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On Soft Generalized ω-Closed Sets and Soft $T_{1/2}$ Spaces in Soft Topological Spaces

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Abstract: In this paper, we define a soft generalized ω-closed set, which is a generalization of both the soft ω-closed set and the soft generalized closed set. We show that the classes of generalized closed sets and generalized ω-closed sets coincide in soft anti-locally countable soft topological spaces. Additionally, in soft locally countable soft topological spaces, we show that every soft set is a soft generalized ω-closed set. Furthermore, we prove that the classes of soft generalized closed sets and soft generalized ω-closed sets coincide in the soft topological space (X, τ_ω, A). In addition to these, we determine the behavior of soft generalized ω-closed sets relative to soft unions, soft intersections, soft subspaces, and generated soft topologies. Furthermore, we investigate soft images and soft inverse images of soft generalized closed sets and soft generalized ω-closed sets under soft continuous, soft closed soft transformations. Finally, we continue the study of soft $T_{1/2}$ spaces, in which we obtain two characterizations of these soft spaces, and investigate their behavior with respect to soft subspaces, soft transformations, and generated soft topologies.

Keywords: soft ω-open set; generalized closed sets; $T_{1/2}$ spaces; generalized ω-closed sets; soft $T_{1/2}$ spaces; soft generated soft topological spaces; soft induced topological spaces

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1. Introduction and Preliminaries

In this paper, we follow the notions and terminologies that appeared in [1,2]. Throughout this paper, topological space and soft topological space will be denoted by TS and STS, respectively. Soft set theory was first introduced by Molodtsov in 1999 [3] as a general mathematical tool for dealing with problems that contain uncertainties. In 2011, the concept of soft topology was defined by Shabir and Naz [4]. Many classical TS concepts were then incorporated into STSs in [1,2,5–21] and other works.

The generalizations of soft open and soft closed sets play an effective role in the structure of soft topology by using them to redefine and investigate some soft topological concepts such as soft continuity, soft compactness, or soft separation axioms. The author in [22] defined the class of soft g-closed sets, which strictly contains the class of soft closed sets. Then, several authors continued the study of soft g-closed sets.

In this paper, we define a soft generalized ω-closed set, which is a generalization of both the soft ω-closed set and the soft generalized closed set. We show that the classes of generalized closed sets and generalized ω-closed sets coincide in soft anti-locally countable soft topological spaces. In addition, in soft locally countable soft topological spaces, we show that every soft set is a soft generalized ω-closed set. Furthermore, we prove that the classes of soft generalized closed sets and soft generalized ω-closed sets coincide in the soft topological space (X, τ_ω, A). In addition to these, we determine the behavior of soft generalized ω-closed sets relative to soft unions, soft intersections, soft subspaces, and generated soft topologies. Furthermore, we investigate soft images and soft inverse images of soft generalized closed sets and soft generalized ω-closed sets under soft continuous, soft closed soft transformations. Finally, we continue the study of soft $T_{1/2}$ spaces, in which we obtain two characterizations of these soft spaces and investigate their behavior with respect to soft subspaces, soft transformations, and generated soft topologies.

The authors in [23,24] proved that soft sets are a class of special information systems. This is a strong motivation to study the structures of soft sets for information systems. Thus, this paper not only constitutes a theoretical basis for further applications of soft topology but also leads to the development of information systems.

Let X be an initial universe and A be a set of parameters. In this paper, the family of all soft sets over X relative to A will be denoted by $SS(X,A)$. Let (X,τ,A) be an STS, $(X,\Im)$ be a TS, $H \in SS(X,A)$, and $D \subseteq X$. Throughout this paper, $\Im^c$ will denote the collection of all closed subsets of $(X,\Im)$, and τ^c will denote the collection of all soft closed sets in (X,τ,A), with $Cl_\tau(H)$ and $Cl_\Im(D)$ denoting the soft closure of H in (X,τ,A) and the closure of D in $(X,\Im)$, respectively.

By the end of this section, we recall some concepts that will be used in the sequel.

Definition 1 ([25])**.** *Let $(X,\Im)$ be a TS, and let $S \subseteq X$. Then, S is said to be a generalized closed (briefly: g-closed) set in $(X,\Im)$ if $Cl_\Im(S)\widetilde{\subseteq}U$ whenever $U \in \Im$ and $S\widetilde{\subseteq}U$. The collection of all g-closed sets in $(X,\Im)$ will be denoted by $GC(X,\Im)$.*

Definition 2 ([25])**.** *A TS $(X,\Im)$ is said to be $T_{1/2}$ if $GC(X,\Im) \subseteq \Im^c$.*

Definition 3 ([26])**.** *Let $(X,\Im)$ be a TS, and let $S \subseteq X$. Then, S is said to be a generalized ω-closed (briefly: $g\omega$-closed) set in $(X,\Im)$ if $Cl_{\Im_\omega}(S)\widetilde{\subseteq}U$ whenever $U \in \Im$ and $S\widetilde{\subseteq}U$. The collection of all $g\omega$-closed sets in $(X,\Im)$ will be denoted by $G\omega C(X,\Im)$.*

Definition 4 ([22])**.** *Let (X,τ,A) be an STS, and let $F \in SS(X,A)$. Then, F is said to be a soft generalized closed (briefly: soft g-closed) set in (X,τ,A) if $Cl_\tau(F)\widetilde{\subseteq}G$ whenever $G \in \tau$ and $F\widetilde{\subseteq}G$. The collection of all soft g-closed sets in (X,τ,A) will be denoted by $GC(X,\tau,A)$.*

Definition 5 ([22])**.** *An STS (X,τ,A) is said to be soft $T_{1/2}$ if $GC(X,\tau,A) \subseteq \tau^c$.*

Definition 6 ([21])**.** *An STS (X,τ,A) is said to be soft door space if $M \in \tau \cup \tau^c$ for each $M \in SS(X,A)$.*

2. Soft Generalized ω-Closed Sets

In this section, we introduce the notion of soft generalized ω-closed and illustrate its relationships with each of the soft generalized closed sets and soft ω-closed sets as two famous classes of soft sets. Additionally, we determine the behavior of soft generalized ω-closed sets relative to soft unions, soft intersections, soft subspaces, and generated soft topologies, and we explore soft images and soft inverse images of soft generalized closed sets and soft generalized ω-closed sets under soft continuous, soft closed soft transformations.

The following definition is the main concept of this section:

Definition 7 ([21])**.** *Let (X,τ,A) be an STS, and let $F \in SS(X,A)$. Then, F is said to be a soft generalized ω-closed (briefly: soft $g\omega$-closed) set in (X,τ,A) if $Cl_{\tau_\omega}(F)\widetilde{\subseteq}G$ whenever $G \in \tau$ and $F\widetilde{\subseteq}G$. The collection of all soft $g\omega$-closed sets in (X,τ,A) will be denoted by $G\omega C(X,\tau,A)$.*

In Theorems 1 and 2 and Example 1, we discuss the relationships between the classes of soft $g\omega$-closed sets and soft g-closed sets:

Theorem 1. *For any STS (X,τ,A), $GC(X,\tau,A) \subseteq G\omega C(X,\tau,A)$.*

Proof. Let $F \in GC(X,\tau,A)$, and let $G \in \tau$ such that $F\widetilde{\subseteq}G$. Then, $Cl_\tau(F)\widetilde{\subseteq}G$, and so $Cl_{\tau_\omega}(F)\widetilde{\subseteq}Cl_\tau(F)\widetilde{\subseteq}G$. Thus, $F \in G\omega C(X,\tau,A)$. □

The following example will show that the inclusion in Theorem 1 cannot be replaced by equality in general:

Example 1. *Let $X = \mathbb{Q}$, $A = \mathbb{R}$, and $\tau = \{M \in SS(X,A)$: for each $a \in A$, either $M(a) = \emptyset$ or $X - M(a)$ is finite$\}$. Let $F = C_{\mathbb{Q}-\{1\}}$. Then, $F \in \tau$, $Cl_\tau(F) = 1_A$, and $Cl_{\tau_\omega}(F) = F$. To see that $F \in G\omega C(X,\tau,A)$, let $G \in \tau$ such that $F\widetilde{\subseteq}G$. Then, $Cl_{\tau_\omega}(F) = F\widetilde{\subseteq}G$, and since $F \in \tau$ and $F\widetilde{\subseteq}F$, while $Cl_\tau(F) = 1_A$, which is not a soft subset of F, then $F \notin GC(X,\tau,A)$.*

Theorem 2. *If (X,τ,A) is a soft anti-locally countable STS, then $GC(X,\tau,A) = G\omega C(X,\tau,A)$.*

Proof. Suppose that (X,τ,A) is soft anti-locally countable. Then, by Theorem 1, $GC(X,\tau,A) \subseteq G\omega C(X,\tau,A)$. To see that $G\omega C(X,\tau,A) \subseteq GC(X,\tau,A)$, let $F \in G\omega C(X,\tau,A)$, and let $G \in \tau$ such that $F\widetilde{\subseteq}G$. Then, $Cl_{\tau_\omega}(F)\widetilde{\subseteq}G$. By Theorem 14 of [2], $Cl_{\tau_\omega}(F) = Cl_\tau(F)$, and so $Cl_\tau(F)\widetilde{\subseteq}G$. Therefore, $F \in GC(X,\tau,A)$. □

In Theorem 3 and Example 2, we discuss the relationships between the classes of the soft $g\omega$-closed sets and soft ω-closed sets:

Theorem 3. *For any STS (X,τ,A), $(\tau_\omega)^c \subseteq G\omega C(X,\tau,A)$.*

Proof. Let $F \in (\tau_\omega)^c$, and let $G \in \tau$ such that $F\widetilde{\subseteq}G$. Since $F \in (\tau_\omega)^c$, then $Cl_{\tau_\omega}(F) = F\widetilde{\subseteq}G$. Hence, $F \in G\omega C(X,\tau,A)$. □

The following example will show that the inclusion in Theorem 3 cannot be replaced by an equality in general:

Example 2. *Let $X = \mathbb{R}$, $A = [0,1]$, $\Im$ be the usual topology on $\mathbb{R}$, and $\tau = \{M \in SS(X,A)$: $M(a) \in \Im$ for all $a \in A\}$. Then, $C_{[0,\infty)} \in G\omega C(X,\tau,A) - (\tau_\omega)^c$.*

In the next theorem, we classify the class of soft $g\omega$-closed sets in soft locally countable STSs:

Theorem 4. *If (X,τ,A) is a soft locally countable STS, then $G\omega C(X,\tau,A)$ is a discrete STS.*

Proof. Let $F \in SS(X,A)$, and let $G \in \tau$ such that $F\widetilde{\subseteq}G$. Since (X,τ,A) is soft locally countable, then by Corollary 5 of [2], $Cl_{\tau_\omega}(F) = F$. Therefore, $Cl_{\tau_\omega}(F)\widetilde{\subseteq}G$. Hence, $F \in G\omega C(X,\tau,A)$. □

In Theorem 5 and Corollary 1, we show that the classes of soft $g\omega$-closed sets, soft ω-closed sets, and soft ω-closed sets coincide in soft ω-topological spaces:

Theorem 5. *For any STS (X,τ,A), $G\omega C(X,\tau_\omega,A) \subseteq (\tau_\omega)^c$.*

Proof. Let $F \in G\omega C(X,\tau_\omega,A)$. We are going to show that $Cl_{\tau_\omega}(F)\widetilde{\subseteq}F$. Suppose to the contrary that there exists $a_x\widetilde{\in}Cl_{\tau_\omega}(F) - F$. Since $F\widetilde{\subseteq}1_A - a_x \in \tau_\omega$, and $F \in G\omega C(X,\tau_\omega,A)$, then $Cl_{(\tau_\omega)_\omega}(F)\widetilde{\subseteq}1_A - a_x$. On the other hand, by Theorem 5 of [2], $Cl_{(\tau_\omega)_\omega}(F) = Cl_{\tau_\omega}(F)$. Thus, $a_x\widetilde{\in}1_A - Cl_{\tau_\omega}(F)$, which is a contradiction. □

Corollary 1. *For any STS (X,τ,A), $(\tau_\omega)^c = GC(X,\tau_\omega,A) = G\omega C(X,\tau_\omega,A)$.*

Proof. Clearly, $(\tau_\omega)^c \subseteq GC(X,\tau_\omega,A)$. Additionally, by Theorems 1 and 5, $GC(X,\tau_\omega,A) \subseteq G\omega C(X,\tau_\omega,A) \subseteq (\tau_\omega)^c$. Therefore, $(\tau_\omega)^c = GC(X,\tau_\omega,A) = G\omega C(X,\tau_\omega,A)$. □

It is natural to inquire about the relationships between the classes $G\omega C(X,\tau,A)$ and $GC(X,\tau_\omega,A)$ of a given STS (X,τ,A). In Theorem 6 and Example 3, we provide answers to this inquiry:

Theorem 6. *For any STS (X,τ,A), $GC(X,\tau_\omega,A) \subseteq G\omega C(X,\tau,A)$.*

Proof. Let $F \in GC(X, \tau_\omega, A)$, and let $G \in \tau \subseteq \tau_\omega$ such that $F \widetilde{\subseteq} G$. Then, $Cl_{\tau_\omega}(F) \widetilde{\subseteq} G$. Therefore, $F \in G\omega C(X, \tau, A)$. □

The following example will show that the inclusion in Theorem 6 cannot be replaced by an equality in general:

Example 3. *Let $X = \mathbb{R}$, $A = \{a, b\}$ and $\tau = \{M \in SS(X, A) : M(c) \in \{\emptyset, X, \{2\}\}$ for all $c \in A\}$. Let $F = C_{\mathbb{R}-\mathbb{N}}$. If $G \in \tau$ such that $F \widetilde{\subseteq} G$, then $G = 1_A$, and so $Cl_{\tau_\omega}(F) \widetilde{\subseteq} G$. Hence, $F \in G\omega C(X, \tau, A)$.*

Claim 1. $a_0 \widetilde{\in} 1_A - Cl_{\tau_\omega}(F)$.

Proof of Claim. Suppose to the contrary that $a_0 \widetilde{\in} Cl_{\tau_\omega}(F)$. Then, there exists $G \in \tau$ such that $a_0 \widetilde{\in} G$ and $G \widetilde{\cap} F = 0_A$. Since $0 \in G(a) \in \{\emptyset, X, \{2\}\}$, then $G(a) = X$, and so

$$\begin{aligned} (G \widetilde{\cap} F)(a) &= G(a) \cap F(a) \\ &= \mathbb{R} \cap (\mathbb{R} - \mathbb{N}) \\ &= \mathbb{R} - \mathbb{N} \\ &\neq \emptyset. \end{aligned}$$

This is a contradiction.

By the above claim, we have $Cl_{\tau_\omega}(F) \neq F$. Since $F \in \tau_\omega$ and $F \widetilde{\subseteq} F$, but $Cl_{\tau_\omega}(F)$ is not a soft subset of F, then $F \notin GC(X, \tau_\omega, A)$. □

The following question is natural:

Question 1. *Let (X, τ, A) be an STS. Is it true that $GC(X, \tau_\omega, A) \subseteq GC(X, \tau, A)$?*

The following example gives a negative answer for Question 1:

Example 4. *Let $X = \mathbb{R}$, $A = [0, 1]$, $\Im$ be the usual topology on $\mathbb{R}$, and $\tau = \{M \in SS(X, A) : M(a) \in \Im$ for all $a \in A\}$. Let $F = C_{\mathbb{Q} \cap (1, \infty)}$. Since $F \in CSS(X, A)$, then by Theorem 2 (d) of [2], $F \in (\tau_\omega)^c$, and so $Cl_{\tau_\omega}(F) = F$. Thus, for every $G \in \tau_\omega$ such that $F \widetilde{\subseteq} G$, we have $Cl_{\tau_\omega}(F) = F \widetilde{\subseteq} G$. Hence, $F \in GC(X, \tau_\omega, A)$. On the other hand, since $F \widetilde{\subseteq} C_{(1,\infty)} \in \tau$ while $Cl_\tau(F) = C_{[1,\infty)}$, which is not a soft subset of F, then $F \notin GC(X, \tau, A)$.*

In the next theorem, we introduce an important property of $g\omega$-closed sets:

Theorem 7. *Let (X, τ, A) be an STS. If $F \in G\omega C(X, \tau, A)$, then the only soft closed set in (X, τ, A) which is soft contained in $Cl_{\tau_\omega}(F) - F$ is 0_A.*

Proof. Let $F \in G\omega C(X, \tau, A)$ and $M \in \tau^c$ such that $M \widetilde{\subseteq} Cl_{\tau_\omega}(F) - F$. Then, $M \widetilde{\subseteq} Cl_{\tau_\omega}(F)$ and $F \widetilde{\subseteq} 1_A - M \in \tau$. Since $F \in G\omega C(X, \tau, A)$ and $F \widetilde{\subseteq} 1_A - M \in \tau$, then $Cl_{\tau_\omega}(F) \widetilde{\subseteq} 1_A - M$, and so $Cl_{\tau_\omega}(F) \widetilde{\cap} M = 0_A$. On the other hand, since $M \widetilde{\subseteq} Cl_{\tau_\omega}(F)$, then $Cl_{\tau_\omega}(F) \widetilde{\cap} M = M$. Therefore, $M = 0_A$. □

We leave the question about the converse of Theorem 7 open:

Question 2. *Let (X, τ, A) be an STS, and let $F \in SS(X, A)$ such that the only soft closed set in (X, τ, A) which is soft contained in $Cl_{\tau_\omega}(F) - F$ is 0_A. Is it true that $F \in G\omega C(X, \tau, A)$?*

In the next theorem, we introduce a sufficient condition for a soft set to be soft $g\omega$-closed:

Theorem 8. *Let (X, τ, A) be an STS, and let $F \in SS(X, A)$ such that the only soft ω-closed set in (X, τ, A) which is soft contained in $Cl_{\tau_\omega}(F) - F$ is 0_A. Then, $F \in G\omega C(X, \tau, A)$.*

Proof. Let $F \in SS(X, A)$ such that the only soft ω-closed set in (X, τ, A) which is soft contained in $Cl_{\tau_\omega}(F) - F$ is 0_A. Suppose to the contrary that $F \notin G\omega C(X, \tau, A)$. Then, there exists $G \in \tau$ such that $F \widetilde{\subseteq} G$, but $Cl_{\tau_\omega}(F)$ is not a soft subset of G. Therefore, we have $Cl_{\tau_\omega}(F) - G \in (\tau_\omega)^c - \{0_A\}$ with $Cl_{\tau_\omega}(F) - G \widetilde{\subseteq} Cl_{\tau_\omega}(F) - F$, which is a contradiction. □

We leave the question about the converse of Theorem 8 open:

Question 3. *Let (X, τ, A) be an STS, and let $F \in G\omega C(X, \tau, A)$. Is it true that the only soft ω-closed set which is soft contained in $Cl_{\tau_\omega}(F) - F$ is 0_A?*

Now, we are going to discuss soft operations on $g\omega$-closed sets in Theorems 9–14, and Examples 5–7:

Theorem 9. *Let (X, τ, A) be an STS. If $F \in G\omega C(X, \tau, A)$ and $K \in SS(X, A)$ such that $F \widetilde{\subseteq} K \widetilde{\subseteq} Cl_{\tau_\omega}(F)$, then $K \in G\omega C(X, \tau, A)$.*

Proof. Let $F \in G\omega C(X, \tau, A)$ and $K \in SS(X, A)$ such that $F \widetilde{\subseteq} K \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. Let $G \in \tau$ such that $K \widetilde{\subseteq} G$. Since $F \widetilde{\subseteq} K$, then $F \widetilde{\subseteq} G$. Since $F \in G\omega C(X, \tau, A)$, then $Cl_{\tau_\omega}(F) \widetilde{\subseteq} G$. Since $F \widetilde{\subseteq} K \widetilde{\subseteq} Cl_{\tau_\omega}(F)$, then $Cl_{\tau_\omega}(F) \widetilde{\subseteq} Cl_{\tau_\omega}(K) \widetilde{\subseteq} Cl_{\tau_\omega}(Cl_{\tau_\omega}(F)) = Cl_{\tau_\omega}(F)$, and so $Cl_{\tau_\omega}(F) = Cl_{\tau_\omega}(K)$. Thus, $Cl_{\tau_\omega}(K) \widetilde{\subseteq} G$, and hence, $K \in G\omega C(X, \tau, A)$. □

Theorem 10. *For any STS (X, τ, A), the family $G\omega C(X, \tau, A)$ is closed under a finite soft union.*

Proof. It is sufficient to show that the soft union of two elements of $G\omega C(X, \tau, A)$ is an element of $G\omega C(X, \tau, A)$. Let $F, M \in G\omega C(X, \tau, A)$. Let $G \in \tau$ such that $F \widetilde{\cup} M \widetilde{\subseteq} G$. Then, $F \widetilde{\subseteq} G$ and $M \widetilde{\subseteq} G$. Therefore, $Cl_{\tau_\omega}(F) \widetilde{\subseteq} G$ and $Cl_{\tau_\omega}(M) \widetilde{\subseteq} G$. Thus, $Cl_{\tau_\omega}(F \widetilde{\cup} M) = Cl_{\tau_\omega}(F) \widetilde{\cup} Cl_{\tau_\omega}(M) \widetilde{\subseteq} G$, and therefore, $F \widetilde{\cup} M \in G\omega C(X, \tau, A)$. □

The following example shows that the family $G\omega C(X, \tau, A)$ need not be closed under a countable soft union:

Example 5. *Let $X = \mathbb{R}$, $A = \mathbb{Z}$, $\Im$ be the usual topology on $\mathbb{R}$ and $\tau = \{M \in SS(X, A) : M(a) \in \Im$ for all $a \in A\}$. For each $n \in \mathbb{N}$, let $F_n = C_{[\frac{1}{n+1}, 1]}$. Then, $\{F_n : n \in \mathbb{N}\} \subseteq G\omega C(X, \tau, A)$ and $\widetilde{\cup}\{F_n : n \in \mathbb{N}\} = C_{(0,1]}$. Since $C_{(0,2)} \in \tau$ with $C_{(0,1]} \widetilde{\subseteq} C_{(0,2)}$ while $Cl_{\tau_\omega}(C_{(0,1]}) = C_{[0,1]}$ is not a soft subset of $C_{(0,1]}$, then $C_{(0,1]} \notin G\omega C(X, \tau, A)$.*

Theorem 11. *Let (X, τ, A) be an STS, and let $\{F_\alpha : \alpha \in \Delta\} \subseteq G\omega C(X, \tau, A)$. If $\{F_\alpha : \alpha \in \Delta\}$ is soft locally finite in (X, τ, A), then $\widetilde{\cup}\{F_\alpha : \alpha \in \Delta\} \in G\omega C(X, \tau, A)$.*

Proof. Let $G \in \tau$ such that $\widetilde{\cup}\{F_\alpha : \alpha \in \Delta\} \widetilde{\subseteq} G$. For each $\alpha \in \Delta$, $F_\alpha \in G\omega C(X, \tau, A)$ with $F_\alpha \widetilde{\subseteq} G \in \tau$, $Cl_{\tau_\omega}(F_\alpha) \widetilde{\subseteq} G$. Since $\{F_\alpha : \alpha \in \Delta\}$ is soft locally finite in (X, τ, A), then by Lemma 3.5 of [10], $Cl_{\tau_\omega}(\widetilde{\cup}\{F_\alpha : \alpha \in \Delta\}) = \widetilde{\cup}\{Cl_{\tau_\omega}(F_\alpha) : \alpha \in \Delta\} \widetilde{\subseteq} G$. Hence, $\widetilde{\cup}\{F_\alpha : \alpha \in \Delta\} \in G\omega C(X, \tau, A)$. □

The following example shows that the finite soft intersection of soft $g\omega$-closed sets of an STS (X, τ, A) need not be soft $g\omega$-closed in (X, τ, A):

Example 6. *Let $X = \mathbb{R}$, $A = \{a, b\}$ and $\tau = \left\{0_A, 1_A, C_{[3,\infty)}\right\}$. Let $F = C_{\{1\} \cup [3,\infty)}$ and $M = C_{\{2\} \cup [3,\infty)}$. Then $F, M \in G\omega C(X, \tau, A)$, while $F \widetilde{\cap} M = C_{[3,\infty)} \notin G\omega C(X, \tau, A)$.*

Theorem 12. *Let (X, τ, A) be an STS. If $F \in G\omega C(X, \tau, A)$ and $M \in \tau^c$, then $F \widetilde{\cap} M \in G\omega C(X, \tau, A)$.*

Proof. Suppose that $F \in G\omega C(X,\tau,A)$ and $M \in \tau^c$. Let $G \in \tau$ such that $F\tilde{\cap}M\tilde{\subseteq}G$. Put $N = 1_A - M$. Then, $G\tilde{\cup}N \in \tau$ and $F\tilde{\subseteq}G\tilde{\cup}N$. Since $F \in G\omega C(X,\tau,A)$, then $Cl_{\tau_\omega}(F)\tilde{\subseteq}G\tilde{\cup}N$, and thus

$$\begin{array}{rcl} Cl_{\tau_\omega}(F\tilde{\cap}M) & \tilde{\subseteq} & Cl_{\tau_\omega}(F)\tilde{\cap}Cl_{\tau_\omega}(M) \\ & \tilde{\subseteq} & Cl_{\tau_\omega}(F)\tilde{\cap}Cl_{\tau}(M) \\ & \tilde{\subseteq} & (G\tilde{\cup}N)\tilde{\cap}M \\ & = & (G\tilde{\cup}(1_A - M))\tilde{\cap}M \\ & = & G\tilde{\cap}M \\ & \tilde{\subseteq} & G. \end{array}$$

Therefore, $F\tilde{\cap}M \in G\omega C(X,\tau,A)$. □

Theorem 13. *Let (X,τ,A) be an STS, and let Y be a non-empty subset of X. Let $F \in SS(Y,A)$. If $F \in G\omega C(X,\tau,A)$, then $F \in G\omega C(Y,\tau_Y,A)$.*

Proof. Let $F \in G\omega C(X,\tau,A)$, and let $K \in \tau_Y$ such that $F\tilde{\subseteq}K$. Choose $G \in \tau$ such that $K = G\tilde{\cap}C_Y$. Since $F \in G\omega C(X,\tau,A)$ and $F\tilde{\subseteq}G \in \tau$, then $Cl_{\tau_\omega}(F)\tilde{\subseteq}G$, and thus $Cl_{(\tau_\omega)_Y}(F) = Cl_{\tau_\omega}(F)\tilde{\cap}C_Y\tilde{\subseteq}G\tilde{\cap}C_Y = K$. On the other hand, by Theorem 15 of [2], $Cl_{(\tau_\omega)_Y}(F) = Cl_{(\tau_Y)_\omega}(F)$, and thus $Cl_{(\tau_Y)_\omega}(F)\tilde{\subseteq}K$. Therefore, $F \in G\omega C(Y,\tau_Y,A)$. □

Theorem 14. *Let (X,τ,A) be an STS, and let Y be a non-empty subset of X such that $C_Y \in (\tau_\omega)^c$. Let $F \in SS(Y,A)$. If $F \in G\omega C(Y,\tau_Y,A)$, then $F \in G\omega C(X,\tau,A)$.*

Proof. Let $F \in G\omega C(Y,\tau_Y,A)$, and let $G \in \tau$ such that $F\tilde{\subseteq}G$. Then, $F\tilde{\subseteq}G\tilde{\cap}C_Y \in \tau_Y$. Since $F \in G\omega C(Y,\tau_Y,A)$, then $Cl_{(\tau_Y)_\omega}(F)\tilde{\subseteq}G\tilde{\cap}C_Y$. Since by Theorem 15 of [2] we have $Cl_{(\tau_\omega)_Y}(F) = Cl_{(\tau_Y)_\omega}(F)$, then $Cl_{(\tau_\omega)_Y}(F) = Cl_{\tau_\omega}(F)\tilde{\cap}C_Y\tilde{\subseteq}G\tilde{\cap}C_Y$, and finally

$$Cl_{\tau_\omega}(F) = Cl_{\tau_\omega}(F\tilde{\cap}C_Y)\tilde{\subseteq}Cl_{\tau_\omega}(F)\tilde{\cap}Cl_{\tau_\omega}(C_Y) = Cl_{\tau_\omega}(F)\tilde{\cap}C_Y\tilde{\subseteq}G\tilde{\cap}C_Y\tilde{\subseteq}G.$$

Thus, $F \in G\omega C(X,\tau,A)$. □

The condition "$C_Y \in (\tau_\omega)^c$" cannot be dropped in Theorem 14, as demonstrated by the following example:

Example 7. *Let (X,τ,A) be as in Example 6, and let $Y = [3,\infty)$. Then, $C_Y \in G\omega C(Y,\tau_Y,A) - G\omega C(X,\tau,A)$.*

At this stage, we believe that the following two questions are natural:

Question 4. *Let (X,τ,A) be an STS, and let $F \in GC(X,\tau,A)$. Is it true that $F(a) \in GC(X,\tau_a)$ for all $a \in A$?*

Question 5. *Let (X,τ,A) be an STS, and let $F \in G\omega C(X,\tau,A)$. Is it true that $F(a) \in G\omega C(X,\tau_a)$ for all $a \in A$?*

The following example provides negative responses to Questions 4 and 5:

Example 8. *Let $X = \mathbb{R}$ and $A = \{a,b\}$. Let $F, M \in SS(X,A)$ be defined by $F(a) = \mathbb{R}$, $F(b) = (-\infty,0)$, $M(a) = [1,\infty)$, and $M(b) = (-\infty,1)$. Let $\tau = \{0_A, 1_A, M\}$. To see that $F \in GC(X,\tau,A)$, let $G \in \tau$ such that $F\tilde{\subseteq}G$. Then, $F(a) = \mathbb{R} \subseteq G(b)$, and so $G(b) = \mathbb{R}$. Therefore, $G = 1_A$. Hence, $F \in GC(X,\tau,A) \subseteq G\omega C(X,\tau,A)$ (see Theorem 1). On the other hand, since $F(b) = (-\infty,0) \subseteq (-\infty,1) \in \tau_b$ while $Cl_{\tau_b}((-\infty,0)) = \mathbb{R} \nsubseteq (-\infty,1)$, then $F(b) \notin GC(X,\tau_b)$. Moreover, since (X,τ_b) is anti-locally countable, then by Proposition 2.2 of [26], $F(b) \notin G\omega C(X,\tau_b)$.*

If the STS (X, τ, A) is an extended STS, then we can easily apply Theorem 3 of [3] to obtain positive answers to Questions 4 and 5.

In the next four theorems, we investigate the correspondence between g-closed sets (resp. $g\omega$-closed) and soft g-closed sets (resp. soft $g\omega$-closed) of a family of TSs and their generated STSs:

Theorem 15. *Let* $\{(X, \Im_a) : a \in A\}$ *be an indexed family of TSs. Let* $F \in SS(X, A)$*. Then,* $F \in GC(X, \oplus_{a \in A} \Im_a, A)$ *if and only if* $F(a) \in GC(X, \Im_a)$ *for all* $a \in A$*.*

Proof. Necessity. Suppose that $F \in GC(X, \oplus_{a \in A} \Im_a, A)$, and let $b \in A$. Let $U \in \Im_b$ such that $F(b) \subseteq U$. Let $G \in SS(X, A)$ be defined by $G(b) = U$ and $G(e) = X$ if $e \neq b$. Then, $G \in \oplus_{a \in A} \Im_a$ and $F \widetilde{\subseteq} G$. Since $F \in GC(X, \oplus_{a \in A} \Im_a, A)$, then $Cl_{\oplus_{a \in A} \Im_a}(F) \widetilde{\subseteq} G$, and so $\left(Cl_{\oplus_{a \in A} \Im_a}(F)\right)(b) \widetilde{\subseteq} G(b) = U$. On the other hand, by Lemma 4.9 of [5], $Cl_{\Im_b}(F(b)) = \left(Cl_{\oplus_{a \in A} \Im_a}(F)\right)(b)$. This ends the proof.

Sufficiency. Suppose that $F(a) \in GC(X, \tau_a)$ for all $a \in A$, and let $G \in \oplus_{a \in A} \Im_a$ such that $F \widetilde{\subseteq} G$. Then, for each $a \in A$, $F(a) \subseteq G(a) \in \Im_a$, and so $Cl_{\Im_a}(F(a)) \subseteq G(a)$. Thus, by Lemma 4.9 of [5], $\left(Cl_{\oplus_{a \in A} \Im_a}(F)\right)(a) \subseteq G(a)$ for all $a \in A$. Hence, $Cl_{\oplus_{a \in A} \Im_a}(F) \widetilde{\subseteq} G$. It follows that $F \in GC(X, \oplus_{a \in A} \Im_a, A)$. □

Theorem 16. *Let* $(X, \Im)$ *be a TS and* A *be any set of parameters. Let* $F \in SS(X, A)$*. Then,* $F \in GC(X, \tau(\Im), A)$ *if and only if* $F(a) \in GC(X, \Im)$ *for all* $a \in A$*.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then, $\tau(\Im) = \oplus_{a \in A} \Im_a$, and by Theorem 15, we obtain the result. □

Theorem 17. *Let* $\{(X, \Im_a) : a \in A\}$ *be an indexed family of TSs. Let* $F \in SS(X, A)$*. Then,* $F \in G\omega C(X, \oplus_{a \in A} \Im_a, A)$ *if and only if* $F(a) \in G\omega C(X, \Im_a)$ *for all* $a \in A$*.*

Proof. Necessity. Suppose that $F \in G\omega C(X, \oplus_{a \in A} \Im_a, A)$, and let $b \in A$. Let $U \in \Im_b$ such that $F(b) \subseteq U$. Let $G \in SS(X, A)$ be defined by $G(b) = U$ and $G(e) = X$ if $e \neq b$. Then, $G \in \oplus_{a \in A} \Im_a$ and $F \widetilde{\subseteq} G$. Since $F \in G\omega C(X, \oplus_{a \in A} \Im_a, A)$, then $Cl_{(\oplus_{a \in A} \Im_a)_\omega}(F) \widetilde{\subseteq} G$, and so $\left(Cl_{(\oplus_{a \in A} \Im_a)_\omega}(F)\right)(b) \widetilde{\subseteq} G(b) = U$. On the other hand, according to Lemma 4.7 of [5], $Cl_{(\Im_b)_\omega}(F(b)) = \left(Cl_{(\oplus_{a \in A} \Im_a)_\omega}(F)\right)(b)$. This ends the proof.

Sufficiency. Suppose that $F(a) \in G\omega C(X, \tau_a)$ for all $a \in A$, and let $G \in \oplus_{a \in A} \Im_a$ such that $F \widetilde{\subseteq} G$. Then, for each $a \in A$, $F(a) \subseteq G(a) \in \Im_a$, and so $Cl_{(\Im_a)_\omega}(F(a)) \subseteq G(a)$. Thus, following Lemma 4.7 of [5], $\left(Cl_{(\oplus_{a \in A} \Im_a)_\omega}(F)\right)(a) \subseteq G(a)$ for all $a \in A$. Hence, $Cl_{(\oplus_{a \in A} \Im_a)_\omega}(F) \widetilde{\subseteq} G$. It follows that $F \in G\omega C(X, \oplus_{a \in A} \Im_a, A)$. □

Theorem 18. *Let* $(X, \Im)$ *be a TS and* A *be any set of parameters. Let* $F \in SS(X, A)$*. Then,* $F \in G\omega C(X, \tau(\Im), A)$ *if and only if* $F(a) \in G\omega C(X, \Im)$ *for all* $a \in A$*.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then, $\tau(\Im) = \oplus_{a \in A} \Im_a$, and through Theorem 17, we obtain the result. □

In the rest of this section, we study the behavior of soft g-closed sets (soft $g\omega$-closed sets) under soft mappings:

Theorem 19. *Let* $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ *be soft continuous and soft closed. Then, for every* $H \in GC(X, \tau, A)$*,* $f_{pu}(H) \in GC(Y, \sigma, B)$*.*

Proof. Suppose that $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ is soft continuous and soft closed. Let $H \in GC(X, \tau, A)$, and let $M \in \sigma$ such that $f_{pu}(H) \widetilde{\subseteq} M$. Since f_{pu} is soft continuous, then $f_{pu}^{-1}(M) \in \tau$. Since $H \in GC(X, \tau, A)$ and $H \widetilde{\subseteq} f_{pu}^{-1}(M) \in \tau$, then $Cl_\tau(H) \widetilde{\subseteq} f_{pu}^{-1}(M)$, and so $f_{pu}(Cl_\tau(H))\widetilde{\subseteq} f_{pu}\left(f_{pu}^{-1}(M)\right)\widetilde{\subseteq} M$. Since f_{pu} is soft closed, and $Cl_\tau(H)\in\tau^c$, then $f_{pu}(Cl_\tau(H))\in\sigma^c$. Since $f_{pu}(H)\widetilde{\subseteq} f_{pu}(Cl_\tau(H))\in\sigma^c$, then $Cl_\sigma((f_{pu}(H)))\widetilde{\subseteq} f_{pu}(Cl_\tau(H))\widetilde{\subseteq} M$. Hence, $f_{pu}(H)\in GC(Y,\sigma,B)$. □

Theorem 20. *Let $f_{pu}:(X,\tau,A)\longrightarrow(Y,\sigma,B)$ be soft continuous and soft closed. Then, for every $K\in GC(Y,\sigma,B)$, $f_{pu}^{-1}(K)\in GC(X,\tau,A)$.*

Proof. Suppose that $f_{pu}:(X,\tau,A)\longrightarrow(Y,\sigma,B)$ is soft continuous and soft closed. Let $K\in GC(Y,\sigma,B)$, and let $N\in\tau$ such that $f_{pu}^{-1}(K)\widetilde{\subseteq} N$. We are going to show that $Cl_\tau\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A-N)=0_A$. Since f_{pu} is soft closed, and $Cl_\tau\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A-N)\in\tau^c$, then $f_{pu}(Cl_\tau\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A-N))\in\sigma^c$. Since f_{pu} is soft continuous, then $f_{pu}(Cl_\tau\left(f_{pu}^{-1}(K)\right)\widetilde{\subseteq} Cl_\sigma\left(f_{pu}\left(f_{pu}^{-1}(K)\right)\right)\widetilde{\subseteq} Cl_\sigma(K)$. Since $f_{pu}^{-1}(K)\widetilde{\subseteq} N$, then $1_A-N\widetilde{\subseteq} 1_A-f_{pu}^{-1}(K)=f_{pu}^{-1}(1_B-K)$, $f_{pu}(1_A-N)\widetilde{\subseteq} f_{pu}\left(f_{pu}^{-1}(1_B-K)\right)\widetilde{\subseteq} 1_B-K$, and thus

$$\begin{array}{rcl} f_{pu}(Cl_\tau\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A-N)) & \widetilde{\subseteq} & f_{pu}(Cl_\tau\left(f_{pu}^{-1}(K)\right))\widetilde{\cap} f_{pu}(1_A-N) \\ & \widetilde{\subseteq} & Cl_\sigma(K)\widetilde{\cap}(1_B-K) \\ & = & Cl_\sigma(K)-K. \end{array}$$

This is true since $K\in GC(Y,\sigma,B)$ and $f_{pu}(Cl_\tau\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A-N))\in\sigma^c$. Hence, by Theorem 3.6 of [22], $f_{pu}(Cl_\tau\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A-N))=0_B$. Therefore, $f_{pu}^{-1}(K)\in GC(X,\tau,A)$. □

Theorem 21. *Let $f_{pu}:(X,\tau,A)\longrightarrow(Y,\sigma,B)$ be a soft continuous function such that $f_{pu}:(X,\tau_\omega,A)\longrightarrow(Y,\sigma_\omega,B)$ is soft closed. Then, for every $H\in G\omega C(X,\tau,A)$, $f_{pu}(H)\in G\omega C(Y,\sigma,B)$.*

Proof. Suppose that $f_{pu}:(X,\tau,A)\longrightarrow(Y,\sigma,B)$ is a soft continuous soft function such that $f_{pu}:(X,\tau_\omega,A)\longrightarrow(Y,\sigma_\omega,B)$ is soft closed. Let $H\in G\omega C(X,\tau,A)$, and let $M\in\sigma$ such that $f_{pu}(H)\widetilde{\subseteq} M$. Since $f_{pu}:(X,\tau,A)\longrightarrow(Y,\sigma,B)$ is soft continuous, then $f_{pu}^{-1}(M)\in\tau$. Since $H\in G\omega C(X,\tau,A)$ and $H\widetilde{\subseteq} f_{pu}^{-1}(M)\in\tau$, then $Cl_{\tau_\omega}(H)\widetilde{\subseteq} f_{pu}^{-1}(M)$, and so $f_{pu}(Cl_{\tau_\omega}(H))\widetilde{\subseteq} f_{pu}\left(f_{pu}^{-1}(M)\right)\widetilde{\subseteq} M$. Since $f_{pu}:(X,\tau_\omega,A)\longrightarrow(Y,\sigma_\omega,B)$ is soft closed, and $Cl_{\tau_\omega}(H)\in(\tau_\omega)^c$, then $f_{pu}(Cl_{\tau_\omega}(H))\in(\sigma_\omega)^c$. Since $f_{pu}(H)\widetilde{\subseteq} f_{pu}(Cl_\tau(H))\in(\sigma_\omega)^c$, then $Cl_{\sigma_\omega}((f_{pu}(H)))\widetilde{\subseteq} f_{pu}(Cl_\tau(H))\widetilde{\subseteq} M$. Hence, $f_{pu}(H)\in G\omega C(Y,\sigma,B)$. □

Theorem 22. *Let $f_{pu}:(X,\tau,A)\longrightarrow(Y,\sigma,B)$ be a soft function such that $f_{pu}:(X,\tau_\omega,A)\longrightarrow(Y,\sigma_\omega,B)$ is soft continuous and $f_{pu}:(X,\tau_\omega,A)\longrightarrow(Y,\sigma,B)$ is soft closed. Then, for every $K\in G\omega C(Y,\sigma,B)$, $f_{pu}^{-1}(K)\in G\omega C(X,\tau,A)$.*

Proof. Suppose that $f_{pu}:(X,\tau_\omega,A)\longrightarrow(Y,\sigma_\omega,B)$ is soft continuous and $f_{pu}:(X,\tau_\omega,A)\longrightarrow(Y,\sigma,B)$ is soft closed. Let $K\in G\omega C(Y,\sigma,B)$, and let $N\in\tau$ such that $f_{pu}^{-1}(K)\widetilde{\subseteq} N$. We are going to show that $Cl_{\tau_\omega}\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A-N)=0_A$. Since $f_{pu}:(X,\tau_\omega,A)\longrightarrow(Y,\sigma,B)$ is soft closed, and $Cl_{\tau_\omega}\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A-N)\in(\tau_\omega)^c$, then $f_{pu}(Cl_{\tau_\omega}\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A-N))\in\sigma^c$. Since $f_{pu}:(X,\tau_\omega,A)\longrightarrow(Y,\sigma_\omega,B)$ is soft continuous, then $f_{pu}(Cl_{\tau_\omega}\left(f_{pu}^{-1}(K)\right)\widetilde{\subseteq} Cl_{\sigma_\omega}\left(f_{pu}\left(f_{pu}^{-1}(K)\right)\right)\widetilde{\subseteq} Cl_{\sigma_\omega}(K)$. Since $f_{pu}^{-1}(K)\widetilde{\subseteq} N$, then $1_A-N\widetilde{\subseteq} 1_A-f_{pu}^{-1}(K)=f_{pu}^{-1}(1_B-K)$, $f_{pu}(1_A-N)\widetilde{\subseteq} f_{pu}\left(f_{pu}^{-1}(1_B-K)\right)\widetilde{\subseteq} 1_B-K$, and therefore

$$\begin{array}{rcl} f_{pu}(Cl_{\tau_\omega}\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A - N)) & \widetilde{\subseteq} & f_{pu}(Cl_{\tau_\omega}\left(f_{pu}^{-1}(K)\right))\widetilde{\cap} f_{pu}(1_A - N) \\ & \widetilde{\subseteq} & Cl_{\sigma_\omega}(K)\widetilde{\cap}(1_B - K) \\ & = & Cl_{\sigma_\omega}(K) - K. \end{array}$$

Since $K \in G\omega C(Y, \sigma, B)$ and $f_{pu}(Cl_{\tau_\omega}\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A - N)) \in \sigma^c$, then by Theorem 7, $f_{pu}(Cl_{\tau_\omega}\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A - N)) = 0_B$. Therefore, $Cl_{\tau_\omega}\left(f_{pu}^{-1}(K)\right)\widetilde{\cap}(1_A - N) = 0_A$. □

3. Separation Axioms

In this section, we continue the study of soft $T_{1/2}$ spaces, in which we obtain two characterizations of these soft spaces and investigate their behavior with respect to soft subspaces, soft transformations, and generated soft topologies.

We start this section by characterizing soft $T_{1/2}$ spaces via soft points:

Theorem 23. *An STS (X, τ, A) is soft $T_{1/2}$ if and only if for each $a_x \in SP(X, A)$, $a_x \in \tau \cup \tau^c$.*

Proof. Necessity. Suppose that (X, τ, A) is soft $T_{1/2}$. Let $a_x \in SP(X, A)$. Suppose that $a_x \notin \tau^c$. Then, $1_A - a_x \notin \tau$. We are going to show that $1_A - a_x \in GC(X, \tau, A)$. Let $G \in \tau$ such that $1_A - a_x \widetilde{\subseteq} G$. Since $1_A - a_x \notin \tau$, then $G = 1_A$, and so $Cl_\tau(1_A - a_x)\widetilde{\subseteq} G = 1_A$. Thus, $1_A - a_x \in GC(X, \tau, A)$. Since (X, τ, A) is soft $T_{1/2}$, then $1_A - a_x \in \tau^c$. Therefore, $a_x \in \tau$.

Sufficiency. Suppose that for each $a_x \in SP(X, A)$, $a_x \in \tau \cup \tau^c$. Let $F \in GC(X, \tau, A)$. We are going to show that $Cl_\tau(F)\widetilde{\subseteq} F$. Let $a_x \widetilde{\in} Cl_\tau(F)$. By assumption, $a_x \in \tau \cup \tau^c$. If $a_x \in \tau$, then $a_x \widetilde{\cap} F \neq 0_A$, and so $a_x \widetilde{\in} F$. If $a_x \in \tau^c$, then through Theorem 3.6 of [22], we have $a_x \widetilde{\notin} Cl_\tau(F) - F$ with $a_x \widetilde{\in} Cl_\tau(F)$, and hence $a_x \widetilde{\in} F$.

The next result is a consequence of Theorem 23: □

Corollary 2. *Every soft door space is soft $T_{1/2}$.*

The following example will show that the converse of Corollary 2 need not be true in general:

Example 9. *Let $X = \{a, b, c, d\}$, $A = \mathbb{Z}$, and $\Im = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then, $(X, \tau(\Im), A)$ is a soft $T_{1/2}$ but not a soft door space.*

Another characterization of soft $T_{1/2}$ spaces is stated in the next theorem:

Theorem 24. *An STS (X, τ, A) is soft $T_{1/2}$ if and only if for every $H \in SS(X, A)$*

$$H = \widetilde{\cap}\{M \in \tau \cup \tau^c : H \widetilde{\subseteq} M\}.$$

Proof. Necessity. Suppose that (X, τ, A) is soft $T_{1/2}$. Suppose to the contrary that there exists $H \in SS(X, A)$ such that $H \neq \widetilde{\cap}\{M \in \tau \cup \tau^c : H \widetilde{\subseteq} M\}$. Since it is clear that $H \widetilde{\subseteq} \widetilde{\cap}\{M \in \tau \cup \tau^c : H \widetilde{\subseteq} M\}$, then there exists $a_x \widetilde{\in} \left(\widetilde{\cap}\{M \in \tau \cup \tau^c : H \widetilde{\subseteq} M\} - H\right)$. Since (X, τ, A) is soft $T_{1/2}$, then by Theorem 23, $a_x \in \tau \cup \tau^c$, and so $1_A - a_x \in \tau \cup \tau^c$. Since $a_x \widetilde{\in} 1_A - H$, then $H \widetilde{\subseteq} 1_A - a_x \in \tau \cup \tau^c$, and so $\widetilde{\cap}\{M \in \tau \cup \tau^c : H \widetilde{\subseteq} M\} \widetilde{\subseteq} 1_A - a_x$. Therefore, $a_x \widetilde{\in} 1_A - \widetilde{\cap}\{M \in \tau \cup \tau^c : H \widetilde{\subseteq} M\}$, a contradiction.

Sufficiency. Suppose that for every $H \in SS(X, A)$, $H = \widetilde{\cap}\{M \in \tau \cup \tau^c : H \widetilde{\subseteq} M\}$. Let $F \in GC(X, \tau, A)$. By assumption, we have $F = \widetilde{\cap}\{M \in \tau \cup \tau^c : F \widetilde{\subseteq} M\}$. To see that $Cl_\tau(F) \subseteq F$, it is sufficient to show that $Cl_\tau(F) \subseteq M$ for all $M \in \tau \cup \tau^c$ with $F \widetilde{\subseteq} M$. Let $M \in \tau \cup \tau^c$ such that $F \widetilde{\subseteq} M$. If $M \in \tau$, then $Cl_\tau(F) \widetilde{\subseteq} M$ because $F \in GC(X, \tau, A)$. If $M \in \tau^c$, then $Cl_\tau(F) = F \widetilde{\subseteq} M$. □

In the next three theorems, we discuss the behavior of soft $T_{1/2}$ spaces with respect to induced TSs and generated STSs:

Theorem 25. *If (X, τ, A) is a soft $T_{1/2}$ STS, then (X, τ_a) is $T_{1/2}$ for all $a \in A$.*

Proof. Suppose that (X, τ, A) is soft $T_{1/2}$, and let $a \in A$. Let $x \in X$. Since (X, τ, A) is soft $T_{1/2}$, then by Theorem 23, $a_x \in \tau \cup \tau^c$, and thus $\{x\} \in \tau_a \cup (\tau_a)^c$. Hence, by Theorem 2.5 of [27], (X, τ_a) is $T_{1/2}$. □

The following example will show that the converse of Theorem 25 is not true in general:

Example 10. *Let $X = \{1, 2\}$, $A = \{a, b\}$, and $\tau = \{0_A, 1_A, a_1, b_1, a_1 \tilde{\cup} b_1\}$. Since $\tau_a = \tau_b = \{\emptyset, X, \{1\}\}$, then (X, τ_a) and (X, τ_b) are both $T_{1/2}$. On the other hand, since $a_2 \notin \tau \cup \tau^c$, then according to Theorem 23, (X, τ, A) is not soft $T_{1/2}$.*

Theorem 26. *Let $\{(X, \Im_a) : a \in A\}$ be an indexed family of TSs. Then, $(X, \oplus_{a \in A} \Im_a, A)$ is soft $T_{1/2}$ if and only if $(X, \Im_a)$ is $T_{1/2}$ for all $a \in A$.*

Proof. Necessity. Suppose that $(X, \oplus_{a \in A} \Im_a, A)$ is soft $T_{1/2}$, and let $b \in A$. Then, under Theorem 25, $(X, (\oplus_{a \in A} \Im_a)_b)$ is $T_{1/2}$. On the other hand, according to Theorem 3.7 of [1], $(\oplus_{a \in A} \Im_a)_b = \Im_b$. Hence, $(X, \Im_b)$ is $T_{1/2}$.

Sufficiency. Suppose that $(X, \Im_a)$ is $T_{1/2}$ for all $a \in A$. Let $b_x \in SP(X, A)$. Since $(X, \Im_b)$ is $T_{1/2}$, then by Theorem 2.5 of [27], $\{x\} \in \Im_b \cup (\Im_b)^c$. Therefore, $b_x \in (\oplus_{a \in A} \Im_a) \cup (\oplus_{a \in A} \Im_a)^c$. Hence, by Theorem 23, $(X, \oplus_{a \in A} \Im_a, A)$ is soft $T_{1/2}$. □

Theorem 27. *Let $(X, \Im)$ be a TS and A be any set of parameters. Then, $(X, \tau(\Im), A)$ is soft $T_{1/2}$ if and only if $(X, \Im)$ is $T_{1/2}$.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then, $\tau(\Im) = \oplus_{a \in A} \Im_a$, and through Theorem 4, we obtain the result. □

Theorem 28. *If (X, τ, A) is soft $T_{1/2}$, then $G\omega C(X, \tau, A) \subseteq (\tau_\omega)^c$.*

Proof. Suppose that (X, τ, A) is soft $T_{1/2}$. Suppose to the contrary that there exists $F \in G\omega C(X, \tau, A) - (\tau_\omega)^c$. Since $F \notin (\tau_\omega)^c$, then there exists $a_x \tilde{\in} Cl_{\tau_\omega}(F) - F$. Since $F \in G\omega C(X, \tau, A)$, then by Theorem 7, $a_x \notin \tau^c$. Since (X, τ, A) is soft $T_{1/2}$, then by Theorem 23, $a_x \in \tau$. Since $a_x \tilde{\in} Cl_{\tau_\omega}(F)$, and $a_x \in \tau \subseteq \tau_\omega$, then $a_x \tilde{\cap} F \neq 0_A$. Hence, $a_x \tilde{\in} F$, a contradiction. □

The following example will show that the converse of Theorem 28 need not be true in general:

Example 11. *Let $X = \{1, 2, 3\}$ and $\Im = \{\emptyset, X, \{1\}, \{1, 2\}\}$. Consider $(X, \tau(\Im), \mathbb{N})$. Then, $G\omega C(X, \tau(\Im), \mathbb{N}) \subseteq ((\tau(\Im))_\omega)^c$, while $(X, \tau(\Im), \mathbb{N})$ is not soft $T_{1/2}$.*

We shall now establish an interesting characterization of soft T_1 STSs, which are soft anti-locally countable in terms of soft $g\omega$-closed sets:

Theorem 29. *Let (X, τ, A) be a soft anti-locally countable STS. Then, (X, τ, A) is soft T_1 if and only if $G\omega C(X, \tau, A) \subseteq (\tau_\omega)^c$.*

Proof. Necessity. Suppose that (X, τ, A) is soft T_1. Then, by Theorem 5.3 of [22], (X, τ, A) is soft $T_{1/2}$. Thus, by Theorem 28, $G\omega C(X, \tau, A) \subseteq (\tau_\omega)^c$.

Sufficiency. Suppose that $G\omega C(X, \tau, A) \subseteq (\tau_\omega)^c$. Suppose to the contrary that (X, τ, A) is not soft T_1. Then, there exists $a_x \in SP(X, A) - \tau^c$, and so $1_A - a_x \notin \tau$. If $G \in \tau$ such that $1_A - a_x \tilde{\subseteq} G$, then $G = 1_A$, and hence $Cl_{\tau_\omega}(1_A - a_x) \tilde{\subseteq} G$. Therefore, $1_A - a_x \in$

$G\omega C(X,\tau,A) \subseteq (\tau_\omega)^c$. Hence, $a_x \in \tau_\omega$. Thus, there exists $M \in \tau$ and $H \in CSS(X,A)$ such that $a_x \widetilde{\in} M$ and $M - H \widetilde{\subseteq} a_x$. Therefore, $M \in (\tau - \{0_A\}) \cap CSS(X,A)$. Hence, (X,τ,A) is not soft anti-locally countable, a contradiction. □

In the next theorem, we discuss the behavior of soft $T_{1/2}$ spaces with respect to soft subspaces:

Theorem 30. *Let (X,τ,A) be an STS, and let Y be a non-empty subset of X. If (X,τ,A) is soft $T_{1/2}$, then (Y,τ_Y,A) is soft $T_{1/2}$.*

Proof. Proof. Suppose that (X,τ,A) is soft $T_{1/2}$. Let $a_y \in SP(Y,A)$. Then, $a_y \in SP(X,A)$. Since (X,τ,A) is soft $T_{1/2}$, then $a_y \in \tau \cup \tau^c$. Thus, $a_y \in \tau_Y \cup (\tau_Y)^c$. Hence, under Theorem 23, (Y,τ_Y,A) is soft $T_{1/2}$. □

In Theorems 31 and 32, we study the behavior of the class of soft $T_{1/2}$ spaces under soft mappings:

Theorem 31. *Let $f_{pu} : (X,\tau,A) \longrightarrow (Y,\sigma,B)$ be soft continuous, soft closed, and onto. If (X,τ,A) is soft $T_{1/2}$, then (Y,σ,B) is soft $T_{1/2}$.*

Proof. Let $K \in GC(Y,\sigma,B)$. Then, under Theorem 20, $f_{pu}^{-1}(K) \in GC(X,\tau,A)$. Since (X,τ,A) is soft $T_{1/2}$, then $f_{pu}^{-1}(K) \in \tau^c$. Since $f_{pu} : (X,\tau,A) \longrightarrow (Y,\sigma,B)$ is soft closed, then $f_{pu}\left(f_{pu}^{-1}(K)\right) \in \sigma^c$. Since f_{pu} is onto, then $f_{pu}\left(f_{pu}^{-1}(K)\right) = K$. Thus, $K \in \sigma^c$. Hence, (Y,σ,B) is soft $T_{1/2}$. □

In Theorem 31, the condition "soft closed" cannot be replaced by the condition "soft open", as the following example shows:

Example 12. *Let $X = \mathbb{N}$, $Y = \{a,b,c\}$, $A = B = [0,1]$, $\Im = \{\emptyset, \{1\}\} \cup \{V \subseteq X : 1 \in V$ and $X - V$ is finite$\}$, and $\aleph = \{\emptyset, \{a\}, Y\}$. Let $p : X \longrightarrow Y$ and $u : A \longrightarrow B$, defined by*

$$p(x) = \begin{cases} a & \text{if } x = 1 \\ b & \text{if } x \text{ is even} \\ c & \text{if } x \text{ is odd and } x \neq 1 \end{cases},$$

$$u(a) = a \text{ for all } a \in A.$$

Then, $f_{pu} : (X,\tau(\Im),A) \longrightarrow (Y,\tau(\aleph),B)$ is soft continuous, soft open, and onto. However, according to Theorem 23, $(X,\tau(\Im),A)$ is soft, $T_{1/2}$ while $(Y,\tau(\aleph),B)$ is not.

Theorem 32. *Let $f_{pu} : (X,\tau,A) \longrightarrow (Y,\sigma,B)$ be soft open and onto such that for each $b_y \in SP(Y,B)$, $\left\{a_x \in SP(X,A) : f_{pu}(a_x) = y_b\right\}$ is a finite set. If (X,τ,A) is soft $T_{1/2}$, then (Y,σ,B) is soft $T_{1/2}$.*

Proof. We shall use Theorem 23. Let $b_y \in SP(Y,B)$, and let $\mathcal{H} = \left\{a_x \in SP(X,A) : f_{pu}(a_x) = y_b\right\}$. By assumption, $\mathcal{H}$ is a non-empty finite set. If there exists $a_x \in \tau \widetilde{\cap} \mathcal{H}$, then $b_y = f_{pu}(a_x) \in \sigma$, since f_{pu} is soft open. Otherwise, $1_A - a_x \in \tau$ for all $a_x \in \mathcal{H}$, and thus $1_B - b_y = f_{pu}(\widetilde{\cap}\mathcal{H}) \in \sigma$. It follows that (Y,σ,B) is soft $T_{1/2}$. □

4. Conclusions

We defined soft generalized ω-closed sets as a new class of soft sets. We proved that this class of sets contains strictly the classes of soft ω-closed and soft generalized closed sets. In addition, we determined the behavior of soft generalized ω-closed sets relative to soft unions, soft intersections, and soft subspaces as well as generated soft topologies. Moreover, we explored soft images and soft inverse images of soft generalized closed and soft generalized ω-closed sets under soft continuous, soft closed soft transformations.

In addition to these, we continued the study of soft $T_{1/2}$ spaces, in which we obtained two characterizations of these soft spaces and investigated their behavior with respect to soft subspaces, soft transformations, and generated soft topologies. Future research could focus on the following topics: (1) defining soft generalized ω-open sets and investigating related concepts such as soft continuity, (2) investigating the behavior of soft generalized ω-closed sets under product STSs, or (3) solving the two open questions raised in this paper.

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