axioms

Article

The Classification of All Singular Nonsymmetric Macdonald Polynomials

Charles F. Dunkl

Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA; cfd5z@virginia.edu

Abstract: The affine Hecke algebra of type A has two parameters (q,t) and acts on polynomials in N variables. There are two important pairwise commuting sets of elements in the algebra: the Cherednik operators and the Jucys–Murphy elements whose simultaneous eigenfunctions are the nonsymmetric Macdonald polynomials, and basis vectors of irreducible modules of the Hecke algebra, respectively. For certain parameter values, it is possible for special polynomials to be simultaneous eigenfunctions with equal corresponding eigenvalues of both sets of operators. These are called singular polynomials. The possible parameter values are of the form $q^m = t^{-n}$ with $2 \leq n \leq N$. For a fixed parameter, the singular polynomials span an irreducible module of the Hecke algebra. Colmenarejo and the author (SIGMA 16 (2020), 010) showed that there exist singular polynomials for each of these parameter values, they coincide with specializations of nonsymmetric Macdonald polynomials, and the isotype (a partition of N) of the Hecke algebra module is $(dn-1, n-1, \ldots, n-1, r)$ for some $d \geq 1$. In the present paper, it is shown that there are no other singular polynomials.

Keywords: nonsymmetric Macdonald polynomials; the affine Hecke algebra of type A; Young tableaux; Jucys–Murphy operators

MSC: 33D52; 20C08; 05E10

check for updates

Citation: Dunkl, C.F. The Classification of All Singular Nonsymmetric Macdonald Polynomials. *Axioms* **2022**, *11*, 208. https://doi.org/10.3390/axioms11050208

Academic Editor: Clemente Cesarano

Received: 30 March 2022
Accepted: 24 April 2022
Published: 29 April 2022

1. Introduction

Many structures arise from the action of the symmetric group on polynomials in N variables. Among them are the Hecke algebra and the affine Hecke algebra of type A. This paper concerns polynomials with noteworthy properties with respect to these algebras. The symmetric group $\mathcal{S}_N$ is generated by the simple reflections $s_i, 1 \leq i < N$, where

$$xs_i := \left(x_1, \ldots, \overset{i}{x_{i+1}}, \overset{i+1}{x_i}, \ldots, x_N\right);$$

they satisfy the braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ for $|i-j| \geq 2$. Let q,t be parameters satisfying $t^n \neq 1$ for $2 \leq n \leq N$ and $q,t \neq 0$. Define $\mathcal{P} = \mathbb{K}[x_1, \ldots, x_N]$ where $\mathbb{K}$ is a field containing $\mathbb{Q}(q,t)$. The Hecke algebra $\mathcal{H}_N(t)$ is generated by Demazure operators (with $p \in \mathcal{P}$ and $1 \leq i < N$)

$$T_i p(x) := (1-t)x_{i+1} \frac{p(x) - p(xs_i)}{x_i - x_{i+1}} + tp(xs_i);$$

they satisfy the same braid relations $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ and $T_i T_j = T_j T_i$ for $|i-j| \geq 2$, as well as the quadratic relations $(T_i - t)(T_i + 1) = 0$. The affine Hecke algebra $\mathcal{H}_N(t;q)$ is obtained by adjoining the q-shift

$$wp(x) := p(qx_N, x_1, x_2, \ldots, x_{N-1})$$

and defining

$$T_0 p(x) := wT_1w^{-1}p(x) = (1-t)x_1\frac{p(x)-p(xs_0)}{qx_N - x_1} + tp(xs_0),$$
$$xs_0 := (qx_N, x_2, \ldots, x_{N-1}, x_1/q).$$

Then $wT_{i+1} = T_i w$ where the indices are taken mod N. (That is, $w^2T_1 = T_{N-1}w^2$.) The quadratic relations imply $T_i^{-1} = t^{-1}(T_i + (1-t))$. There are two commutative families of operators in $\mathcal{H}_N(t;q)$ (each indexed $1 \leq i \leq N$): the Cherednik operators (see [1])

$$\xi_i := t^{i-1}T_iT_{i+1}\cdots T_{N-1}wT_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1}$$

and the Jucys–Murphy operators

$$\omega_i = t^{i-N}T_iT_{i+1}\cdots T_{N-1}T_{N-1}T_{N-2}\cdots T_i.$$

Note that $\xi_i = t^{-1}T_i\xi_{i+1}T_i$ and $\omega_i = t^{-1}T_i\omega_{i+1}T_i$ for $i < N$. The simultaneous eigenfunctions of the Cherednik operators are the nonsymmetric Macdonald polynomials and the simultaneous eigenvectors of the Jucys–Murphy operators span irreducible representations of $\mathcal{H}_N(t)$. Our concern is to determine all polynomials which are simultaneous eigenfunctions of both sets of operators, more specifically, when q,t satisfy a relation of the form $q^mt^n = 1$ to determine the homogeneous polynomials p such that $\xi_i p = \omega_i p$ for all i. These are called *singular polynomials* with singular parameter $q^m = t^{-n}$. In a previous paper [2] Colmenarejo and the author found a large class of such polynomials associated with tableaux of quasi-staircase shape. In this paper, we will show that there are no other occurrences.

Affine Hecke algebras were used by Kirillov and Noumi [3] to derive important results about the coefficients of Macdonald polynomials. Mimachi and Noumi [4] found double sums for reproducing kernels for series in nonsymmetric Macdonald polynomials. The paper [5] by Baker and Forrester is a source of some background for the present paper.

In Section 2, we collect the needed definitions and results about the Hecke algebra action on polynomials, Cherednik operators, nonsymmetric Macdonald polynomials, and the representation theory of Hecke algebra of type A. The definition of singular polynomials and its consequences, that is, necessary conditions, are presented in Section 3. This section also explains the known existence theorem. Section 4 concerns the method of restriction to produce singular polynomials with a smaller number of variables and this leads into Section 5 where our main nonexistence theorem is proved.

2. Preliminary Results

In this section, we present background information and computational results dealing with $\mathcal{H}_N(t)$ and the action on polynomials.

Lemma 1. *If $j > i+1$ or $j < i$ then $T_i\omega_j = \omega_jT_i$, and $T_i\omega_i = (t-1)\omega_i + \omega_{i+1}T_i$, $T_i\omega_{i+1} = \omega_iT_i - (t-1)\omega_i$.*

Proof. If $j > i+1$ then T_i commutes with each factor of ω_i. Suppose $j = i-1$ then by the braid relations

$$\begin{aligned}T_i\omega_{i-1} &= t^{i-1-N}T_iT_{i-1}T_iT_{i+1}\cdots T_iT_{i-1} = t^{i-1-N}T_{i-1}T_iT_{i-1}T_{i+1}\cdots T_iT_{i-1}\\ &= t^{i-1-N}T_{i-1}T_{i+1}T_{i+2}\cdots T_{i-1}T_iT_{i-1}\\ &= t^{i-1-N}T_{i-1}T_iT_{i+1}\cdots T_iT_{i-1}T_i = \omega_{i-1}T_i.\end{aligned}$$

Suppose $j < i-1$ then $\omega_j = t^{j-i+1}T_jT_{j+1}\cdots T_{i-2}\omega_{i-1}T_{i-2}\cdots T_j$ and T_i commutes with each factor in this product. If $j = i$ then

$$\begin{aligned} T_i\omega_i &= t^{-1}T_i^2\omega_{i+1}T_i = t^{-1}\{(t-1)T_i + t\}\omega_{i+1}T_i \\ &= (t-1)\omega_i + \omega_{i+1}T_i, \end{aligned}$$

and similarly $\omega_iT_i = t^{-1}T_i\omega_{i+1}T_i^2 = (t-1)\omega_i + T_i\omega_{i+1}$. □

Lemma 2. *If $j > i+1$ or $j < i$ then $T_i\xi_j = \xi_jT_i$, and $T_i\xi_i = (t-1)\xi_i + \xi_{i+1}T_i$, $T_i\xi_{i+1} = \xi_iT_i - (t-1)\xi_i$.*

Proof. Recall $wT_{i+1} = T_iw$, $w^2T_1 = T_{N-1}w^2$. Suppose $j = i-1$ then

$$\begin{aligned} T_i\xi_{i-1} &= t^{i-1-N}T_iT_{i-1}T_iT_{i+1}\cdots T_{N-1}wT_1^{-1}\cdots T_{i-2}^{-1} \\ &= t^{i-1-N}T_{i-1}T_iT_{i-1}T_{i+1}\cdots T_{N-1}wT_1^{-1}\cdots T_{i-2}^{-1} \\ &= t^{i-1-N}T_{i-1}T_iT_{i+1}\cdots T_{N-1}T_{i-1}wT_1^{-1}\cdots T_{i-2}^{-1} \\ &= t^{i-1-N}T_{i-1}T_iT_{i+1}\cdots T_{N-1}wT_iT_1^{-1}\cdots T_{i-2}^{-1} = \xi_{i-1}T_i. \end{aligned}$$

The analogous argument as in the previous lemma shows $T_i\xi_j = \xi_jT_i$ for $j < i-1$. Suppose $j > i+1$ then

$$\begin{aligned} T_i\xi_j &= t^{j-N}T_iT_jT_{j+1}\cdots T_{N-1}wT_1^{-1}\cdots T_{j-1}^{-1} = t^{j-N}T_jT_{j+1}\cdots T_{N-1}T_iwT_1^{-1}\cdots T_{j-1}^{-1} \\ &= t^{j-N}T_jT_{j+1}\cdots T_{N-1}wT_{i+1}T_1^{-1}\cdots T_{j-1}^{-1} \\ &= t^{j-N}T_j\cdots T_{N-1}wT_1^{-1}\cdots T_{i-1}T_{i-2}^{-1}T_{i-1}^{-1}\cdots T_{j-1}^{-1}. \end{aligned}$$

The modified braid relations $aba = bab \Leftrightarrow ab^{-1}a^{-1} = b^{-1}a^{-1}b$ imply $T_{i+1}T_i^{-1}T_{i+1}^{-1} = T_i^{-1}T_{i+1}^{-1}T_i$ and thus $T_i\xi_j = \xi_jT_i$. As before

$$\begin{aligned} &T_i\xi_i = t^{-1}T_i^2\xi_{i+1}T_i = t^{-1}\{(t-1)T_i + t\}\xi_{i+1}T_i = (t-1)\xi_i + \xi_{i+1}T_i. \\ &\xi_iT_i = (t-1)\xi_i + T_i\xi_{i+1}. \end{aligned}$$

□

Polynomials are spanned by monomials $x^\alpha = \prod_{i=1}^{N} x_i^{\alpha_i}, \alpha \in \mathbb{N}_0^N$. For $\alpha \in \mathbb{N}_0^N$ set $s_i\alpha = \left(\alpha_1, \ldots, \overset{i}{\alpha_{i+1}}, \overset{i+1}{\alpha_i}, \ldots\right)$ for $1 \le i < N$, and $|\alpha| = \sum_{j=1}^N \alpha_j$ (the degree of x^α). Let $\mathbb{N}_0^{N,+} = \left\{\alpha \in \mathbb{N}_0^N : \alpha_1 \ge \alpha_2 \ge \ldots \ge \alpha_N\right\}$, the set of partitions of length $\le N$. Let α^+ denote the nonincreasing rearrangement of α (thus $\alpha^+ \in \mathbb{N}_0^{N,+}$). There is a partial order on $\mathbb{N}_0^N$

$$\begin{aligned} &\alpha \prec \beta \Longleftrightarrow \sum_{j=1}^{i}\alpha_j \le \sum_{j=1}^{i}\beta_j,\ 1 \le i \le N,\ \alpha \ne \beta, \\ &\alpha \lhd \beta \Longleftrightarrow (|\alpha| = |\beta|) \wedge \left[\left(\alpha^+ \prec \beta^+\right) \vee \left(\alpha^+ = \beta^+ \wedge \alpha \prec \beta\right)\right], \end{aligned}$$

and a rank function ($1 \le i \le N$)

$$r_\alpha(i) := \#\left\{j : \alpha_j > \alpha_i\right\} + \#\left\{j : 1 \le j \le i, \alpha_j = \alpha_i\right\}.$$

Note $\alpha_i = \alpha^+_{r_\alpha(i)}$.

2.1. Nonsymmetric Macdonald Polynomials

The key fact about the Cherednik operators is the triangular property (see [5])

$$\xi_i x^\alpha = q^{\alpha_i} t^{N-r_\alpha(i)} x^\alpha + \sum_{\beta \lhd \alpha} c_{\alpha,\beta}(q,t) x^\beta, \tag{1}$$

where the coefficients $c_{\alpha,\beta}(q,t)$ are polynomials in q,t. For generic (q,t) (this means $q^m t^n \neq 1, 0$ for $m \geq 0$ and $1 \leq n \leq N$) there is a basis of $\mathcal{P}$, for $\alpha \in \mathbb{N}_0^N$

$$M_\alpha(x) = q^{b(\alpha)} t^{e(\alpha)} x^\alpha + \sum_{\beta \lhd \alpha} A_{\alpha,\beta}(q,t) x^\beta$$

(where $A_{\alpha,\beta}(q,t)$ is a rational function of (q,t) with no poles when (q,t) is generic) and for $1 \leq i \leq N$

$$\xi_i M_\alpha = q^{\alpha_i} t^{N-r_\alpha(i)} M_\alpha.$$

The exponents are $b(\alpha) = \frac{1}{2} \sum_{i=1}^N \alpha_i(\alpha_i - 1)$ and $e(\alpha) = \sum_{i=1}^N \alpha_i^+ (N - 2i + 1) - \operatorname{inv}(\alpha)$, with $\operatorname{inv}(\alpha) := \#\{(i,j) : 1 \leq i < j \leq N, \alpha_i < \alpha_j\}$; there is an equivalent formula:

$$e(\alpha) = \frac{1}{2} \sum_{1 \leq i < j \leq N} \left(|\alpha_i - \alpha_j| + |\alpha_i - \alpha_j + 1| - 1 \right).$$

These powers arise from the Yang-Baxter graph method of constructing the M_α, and are not actually needed here. The *spectral vector* of M_α is $[\zeta_\alpha(i)]_{i=1}^N$ with $\zeta_\alpha(i) = q^{\alpha_i} t^{N-r_\alpha(i)}$. We will need the formulas for the action of T_i on M_α. Suppose $\alpha_i < \alpha_{i+1}$ and $z = \zeta_\alpha(i+1)/\zeta_\alpha(i) = q^{\alpha_{i+1}-\alpha_i} t^{r_\alpha(i) - r_\alpha(i+1)}$ then

$$T_i M_\alpha = M_{s_i\alpha} - \frac{1-t}{1-z} M_\alpha, \tag{2}$$

$$T_i M_{s_i\alpha} = \frac{(1-zt)(t-z)}{(1-z)^2} M_\alpha + \frac{z(1-t)}{(1-z)} M_{s_i\alpha}. \tag{3}$$

If $\alpha_i = \alpha_{i+1}$ then $T_i M_\alpha = t M_\alpha$. The quadratic relation appears as

$$\left(T_i + \frac{1-t}{1-z} \right) \left(T_i - \frac{z(1-t)}{1-z} \right) = \frac{(1-zt)(t-z)}{(1-z)^2}.$$

2.2. Action of T_i on Polynomials and $\rhd$-Maximal Terms

The following are routine computations:

Lemma 3. *Suppose $\gamma \in \mathbb{N}_0^N$ and $1 \leq i < N$. Set $x' = \prod_{j \neq i,i+1} x^{\gamma_j}$. Then*

(1) $\gamma_i > \gamma_{i+1} + 1$ implies $T_i x^\gamma = (1-t) x' \sum_{j=0}^{\gamma_i - \gamma_{i+1} - 1} x_i^{\gamma_i - j - 1} x_{i+1}^{\gamma_{i+1} + j + 1} + t x^{s_i\gamma}$;

(2) $\gamma_i = \gamma_{i+1} + 1$ implies $T_i x^\gamma = x^{s_i\gamma}$;

(3) $\gamma_i = \gamma_{i+1}$ implies $T_i x^\gamma = t x^\gamma$;

(4) $\gamma_i = \gamma_{i+1} - 1$ implies $T_i x^\gamma = t x^{s_i\gamma} + (t-1) x^\gamma$;

(5) $\gamma_i < \gamma_{i+1} - 1$ implies $T_i x^\gamma = (t-1) x' \sum_{j=0}^{\gamma_{i+1} - \gamma_i - 1} x_i^{\gamma_i + j} x_{i+1}^{\gamma_{i+1} - j} + t x^{s_i\gamma}$.

Lemma 4. *Suppose $\lambda \in \mathbb{N}_0^{N,+}, \lambda_i > \lambda_j + 1$ ($i > j$) and $1 \leq s < \lambda_i - \lambda_j$, $\mu \in \mathbb{N}_0^N$ such that $\mu_k = \lambda_k$ for $k \neq i, j$, $\mu_i = \lambda_i - s, \mu_j = \lambda_j + s$ then $\lambda \succ \mu^+$.*

(The proof is left as an exercise.)

In (1) above let $\alpha_k = \gamma_k$ for $k \neq i, i+1$ and $\alpha_i = \gamma_i - j - 1, \alpha_{i+1} = \gamma_{i+1} + j + 1$ with $1 \leq j + 1 < \gamma_i - \gamma_{i+1}$ then the Lemma with $\lambda = \gamma^+$ and $\mu^+ = \alpha^+$ shows $\gamma^+ \succ a^+$ (the other term in (1) is $x^{s_i\gamma}$ and $\gamma \succ s_i\gamma$). Similarly in (5) let $\alpha_i = \gamma_i + j, \alpha_{i+1} = \gamma_{i+1} - j$ with $1 \leq j \leq \gamma_{i+1} - \gamma_i - 1$, thus $\gamma^+ \succ a^+$ (the other term in (5) for $j = 0$ is x^γ and $s_i\gamma \succ \gamma$.

Proposition 1. *Suppose α is $\rhd$-maximal in $p = \sum_\delta c_\delta x^\delta$ (a homogeneous polynomial, $|\delta| = |\alpha|$), that is, $c_\alpha \neq 0$ and if some $\delta \unrhd \alpha$ with $c_\delta \neq 0$ then $\delta = \alpha$. Furthermore suppose $\alpha_{i+1} > \alpha_i$ for some i and x^β with $\beta \rhd s_i\alpha$ appears in $(T_i + c)p$ then $\beta^+ = \alpha^+$ and $\beta \succ s_i\alpha$.*

Proof. Suppose x^β appears in $T_i x^\gamma$ (with $c_\gamma \neq 0$) in one of the five cases of Lemma 3 and $\beta^+ \succ (s_i\alpha)^+ = \alpha^+$. Every term satisfies $\gamma^+ \succ \beta^+$ or $\gamma^+ = \beta^+$ but then $\gamma^+ \succeq \beta^+ \succ \alpha^+$ and $\gamma \rhd \alpha$, a contradiction. Suppose $\beta^+ = \alpha^+$ then $\beta \rhd s_i\alpha$ implies $\beta \succ s_i\alpha$. $\square$

Corollary 1. *If α is $\rhd$-maximal in $p = \sum_\delta c_\delta x^\delta$ and x^β appears in $(T_i + c)p$ with $\beta \unrhd s_i\alpha$ then either $\beta = s_i\alpha$ or $\beta^+ = \alpha^+$ and $\beta \succ s_i\alpha$ with $\beta = s_i\gamma$ where x^γ appears in p.*

Proof. If β occurs in case (1) or case (5) of Lemma 3 and $\beta \neq \gamma, s_i\gamma$ (for x^γ appearing in p) then $\gamma^+ \succ \beta^+ \succ s_i\alpha \succ \alpha$ which violates the $\rhd$-maximality of α, this leaves only $\beta = s_i\gamma$. $\square$

Note $\beta = s_i\gamma$ does not imply $s_i\beta \succ \alpha$, for example let $\beta = (4,1,3,2)$ and $s_1\alpha = (3,2,1,4)$ then $\beta \succ s_1\alpha$ but $s_1\beta = (1,4,3,2)$ and $\alpha = (2,3,1,4)$ are not $\rhd$-comparable.

2.3. Irreducible Representations of the Hecke Algebra

Irreducible representations of $\mathcal{H}_N(t)$ are indexed by partitions of N (for background see Dipper and James [6]). Given a partition $\tau \in \mathbb{N}_0^{N,+}$ with $|\tau| = N$ there is a *Ferrers diagram*: boxes at (i,j) with $1 \leq i \leq \ell(\tau) = \max\{j : \tau_j > 0\}$ and $1 \leq j \leq \tau_i$. The module is spanned by reverse standard Young tableaux (abbr. RSYT) of shape τ (denoted $\mathcal{Y}_\tau$): the numbers $1,\ldots,N$ are inserted into the Ferrers diagram so that the entries in each row and in each column are decreasing. The module $\mathrm{span}_{\mathbb{K}}\{Y : Y \in \mathcal{Y}_\tau\}$ is said to be of isotype τ. If k is in cell (i,j) of RSYT Y (denoted $Y[i,j] = k$) then the *content* $c(k,Y) := j - i$; the *content vector* $[c(k,Y)]_{k=1}^N$ determines Y uniquely. The action of $\mathcal{H}_N(t)$ is specified by the formulas for T_iY:

- If $c(i,Y) - c(i+1,Y) = 1$ then $T_iY = tY$;
- If $c(i,Y) - c(i+1,Y) = -1$ then $T_iY = -Y$;
- If $|c(i,Y) - c(i+1,Y)| \geq 2$ then let $Y^{(i)}$ denote the RSYT obtained by interchanging i and $i+1$ in Y and set $z = t^{c(i+1,Y)-c(i,Y)}$: if $c(i,Y) - c(i+1,Y) \geq 2$, then
$$T_iY = Y^{(i)} - \frac{1-t}{1-z}Y;$$
if $c(i,Y) - c(i+1,Y) \leq -2$, then
$$T_iY = \frac{(1-zt)(t-z)}{(1-z)^2}Y^{(i)} - \frac{1-t}{1-z}Y.$$

From these relations it follows that $\omega_iY = t^{c(i,Y)}Y$ for $1 \leq i \leq N$. Call the vector $\left[t^{c(i,Y)}\right]_{i=1}^N$ the t-exponential content vector of Y, or the t^C-vector for short. Note $c(N,Y) = 0$ always and $\omega_N := 1$.

So if one finds a simultaneous eigenfunction of $\{\omega_i\}$ then the eigenvalues determine an RSYT and the isotype (partition) of an irreducible representation.

2.4. Singular Parameters

For integers m and n such that $m \geq 1$ and $2 \leq n \leq N$ we consider *singular* parameters (q,t) satisfying $q^mt^n = 1$ with the property that if $q^at^b = 1$ then $a = rm, b = rn$ for some $r \in \mathbb{Z}$.

Definition 1. *Let* $g = \gcd(m,n)$ *and let* $z = \exp\left(\frac{2\pi \mathrm{i}k}{m}\right)$ *with* $\gcd(k,g) = 1$*, that is,* $z^{m/g}$ *is a primitive* g^{th} *root of unity. If* $g = 1$ *then set* $z = 1$*. Define* $\varpi := (q,t) = \left(zu^{-n/g}, u^{m/g}\right)$ *where* u *is not a root of unity and* $u \neq 0$*.*

Lemma 5. *If* $q^a t^b|_{\varpi} = 1$ *for some integers* a, b *then* $a = rm, b = rn$ *for some* $r \in \mathbb{Z}$*.*

Proof. By hypothesis $z^a u^{-an/g+bm/g} = 1$ and, since u is not a root of unity, $-a\frac{n}{g} + b\frac{m}{g} = 0$. From $\gcd\left(\frac{n}{g}, \frac{m}{g}\right) = 1$, it follows that $a = p'\frac{m}{g}$ and $b = p'\frac{n}{g}$, for some $p' \in \mathbb{Z}$. Thus, $1 = z^a = \exp\left(\frac{2\pi \mathrm{i}k}{m}\frac{mp'}{g}\right) = \exp\left(\frac{2\pi \mathrm{i}k}{g}p'\right)$. Moreover, since $\gcd(k,g) = 1$, $p' = pg$ with $p \in \mathbb{Z}$. Hence $a = pm$ and $b = pn$. □

In fact, to describe all the possibilities for ϖ, it suffices to let $1 \leq k < g$. To be precise, ϖ is not a single point but a variety in $(\mathbb{C}\backslash\{0\})^2$.

3. Necessary Conditions for Singular Polynomials

By using the degree-lowering (q-Dunkl) operators defined by Baker and Forrester [5] we find another characterization of singular polynomials.

Definition 2. *Suppose* $p \in \mathcal{P}$ *then*

$$\begin{aligned} D_N p(x) &:= \frac{1}{x_N}(1 - \xi_N)p(x), \\ D_i p(x) &:= \frac{1}{t} T_i D_{i+1} T_i p(x),\ i < N. \end{aligned}$$

Proposition 2. *A polynomial* p *is singular if and only if* $D_i p = 0$ *for* $1 \leq i \leq N$*.*

Proof. The proof is by downward induction on i. Since $\omega_N = 1$, it follows that $D_N p = 0$ iff $\xi_N p = p = \omega_N p$. Suppose that $D_i p = 0$ iff $\xi_i p = \omega_i p$ for all p and $k \leq i \leq N$. Then $D_{k-1}p = 0$ iff $t^{-1}T_{k-1}D_k T_{k-1}p = 0$ iff $D_k T_{k-1} p = 0$ iff $\xi_k T_{k-1} p = \omega_k T_{k-1} p$ iff $t^{-1}T_{k-1}\xi_k T_{k-1}p = t^{-1}T_{k-1}\omega_k T_{k-1}p$. □

First we show that any singular polynomial generates an $\mathcal{H}_N(t)$-module consisting of singular polynomials. This allows the use of the representation theory of $\mathcal{H}_N(t)$.

Proposition 3. *Suppose* p *is singular and* $1 \leq i < N$*, then* $T_i p$ *is singular.*

Proof. The commutation relations from Lemmas 1 and 2 are used. Suppose $j < i$ or $j > i+1$ then $\xi_j T_i p = T_i \xi_j p = T_i \omega_j p = \omega_j T_i p$. Case $j = i$:

$$\begin{aligned} \xi_i T_i p &= \{(t-1)\xi_i + T_i \xi_{i+1}\}p = (t-1)\omega_i p + T_i \omega_{i+1} p \\ &= \{(t-1)\omega_i + T_i \omega_{i+1}\}p = \omega_i T_i p. \end{aligned}$$

Case $j = i+1$

$$\begin{aligned} \xi_{i+1} T_i p &= \{T_i \xi_i - (t-1)\xi_i\}p = T_i \omega_i p - (t-1)\omega_i p \\ &= \{T_i \omega_i - (t-1)\omega_i\}p = \omega_{i+1} T_i p. \end{aligned}$$

□

Proposition 4. *Suppose* p *is singular then* $\mathcal{M} = \mathcal{H}_N(t)p$ *is a linear space of singular polynomials, and it is closed under the actions of* ξ_i, ω_i. *for* $1 \leq i \leq N$*, and* w*.*

Proof. By definition of ω_i we see that $f \in \mathcal{M}$ implies $\omega_i f \in \mathcal{M}$, and by definition $\xi_i f = \omega_i f \in \mathcal{M}$. Also

$$\begin{aligned}\xi_1 p &= T_1 T_2 \cdots T_{N-1} w p \\ &= \omega_1 p = t^{1-N} T_1 T_2 \ldots \cdots T_{N-1} T_{N-1} T_{N-2} \cdots T_1 p\end{aligned}$$

thus $wp = t^{1-N} T_{N-1} T_{N-2} \cdots T_1 p$. $\square$

Note that $\mathcal{M}$ is also a module of the affine Hecke algebra. By the representation theory of $\mathcal{H}_N(t)$ the module has a basis of $\{\omega_i\}$-simultaneous eigenfunctions and by definition these are $\{\xi_i\}$-simultaneous eigenfunctions - note we are not claiming they are specializations of nonsymmetric Macdonald polynomials at ϖ. Suppose f is such an eigenfunction and let α be $\rhd$-maximal in the expression $f(x) = \sum_\beta c_\beta x^\beta$. Then $\xi_i f = q^{\alpha_i} t^{N-r_\alpha(i)} f$ because by the triangularity property of ξ_i (see (1)) x^α can only appear in $\xi_i f$ in the term $\xi_i x^\alpha$. Furthermore $\xi_i f = \omega_i f$ implies $q^{\alpha_i} t^{N-r_\alpha(i)} = t^{c(i,Y)}$ for some RSYT Y, at ϖ. As well we can conclude $\alpha_i = mr, N - r_\alpha(i) - c(i,Y) = nr$ for some $r \in \mathbb{N}$ (Lemma 5). The next step is to produce a simultaneous eigenfunction which has a $\rhd$-maximal term x^λ with $\lambda \in \mathbb{N}_0^{N,+}$.

Proposition 5. *There exists $f \in \mathcal{M}$ which is a simultaneous $\{\omega_i\}$-eigenfunction and $f = c_\lambda x^\lambda + \sum_{\beta \lhd \lambda} c_\beta x^\beta + \sum_\gamma c_\gamma x^\gamma$ where γ is not $\rhd$-comparable to λ, and $\lambda \in \mathbb{N}_0^{N,+}$.*

Proof. Suppose $f = \sum c_\alpha x^\alpha$ is an eigenfunction and there is a $\rhd$-maximal α with x^α (i.e., $c_\alpha \neq 0$) appearing in f, and $\alpha_i < \alpha_{i+1}$ then $T_i f \neq f$ and the coefficient of $x^{s_i \alpha}$ is $t c_\alpha$; let $\omega_j f = \mu_j f$ for $1 \le j \le N$ and $\mu_{i+1} \neq \mu_i$ (because $c(i,Y) \neq c(i+1,Y)$ for any RSYT) so that

$$g := T_i f + \frac{t-1}{\mu_{i+1}/\mu_i - 1} f$$

is a simultaneous eigenfunction with $\rhd$-maximal β such that $\beta^+ = \alpha^+$ and $\beta \succeq s_i \alpha$, (by Proposition 1) and eigenvalues $\ldots \mu_{i+1}, \mu_i \ldots$In general this formula could produce a zero function g but this does not happen here because the coefficient of $x^{s_i \alpha}$ in g is not zero. Repeating these steps eventually produces a $\rhd$-maximal term x^λ with $\lambda \in \mathbb{N}_0^{N,+}$ (at most $\operatorname{inv}(\alpha)$ steps). $\square$

At this point we have shown if there is a singular polynomial then there is a partition $\lambda \in \mathbb{N}_0^{N,+}$ and an RSYT Y such that $q^{\lambda_i} t^{N-i} = t^{c(i,Y)}$ at ϖ, for $1 \le i \le N$. Next we determine necessary conditions on λ for the existence of Y, in other words, when $\left[q^{\lambda_i} t^{N-i}\right]_{i=1}^N$ at ϖ is a valid t^C-vector. The equations $\lambda_i = mr_i, N - i - c(i,Y) = nr_i$ for $1 \le i \le N$ show that λ can be replaced by $\frac{1}{m}\lambda$ and ϖ by $qt^n = 1$ (simply $q = t^{-n}$), also $n\lambda_i = N - i - c(i,Y)$.

The following is a restatement of the development in [2] with significant differences in notation. First there is an informal discussion of the beginning of the process of building Y by placing $N, N-1, N-2, \ldots$ in possible locations and determining $\lambda_N, \lambda_{N-1}, \lambda_{N-2}, \ldots$accordingly. Abbreviate $c_i = c(i,Y)$.

Suppose λ_{N-k} is the last nonzero entry of λ ($\lambda_i = 0$ for $i > N-k$) then $k - c_{N-k} = n\lambda_{N-k}$ ($c_{N-j} = j$ for $0 \le j < k$ implies $Y[1,j] = N - j - 1$); the entry $N-k$ in Y is at $[1,k+1]$ or $[2,1]$ thus $c_{N-k} = k, \lambda_{N-k} = 0$ (contra) or $c_{N-k} = -1, n\lambda_{N-k} = k+1$. Set $\lambda_{N-k} = d_1$ and $k = nd_1 - 1$.The entry $N-k-1$ in Y is in one of $[3,1], [2,2], [1,k+1]$ with contents $-2, 0, k$, respectively, yielding the equations $n\lambda_{N-k-1} = k + 1 - c_{N-k-1} = k-1, k+1, 1 = nd_1 - 2, nd_1, 1$, respectively. If $n > 2$ then only $[2,2]$ is possible and $\lambda_{N-k-1} = d_1$. If $n = 2$ then $[3,1], \lambda_{N-k-1} = d_1 + 1$ and $[2,2], \lambda_{N-k-1} = d_1$ are possible.

Theorem 1. *There are numbers $d_1 \ge d_2 \ge \ldots \ge d_L \ge 1$ such that with $\gamma_s := \sum_{i=1}^{s-1} d_i$ and $0 \le r_{L+1} < N - n\gamma_{L+1} + L \le nd_L - 1$ the entries in row s of Y are $R_s := \{i : n\gamma_s - s + 1 \le$*

$N-i \le n\gamma_{s+1}-s-1\}$ for $1 \le s \le L$, $R_{L+1}=\{i : n\gamma_{L+1}-L \le N-i \le N-1\}$ and $\lambda_i=\gamma_s$ for $i \in R_s$. The isotype of Y is $\tau := (nd_1-1, nd_2-1, \ldots, nd_L-1, r_{L+1})$.

Proof. By way of induction suppose there are numbers $d_1 \ge d_2 \ge \ldots \ge d_{k-1} > 0$ such that the entries in row s of Y are $R_s=\{i : n\gamma_s-s+1 \le N-i \le n\gamma_{s+1}-s-1\}$ and $\lambda_i=\gamma_s$ for $i \in R_s$. Assume this has been proven for $1 \le s < k$ and for row k up to $n\gamma_k-k+1 \le N-i \le n\gamma_k-k+\ell$ with $\ell \le nd_{k-1}-1$ (the length $\#R_{k-1}$ of row $k-1$). Consider the possible locations for the next entry $p=N-(n\gamma_k-k+\ell+1)$. The possible boxes are (1) $[s, nd_s]$ ($s<k$ and $d_s<d_{s-1}$ or $s=1$), (2) $[k,\ell+1]$, (3) $[k+1,1]$ with contents $nd_s-s, \ell+1-k, -k$, respectively. The equations

$$\begin{aligned} n\lambda_p &= N-p-c_p = n\gamma_k-k+\ell+1-c_p \\ n\left(\lambda_p-\gamma_k\right) &= -k+\ell+1-c_p \end{aligned}$$

must hold;
case (1): (note $\ell+1 \le nd_{k-1}$)

$$\begin{aligned} n\left(\lambda_p-\gamma_k\right) &= -k+\ell+1-nd_s+s \\ n\left(\lambda_p-\gamma_k+d_s\right) &= -k+s+1+\ell \le -k+s+nd_{k-1} \\ n\left(\lambda_p-\gamma_k+d_s-d_{k-1}\right) &\le s-k<0 \end{aligned}$$

$\lambda_p \ge \gamma_k=\lambda_{p+1}$ and $d_s \ge d_{k-1}$ by inductive hypothesis, so the left side ≥ 0 and there is a contradiction.
case (2):

$$\begin{aligned} n\left(\lambda_p-\gamma_k\right) &= -k+\ell+1-(\ell+1-k)=0 \\ \lambda_p &= \gamma_k \end{aligned}$$

and the inductive hypothesis is proved for $n\gamma_k-k+1 \le N-i \le n\gamma_k-k+\ell+1$, entries in row k.
case (3)

$$n\left(\lambda_p-\gamma_k\right) = -k+\ell+1+k = \ell+1$$

set $\ell=nd_k-1$ and $\gamma_{k+1}=\gamma_k+d_k, \lambda_p=\gamma_{k+1}$. The inductive step has been proven for k and for $k+1$ with $Y[k+1,1]=N-n\gamma_{k+1}+k$. By induction this uses up all the entries. Let row $L+1$ be the last row of Y and of length r_{L+1}, then $N=\sum_{i=1}^{L}(nd_i-1)+r_{L+1}$ and $r_{L+1} \le nd_L-1$. $\square$

Corollary 2. *Suppose $\varpi=(q,t)$ as in Definition 1 and p is singular. Then $\mathcal{H}_N(t)p$ contains a $\{\omega_i,\xi_i\}$ simultaneous eigenfunction $f=c_\lambda x^\lambda+\sum_{\beta\lhd\lambda} c_\beta x^\beta+\sum_\gamma c_\gamma x^\gamma$ with γ not $\rhd$-comparable to λ so that $\lambda_i=m\gamma_s$ if $i \in R_s$, in the notation of the Theorem.*

We have shown if α is $\rhd$-maximal in a simultaneous $\{\omega_i,\xi_i\}$ eigenfunction then there is an eigenfunction in which α^+ is $\rhd$-maximal. Now the eigenvalues are determined by Y and it follows that $\alpha^+=\lambda$ as constructed above. Hence each term x^γ in an eigenfunction satisfies $\gamma \unlhd \lambda$. (Suppose at some stage γ is $\rhd$-maximal then there is a simultaneous eigenfunction with γ^+ being $\rhd$-maximal and the construction produces an RSYT of the same isotype τ and the numbers $N, N-1, \ldots$ are entered row-by-row forcing $\gamma^+=\lambda$.)

Theorem 2 ([2])**.** *In the notation of Theorem 1 if $d_i=1$ for $i \ge 2$ then $M_\lambda(x)$ specialized to ϖ has no poles and is singular. The module $\mathcal{H}_N(t)M_\lambda$ is spanned by $M_{\alpha(Y)}$ where $Y \in \mathcal{Y}_\tau$, $\tau=\left(nd_1-1,(n-1)^{L-1},r_{L+1}\right)$ and $\alpha(Y)_i=m(d_1+s-2)$ if $Y[s,k]=i$ for $s \ge 2$ and some k, otherwise ($Y[1,k]=i$) $\alpha(Y)_i=0$.*

The Ferrers diagram of λ (from Theorem 1) is called a quasi-staircase, the shape suggested when French notation with row 1 on the bottom is used.

We have reached the main purpose of this paper: to show there are no other singular polynomials.

4. Restrictions

In this section, we show that the desired nonexistence result can be reduced to the simpler two-row situation.

Suppose $\alpha \in \mathbb{N}_0^N$ and $r_\alpha(1) = 1$ (that is, $\alpha_i \leq \alpha_1$ for all i). Let $\alpha' = (\alpha_2, \ldots, \alpha_N)$ and $Y' = Y\backslash\{1\}$ (the RSYT where the entry 1 is deleted) and f satisfies $\xi_i f = q^{\alpha_i} t^{N-r_\alpha(i)} f$, at ϖ. First we will show that $f_{\alpha'} := \mathrm{coeff}(x_1^{\alpha_1}, f)$ is an eigenfunction of ξ_i' with eigenvalue $q^{\alpha_i} t^{N-r_\alpha(i)}$ for $2 \leq i \leq N$ where

$$\begin{aligned} w'p(x) &:= p(qx_N, x_2, x_3, \ldots, x_{N-1}), \\ \xi_i' p(x) &:= t^{i-2} T_i T_{i+1} \cdots T_{N-1} w' T_2^{-1} \cdots T_{i-1}^{-1} p(x) \end{aligned}$$

Lemma 6. *Let $f = x_1^{\alpha_1} x_2^{\alpha_2} p(x_3, \ldots, x_N)$ with $\alpha_1 \geq \alpha_2$ then*

$$\mathrm{coeff}\left(x_1^{\alpha_1}, wT_1^{-1} f\right) = t^{-1} w' \mathrm{coeff}\left(x_1^{\alpha_1}, f\right).$$

Proof. By definition

$$\begin{aligned} T_1^{-1} f &= \frac{1-t}{t} x_1 \frac{f(x) - f(xs_1)}{x_1 - x_2} + t^{-1} f(xs_1) \\ &= \frac{1-t}{t} x_1^{1+\alpha_2} x_2^{\alpha_2} \frac{x_1^{\alpha_1-\alpha_2} - x_2^{\alpha_1-\alpha_2}}{x_1 - x_2} p + t^{-1} x_1^{\alpha_2} x_2^{\alpha_1} p(x_3, \ldots, x_N) \\ &= \frac{1-t}{t} \sum_{i=0}^{\alpha_1-\alpha_2-1} x_1^{\alpha_1-i} x_2^{\alpha_2+i} p + t^{-1} x_1^{\alpha_2} x_2^{\alpha_1} p(x_3, \ldots, x_N) \end{aligned}$$

then

$$\begin{aligned} wT_1^{-1} f = \frac{1-t}{t} &\sum_{i=0}^{\alpha_1-\alpha_2-1} (qx_N)^{\alpha_1-i} x_1^{\alpha_2+i} p(x_2, x_3, \ldots, x_{N-1}) \\ &+ x_1^{\alpha_1} (qx_N)^{\alpha_2} t^{-1} p(x_2, x_3, \ldots, x_{N-1}). \end{aligned}$$

The highest power of x_1 in the first term is $\alpha_1 - 1$ thus

$$\mathrm{coeff}\left(x_1^n, wT_1^{-1} f\right) = (qx_N)^{\alpha_2} t^{-1} p(x_2, x_3, \ldots, x_{N-1})$$

and the right hand side is $t^{-1} w' x_2^{\alpha_2} p(x_3, \ldots, x_N)$. □

Let $\pi_n f := \mathrm{coeff}(x_1^n, f)$.

Theorem 3. *Suppose $f = \sum_\alpha c_\alpha x^\alpha$ with $\max_i \alpha_i = n$ then $\pi_n \xi_i f = \xi_i' \pi_n f$ for $2 \leq i \leq N$.*

Proof. Let $i > 1$ then

$$\begin{aligned}\pi_n \xi_i f &= t^{i-1}\pi_n T_i T_{i+1}\cdots T_{N-1} w T_1^{-1} T_2^{-1}\cdots T_{i-1}^{-1} f(x)\\ &= t^{i-1} T_i T_{i+1}\cdots T_{N-1}\pi_n w T_1^{-1} T_2^{-1}\cdots T_{i-1}^{-1} f(x)\\ &= t^{i-2} T_i T_{i+1}\cdots T_{N-1} w' \pi_n T_2^{-1}\cdots T_{i-1}^{-1} f(x)\\ &= t^{i-2} T_i T_{i+1}\cdots T_{N-1} w' T_2^{-1}\cdots T_{i-1}^{-1}\pi_n f(x)\\ &= \xi_i' \pi_n f;\end{aligned}$$

this uses the Lemma and the fact that $\xi_i f$ and $T_2^{-1}\cdots T_{i-1}^{-1} f$ are sums of monomials x^β with $\beta_j \leq n$ for $j \geq 1$ (properties of the order $\rhd$ and of T_j^{-1}). If $i = 2$ then the empty product $T_2^{-1}\cdots T_{i-1}^{-1}$ reduces to 1. $\square$

Suppose $\alpha, \beta \in \mathbb{N}_0^{N-1}$ (indexed $2 \leq i \leq N$) and $|\alpha| = |\beta|$, set $\alpha' = (n, \alpha), \beta' := (n, \beta)$ (so that $|\alpha'| = |\beta'|$).

Lemma 7. *Suppose* $\max_i \alpha_i \leq n$ *and* $\max_i \beta_i \leq n$ *then* $\alpha'^+ = (n, \alpha^+), \beta'^+ = (n, \beta^+)$ *and* $\alpha' \succ \beta'$ *iff* $\alpha \succ \beta$, $\alpha' \rhd \beta'$ *iff* $\alpha \rhd \beta$.

Proof. By hypothesis $(\alpha'^+)_1 = n$ and $\alpha'^+ = (n, \alpha^+)$, similarly $\beta'^+ = (n, \beta^+)$. Furthermore

$$\begin{aligned}\alpha' \succ \beta' &\iff n + \sum_{j=2}^{i} \alpha_j \geq n + \sum_{j=2}^{i} \beta_j \ \forall i \geq 2\\ &\iff \alpha \succ \beta\end{aligned}$$

Then

$$\begin{aligned}\alpha \rhd \beta &\iff \left(\alpha^+ \succ \beta^+\right) \vee \left(\alpha^+ = \beta^+ \wedge \alpha \succ \beta\right)\\ \alpha' \rhd \beta' &\iff \left(\alpha'^+ \succ \beta'^+\right) \vee \left(\alpha'^+ = \beta'^+ \wedge \alpha' \succ \beta'\right)\end{aligned}$$

and $\alpha \rhd \beta \iff \alpha' \rhd \beta'$. $\square$

Proposition 6. *Let f be the $\{\omega_i, \xi_i\}$ simultaneous eigenfunction from Corollary 2 with eigenvalues $q^{\lambda_i} t^{N-i} = t^{c(i,Y)}$ at $q^m t^n = 1$ for $1 \leq i \leq N$ and $\lambda_2 > 0$. Then $\pi_{\lambda_1} f$ is a nonzero $\{\omega_i, \xi_i' : i \geq 2\}$ simultaneous eigenfunction with the same eigenvalues as f for $i \geq 2$ with $c(i, Y) = c(i, Y\backslash\{1\})$. Here $Y\backslash\{1\}$ is the RSYT obtained by removing the box containing 1 from Y.*

Proof. We showed that each term x^α appearing in f satisfies $\lambda \unrhd \alpha$ and $\alpha_1 \leq \lambda_1$ for all i. Apply π_{λ_1} to f then by Lemma 7 $\beta \unlhd (\lambda_2, \lambda_3, \ldots, \lambda_N)$ for each x^β appearing in $\pi_{\lambda_1} f$. For $i \geq 2$ ω_i commutes with π_{λ_1} and by Theorem 3 $\pi_{\lambda_1}\xi_i f = \xi_i' \pi_{\lambda_1} f$. Thus $\omega_i \pi_{\lambda_1} f = \xi_i' \pi_{\lambda_1} f$ for $i \geq 2$. Furthermore, $(\lambda_2, \lambda_3, \ldots, \lambda_N) \in \mathbb{N}_0^{N-1,+}$ is $\rhd$-maximal in $\pi_{\lambda_1} f$. $\square$

The definition of RSYT has been slightly modified to allow filling with $2, 3, \ldots, N$. The isotype of $\pi_{\lambda_1} f$ is $\tau' := (nd_1 - 1, nd_2 - 1, \ldots, nd_L - 1, r_{L+1} - 1)$.

Theorem 4. *In the notation of Theorem 1 if $d_2 \geq 2$ then there is a singular polynomial for the parameter ϖ in $n(d_1 + 1) - 1$ variables with $\lambda = \left((md_1)^n, 0^{nd_1 - 1}\right)$, of isotype $(nd_1 - 1, n)$.*

Proof. Apply Proposition 6 repeatedly, and by hypothesis $nd_2 - 1 \geq 2n - 1 > n$. The remaining RSYT is

$$Y' = \begin{bmatrix} N & N-1 & \ldots & \ldots & \ldots & N - nd_1 + 2 \\ N - nd_1 + 1 & \ldots & N - nd_1 - n + 2 & & & \end{bmatrix},$$

and has the t^C-vector $\left[t^{n-2}, t^{n-3}, \ldots, 1, t^{-1}, t^{nd_1-2}, t^{nd_1-3}, \ldots, t, 1\right]$. $\square$

5. Concluding Argument

Re-index the variables by replacing $d_1 \geq 2$ (implied by $d_2 \geq 2$) by d, N by $N = nd - 1 + n$ and

$$\Upsilon'' = \begin{bmatrix} nd-1+n & nd-2+n & \ldots & \ldots & \ldots & n+1 \\ n & & \ldots & 1 & & \end{bmatrix}.$$

Proposition 7. *Suppose $\lambda = \left(d^n, 0^{nd-1}\right)$ and $\gamma \in \mathbb{N}_0^K$ for some $K \geq N$ satisfies $|\gamma| = nd$ and $C_i : n(\lambda_i - \gamma_i) = r_\gamma(i) - i$ for $1 \leq i \leq K$ (setting $\lambda_i = 0$ for $i > N$) then $\gamma = \lambda$ or $\gamma = \beta := \left(0^n, 1^{nd}\right)$.*

Proof. By condition C_{n+1} we have $(r_\gamma(n+1) - n - 1) = -n\gamma_{n+1}$ so that $\gamma_{n+1} = 1 - \frac{1}{n}(r_\gamma(n+1) - 1) \leq 1$ and thus $\gamma_{n+1} = 1$ or $\gamma_{n+1} = 0$. If $\gamma_{n+1} = 1$ then $r_\gamma(n+1) = 1$, which implies $\gamma_i = 0$ for $1 \leq i \leq n$ and $\gamma_i \leq 1$ for $i > n+1$. If $j > n$ and $\gamma_j = 0$ then by C_j $r_\gamma(j) = j = \#\{k <= j : \gamma_k \geq 0\} + \#\{k > j : \gamma_k > 0\}$ so that $k > j$ implies $\gamma_k = 0$. Since $|\gamma| = |\lambda| = nd$ we see that $\gamma_{n+1} = 1$ implies $\gamma^+ = \left(1^{nd}\right)$ and in fact $\gamma_i = 1$ for $n+1 \leq i \leq n(d+1)$, since $\gamma_j = 0$ and $\gamma_{j+1} = 1$ is impossible for any $j > n$. If $1 \leq j \leq n$ then $r_\gamma(j) = nd + j$ and $n(\lambda_i - \gamma_i) = nd = r_\gamma(j) - j$, thus satisfying C_j. The other conditions C_i are verified similarly. Thus, $\gamma = \beta$.

If $\gamma_{n+1} = 0$ then $r_\gamma(n+1) = n+1$ and $\ell(\gamma) = n$. Suppose $1 \leq j \leq n$ then C_j states $n(\lambda_j - \gamma_j) = r_\gamma(j) - j$ and the bounds $1 \leq j, r_\gamma(j) \leq n$ imply $|r_\gamma(j) - j| \leq n-1$ and thus $\gamma_j = \lambda_j$. $\square$

Corollary 3. *Suppose $\lambda = \left((md)^n, 0^{nd-1}\right) \in \mathbb{N}_0^{N,+}$. The coefficients of $M_\lambda(x)$ have no poles at ϖ.*

Proof. $M_\lambda(x)$ is a nonzero multiple of $x^\lambda + \sum_{\beta \lhd \lambda} A_{\lambda,\beta} x^\beta$. For each $\beta \lhd \lambda$ there is at least one index j_β such that $\zeta_\lambda(i_\beta) \neq \zeta_\beta(i_\beta)$ at ϖ or else $q^{\lambda_i - \beta_i} t^{r_\beta(i) - i} = 1$ for all $i \leq N$. In this case by Lemma 5 $(\lambda_i - \beta_i) = ms_i, r_\beta(i) - i = ns_i$ for some $s_i \in \mathbb{Z}$. Set $\lambda' = \frac{1}{m}\lambda, \beta' = \frac{1}{m}\beta$ then $n(\lambda'_i - \beta'_i) = r_\beta(i) - i$ for all i and by the Proposition $\beta' = \lambda'$ or $\beta' = \left(0^n, 1^{nd}\right)$ but the latter is impossible because $\left(0^n, 1^{nd}\right) \notin \mathbb{N}_0^N$. Finally (this works because there is a triangular expansion $x^\lambda = cM_\lambda + \sum\limits_{\beta \lhd \lambda} A'_{\beta,\lambda} M_\beta$ which holds for generic (q,t))

$$M_\lambda(x) = c \prod_{\beta \lhd \lambda} \frac{\xi_{i_\beta} - \zeta_\beta(i_\beta)}{\zeta_\lambda(i_\beta) - \zeta_\beta(i_\beta)} x^\lambda.$$

This shows that the poles of M_λ are of the form $q^a t^b - 1 = 0$ and ϖ is not a pole. $\square$

Proposition 8. *Suppose f is as in Theorem 4 then $f(x) = cM_\lambda(x)$ at ϖ for some constant $c \neq 0$.*

Proof. By matching coefficients of x^λ find c so that $\mathrm{coeff}\left(x^\lambda, f - cM_\lambda\right) = 0$. If $g := f - cM_\lambda \neq 0$ then there exists β such that x^β is $\rhd$-maximal in g. By $\rhd$-triangularity $\xi_i g = q^{\beta_i} t^{N - r_\beta(i)} g$ (at ϖ) for all i. However, g has the same eigenvalues as M_λ, that is, $q^{\beta_i} t^{N - r_\beta(i)} = q^{\lambda_i} t^{N-i}$ at ϖ and the proof of the Corollary showed that $\beta = \lambda$, contradicting $g \neq 0$. $\square$

Recall the transformation Formula (3) for M_α for $\alpha_i > \alpha_{i+1}$ with $z = \frac{\zeta_\alpha(i+1)}{\zeta_\alpha(z)}$

$$M_{s_i\alpha} = \frac{(1-z)^2}{(1-zt)(t-z)}\left(T_i + \frac{1-t}{1-z}\right)M_\alpha.$$

If M_α has no pole at ϖ and $z \neq 1, t, t^{-1}$ then $M_{s_i\alpha}$ has no pole at ϖ. When $\alpha^+ = \lambda$ then $\alpha_i > \alpha_{i+1}$ implies $\alpha_i = md$ and $\alpha_{i+1} = 0, z = q^{-md}t^{r_\alpha(i)-r_\alpha(i+1)} = t^{nd+r_\alpha(i)-r_\alpha(i+1)}$ at ϖ. In the substring $(\alpha_1, \dots, \alpha_i, \alpha_{i+1})$ there are $r_\alpha(i)$ values md and $i+1-r_\alpha(i)$ zeros, thus $r_\alpha(i+1) = n+i+1-r_\alpha(i)$. Thus, $z = t^b$ with $b = nd+2r_\alpha(i)-n-i-1$. Suppose $r_\alpha(i) = n$, thus $i \geq n$ and s_i can act on α without introducing a pole at ϖ if $nd+n-i-1 > 1$, that is $i < nd+n-2 = N-1$. The last permitted occurrence of md in α is $i = N-2$. Next move the second last occurrence of md in α as far as possible without a pole: set $r_\alpha(i) = n-1$ and require $nd+2(n-1)-n-i-1 > 1$, that is, $i < nd+n-4 = N-3$, thus $i = N-4$ is the last permitted value. More generally let $r_\alpha(i) = n-j$ (with $0 \leq j \leq n-1$) then require $nd+2(n-j)-n-i-1 > 1$, that is, $nd+n-2j-2 > i$ or $i < N-1-2j$; the last permitted value is $i = N-2(j+1)$.

Let

$$\begin{aligned}\alpha &= \left(0^{nd-n-1}, md, 0, md, 0\dots, md, 0\right)\\ \zeta_\alpha &= \left[t^{N-n-1}, \dots, t^n, q^{md}t^{N-1}, t^{n-1}, \dots, q^{md}t^{N-n}, 1\right].\end{aligned}$$

We showed that M_α has no poles at ϖ, and if M_λ at ϖ is singular then so is M_α. The spectral vector ζ_α at ϖ coincides with the t^C-vector of the RSYT

$$Y_0 = \begin{bmatrix} N & N-2 & \cdots & N-2n+2 & N-2n & \cdots & 1 \\ N-1 & N-3 & \cdots & N-2n+1 & & & \end{bmatrix},$$

and thus $\omega_{N-1}Y_0 = t^{-1}Y_0$; by construction $\zeta_\alpha(N-1) = q^{md}t^{N-n} = t^{-nd+N-n} = t^{-1}$. If M_α at ϖ is singular then $\omega_{N-1}M_\alpha = \xi_{N-1}M_\alpha = t^{-1}M_\alpha$; this means

$$\begin{aligned}t^{-1}T_{N-1}T_{N-1}M_\alpha &= t^{-1}M_\alpha\\ ((t-1)T_{N-1}+t)M_\alpha &= M_\alpha\\ (t-1)T_{N-1}M_\alpha &= (1-t)M_\alpha\\ (T_{N-1}+1)M_\alpha &= 0.\end{aligned}$$

For the next step we recall some standard definitions: the q-Pochhammer symbol is $(a;q)_k = \prod_{i=1}^{k}\left(1-aq^{i-1}\right)$ and the generalized (q,t)-Pochhammer symbol for $\lambda \in \mathbb{N}_0^{N,+}$ is

$$(v;q,t) = \prod_{i=1}^{N}\left(vt^{1-i};q\right)_{\lambda_i}.$$

In the context of the Ferrers diagram representation of a composition $\alpha \in \mathbb{N}_0^N$, $\{(i,j) : 1 \leq i \leq N, 1 \leq j \leq \alpha_i\}$ (the rows with $\alpha_i = 0$ are empty) define the arm-length and leg-length of a box in the diagram ($\lambda \in \mathbb{N}_0^{N,+}$)

$$\begin{aligned}\mathrm{arm}(i,j;\lambda) &:= \lambda_i - j,\\ \mathrm{arm}(i,j;\alpha) &:= \alpha_i - j,\\ \mathrm{leg}(i,j;\lambda) &:= \#\{l : i < l \leq N, j \leq \lambda_l\},\end{aligned}$$

$$\mathrm{leg}(i,j;\alpha) := \#\{r : r > i, j \leq \alpha_r \leq \alpha_i\} + \#\{r : r < i, j \leq \alpha_r + 1 \leq \alpha_i\}.$$

The (q,t)-hook product is

$$h_{q,t}(v;\alpha)=\prod_{(i,j)\in\alpha}\left(1-vq^{\operatorname{arm}(i,j;\alpha)}t^{\operatorname{leg}(i,j;\alpha)}\right).$$

There is an evaluation at a special point (see [Cor. 7] [7]): let $x^{(0)}:=\left(1,t,t^2,\ldots,t^{N-1}\right)$, then for any $\beta\in\mathbb{N}_0^N$

$$M_\beta\left(x^{(0)}\right)=q^{b(\beta)}t^{e'(\beta^+)}\frac{\left(qt^N;q,t\right)_{\beta^+}}{h_{q,t}(qt;\beta)},$$

where $b(\beta)=\sum_{i=1}^N\binom{\beta_i}{2}, e'(\beta^+)=\sum_{i=1}^N\beta_i^+(N-i)$.

Theorem 5. *$(T_{N-1}+1)M_\alpha\neq 0$ at ϖ and M_α is not singular.*

Proof. For any polynomial p let $x=x^{(0)}$ in $T_ip(x)=(1-t)x_{i+1}\frac{p(x)-p(xs_i)}{x_i-x_{i+1}}+tp(xs_i)$ then $T_ip\left(x^{(0)}\right)=t\left(p\left(x^{(0)}\right)-p\left(x^{(0)}s_i\right)\right)+tp\left(x^{(0)}s_i\right)=tp\left(x^{(0)}\right)$ (since $x_{i+1}^{(0)}=tx_i^{(0)}$). Set $b_0=b(\alpha)=n\binom{md}{2}$, $e_0=e'(\alpha^+)=\frac{1}{2}mdn(2N-n-1)$ then

$$\begin{aligned}T_{N-1}M_\alpha\left(x^{(0)}\right)+M_\alpha\left(x^{(0)}\right)&=(t+1)M_\alpha\left(x^{(0)}\right)\\&=q^{b_0}t^{e_0}(t+1)\frac{\left(q^Nt;q,t\right)_{\alpha^+}}{h_{q,t}(qt;\alpha)}.\end{aligned}$$

The numerator is

$$\left(q^Nt;q,t\right)_{\alpha^+}=\prod_{i=1}^n\left(qt^{N-i+1};q\right)_{md}=\prod_{i=1}^n\prod_{j=1}^{dm}\left(1-q^jt^{nd+n-i}\right),$$

where the only term vanishing at ϖ is for $i=n,j=dm$ (for suppose $j=rm$ with $r\leq d, nd+n-i=rn$ for some $r\in\mathbb{N}$ then $n\geq i=n(d-r+1)$ and $d-r+1\leq 1$, that is, $r\geq d$, hence $r=d,i=n$). For the hook product observe that if $1\leq j\leq n$ then $\operatorname{leg}(\alpha;N-2j+1,1)=nd-2$ because there are $nd-1-j$ zero values in $\left(\alpha_1,\ldots,\alpha_{N-2j+1}\right)$ and $j-1$ values of md in $\left(\alpha_{N-2j+2},\ldots,\alpha_N\right)$. Since $\operatorname{arm}(\alpha;N-2j+1,1)=dm-1$ we find that the boxes $\{[N-2j+1,1]:1\leq j\leq n\}$ contribute $\left(1-q^{dm}t^{nd-1}\right)^n$ to $h_{q,t}(qt;\alpha)$. This term becomes $\left(1-t^{-1}\right)^n$ at ϖ. The other boxes in the diagram of α are $\{[N-2j+1,k]:1\leq j\leq n, 2\leq k\leq md\}$ and $\operatorname{leg}(\alpha;N-2j+1,k)=j-1$, $\operatorname{arm}(\alpha;N-2j+1,k)=dm-k$. Thus

$$\begin{aligned}h_{q,t}(qt;\alpha)&=\left(1-q^{dm}t^{nd-1}\right)^n\prod_{j=1}^n\prod_{k=1}^{dm}\left(1-q^{dm-k+1}t^j\right)\\&=\left(1-q^{dm}t^{nd-1}\right)^n\prod_{j=1}^n\prod_{i=1}^{dm}\left(1-q^it^j\right).\end{aligned}$$

The only term in the product vanishing at ϖ is for $i=m,j=n$. Thus, the term $(1-q^mt^n)$ cancels out in $\dfrac{\left(q^Nt;q,t\right)_{\alpha^+}}{h_{q,t}(qt;\alpha)}$ and $(T_{N-1}+1)M_\alpha\left(x^{(0)}\right)\neq 0$. □

Example 1. *Let $N=5$, $n=2$, $m=1$, $d=2$ then $\alpha=(0,2,0,2,0)$ and $\varpi=\left(t^{-2},t\right)$ (that is, $qt^2=1$) The spectral vector of α is $\left[t^2,q^2t^4,t,q^2t^3,1\right]$ which equals $\left[t^2,1,t,t^{-1},1\right]$ at $q=t^{-2}$.*

The expression for M_α is too large to display here (32 monomials); the denominators of the coefficients are factors of $qt-1$, $\left(q^2t^3-1\right)^2$ and

$$M_\alpha\left(1,t,t^2,t^3,t^4\right)=q^2t^{14}\frac{\left(qt^2+1\right)\left(qt^4-1\right)\left(qt^5-1\right)\left(q^2t^5-1\right)}{\left(q^2t^3-1\right)^2(qt-1)}$$

which does not vanish at $q=t^{-2}$. However, the same polynomial is singular with $n=4$, $d=1$, $m=2$ and $q=-t^{-2}$ (that is, $q^2t^4=1$ but $qt^2\neq 1$). The singularity can be proven by direct computation and the vanishing of $M_\alpha\left(1,\ldots,t^4\right)$ is only a necessary condition.

We have shown if there is a singular polynomial as described in Theorem 1 and $d_2\geq 2$ then by using the restriction Proposition 6 repeatedly there is a singular polynomial of isotype (nd_1-1,n), which in turn implies that M_α is singular. This is impossible and we conclude that $d_2=1$ is necessary, and all singular polynomials have been determined.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Cherednik, I. Nonsymmetric Macdonald polynomials. *Int. Math. Res. Not.* **1995**, *1995*, 483–515. [CrossRef]
2. Colmenarejo, L.; Dunkl, C.F. Singular nonsymmetric Macdonald polynomials and quasistaircases. *SIGMA Symmetry Integr. Geom. Methods Appl.* **2020**, *16*, 010. [CrossRef]
3. Kirillov, A.N.; Noumi, M. Affine Hecke algebras and raising operators for Macdonald polynomials. *Duke Math. J.* **1998**, *93*, 1–39. [CrossRef]
4. Mimachi, K.; Noumi, M. A reproducing kernel for nonsymmetric Macdonald polynomials. *Duke Math. J.* **1998**, *91*, 621–634. [CrossRef]
5. Baker, T.H.; Forrester, P.J. A q-analogue of the type A Dunkl operator and integral kernel. *Int. Math. Res. Not.* **1997**, *14*, 667–686. [CrossRef]
6. Dipper, R.; James, G. Representations of Hecke algebras of general linear groups. *Proc. Lond. Math. Soc.* **1986**, *52*, 2–52. [CrossRef]
7. Dunkl, C.F.; Luque, J.-G. Clustering properties of rectangular Macdonald polynomials. *Annales de l'Institut Henri Poincaré D* **2015**, *2*, 263–307. [CrossRef]