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Approximate Optimal Control for a Parabolic System with Perturbations in the Coefficients on the Half-Axis

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Abstract: In this paper, we use the averaging method to find an approximate solution in the optimal control problem of a parabolic system with non-linearity of the form $f(t/\varepsilon, y)$ on an infinite time interval.

Keywords: parabolic system; optimal control; averaging method; approximate solution

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1. Introduction

Many results in the theory of asymptotic approximations have been obtained from 1930 onwards. Indeed, there were a lot of results on integral manifolds, equations with retarded argument, quasi- or almost-periodic equations etc. Earlier work on this theory has been presented in the famous book [1].

Averaging is a valuable method to understand the long-term evolution of dynamical systems characterized by slow dynamics and fast periodic or quasi-periodic dynamics. In [2], a transparent proof of the validity of averaging in the periodic case is presented. Different proofs for both the periodic and the general case are provided by [3,4]. In the last paper, moreover, the relation between averaging and the multiple time-scales method is established.

The averaging method for constructing approximate solutions in the theory of ODEs is presented in [5,6]. In [7], the asymptotic analysis of nonlinear dynamical systems is developed.

The work [8] is devoted to using an asymptotic method for studying the Cauchy problem for a 1D Euler–Poisson system, which represents a physically relevant hydrodynamic model but also a challenging case for a bipolar semiconductor device by considering two different pressure functions. In [9], the averaging results for ordinary differential equations perturbed by a small parameter are proved. Here, authors assume only that the right-hand sides of the equations are bounded by some locally Lebesgue integrable functions with the property that their indefinite integrals satisfy a Lipschitz-type condition.

In [10], the authors prove that averaging can be applied to the extremal flow of optimal control problems with two fast variables, that is considerably more complex because of resonances.

The averaging method is one of the most effective tools for constructing approximate solutions, including optimal control problems for ODEs [11] and PDEs [12], where the autors consider the optimal control problem in coefficients in the so-called class of H-admissible solutions.

The Krasnoselski–Krein theorem and its various modifications [13–15] play an essential role in all such considerations, since it guarantees the limit transition in perturbed problem with fast-oscillating coefficients of the form $a\left(\frac{t}{\varepsilon}\right)$ as $\varepsilon \to 0$.

The typical averaging problem may be defined as follows: one considers an unperturbed problem in which the slow variables remain fixed. Upon perturbation, a slow drift appears in these variables which one would like to approximate independently of the fast variables.

In the present paper we use this approach to nonlinear parabolic system with fast-oscillating (w.r.t. time variable) coefficients $f\left(\frac{t}{\varepsilon}, y\right)$ on an infinite time interval. We prove that the optimal control of the problem with averaging coefficients can be considered to be "approximately" optimal for the initial perturbed system.

2. Statement of the Problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. In cylinders $Q = (0, +\infty) \times \Omega$ we consider an initial boundary-value problem for a parabolic system [16,17]

$$\begin{cases} \frac{\partial y}{\partial t} = A\Delta y + f\left(\frac{t}{\varepsilon}, y\right) + g(y) \cdot u(t,x), (t,x) \in Q, \\ y|_{\partial\Omega} = 0, \\ y|_{t=0} = y_0(x). \end{cases} \tag{1}$$

Here $\varepsilon > 0$ is a small parameter, A is a real $N \times N$ matrix, f is a given vector-valued mapping, g is a given matrix-valued mapping, $y = (y_1, \dots, y_N)$ is an unknown state function, $u = (u_1, \dots, u_M)$ is an unknown control function, which are determined by requirements

$$u \in U \subseteq (L^2(Q))^M, \tag{2}$$

$$J(y,u) = \int_Q e^{-\gamma \cdot t} \cdot q(x, y(t,x)) dt dx + \int_Q \sum_{i=1}^M \alpha_i \cdot u_i^2(t,x) dt dx \to \inf, \tag{3}$$

where $\gamma, \alpha_1, \dots, \alpha_M$ are positive constants.

Under the natural assumptions on A, f, g, U, q we prove, that the optimal control problem (1)–(3) has a solution $\{\bar{y}^\varepsilon, \bar{u}^\varepsilon\}$, i.e., for every $u \in U$ and for any solution y^ε of (1) with control u we have

$$J(\bar{y}^\varepsilon, \bar{u}^\varepsilon) \leq J(y^\varepsilon, u)$$

In what follows we consider the problem of finding an approximate solution of (1)–(3) by transition to averaged coefficients. For this purpose we assume that uniformly w.r.t. $y \in \mathbb{R}^N$ there exists

$$\bar{f}(y) := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(s,y) ds. \tag{4}$$

We consider the following optimal control problem

$$\begin{cases} \frac{\partial y}{\partial t} = A\Delta y + \bar{f}(y) + g(y) \cdot u(t,x), (t,x) \in Q, \\ y|_{\partial\Omega} = 0, \\ y|_{t=0} = y_0(x), \end{cases} \tag{5}$$

$$u \in U \subseteq (L^2(Q))^M, \tag{6}$$

$$J(y,u) = \int_Q e^{-\gamma \cdot t} \cdot q(x, y(t,x)) dt dx + \int_Q \sum_{i=1}^M \alpha_i \cdot u_i^2(t,x) dt dx \to \inf. \tag{7}$$

It should be noted that the transition to the averaging parameters can essentially simplify the problem. In particular, if $\bar{f}$ does not depend on y then in some cases exact solution of (1)–(3) can be found [18,19]. Another approaches for finding exact solutions of optimal control problems and approximate procedures can be found in [20,21].

Assume that $\{\bar{y}, \bar{u}\}$ is a solution of (5)–(7). The main goal of the paper is to prove the limit equality

$$J(\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}) - J(\bar{y}, \bar{u}) \to 0, \varepsilon \to 0. \tag{8}$$

As a consequence of (8) we will prove that the control $\bar{u}$ is approximately optimal for the problem (1)–(3) in the following sense:

$$J(\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}) - J(y^{\varepsilon}, \bar{u}) \to 0, \varepsilon \to 0,$$

where y^{ε} is a solution of (1) with control $\bar{u}$.

3. Assumptions, Notations and Basic Results

We assume the following conditions hold.

Assumption 1. *$\frac{1}{2}(A + A^*) \geq \nu \cdot I$, where $\nu > 0$ and I is a unit matrix;*

Assumption 2. *$f : \mathbb{R}_+ \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is continuous and bounded:*

$$\exists C_1 > 0 \quad \forall t \geq 0 \quad \forall y \in \mathbb{R}^N \quad \|f(t, y)\|_{\mathbb{R}^N} \leq C_1;$$

Assumption 3. *$g : \mathbb{R}^N \mapsto \mathbb{R}^{N \times M}$ is continuous and bounded:*

$$\exists C_2 > 0 \quad \forall y \in \mathbb{R}^N \quad \|g(y)\|_{\mathbb{R}^{N \times M}} \leq C_2;$$

Assumption 4. *$U \subseteq \left(L^2(Q)\right)^M$ is closed and convex, $0 \in U$;*

Assumption 5. *$q : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}$ is a Carathéodory function, $\exists K_1, K_2 \in L^1(\Omega)$, $\exists C_3 > 0$ such that $\forall x \in \Omega$, $\forall \xi \in \mathbb{R}^N$*

$$|q(x, \xi)| \leq C_3 \|\xi\|^2_{\mathbb{R}^N} + K_2(x), \quad q(x, \xi) \geq K_1(x).$$

Here, $\|\xi\|_{\mathbb{R}^N}$ denotes the Euclidean norm of $\xi \in \mathbb{R}^N$.

For $u \in U$ and $y_0 \in \left(L^2(\Omega)\right)^N$ we understand solution of (1) in weak (or generalized) sense on every finite time interval, i.e., y is a solution of (1) if

$$y \in L^2_{loc}\left(0, +\infty, (H^1_0(\Omega))^N\right) \bigcap L^{\infty}_{loc}\left(0, +\infty, (L^2(\Omega))^N\right)$$

such that $\forall T > 0$, $\forall \varphi \in (H^1_0(\Omega))^N$, $\forall \eta \in C^{\infty}_0(0, T)$ the following equality holds:

$$-\int_0^T (y, \varphi)_H \cdot \eta' dt + \int_0^T (A\nabla y, \nabla \varphi)_H \eta dt = \int_0^T \left(f\left(\frac{t}{\varepsilon}, y\right), \varphi\right)_H \eta dt + \int_0^T (g(y) \cdot u, \varphi)_H \eta dt. \tag{9}$$

Here and after we denote by $\|\cdot\|_H$ and $(\cdot, \cdot)_H$ the classical norm and scalar product in $H := (L^2(\Omega))^N$, by $\|\cdot\|_V$ and $(\cdot, \cdot)_V$ the classical norm and scalar product in $V := (H^1_0(\Omega))^N$, by $\|\cdot\|_U$ the norm in $L^2(Q))^M$, and by V^* the dual space to V.

Due to the Assumptions 1–3, every solution of (1) satisfies

$$\frac{\partial y}{\partial t} \in L^2_{loc}(0, +\infty, V^*).$$

It means that $\forall T > 0$ every solution of (1) is an absolutely continuous function from $[0, T]$ to H, and equality (9) is equivalent to the following one [16]:

$$\frac{d}{dt}(y, \varphi)_H + (A\nabla y, \nabla \varphi)_H = \left(f\left(\frac{t}{\varepsilon}, y\right), \varphi\right)_H + (g(y) \cdot u, \varphi)_H \tag{10}$$

for almost all (a.a.) $t > 0$.

It is known [16,17] that, under Assumptions 1–3, for every $y_0 \in H$, $u \in U$ there exists at least one solution of (1), and for a.a. $t > 0$

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + v \cdot \|y(t)\|_V^2 \leq C_1\|y(t)\|_H + C_2\|y(t)\|_H \cdot \|u(t)\|_{(L^2(\Omega))^M}. \tag{11}$$

Remark 1. *Uniqueness of solution of (1) is not guaranteed. This can be done under some additional assumptions, e.g., [16]* $\forall s \geq 0$, $\forall y \in \mathbb{R}^N$, $\forall \omega \in \mathbb{R}^N$

$$\left(f_y'(s,y)\omega,\omega\right)_{\mathbb{R}^N} \geq -C_4 \cdot \|\omega\|_{\mathbb{R}^N}$$

In the sequel, we denote by $\mathcal{F}^\varepsilon$ (or $\bar{\mathcal{F}}$) a set of all pairs $\{y,u\}$, where y is a solution of (1) (or (5)) with control u.

The following Lemma gives us a result about the solvability of the optimal control problem (1)–(3) and it also provides some useful inequalities.

Lemma 1. *Under the Assumptions 1–5 for every* $\varepsilon > 0$ *the problem (1)–(3) has a solution* $\{\bar{y}^\varepsilon, \bar{u}^\varepsilon\}$, *that is,*

$$J(\bar{y}^\varepsilon, \bar{u}^\varepsilon) \leq J(y,u) \quad \forall \{y,u\} \in \mathcal{F}^\varepsilon.$$

Proof of Lemma 1. Fix $\varepsilon > 0$ and suppress index ε throughout the proof. The idea of the proof is to derive a priori estimates for the minimizing sequence. Obtained estimates allow us to pass to the limit in problem (1)–(3).

From (11), Poincare inequality $\|y\|_V^2 \geq \lambda\|y\|_H^2$, $y \in V$, and Young inequality we derive that for some $\delta > 0$, $C_5 > 0$ (not depending on ε) for every $\{y,u\} \in \mathcal{F}^\varepsilon$ for a.a. $t > 0$

$$\frac{d}{dt}\|y(t)\|_H^2 + \delta\|y(t)\|_H^2 \leq C_5\left(1 + \|u(t)\|_{(L^2(\Omega))^M}^2\right).$$

Therefore using Gronwall inequality we get for all $t > 0$

$$\|y(t)\|_H^2 \leq e^{-\delta\cdot t}\left\{\|y_0(t)\|_H^2 + C_5\int_0^t \left(1 + \|u(s)\|_{(L^2(\Omega))^M}^2\right)e^{\delta\cdot s}ds\right\}, \tag{12}$$

$$\|y(t)\|_H^2 \leq e^{-\delta\cdot t}\|y_0\|_H^2 + \frac{C_5}{\delta} + C_5 \cdot \|u\|_U^2. \tag{13}$$

From the inequality (13) and the first inequality from the Assumption 5 we have that for some $C_6 > 0$

$$J(y,u) \leq C_6\left(1 + \|y_0\|_H^2 + \|u\|_U^2\right). \tag{14}$$

Now let $\{y_n, u_n\}$ be a minimizing sequence, that is,

$$\lim_{n\to\infty} J(y_n,u_n) = \inf_{\{y,u\}\in\mathcal{F}^\varepsilon} J(y,u) =: \bar{J}^\varepsilon. \tag{15}$$

Note that due to the Assumption 5 $\forall\{y,u\} \in \mathcal{F}^\varepsilon$

$$J(y,u) \geq \frac{\|K_1\|_{L^1}}{\gamma} \Rightarrow \bar{J}^\varepsilon \geq \frac{\|K_1\|_{L^1}}{\gamma} > -\infty.$$

From (15) for sufficiently large n

$$J(y_n,u_n) \leq \bar{J}^\varepsilon + 1. \tag{16}$$

On the other hand, for $\alpha := \min\limits_{1\le i\le M} \alpha_i > 0$

$$J(y_n, u_n) \ge \frac{\|K_1\|_{L^1}}{\gamma} + \alpha \cdot \|u_n\|^2_U. \tag{17}$$

Inequalities (16) and (17) imply that $\{u_n\}$ is bounded in $(L^2(Q))^M$, so for subsequence

$$u_n \to u \text{ weakly in } (L^2(Q))^M. \tag{18}$$

Due to convexity of U we have inclusion $u \in U$. From (11) over $(0, T)$ and using (13) we we obtain from (5) that $\{y_n\}$ is bounded in

$$L^2(0, T; V) \bigcap L^\infty(0, T; H),$$

$\left\{\frac{\partial y_n}{\partial t}\right\}$ is bounded in $L^2(0, T; V^*)$. Using Compactness Lemma [22] we conclude that up to subsequence $\forall T > 0$

$$\begin{gathered} y_n \to y \text{ weakly in } L^2(0, T; V), \\ y_n \to y \text{ in } L^2(0, T; H), \\ \forall t \ge 0 \;\; y_n(t) \to y(t) \text{ weakly in } H, \\ y_n(t, x) \to y(t, x) \text{ a.a. in } Q. \end{gathered} \tag{19}$$

From (19) and Lebesgue's Dominated Convergence Theorem we can pass to the limit in the equality (9) applied to $\{y_n, u_n\}$, and obtain that $\{y, u\} \in \mathcal{F}^\varepsilon$. Due to pointwise convergence

$$e^{-\gamma \cdot t} \cdot q(x, y_n(t, x)) \to e^{-\gamma \cdot t} q(x, y(t, x)) \text{ a.a. in } Q,$$

Fatou's lemma and weak convergence (18) we have

$$\bar{J}^\varepsilon = \lim_{n\to\infty} J(y_n, u_n) \ge \underline{\lim} \int_Q e^{-\gamma \cdot t} q(x, y_n(t, x)) dt dx + \underline{\lim} \int_Q \sum_{i=1}^M \alpha_i (u_i^n(t, x))^2 dt dx \ge J(y, u).$$

Therefore $\{y, u\}$ is a solution of (1)–(3). $\square$

4. Main Results

We assume that $\forall \eta > 0\ \exists \delta > 0\ \forall t \ge 0, \forall y, z \in \mathbb{R}^N$

$$\|y - z\|_{\mathbb{R}^N} < \delta \Rightarrow \|f(t, y) - f(t, z)\|_{\mathbb{R}^N} < \eta. \tag{20}$$

Assumption (20) implies that the averaged function $\bar{f} : \mathbb{R}^N \mapsto \mathbb{R}^N$ from (4) is a continuous function and the Assumption 2 holds. It means that under conditions (4), (20) the optimal control problem (5)–(7) has a solution $\{\bar{y}, \bar{u}\}$.

The main result of the paper is the following

Theorem 1. *Suppose that the Assumptions 1–5 and (4), (20) hold and, moreover, the problem (5) has a unique solution for every $u \in U$. Let $\{\bar{y}^\varepsilon, \bar{u}^\varepsilon\}$ be a solution of (1)–(3). Then*

$$J(\bar{y}^\varepsilon, \bar{u}^\varepsilon) \to J(\bar{y}, \bar{u}), \varepsilon \to 0, \tag{21}$$

and up to subsequence

$$\begin{gathered} \bar{y}^\varepsilon \to \bar{y} \text{ in } L^2_{loc}(0, +\infty; H), \\ \bar{u}^\varepsilon \to \bar{u} \text{ in } (L^2(Q))^M, \end{gathered} \tag{22}$$

where $\{\bar{y}, \bar{u}\}$ is a solution of (5)–(7).

Proof. Let $\varepsilon_n \to 0$, $\{\bar{y}^n, \bar{u}^n\}$ be a solution of (1)–(3) for $\varepsilon = \varepsilon_n$. Due to the optimality of $\{\bar{y}^n, \bar{u}^n\}$ we have

$$J(\bar{y}^n, \bar{u}^n) \le J(y_n, 0),$$

where y_n is a solution of (1) with $\varepsilon = \varepsilon_n$ and $u \equiv 0$. Then from (14)

$$\frac{1}{\gamma}\|K_1\|_{L^1} + \alpha\|\bar{u}^n\|_U^2 \le C_6 \cdot (1 + \|y_0\|_H^2).$$

Repeating arguments used in the prof of Lemma 1 conclude that on subsequence for some $\hat{y}$, $\hat{u}$:

$$\begin{aligned} &\bar{u}^n \to \hat{u} \text{ weakly in } (L^2(Q))^M, n \to \infty, \\ &\bar{y}^n \to \hat{y} \text{ in the sense of (19)}, n \to \infty, \end{aligned} \tag{23}$$

Let us prove that $\{\hat{y}, \hat{u}\} \in \bar{\mathcal{F}}$, i.e., $\hat{y}$ is a solution of the averaged problem (5) with control $\hat{u}$. For this purpose it is sufficient to make a limit transition in the equality

$$(\bar{y}^n, \varphi)_H - (y_0, \varphi)_H + \int_0^T (A\nabla\bar{y}^n, \nabla\varphi)_H = \int_0^T \left(f\left(\frac{t}{\varepsilon_n}, \bar{y}^n\right), \varphi\right)_H + \int_0^T (g(\bar{y}^n)\bar{u}^n, \varphi))_H, \tag{24}$$

for arbitrary $\varphi \in V$ and $T > 0$.

Limit transition in the left part of (24) is a direct consequence of (23). From the Dominated Convergence Theorem we see that

$$g(\bar{y}^n) \to g(\hat{y}) \text{ in } L^2(0, T; H), n \to \infty.$$

Then (23) implies convergence in the last term of (24).

Let us prove that $\forall T > 0, \forall \varphi \in V$

$$\int_{Q_T} \sum_{i=1}^N f_i\left(\frac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right)\varphi_i(x)dtdx \to \int_{Q_T} \sum_{i=1}^N \bar{f}_i(\hat{y}(t,x))\varphi_i(x)dtdx, n \to \infty, \tag{25}$$

where $Q_T = (0, T) \times \Omega$. Due to the Dominated Convergence Theorem $\forall 0 < a < b, \forall \psi \in H$

$$\int_a^b \int_\Omega \sum_{i=1}^N \left(f_i\left(\frac{t}{\varepsilon_n}, \psi(x)\right) - \bar{f}_i(\psi(x))\right)\varphi_i(x)dxdt \to 0, n \to \infty. \tag{26}$$

Due to Egorov's theorem $\forall \delta > 0\ \exists Q_1^\delta \subset Q_T$ such that $\mu(Q_1^\delta) < \delta$ and

$$\bar{y}^n \to \hat{y} \text{ uniformly on } Q_T \setminus Q_1^\delta \text{ as } n \to \infty. \tag{27}$$

Here μ is Lebesgue's measure on $\mathbb{R}^2$. On the other hand there exists a sequence of step functions

$$y^m(t,x) = \sum_{k=1}^m y_k^m(x) \cdot \chi_{A_k^m}(t), \{y_k^m\} \subset H,$$

$\{A_k^m = (a_k^m, b_k^m)\}$ is a covering of $(0, T)$ such that

$$y^m \to \hat{y} \text{ in } L^2(0, T; H) \text{ and a.e. in } Q_T.$$

Moreover $\forall \delta > 0\ \exists Q_2^\delta \subset Q_T$ such that $\mu(Q_2^\delta) < \delta$ and

$$y^m \to \hat{y} \text{ uniformly on } Q_T \setminus Q_2^\delta \text{ as } m \to \infty.$$

Let us denote

$$I_1^{(n)} := \int_{Q_T} \sum_{i=1}^{N} \left(f_i\left(\frac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right) - f_i\left(\frac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\right) \varphi_i(x) dtdx,$$

$$I_2^{(n)} := \int_{Q_T} \sum_{i=1}^{N} \left(f_i\left(\frac{t}{\varepsilon_n}, \hat{y}(t,x)\right) - \bar{f}(\hat{y}(t,x))\right) \varphi_i(x) dtdx.$$

Then due to (27)

$$I_1^{(n)} \leq \int_{Q_T \setminus Q_1^\delta} \left\| f\left(\frac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right) - f\left(\frac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\right\|_{\mathbb{R}^N} \cdot \|\varphi(x)\|_{\mathbb{R}^N} dtdx + 2C_1 \cdot \|\varphi\|_H^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}}. \tag{28}$$

Due to (20) for a given $\delta > 0$ $\exists \lambda$ $\forall n \geq 1$, $\forall t \geq 0$

$$\|y - z\|_{\mathbb{R}^N} < \lambda \Rightarrow \left\| f\left(\frac{t}{\varepsilon_n}, y\right) - f\left(\frac{t}{\varepsilon_n}, z\right)\right\| \leq \delta^{\frac{1}{2}}.$$

Therefore, choosing n_1 such that $\forall n \geq n_1$

$$\sup_{(t,x) \in Q_T \setminus Q_1^\delta} \|\bar{y}^n(t,x) - \hat{y}(t,x)\|_{\mathbb{R}^N} < \lambda$$

we get from (28) that $\forall n \geq n_1$

$$I_1^{(n)} \leq \delta^{\frac{1}{2}} \cdot \mu^{\frac{1}{2}}(Q_T) \cdot \|\varphi\|_H^{\frac{1}{2}} + 2C_1 \cdot \|\varphi\|_H^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}} \leq C_7(T)\delta^{\frac{1}{2}}. \tag{29}$$

On the other hand, for every step function $y^m(t,x)$ we have due to (26): $\forall m \geq 1$

$$\begin{aligned} &\int_{Q_T} \sum_{i=1}^{N} \left(f_i\left(\tfrac{t}{\varepsilon_n}, y^m(t,x)\right) - \bar{f}_i(y^m(t,x))\right) \varphi_i(x) dtdx \\ &= \sum_{k=1}^{m} \int_{A_k^m} \int_{\Omega} \sum_{i=1}^{N} \left(f_i\left(\frac{t}{\varepsilon_n}, y_k^m(x)\right) - \bar{f}_i(y_k^m(x))\right) \varphi_i(x) dtdx \to 0, n \to \infty. \end{aligned} \tag{30}$$

So $\forall m \geq 1$, $\exists n_2 = n_2(m)$, $\forall n \geq n_2$

$$\left| \int_{Q_T} \sum_{i=1}^{N} \left(f_i\left(\frac{t}{\varepsilon_n}, y^m(t,x)\right) - \bar{f}_i(y^m(t,x))\right) \varphi_i(x) dtdx \right| < \delta. \tag{31}$$

Furthermore, $\exists m_0$, $\forall m \geq m_0$, $\forall n \geq 1$

$$\int_{Q^T \setminus Q_2^\delta} \left\| f\left(\frac{t}{\varepsilon_n}, \hat{y}(t,x)\right) - f\left(\frac{t}{\varepsilon_n}, y^m(t,x)\right)\right\|_{\mathbb{R}^N} \cdot \|\varphi(x)\|_{\mathbb{R}^N} dtdx \leq \delta^{\frac{1}{2}} \cdot \mu^{\frac{1}{2}}(Q_T) \cdot \|\varphi\|_H^{\frac{1}{2}}, \tag{32}$$

$$\int_{Q_T \setminus Q_2^\delta} \|\bar{f}(\hat{y}(t,x)) - \bar{f}(y^m(t,x))\|_{\mathbb{R}^N} \cdot \|\varphi(x)\|_{\mathbb{R}^N} dtdx \leq \delta^{\frac{1}{2}} \cdot \mu^{\frac{1}{2}}(Q_T) \cdot \|\varphi\|_H^{\frac{1}{2}}. \tag{33}$$

Combining (30)–(33), we obtain $\forall m \geq m_0$, $\forall n \geq n_2(m)$

$$I_2^{(n)} \leq 2 \cdot \delta^{\frac{1}{2}} \cdot \mu^{\frac{1}{2}}(Q_T) \|\varphi\|_H^{\frac{1}{2}} + \delta \leq C_8(T) \cdot \delta^{\frac{1}{2}}. \tag{34}$$

Inequalities (29), (34) imply (25). So we can pass to the limit in (24) and obtain that $\{\hat{y}, \hat{u}\} \in \bar{\mathcal{F}}$. Now let us prove that $\{\hat{y}, \hat{u}\}$ is an optimal process in (5)–(7).

Fatou's lemma implies
$$\underline{\lim} J(\bar{y}^n, \bar{u}^n) \geq J(\hat{y}, \hat{u}).$$

On the other hand, for every $u \in U$ and any y_n–solution of (1) with control u and $\varepsilon = \varepsilon_n$ we get
$$J(\bar{y}^n, \bar{u}^n) \leq J(\bar{y}_n, u).$$

Using the same arguments as in proof of the Lemma 1 for $\{y_n\}$ we derive that
$$y_n \to y \text{ in the sense of (19)},$$
where y is a unique solution of (5) with control u.

Let us prove that
$$\int\limits_Q e^{-\gamma \cdot t} q(x, y_n(t,x)) dtdx \to \int\limits_Q e^{-\gamma \cdot t} q(x, y(t,x)) dtdx \tag{35}$$

Indeed due to the Assumption 5 and (13) we have
$$\left| e^{-\gamma \cdot t} q(x, y_n(t,x)) \right| \leq C_3 e^{-\gamma \cdot t} \|y_n(t,x)\|^2_{\mathbb{R}^N} + e^{-\gamma \cdot t} \cdot K_2(x). \tag{36}$$

As $y_n \to y$ in $L^2(0,T;H)$ and a.e. in Q, we deduce from Lebesgue's Dominated Convergence theorem:
$$\forall T > 0 \quad \int\limits_{Q_T} e^{-\gamma \cdot t} q(x, y_n(t,x)) dtdx \to \int\limits_{Q_T} e^{-\gamma \cdot t} q(x, y(t,x)) dtdx, n \to \infty. \tag{37}$$

On the other hand, from (12) and (36)
$$\begin{aligned} &\int\limits_T^{+\infty} \int\limits_\Omega e^{-\gamma \cdot t} |q(x, y_n(t,x))| dtdx \leq \int\limits_T^{+\infty} e^{-\gamma \cdot t} \left[C_3 \cdot \|y_n(t)\|^2_H + \|K_2\|_{L^1} \right] dt \\ &\leq \int\limits_T^{+\infty} e^{-\gamma \cdot t} \left[C_3 e^{-\delta \cdot t} \cdot \|y_0(t)\|^2_H + \frac{C_3 \cdot C_5}{\delta} + C_3 \cdot C_5 \cdot \|u\|^2_U + \|K_2\|_{L^1} \right] dt \\ &\leq C_9 \cdot e^{-\gamma \cdot T}, \end{aligned} \tag{38}$$
where C_9 does not depend on T and n. The last inequality together with with (37) leads to (35).

From (35) we conclude the following inequality: $\forall \{y, u\} \in \bar{\mathcal{F}}$
$$J(\hat{y}, \hat{u}) \leq \underline{\lim} J(\bar{y}^n, \bar{u}^n) \leq \underline{\lim} J(y_n, u) = J(y, u). \tag{39}$$

This means that $\{\hat{y}, \hat{u}\}$ is a solution of (5)–(7).

Now we substitute $u = \hat{u}$ in previous arguments. Then $y = \hat{y}$ due to uniqueness. So from (39), we obtain
$$J(\hat{y}, \hat{u}) \leq \underline{\lim} J(\bar{y}^n, \bar{u}^n) \leq J(\hat{y}, \hat{u}).$$

These inequalities mean that up to subsequence
$$J(\bar{y}^n, \bar{u}^n) \to J(\hat{y}, \hat{u}), n \to \infty. \tag{40}$$

Since $J(\hat{y}, \hat{u}) = \inf_{\{y,u\} \in \bar{\mathcal{F}}} J(y, u)$, then convergence in (40) holds for the whole sequence. Therefore (21) is proved.

Moreover, up to subsequence $\bar{y}^n$ tends to $\hat{y}$ in $L^2_{loc}(0, +\infty; H)$. So, repeating arguments (37) and (38) for $\bar{y}^n$, and using boundness of $\{\bar{u}^n\}$, we have

$$\int_Q e^{-\gamma \cdot t} q(x, \bar{y}^n(t,x)) dt dx \to \int_Q e^{-\gamma \cdot t} q(x, \hat{y}(t,x)) dt dx.$$

Then from (40) and weak convergence we deduce (22) □

Corollary 1. *An optimal control $\bar{u} \in U$ of the averaged problem (5)–(7) can serve as an "approximate" optimal control in the initial problem (1), that is:*

$$J(\bar{y}^\varepsilon, \bar{u}^\varepsilon) - J(y^\varepsilon, \bar{u}) \to 0, \varepsilon \to 0, \tag{41}$$

where y^ε is a solution of (1) with control $u = \bar{u}$.

Indeed, for y^{ε_n}, $\varepsilon_n \to 0$, we can repeat arguments of the proof of the Theorem, and due to the uniqueness of the solution of (5) for $u = \bar{u}$ we have up to subsequence

$$y^{\varepsilon_n} \to \bar{y} \text{ in the sense of } (19)$$

Then (35) holds and taking into account strong convergence (22), we obtain (41).

5. Conclusions and Future Research

We sought to obtain a theoretical result that demonstrates the effectiveness of the averaging method of finding an approximate solution of the optimal control problem for a non-linear parabolic system with fast-oscillating coefficients with respect to a time variable. We proved that the optimal control of the problem with averaging coefficients can be considered as an "approximately" optimal for the initial perturbed system. To demonstrate effectiveness of the method we plan to continue research focusing on the practical applications and simulation results using in particular genetic algorithms.

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