. Hadamard Product Properties for Certain Subclasses of p-Valent Meromorphic Functions. *Axioms* **2022**, *11*, 172. https://doi.org/10.3390/axioms11040172

Academic Editor: Georgia Irina Oros

Received: 21 March 2022
Accepted: 11 April 2022
Published: 13 April 2022

1. Introduction

Let $\mathcal{M}_p$ stand for the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=-p+1}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the perforated unit disc $U^* = U\backslash\{0\} = \{z : z \in \mathbb{C} : 0 < |z| < 1\}$. The class $\mathcal{M}_p$ refers to the a class of p-valent meromorphic functions. It is worth noting that $\mathcal{M}_1 = \mathcal{M}$, which is the class of univalent meromorphic functions. If the function $g \in \mathcal{M}_p$ is given by

$$g(z) = z^{-p} + \sum_{k=-p+1}^{\infty} b_k z^k,$$

then the Hadamard product (or convolution) of f and g is provided by

$$(f * g)(z) = z^{-p} + \sum_{k=-p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

Interesting traits such as coefficient estimates, subordination relations and univalence features related some subclasses of p-valent functions were obtained in [1–3] (see also, [4]). With the help of the q-differential operator, a new subclass of meromorphic multivalent functions in the Janowski domain were introduced by Bakhtiar et al. in [5] (see also, [6]). Moreover, new subclasses of meromorphically p-valent functions were defined using q-derivative operator and investigations related to geometric properties of the class are conducted in [7–9].

If f and g are analytic in the open unit disc U, we say that f is subordinate to g, written as $f \prec g$ in U or $f(z) \prec g(z)(z \in U)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$ [10].

For $0 < q < 1$, the q-difference operator, which was introduced by Jackson [11], is characterised with

$$\partial_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

The Jackson q-difference operator is another name for the q-difference operator. Additionally, for f given by (1), one can write

$$\partial_q f(z) = -q^{-p}[p]_q z^{-p-1} + \sum_{k=-p+1}^{\infty} [k]_q a_k z^{k-1} (z \in U^*), \tag{2}$$

where $[k]_q = \left(1 - q^k\right)/(1-q)$ is the well-known q-bracket, $\lim_{q\to 1^-} [k]_q = k$ and $\lim_{q\to 1^-} \partial_q f(z) = f'(z)$.

Now, for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define the operator $\mathfrak{D}^n_{p,q} : \mathcal{M}_p \longrightarrow \mathcal{M}_p$ with the help of the q-difference operator, as follows:

$$\begin{aligned} &\mathfrak{D}^0_{p,q} f(z) = f(z), \\ &\mathfrak{D}^1_{p,q} f(z) = z^{-p}\partial_q\left(z^{p+1} f(z)\right), \\ &\mathfrak{D}^n_{p,q} f(z) = z^{-p}\partial_q\left(z^{p+1}\mathfrak{D}^{n-1}_{p,q} f(z)\right) \ (n \in \mathbb{N}), \end{aligned}$$

then

$$\mathfrak{D}^n_{p,q} f(z) = z^{-p} + \sum_{k=-p+1}^{\infty} [k+p+1]^n_q a_k z^k \ (n \in \mathbb{N}_0), \tag{3}$$

which satisfies the following recurrence relation:

$$q^{p+1} z \partial_q\left(\mathfrak{D}^n_{p,q} f(z)\right) = \mathfrak{D}^{n+1}_{p,q} f(z) - [p+1]_q \mathfrak{D}^n_{p,q} f(z). \tag{4}$$

Definition 1. *Utilising the q-derivative $\partial_q f(z)$, the subclasses$\mathcal{MS}^*_{p,q}(A,B)$ and $\mathcal{MK}_{p,q}(A,B)$ are introduced as follows:*

$$\mathcal{MS}^*_{p,q}(A,B) = \left\{ f \in \mathcal{M}_p \colon \frac{-q^p z \partial_q f(z)}{[p]_q f(z)} \prec \frac{1+Az}{1+Bz} \right\}, \tag{5}$$

$$(0 < q < 1; -1 \leq B < A \leq 1; z \in U),$$

and

$$\mathcal{MK}_{p,q}(A,B) = \left\{ f \in \mathcal{M}_p \colon \frac{-q^p \partial_q\left(z\partial_q f(z)\right)}{[p]_q \partial_q f(z)} \prec \frac{1+Az}{1+Bz}, z \in U \right\}, \tag{6}$$

$$(0 < q < 1; -1 \leq B < A \leq 1; z \in U).$$

Using (5) and (6), we have the following equivalence relation:

$$f(z) \in \mathcal{MK}_{p,q}(A,B) \Longleftrightarrow -\frac{q^p z \partial_q f(z)}{[p]_q} \in \mathcal{MS}^*_{p,q}(A,B). \tag{7}$$

Remark 1. *We list the following subclasses by specialising the parameters p, q, A and B:*
(i) $\mathcal{MS}^*_{p,q}(1-2\alpha,-1) = \mathcal{MS}^*_{p,q}(\alpha) = \{f \in \mathcal{M}_p \colon \operatorname{Re}\left(-\frac{q^p z \partial_q f(z)}{[p]_q f(z)}\right) > \alpha; 0 \leq \alpha < 1, z \in U\}$ the subclass of p-valent meromorphic q-starlike functions, and $\mathcal{MK}_{p,q}(1-2\alpha,-1) = \mathcal{MK}_{p,q}(\alpha) =$

$\{f \in \mathcal{M}_p\colon \operatorname{Re}\left(-\frac{q^p\partial_q(z\partial_q f(z))}{[p]_q\partial_q f(z)}\right) > \alpha; 0 \le \alpha < 1, z \in U\}$ *the subclass of p-valent meromorphic q-convex functions;*

(ii) $\mathcal{MS}^*_{1,q}(1-2\alpha,-1) = \mathcal{MS}^*_q(\alpha) = \{f \in \mathcal{M}\colon \operatorname{Re}\left(-\frac{qz\partial_q f(z)}{f(z)}\right) > \alpha; 0 \le \alpha < 1, z \in U\}$ *the subclass of meromorphic q-starlike functions, and* $\mathcal{MK}_{1,q}(1-2\alpha,-1) = \mathcal{MK}_q(\alpha) = \{f \in \mathcal{M}\colon \operatorname{Re}\left(-\frac{q\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right) > \alpha;\ 0 \le \alpha < 1, z \in U\}$ *the subclass of meromorphic q-convex functions;*

(iii) $\lim_{q\to 1^-} \mathcal{MS}^*_{p,q}(A,B) = \mathcal{MS}^*_p(A,B) = \{f \in \mathcal{M}_p\colon -\frac{zf'(z)}{pf(z)} \prec \frac{1+Az}{1+Bz};\ -1 \le B < A \le 1, z \in U\}$, *and* $\lim_{q\to 1^-} \mathcal{MK}_{p,q}(A,B) = \mathcal{MK}_p(A,B) = \{f \in \mathcal{M}_p\colon -\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+Az}{1+Bz}; -1 \le B < A \le 1, z \in U\}$, *were introduced and studied by Ali and Ravichandran [12];*

(iv) $\lim_{q\to 1^-} \mathcal{MS}^*_{1,q}(1-2\alpha,-1) = \mathcal{MS}^*(\alpha) = \{f \in \mathcal{M}\colon \operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \alpha; 0 \le \alpha < 1, z \in U\}$, *and* $\lim_{q\to 1^-} \mathcal{MK}_{p,q}(1-2\alpha,-1) = \mathcal{MK}(\alpha) = \{f \in \mathcal{M}\colon \operatorname{Re}\left(-1-\frac{zf''(z)}{f'(z)}\right) > \alpha; 0 \le \alpha < 1, z \in U\}$, *were introduced and studied by Kaczmarski [13];*

(v) $\lim_{q\to 1^-} \mathcal{MS}^*_{1,q}(1,-1) = \mathcal{MS}^*$, *and* $\lim_{q\to 1^-} \mathcal{MK}_{1,q}(1,-1) = \mathcal{MK}$, *which are well-known function classes of meromorphic starlike and meromorphic convex functions, respectively; see Pommerenke [14], Clunie [15] and Miller [16] for more details.*

Definition 2. *For* $n \in \mathbb{N}_0$ *and* $0 < q < 1$, *we define the following subclasses:*

$$\mathcal{MS}^*_{p,q}(n;A,B) = \left\{f \in \mathcal{M}_p : \mathfrak{D}^n_{p,q} f(z) \in \mathcal{MS}^*_{p,q}(A,B)\right\}, \tag{8}$$

$$(n \in \mathbb{N}_0; 0 < q < 1; -1 \le B < A \le 1; z \in U),$$

and

$$\mathcal{MK}_{p,q}(n;A,B) = \left\{f \in \mathcal{M}_p : \mathfrak{D}^n_{p,q} f(z) \in \mathcal{MK}_{p,q}(A,B)\right\}, \tag{9}$$

$$(n \in \mathbb{N}_0; 0 < q < 1; -1 \le B < A \le 1; z \in U).$$

It is easy to show that

$$f(z) \in \mathcal{MK}_{p,q}(n;A,B) \iff -\frac{q^p z\partial_q f(z)}{[p]_q} \in \mathcal{MS}^*_{p,q}(n;A,B). \tag{10}$$

There is extensive literature dealing with convolution properties of different families of analytic and meromorphic functions; for details, see [17–23]. More recently, the quantum derivative was utilised by Seoudy and Aouf [24] (see also [25]) to introduce the convolution features for certain classes of analytic functions. Here, we use the quantum derivative to obtain some convolution properties of the meromorphic functions. For this purpose, we defined the new classes $\mathcal{MS}^*_{p,q}(A,B)$ and $\mathcal{MK}_{p,q}(A,B)$. The convolution results are followed by some consequences such as necessary and sufficient conditions, the estimates of coefficients and inclusion characteristics of the subclasses $\mathcal{MS}^*_{p,q}(n;A,B)$ and $\mathcal{MK}_{p,q}(n;A,B)$.

2. Convolution Properties

Theorem 1. *The function f given by (1) is in the class* $\mathcal{MS}^*_{p,q}(A,B)$, *if and only if*

$$z^p\left[f(z) * \frac{1+(C-q)z}{z^p(1-z)(1-qz)}\right] \neq 0\ (z \in U), \tag{11}$$

for all

$$C = \frac{B+e^{-i\theta}}{A-[p]_q B - q[p-1]_q e^{-i\theta}};\ \theta \in [0,2\pi), \tag{12}$$

and also for $C = 0$.

Proof. It is simple to check the following two equalities

$$f(z) * \frac{1}{z^p(1-z)} = f(z) \tag{13}$$

and

$$f(z) * \left(\frac{1}{qz^p(1-z)(1-qz)} - \frac{[1+p]_q}{qz^p(1-z)}\right) = q^p z\partial_q f(z) \tag{14}$$

In view of (5), $f \in \mathcal{MS}^*_{p,q}(A,B)$, if and only if (1.4) holds. Since the function $\frac{1+Az}{1+Bz}$ is analytic function on U, it follows that $f(z) \neq 0$, $z \in U^*$; that is $z^p f(z) \neq 0$, $z \in U$, and using the first identity of (13). That is the same as saying that the relation (11) is satisfied for $C = 0$. According to the concept of subordination of two functions in (14), there exists an analytic function $w(z)$ in U with $w(0) = 0$, $|w(z)| < 1$ in such a way that

$$\frac{-q^p z\partial_q f(z)}{[p]_q f(z)} = \frac{1+Aw(z)}{1+Bw(z)} \ (z \in U),$$

which leads to

$$\frac{-q^p z\partial_q f(z)}{[p]_q f(z)} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}} \ (f(z) \neq 0, z \in U; 0 \leq \theta < 2\pi),$$

or

$$z^p\left[\left(q^p z\partial_q f(z)\right)\left(1+Be^{i\theta}\right) + [p]_q f(z)\left(1+Ae^{i\theta}\right)\right] \neq 0 \tag{15}$$

We may now deduce the following from (13)–(15):

$$z^p\left[\left(f(z) * \frac{1-[1+p]_q(1-qz)}{qz^p(1-z)(1-qz)}\right)\left(1+Be^{i\theta}\right) + \left(1+Ae^{i\theta}\right)\left(f(z) * \frac{1}{z^p(1-z)}\right)\right] \neq 0,$$

$$z^p\left[f(z) * \left(\frac{\left(1-[1+p]_q + q[1+p]_q z\right)(1+Be^{i\theta}) + q(1-qz)(1+Ae^{i\theta})}{qz^p(1-z)(1-qz)}\right)\right] \neq 0,$$

but $1-[1+p]_q = -q[p]_q$; then, the condition became

$$z^p\left[f(z) * \left(\frac{q\left([1+p]_q z - [p]_q\right)(1+Be^{i\theta}) + q(1-qz)(1+Ae^{i\theta})}{qz^p(1-z)(1-qz)}\right)\right] \neq 0,$$

or,

$$z^p\left[f(z) * \left(\frac{\left([1+p]_q z - [p]_q\right)(1+Be^{i\theta}) + (1-qz)(1+Ae^{i\theta})}{z^p(1-z)(1-qz)}\right)\right] \neq 0,$$

or, equivalent to

$$z^p\left[f(z) * \left(\frac{1-[p]_q + \left(A-[p]_q B\right)e^{i\theta} + \left([1+p]_q - q + \left([1+p]_q B - qA\right)e^{i\theta}\right)z}{z^p(1-z)(1-qz)}\right)\right] \neq 0,$$

or,

$$z^p\left[f(z) * \left(\frac{-q[p-1]_q + \left(A-[p]_q B\right)e^{i\theta} + \left([1+p]_q - q + \left([1+p]_q B - qA\right)e^{i\theta}\right)z}{z^p(1-z)(1-qz)}\right)\right] \neq 0,$$

or,

$$z^p\left[f(z)*\left(\frac{1+\frac{\left([1+p]_q-q+\left([1+p]_qB-qA\right)e^{i\theta}\right)z}{-q[p-1]_q+\left(A-[p]_qB\right)e^{i\theta}}}{z^p(1-z)(1-qz)}\left(\left(A-[p]_qB\right)e^{i\theta}-q[p-1]_q\right)\right)\right]\neq 0,$$

by dividing both sides by the non-zero quantity $\left(A-[p]_qB\right)e^{i\theta}-q[p-1]_q$, then we have

$$z^p\left[f(z)*\left(\frac{1+\frac{\left([1+p]_q-q+\left([1+p]_qB-qA\right)e^{i\theta}\right)z}{-q[p-1]_q+\left(A-[p]_qB\right)e^{i\theta}}}{z^p(1-z)(1-qz)}\right)\right]\neq 0,$$

which is the same as

$$z^p\left[f(z)*\left(\frac{1+\left(\frac{[1+p]_q-q+\left([1+p]_qB-qA\right)e^{i\theta}+q\left(-q[p-1]_q+\left(A-[p]_qB\right)e^{i\theta}\right)}{-q[p-1]_q+\left(A-[p]_qB\right)e^{i\theta}}-q\right)z}{z^p(1-z)(1-qz)}\right)\right]\neq 0,$$

or,

$$z^p\left[f(z)*\left(\frac{1+\left(\frac{[1+p]_q-q-q^2[p-1]_q+\left([1+p]_q-q[p]_q\right)Be^{i\theta}}{-q[p-1]_q+\left(A-[p]_qB\right)e^{i\theta}}-q\right)z}{z^p(1-z)(1-qz)}\right)\right]\neq 0,$$

but $[1+p]_q-q-q^2[p-1]_q=[1+p]_q-q[p]_q=1$, then the convolution condition became

$$z^p\left[f(z)*\left(\frac{1+\left(\frac{e^{-i\theta}+B}{A-[p]_qB-q[p-1]_qe^{-i\theta}}-q\right)z}{z^p(1-z)(1-qz)}\right)\right]\neq 0,$$

This leads to (11), proving the first part of Theorem 1.

In contrast, because (11) holds for $C=0$, it follows that $z^pf(z)\neq 0$ for all $z\in U$, and hence the function.

$$\varphi(z)=\frac{-q^pz\partial_qf(z)}{[p]_qf(z)},$$

is analytic in U (i.e., it is regular at $z_0=0$, with $\varphi(0)=1$). We obtain that because the assumption (11) is equivalent to (15), as shown in the first section of the proof.

$$\frac{-q^pz\partial_qf(z)}{[p]_qf(z)}\neq\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\ (\theta\in[0,2\pi),f(z)\neq 0,z\in U),\tag{16}$$

if we denote

$$\psi(z)=\frac{1+Az}{1+Bz},\tag{17}$$

therefore $\varphi(U)\cap\psi(\partial U)=\phi$, with the help of the relation (16). Thus, the simply connected domain $\varphi(U)$ is included in a connected component of $\mathbb{C}\backslash\psi(\partial U)$. As a result, a connected component of $\mathbb{C}\backslash\psi(\partial U)$ includes the simply connected domain $\varphi(U)$. The fact that $\varphi(0)=\psi(0)$ and the univalence of the function ψ lead to the conclusion that

$\varphi(z) \prec \psi(z)$. This completes the proof of the second item of Theorem 1 by representing the subordination (5), i.e., $f \in \mathcal{MS}^*_{p,q}(A,B)$. $\square$

Remark 2. *(i) We obtain the results obtained in the paper of Aouf et al. in [17] (Theorem 4, with $\lambda = 0$ and $b = 1$) by putting $p = 1$ and $q \to 1^-$ in Theorem 1. See also, Bulboacă et al. [20] (Theorem 1, with $b = 1$) and El-Ashwah [21] (Theorem 1, with $p = 1$);*

(ii) Putting $p = 1$, $q \to 1^-$, $A = 1$ and $B = -1$ in Theorem 1, we obtain the result of Aouf et al. [18] (Theorem 1, with $b = m = 1$).

In Theorem 1, we have the following corollary if $A = 1 - 2\alpha$ and $B = -1$.

Corollary 1. *The function f defined by (1) is in the class $\mathcal{MS}^*_{p,q}(\alpha)$, if and only if*

$$z^p\left[f(z) * \frac{1+\left(\frac{\left(1+q^2[p-1]_q\right)e^{-i\theta}-q(1-2\alpha+[p]_q)}{1-2\alpha+[p]_q-q[p-1]_q e^{-i\theta}}\right)z}{z^p(1-z)(1-qz)}\right] \neq 0 \; (z \in U),$$

Taking $q \to 1^-$, $A = 1 - 2\alpha$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 2. *The function f expressed in (1) belongs to $\mathcal{MS}^*_p(\alpha)$, if and only if*

$$z^p\left[f(z) * \frac{1+\left[\frac{2(1-\alpha)+p\left(e^{-i\theta}-1\right)}{1-2\alpha+p-(p-1)e^{-i\theta}}\right]z}{z^p(1-z)^2}\right] \neq 0 \; (z \in U),$$

Theorem 2. *The function f of the form (1) is a member of the class $\mathcal{MK}_{p,q}(A,B)$, if and only if*

$$z^p\left[f(z) * \frac{1-\frac{\left(1-q^{p+2}\right)-q\left(1-q^{p-1}\right)(C-q)}{1-q^p}z-\frac{q\left(1-q^{p+1}\right)(C-q)}{1-q^p}z^2}{z^p(1-z)(1-qz)(1-q^2z)}\right] \neq 0 \; (z \in U), \tag{18}$$

for all C defined by (12), and also for $C = 0$.

Proof. If

$$g(z) = \frac{1+(C-q)z}{z^p(1-z)(1-qz)}, \tag{19}$$

then

$$-\frac{q^p z \partial_q g(z)}{[p]_q} = \frac{-q^p z}{[p]_q}\left[\frac{1}{(q-1)z}(g(qz)-g(z))\right]$$

which leads to

$$-\frac{q^p z \partial_q g(z)}{[p]_q} = \frac{1-\left(\frac{\left(1-q^{p+2}\right)-q\left(1-q^{p-1}\right)(C-q)}{1-q^p}\right)z-\left(\frac{q\left(1-q^{p+1}\right)(C-q)}{1-q^p}\right)z^2}{z^p(1-z)(1-qz)(1-q^2z)} \tag{20}$$

The following identity remains true for two functions, f and g, which belong to $\mathcal{M}_p$.

$$\left(-\frac{q^p z \partial_q f(z)}{[p]_q}\right) * g(z) = f(z) * \left(-\frac{q^p z \partial_q g(z)}{[p]_q}\right). \tag{21}$$

Now, by using equivalence relation (7) and Theorem 1, the proof can be achieved by applying (20) and (21). $\square$

Remark 3. *(i) Putting $p = 1$ and $q \to 1^-$ in Theorem 2, we arrive at the results of Aouf et al. [17] (Theorem 6, with $\lambda = 0$ and $b = 1$) and Bulboacă et al. [20] (Theorem 2, with $b = 1$), and El-Ashwah [21] (Theorem 2, with $p = 1$);*
(ii) Putting $p = 1$, $q \to 1^-$, $A = 1$ and $B = -1$ in Theorem 2, we reach the conclusion of Aouf et al. [18] (Theorem 3, with $b = m = 1$).

As a result, we have the following corollary by taking $A = 1 - 2\alpha$ and $B = -1$ in Theorem 2.

Corollary 3. *The function $f \in \mathcal{MK}_{p,q}(\alpha)$, if and only if*

$$z^p\left[f(z) * \frac{1 - Dz - Ez^2}{z^p(1-z)(1-qz)(1-q^2z)}\right] \neq 0 \; (z \in U),$$

where

$$D = \frac{(1-q^{p+2}) - q(1-q^{p-1})\left(\frac{\left(1+q^2[p-1]_q\right)e^{-i\theta} - q\left(1-2\alpha+[p]_q\right)}{1-2\alpha+[p]_q - q[p-1]_q e^{-i\theta}}\right)}{1-q^p},$$

and

$$E = \frac{q(1-q^{p+1})\left(\left(1+q^2[p-1]_q\right)e^{-i\theta} - q\left(1-2\alpha+[p]_q\right)\right)}{(1-q^p)\left(1-2\alpha+[p]_q - q[p-1]_q e^{-i\theta}\right)}.$$

As a result, we have the following corollary by taking $q \to 1^-$, $A = 1 - 2\alpha$ and $B = -1$ in Theorem 2.

Corollary 4. *The function $f \in \mathcal{MK}_p(\alpha)$, if and only if*

$$z^p\left[f(z) * \frac{1 - \frac{2p(1-2\alpha+p) - (2p^2-p-1)e^{-i\theta}}{p(1-2\alpha+p) - p(p-1)e^{-i\theta}}z - \frac{(p+2)\left(pe^{-i\theta} - (1-2\alpha+p)\right) - 1}{p(1-2\alpha+p) - p(p-1)e^{-i\theta}}z^2}{z^p(1-z)^3}\right] \neq 0 \; (z \in U).$$

Theorem 3. *The following are necessary and sufficient requirements for the function $f \in \mathcal{M}_p$ to be in the class $\mathcal{MS}^*_{p,q}(n; A, B)$:*

$$1 + \sum_{k=-p+1}^{\infty} [k+p+1]_q^n a_k z^{k+p} \neq 0 \; (z \in U), \tag{22}$$

or

$$1 + \sum_{k=-p+1}^{\infty} \left([k+p]_q C + 1\right)[k+p+1]_q^n a_k z^{k+p} \neq 0 \;\; (z \in U), \tag{23}$$

where C is defined by (12).

Proof. Let $f \in \mathcal{M}_p$, then, by using Theorem 1 and (8) we have $f \in \mathcal{MS}^*_{p,q}(n; A, B)$, if and only if

$$z^p\left[\left(\mathfrak{D}_q^n f\right)(z) * \frac{1 + (C-q)z}{z^p(1-z)(1-qz)}\right] \neq 0 \; (z \in U), \tag{24}$$

for all $C = \frac{B + e^{-i\theta}}{A - [p]_q B - q[p-1]_q e^{-i\theta}}$; $\theta \in [0, 2\pi)$, and also for $C = 0$. Since

$$\frac{1 + (0-q)z}{z^p(1-z)(1-qz)} = z^{-p} + \sum_{k=-p+1}^{\infty} z^k, \tag{25}$$

by using (3) and (25) in (24) in case of $C = 0$, then we can obtain (22).

Similarly, it can be shown that

$$\frac{1+(C-q)z}{z^p(1-z)(1-qz)} = z^{-p} + \sum_{k=-p+1}^{\infty}\Big([k+p]_q C + 1\Big)z^k, \tag{26}$$

then using (3) and (26) in (24), we can obtain (23). The proof is complete. □

The next theorem can be established using the same method, and the proof is eliminated.

Theorem 4. *The following are necessary and sufficient requirements for the function $f \in \mathcal{M}_p$ to be in the class $\mathcal{MK}_{p,q}(n; A, B)$:*

$$1 - \sum_{k=-p+1}^{\infty} q[k]_q [k+p+1]_q^n a_k z^{k+p} \neq 0 \; (z \in U), \tag{27}$$

or

$$1 - \sum_{k=-p+1}^{\infty} q[k]_q \Big([k+p]_q C + 1\Big)[k+p+1]_q^n a_k z^{k+p} \neq 0 \; (z \in U). \tag{28}$$

3. Estimates of Coefficients and Inclusion Characteristics

In this section, as an application of Theorems 3 and 4, we introduce some estimates of the coefficients $a_k (k \geq -p+1)$ of functions of the form (1) which belong to the two main classes $\mathcal{MS}^*_{p,q}(n; A, B)$ and $\mathcal{MK}_{p,q}(n; A, B)$, respectively. Moreover, we give the inclusion relationships of the two classes.

Theorem 5. *If the function $f \in \mathcal{M}_p$ fulfills the inequalities*

$$\sum_{k=-p+1}^{\infty} [k+p+1]_q^n |a_k| < 1, \tag{29}$$

and

$$\sum_{k=-p+1}^{\infty} \Big([k+p]_q |C| + 1\Big)[k+p+1]_q^n |a_k| < 1, \tag{30}$$

then $f \in \mathcal{MS}^*_{p,q}(n; A, B)$.

Proof. According to (29), a simple calculation shows that

$$\begin{aligned}\left|1 + \sum_{k=-p+1}^{\infty} [k+p+1]_q^n a_k z^{k+p}\right| &\geq 1 - \left|\sum_{k=-p+1}^{\infty} [k+p+1]_q^n a_k z^{k+p}\right| \\ &\geq 1 - \sum_{k=-p+1}^{\infty} [k+p+1]_q^n |a_k| |z|^{k+p} \\ &> 1 - \sum_{k=-p+1}^{\infty} [k+p+1]_q^n |a_k| > 0\end{aligned}$$

which leads to satisfaction of (22), then $f \in \mathcal{MS}^*_{p,q}(n;A,B)$. Similarly, using the assumption (30), we conclude that

$$\begin{aligned}
&\left|1+\sum_{k=-p+1}^{\infty}\Big([k+p]_q C+1\Big)[k+p+1]_q^n a_k z^{k+p}\right| \\
&\geq 1-\left|\sum_{k=-p+1}^{\infty}\Big([k+p]_q C+1\Big)[k+p+1]_q^n a_k z^{k+p}\right| \\
&\geq 1-\sum_{k=-p+1}^{\infty}\Big([k+p]_q|C|+1\Big)[k+p+1]_q^n |a_k||z|^{k+p} \\
&> 1-\sum_{k=-p+1}^{\infty}\Big([k+p]_q|C|+1\Big)[k+p+1]_q^n |a_k| > 0,
\end{aligned}$$

which shows that (23) holds true and $f \in \mathcal{MS}^*_{p,q}(n;A,B)$; the proof is finished. □

Similarly, results regarding $\mathcal{MK}_{p,q}(n;A,B)$ can be introduced as follows:

Theorem 6. *If the function $f \in \mathcal{M}_p$ fulfills the inequalities*

$$\sum_{k=-p+1}^{\infty} q[k]_q[k+p+1]_q^n|a_k| < 1, \tag{31}$$

and

$$\sum_{k=-p+1}^{\infty} q[k]_q\Big([k+p]_q|C|+1\Big)[k+p+1]_q^n|a_k| < 1, \tag{32}$$

then $f \in \mathcal{MK}_{p,q}(n;A,B)$.

Now, using the appropriate technique due to Ahuja [26], we introduce the inclusion relationships of $\mathcal{MS}^*_{p,q}(n;A,B)$ and $\mathcal{MK}_{p,q}(n;A,B)$, respectively.

Theorem 7. *If $n \in \mathbb{N}_o$, then*

$$\mathcal{MS}^*_{p,q}(n+1;A,B) \subset \mathcal{MS}^*_{p,q}(n;A,B). \tag{33}$$

Proof. If $f \in \mathcal{MS}^*_{p,q}(n+1;A,B)$, then using Theorem 3, we can write

$$1+\sum_{k=-p+1}^{\infty}[k+p+1]_q^{n+1}a_k z^{k+p} \neq 0 \ \ (z \in U), \tag{34}$$

or

$$1+\sum_{k=-p+1}^{\infty}\Big([k+p]_q C+1\Big)[k+p+1]_q^{n+1}a_k z^{k+p} \neq 0 \ \ (z \in U), \tag{35}$$

but (34) and (35) can be written as follows:

$$\left(1+\sum_{k=-p+1}^{\infty}[k+p+1]_q z^{k+p}\right) * \left(1+\sum_{k=-p+1}^{\infty}[k+p+1]_q^n a_k z^{k+p}\right) \neq 0, \tag{36}$$

and

$$\left(1+\sum_{k=-p+1}^{\infty}[k+p+1]_q z^{k+p}\right) * \left(1+\sum_{k=-p+1}^{\infty}\Big([k+p]_q C+1\Big)[k+p+1]_q^n a_k z^{k+p}\right) \neq 0. \tag{37}$$

Let us really define the function

$$h_1(z) = 1 + \sum_{k=-p+1}^{\infty} [k+p+1]_q z^{k+p}. \tag{38}$$

We note that the assumption that $h_1(z) = 0$ leads to $|z| > 1$, Thus, we deduce that $h_1(z) \neq 0$. Using the property that if $h_1 * g \neq 0$ and $h_1 \neq 0$, then $g \neq 0$. Thus from (36) and (37) and using the function $h_1(z) \neq 0$, we obtain

$$1 + \sum_{k=-p+1}^{\infty} [k+p+1]_q^n a_k z^{k+p} \neq 0, \tag{39}$$

and

$$1 + \sum_{k=-p+1}^{\infty} \Big([k+p]_q C + 1\Big) [k+p+1]_q^n a_k z^{k+p} \neq 0, \tag{40}$$

then Theorem 3 tells us that $f \in \mathcal{MS}^*_{p,q}(n; A, B)$. $\square$

The following theorem gives the inclusion relationship regarding $\mathcal{MK}_{p,q}(n; A, B)$.

Theorem 8. *For $n \in \mathbb{N}_0$, we have*

$$\mathcal{MK}_{p,q}(n+1; A, B) \subset \mathcal{MK}_{p,q}(n; A, B). \tag{41}$$

Our results in Theorems 7 and 8 above can be utilised to introduce the following consequences.

Corollary 5. *Suppose that $m = n+1, n+2, \ldots (n \in \mathbb{N}_0)$. Then*

$$f \in \mathcal{MS}^*_{p,q}(m; A, B) \Longrightarrow f \in \mathcal{MS}^*_{p,q}(n; A, B).$$

Equivalently, if

$$\mathfrak{D}_q^m f(z) \in \mathcal{MS}^*_{p,q}(A, B),$$

then

$$f \in \mathcal{MS}^*_{p,q}(n; A, B).$$

Corollary 6. *Suppose that $m = n+1, n+2, \ldots (n \in \mathbb{N}_0)$. Then*

$$f \in \mathcal{MK}_{p,q}(m; A, B) \Longrightarrow f \in \mathcal{MK}_{p,q}(n; A, B).$$

Equivalently, if

$$\mathfrak{D}_q^m f(z) \in \mathcal{MK}_{p,q}(A, B),$$

then

$$f \in \mathcal{MK}_{p,q}(n; A, B).$$

4. Conclusions

We have defined a new operator on the set of meromorphically multivalent functions. With the help of this operator, we introduced the new subclasses $\mathcal{MK}_{p,q}(n; A, B)$ and $\mathcal{MS}^*_{p,q}(n; A, B)$. The study was concentrated on convolution conditions. Our suggestions for future studies on these subclasses is to use them in studies involving the theories of differential subordination and superordination. Additionally, one can define the results concerning the calculation of the bounds of coefficients of the bi-univalent functions, also obtaining the Fekete–Szegö functionals.

Author Contributions: Formal analysis and methodology, A.H.E.-Q.; resources, I.S.E. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to thank the Common First Year Research Unit at King Saud University for giving us the funds for this article.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to give thanks for the help of HM Abbas.

Conflicts of Interest: The authors confirm no competing interests.

References

1. Yousef, A.T.; Salleh, Z.; Al-Hawary, T. On a class of p-valent functions involving generalized differential operator. *Afr. Mat.* **2021**, *32*, 275–287. [CrossRef]
2. Al-Janaby, H.F.; Ghanim, F. A subclass of Noor-type harmonic p-valent functions based on hypergeometric functions. *Kragujev. J. Math.* **2021**, *45*, 499–519. [CrossRef]
3. Oros, G.I.; Oros, G.; Owa, S. Applications of Certain p-Valently Analytic Functions. *Mathematics* **2022**, *10*, 910. [CrossRef]
4. El-Ashwah, R.M.; Hassan, A.H. Properties of certain subclass of p-valent meromorphic functions associated with certain linear operator. *J. Egyptian Math. Soc.* **2016**, *24*, 226–232. [CrossRef]
5. Bakhtiar, A.; Mashwani, W.K.; Serkan, A.; Saima, M.; Khan, M.G.; Khan, B. A subclass of meromorphic Janowski-type multivalent q-starlike functions involving a q-differential operator. *Adv. Differ. Equ.* **2022**, *2022*, 5. [CrossRef]
6. El-Qadeem, A.H.; Mamon, M.A. Comprehensive subclasses of multivalent functions with negative coefficients defined by using a q-difference operator. *Trans. A Razmadze Math. Inst.* **2018**, *172*, 510–526. [CrossRef]
7. Golmohammadi, M.H.; Najafzadeh, S.; Forutan, M.R. On a Generalized Subclass of p-Valent Meromorphic Functions by Defined q-Derivative Operator. *Adv. Math. Finance Appl.* **2021**, *6*, 869–881.
8. Khan, B.; Srivastava, H.M.; Arjika, S.; Khan, S.; Khan, N.; Ahmad, Q.Z. A certain q-Ruscheweyh type derivative operator and its applications involving multivalent functions. *Adv. Differ. Equ.* **2021**, *2021*, 279. [CrossRef]
9. Aldawish, I.; Aouf, M.K.; Frasin, B.A.; Al-Hawary, T. New subclass of analytic functions defined by q-analogue of p-valent Noor integral operator. *AIMS Math.* **2021**, *6*, 10466–10484. [CrossRef]
10. Miller, S.S.; Mocanu, P.T. *Differenatial Subordinations: Theory and Applications*; Series on Monographs and Textbooks in Pure and Appl. Math. No. 255; Marcel Dekker, Inc.: New York, NY, USA, 2000.
11. Jackson, F.H. On q-functions and a certain difference operator. *Trans. Roy. Soc. Edinb* **1909**, *46*, 253–281. [CrossRef]
12. Ali, R.M.; Ravichandran, V. Classes of meromorphic alpha-convex functions. *Taiwan J. Math.* **2010**, *14*, 1479–1490. [CrossRef]
13. Kaczmarski, J. On the coefficients of some classes of starlike functions. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **1969**, *17*, 495–501.
14. Pommerenke, C. On meromorphic starlike functions. *Pac. J. Math.* **1963**, *13*, 221–235. [CrossRef]
15. Clune, J. On meromorphic Schlicht functions. *J. Lond. Math. Soc.* **1959**, *34*, 215–216. [CrossRef]
16. Miller, J.E. Convex meromorphic mappings and related functions. *Proc. Am. Math. Soc.* **1970**, *25*, 220–228. [CrossRef]
17. Aouf, M.K.; Mostafa, A.O.; Zayed, H.M. Convolution properties for some subclasses of meromorphic functions of complex order. *Abstract Appl. Anal.* **2015**, *2015*, 973613. [CrossRef]
18. Aouf, M.K.; Mostafa, A.O.; Zayed, H.M. Convolution conditions for some subclasses of meromorphic bounded functions of complex order. *Thai J. Math.* **2016**, *14*, 249–258.
19. Aouf, M.K.; Seoudy, T.M. Classes of analytic functions related to the Dziok–Srivastava operator. *Integral Transforms Spec. Funct.* **2011**, *22*, 423–430. [CrossRef]
20. Bulboacă, T.; Aouf, M.K.; El-Ashwah, R.M. Convolution properties for subclasses of meromorphic univalent functions of complex order. *Filomat* **2012**, *26*, 153–163. [CrossRef]
21. El-Ashwah, R.M. Some convolution and inclusion properties for subclasses of meromorphic p-valent functions involving integral operator. *Acta Math. Sci.* **2013**, *33*, 1749–1758. [CrossRef]
22. Mostafa, A.O.; Aouf, M.K. On convolution properties for certain classes of p-valent meromorphic functions defined by linear operator. *Lematematiche* **2014**, *69*, 259–266.
23. Mostafa, A.O.; Aouf, M.K. On convolution properties for some classes of meromorphic functions associated with linear operator. *Bull. Iranian Math. Soc.* **2015**, *41*, 325–332.
24. Seoudy, T.M.; Aouf, M.K. Convolution properties for classes of bounded analytic functions with complex order defined by q-derivative operator. *RACSAM* **2019**, *113*, 1279–1288.
25. Seoudy, T.M.; Aouf, M.K. Convolution properties for certain classes of analytic functions defined by q-derivative operator. *Abstract Appl. Anal.* **2014**, *2014*, 846719. [CrossRef]
26. Ahuja, O.P. Families of analytic functions related to Ruscheweyh derivatives and subordinate to convex functions. *Yokohama Math. J.* **1993**, *41*, 39–50.