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Applications of Confluent Hypergeometric Function in Strong Superordination Theory

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Abstract: In the research presented in this paper, confluent hypergeometric function is embedded in the theory of strong differential superordinations. In order to proceed with the study, the form of the confluent hypergeometric function is adapted taking into consideration certain classes of analytic functions depending on an extra parameter previously introduced related to the theory of strong differential subordination and superordination. Operators previously defined using confluent hypergeometric function, namely Kummer–Bernardi and Kummer–Libera integral operators, are also adapted to those classes and strong differential superordinations are obtained for which they are the best subordinants. Similar results are obtained regarding the derivatives of the operators. The examples presented at the end of the study are proof of the applicability of the original results.

Keywords: analytic function; starlike function; convex function; strong differential superordination; best subordinant; confluent (Kummer) hypergeometric function

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1. Introduction

The theory of strong differential subordination was initiated by Antonino and Romaguera [1] as a generalization of the classical concept of differential subordination introduced by Miller and Mocanu [2,3]. The results obtained by Antonino and Romaguera for the case of strong Briot–Bouquet differential subordinations inspired the development of the general theory related to strong differential subordination as seen for the classical case of differential subordination which is synthetized in [4]. The main aspects of strong differential subordination theory were established in a paper published in 2009 [5] by stating the three problems on which the theory is based on and by defining the notions of solution of a strong differential subordination and dominant of the solutions of the strong differential subordination. The class of admissible functions, a basic tool in the study of strong differential subordinations, was also introduced in this paper. The theory developed rapidly especially through studies associated to different operators like Liu–Srivastava operator [6], a generalized operator [7], multiplier transformation [8,9], Komatu integral operator [10], Sălăgean operator and Ruscheweyh derivative [11] or a certain differential operator [12]. The topic is still interesting for researchers as it is obvious from the numerous publications in the last two years when multiplier transformation and Ruscheweyh derivative [13] or integral operators [14] were used for obtaining new strong subordination results. We can refer to [15,16] for applications of differential operators in the analyses of phenomena from mathematical biology.

The dual notion of strong differential superordination was introduced also in 2009 [17] following the pattern set by Miller and Mocanu for the classical notion of differential superordination [18]. The special case of first order strong differential superordinations was next investigated [19]. Strong differential superodinations were applied to a general equation [20] and they were also related to different operators such as generalized Sălăgean and Ruscheweyh operators [21], new generalized derivative operator [22], or certain general operators [23]. This notion is still popular as it can be proved by listing a few more papers than already shown, published recently [24–26].

In 2012 [27], some interesting new classes were introduced related to the theory of strong differential subordination and superordination. They are intensely used for obtaining new results ever since they were connected to the studies.

The study presented in this paper uses those classes which we list as follows:

For $U = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc of the complex plane, there are some notations used: $\overline{U} = \{z \in C : |z| \leq 1\}$ and $\partial U = \{z \in C : |z| = 1\}$. $H(U)$ denotes the class of holomorphic functions in the unit disc.

Let $H(U \times \overline{U})$ denote the class of analytic functions in $U \times \overline{U}$.

The following subclasses of $H(U \times \overline{U})$ are defined in [27]:

$$H_\zeta[a,n] = \left\{ f \in H(U \times \overline{U}) : f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \ldots, z \in U, \zeta \in \overline{U} \right\}$$

with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$, $a \in \mathbb{C}$, $n \in \mathbb{N}$.

$$H\zeta_U(U) = \{ f \in H_\zeta[a,n] : f(\cdot,\zeta) \text{ univalent in } U \text{ for all } \zeta \in \overline{U} \}$$

$$A\zeta_n = \left\{ f \in H(U \times \overline{U}) : f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \ldots, z \in U, \zeta \in \overline{U} \right\}, \text{ with } A\zeta_1 = A\zeta$$

and $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n+1$, $n \in \mathbb{N}$.

$$S^*\zeta = \left\{ f \in A\zeta : Re\, \frac{z f'_z(z,\zeta)}{f(z,\zeta)} > 0, z \in U, \zeta \in \overline{U} \right\}$$

denotes the class of starlike functions in $U \times \overline{U}$.

$$K\zeta = \left\{ f \in A\zeta : Re \left(\frac{z f''_{z^2}(z,\zeta)}{f'_z(z,\zeta)} + 1 \right) > 0, z \in U, \zeta \in \overline{U} \right\}$$

denotes the class of convex functions in $U \times \overline{U}$.

For obtaining the original results of this paper, the following definitions and notations introduced in [27] are necessary:

Definition 1 ([27]). *Let $h(z,\zeta)$ and $f(z,\zeta)$ be analytic functions in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $h(z,\zeta)$, or $h(z,\zeta)$ is said to be strongly superordinate to $f(z,\zeta)$ if there exists a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$ such that $f(z,\zeta) = h(w(z),\zeta)$, for all $\zeta \in \overline{U}$, $z \in U$. In such a case, we write*

$$f(z,\zeta) \prec\prec h(z,\zeta),\ z \in U,\ \zeta \in \overline{U}.$$

Remark 1 ([27]). *(a) If $f(z,\zeta)$ is analytic in $U \times \overline{U}$ and univalent in U for $\zeta \in \overline{U}$, then Definition 1 is equivalent to:*

$$f(0,\zeta) = h(0,\zeta), \text{for all } \zeta \in \overline{U} \text{ and } f(U \times \overline{U}) \subset h(U \times \overline{U}).$$

(b) If $f(z,\zeta) = f(z), h(z,\zeta) = h(z)$, then the strong superordination becomes the usual superordination.

Definition 2 ([27]). *We denote by Q_ζ the set of functions $q(\cdot,\zeta)$ that are analytic and injective, as function of z, on $\overline{U} \backslash E(q(z,\zeta))$ where*

$$E(q(z,\zeta)) = \left\{ \xi \in \partial U : \lim_{z \to \xi} q(z,\zeta) = \infty \right\}$$

and are such that $q'_z(\xi,\zeta) \neq 0$ for $\xi \in \partial U \backslash E(q(z,\zeta))$, $\zeta \in \overline{U}$.

The subclass of Q_ζ for which $q(0,\zeta) = a$ is denoted by $Q_\zeta(a)$.

Definition 3 ([27]). *Let Ω_ζ be a set in $\mathbb{C}$, $q(\cdot,\zeta) \in \Omega_\zeta$, and n a positive integer. The class of admissible functions $\Phi_n[\Omega_\zeta, q(\cdot,\zeta)]$ consists of those functions $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(r,s,t;\xi,\zeta) \in \Omega_\zeta \tag{A}$$

whenever $r = q(z,\zeta)$, $s = \frac{zq'_z(z,\zeta)}{m}$, $Re\left(\frac{t}{s}+1\right) \le \frac{1}{m} Re\left[\frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}+1\right]$, $z \in U$, $\xi \in \overline{U} \backslash E(q(\cdot,\zeta))$ and $m \ge n \ge 1$. When $n = 1$ we write $\Phi_1[\Omega_\zeta, q(\cdot,\zeta)]$ as $\Phi[\Omega_\zeta, q(\cdot,\zeta)]$.
In the special case when $h(\cdot,\zeta)$ is an analytic mapping of $U \times \overline{U}$ onto $\Omega_\zeta \neq \mathbb{C}$ we denote the class $\Phi_n[h(U \times \overline{U}), q(z,\zeta)]$ by $\Phi_n[h(z,\zeta), q(z,\zeta)]$.
If $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, then the admissibility condition (A) reduces to

$$\varphi\left(q(z,\zeta), \frac{zq'_z(z,\zeta)}{m}; \xi, \zeta\right) \in \Omega_\zeta, \tag{A'}$$

where $z \in U, \zeta \in \overline{U}$, $\xi \in \overline{U} \backslash E(q(\cdot,\zeta))$ and $m \ge n \ge 1$.

Miller—Mocanu lemma given in [18] was rewritten in [27] for functions $p(z,\zeta)$ and $q(z,\zeta)$ as follows:

Lemma 1 ([17,27]). *Let $p(z,\zeta) \in Q(a)$ and let $q(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \ldots$ with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \ge n$, $q(z,\zeta) \not\equiv a$ and $n \ge 1$. If $q(z,\zeta)$ is not subordinate to $p(z,\zeta)$, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\xi_0 \in \partial U \backslash E(p(z,\zeta))$ and an $m \ge n \ge 1$ for which $q(U \times \overline{U}_{r_0}) \subset p(U \times \overline{U})$ and*
(i) $q(z_0,\zeta) = p(\xi_0,\zeta)$,
(ii) $z_0 q'_z(z_0,\zeta) = m\xi_0 p'_z(\xi_0,\zeta)$ and
(iii) $Re\left(\frac{z_0 q''_{z^2}(z_0,\zeta)}{q'_z(z_0,\zeta)}+1\right) \ge mRe\left(\frac{\xi_0 p''_{z^2}(\xi_0,\zeta)}{p'_z(\xi_0,\zeta)}+1\right)$.

This lemma will be used in the next section for proving the theorems which contain the original results. Another helpful result which will be used is the next lemma proved in [28].

Lemma 2 ([28]). *Let $h(z,\zeta)$ be convex in U for all $\zeta \in \overline{U}$ with $h(0,\zeta) = a$, $\gamma \neq 0$, $Re\,\gamma > 0$ and $p \in H_\zeta[a,1] \cap Q$. If $p(z,\zeta) + \frac{zp'_z(z,\zeta)}{\gamma}$ is univalent in U for all $\zeta \in \overline{U}$,*

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{zp'_z(z,\zeta)}{\gamma}$$

and

$$q(z,\zeta) = \frac{\gamma}{z^\gamma} \int_0^z h(t,\zeta) t^{\gamma-1} dt,$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U, \zeta \in \overline{U}.$$

The function q is convex and is the best subordinant.

The connection between univalent function theory and hypergeometric functions was established in 1985 when de Branges used the generalized hypergeometric function for proving Bieberbach's conjecture [29]. Once hypergeometric functions were considered in studies regarding univalent functions, confluent hypergeometric function was used in many investigations. One of the first papers which investigated confluent hypergeometric function and gave conditions for its univalence was published in 1990 [30]. Ever since then, aspects of its univalence were further investigated [31,32], it was considered in connection with other important functions [33–37] and it was used in the definition of new operators [38]. This prolific function is used in the present paper for obtaining results related to another topic, strong differential superordinations. The function is considered as follows:

Definition 4 ([30]). *Let a and c be complex numbers with $c \neq 0, -1, -2, \ldots$ and consider*

$$\phi(a,c;z) = 1 + \frac{a}{c}\cdot\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\cdot\frac{z^2}{2!} + \ldots,\ z \in U \tag{1}$$

This function is called confluent (Kummer) hypergeometric function, is analytic in $\mathbb{C}$, and satisfies Kummer's differential equation:

$$z\cdot w''(z) + [c - z]\cdot w'(z) - a\cdot w(z) = 0.$$

If we let

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2)\ldots(d+k-1) \textbf{ and } (d)_0 = 1,$$

then (1) can be written in the form

$$\phi(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\cdot\frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)}\cdot\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(c+k)}\cdot\frac{z^k}{k!} \tag{2}$$

In the study conducted for obtaining the original results presented in the next section of this paper, the operators introduced in [38] are adapted to the subclasses of $H(U \times \overline{U})$ defined in [27] as follows:

Definition 5 ([38]). *Let $\phi(a,c;z)$ be given by (1) and let $\gamma > 0$. The integral operator $B: H_\zeta[1,1] \to H_\zeta[1,1]$,*

$$B[\phi(a(\zeta),c(\zeta);z,\zeta)] = B(a(\zeta),c(\zeta);z,\zeta) = \frac{\gamma}{z^\gamma}\int_0^z \phi(a(\zeta),c(\zeta);t,\zeta)t^{\gamma-1}dt \tag{3}$$

$z \in U$, $\zeta \in \overline{U}$, is called Kummer–Bernardi integral operator.
For $\gamma = 1$ the integral operator $L: H_\zeta[1,1] \to H_\zeta[1,1]$ is defined as

$$L[\phi(a(\zeta),c(\zeta);z,\zeta)] = L(a(\zeta),c(\zeta);z,\zeta) = \frac{1}{z}\int_0^z \phi(a(\zeta),c(\zeta);t,\zeta)dt, \tag{4}$$

$z \in U, \zeta \in \overline{U}$, which is called Kummer–Libera integral operator.

The form of the confluent hypergeometric function adapted to the new classes depending on the extra parameter ζ needed in the studies related to strong differential superordination theory is given in the next section. Strong differential superordinations are proved in the theorems for which the operators given by (3) and (4) and their derivatives with respect to z are the best subordinants considering γ in relation (3) both a real number, $\gamma > 0$, and a complex number with $Re\,\gamma > 0$. Examples are constructed as proof of the applicability of the new results.

2. Main Results

Considering confluent hypergeometric function defined by (1) or (2), if coefficients a and c complex numbers are replaced by holomorphic functions $a(\zeta),\ c(\zeta)$ depending on the parameter $\zeta \in \overline{U}$, the function changes its form into the following:

$$\phi(a(\zeta),c(\zeta);z,\zeta) = 1 + \frac{a(\zeta)}{c(\zeta)}\cdot\frac{z}{1!} + \frac{a(\zeta)[a(\zeta)+1]}{c(\zeta)[c(\zeta)+1]}\cdot\frac{z^2}{2!} + \ldots,\ z \in U, \tag{5}$$

where $(\zeta) \neq 0,\ c(\zeta) \neq 0, -1, -2, \ldots$.

In [32], Corollary 4 the convexity in the unit disc of the function $\phi(a,c;z)$ given by (1) was proved. This property extends to the new form of the function $(a(\zeta),c(\zeta);z,\zeta)$, as seen in (5).

The first original theorem presented in this paper uses the convexity of the function $\phi(a(\zeta),c(\zeta);z,\zeta)$ and the methods related to strong differential superordination theory in order to find necessary conditions for Kummer–Bernardi integral operator presented in Definition 5 to be the best subordinant of a certain strong differential superordination involving confluent hypergeometric function $\phi(a(\zeta),c(\zeta);z,\zeta)$.

Theorem 1. *Consider the confluent hypergeometric function $\phi(a(\zeta),c(\zeta);z,\zeta)$ defined by (5) and Kummer–Bernardi integral operator $B(a(\zeta),c(\zeta);z,\zeta)$ given by (3). Let $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ be an admissible function with the properties seen in Definition 3. Suppose that $\phi(a(\zeta),c(\zeta);z,\zeta)$ is a univalent solution of the equation*

$$\phi(a(\zeta),c(\zeta);z,\zeta) = \varphi\big(B(a(\zeta),c(\zeta);z,\zeta), z\cdot B'_z(a(\zeta),c(\zeta);z,\zeta);z,\zeta\big). \tag{6}$$

If $\varphi \in \Phi_n\big[h\big(U\times\overline{U}\big),q(z,\zeta)\big],\ p(z,\zeta) \in Q_\zeta(1)$ and $\varphi(p(z,\zeta),z\cdot p'_z(z,\zeta);z,\zeta)$ are univalent in U for all $\zeta \in \overline{U}$, then strong superordination

$$\phi(a(\zeta),c(\zeta);z,\zeta) \ll \varphi\big(p(z,\zeta),z\cdot p'_z(z,\zeta);z,\zeta\big) \tag{7}$$

implies

$$B(a(\zeta),c(\zeta);z,\zeta) \ll p(z,\zeta),\ z\in U,\ \zeta \in \overline{U}.$$

The function $q(z,\zeta) = B(a(\zeta),c(\zeta);z,\zeta)$ is the best subordinant.

Proof. Using relation (3) we obtain

$$z^\gamma\cdot B(a(\zeta),c(\zeta);z,\zeta) = \gamma\int_0^z \phi(a(\zeta),c(\zeta);t,\zeta)t^{\gamma-1}dt. \tag{8}$$

Differentiating (8) with respect to z, following a simple calculation, the next equation is obtained:

$$B(a(\zeta),c(\zeta);z,\zeta) + \frac{1}{\gamma}z\cdot B'_z(a(\zeta),c(\zeta);z,\zeta);z,\zeta) = \phi(a(\zeta),c(\zeta);z,\zeta). \tag{9}$$

Using relation (9), strong superordination (7) becomes:

$$B(a(\zeta),c(\zeta);z,\zeta) + \frac{1}{\gamma}z\cdot B'_z(a(\zeta),c(\zeta);z,\zeta);z,\zeta) \ll \varphi\big(p(z,\zeta),z\cdot p'_z(z,\zeta);z,\zeta\big). \tag{10}$$

Let $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ be an admissible function, $\varphi(r,s;z,\zeta) \in \Phi_n\big[h\big(U\times\overline{U}\big),q(z,\zeta)\big]$, defined by:

$$\varphi(r,s;z,\zeta) = r + \frac{1}{\gamma}\,s,\ \ r,s\in\mathbb{C},\ \gamma > 0. \tag{11}$$

Taking $r = B(a(\zeta),c(\zeta);z,\zeta),\ s = z\cdot B'_z(a(\zeta),c(\zeta);z,\zeta);z,\zeta)$ relation (11) becomes:

$$\begin{aligned}&\varphi(B(a(\zeta),c(\zeta);z,\zeta),z\cdot B'_z(a(\zeta),c(\zeta);z,\zeta);z,\zeta)\ )\\&\qquad = B(a(\zeta),c(\zeta);z,\zeta)+\tfrac{1}{\gamma}z\cdot B'_z(a(\zeta),c(\zeta);z,\zeta);z,\zeta).\end{aligned} \tag{12}$$

Using relation (12) in (10) we get:

$$\varphi\big(B(a(\zeta),c(\zeta);z,\zeta),z\cdot B'_z(a(\zeta),c(\zeta);z,\zeta);z,\zeta\big)\ll\varphi\big(p(z,\zeta),z\cdot p'_z(z,\zeta);z,\zeta\big).$$

Using Definition 1 and Remark 1, a), considering strong differential subordination (7) we get:

$$\phi(a(\zeta),c(\zeta);0,\zeta)=\varphi(p(0,\zeta),0;0,\zeta)$$

and

$$\phi\big(U\times\overline{U}\big)\subset\varphi\big(U\times\overline{U}\big). \tag{13}$$

Interpreting relation (13) we conclude that

$$\varphi\big(p(\xi,\zeta),\xi\cdot p'_z(\xi,\zeta);\xi,\zeta\big)\notin\phi\big(U\times\overline{U}\big),\xi\in\partial U,\zeta\in\ \overline{U}. \tag{14}$$

For $\xi=\xi_0\in\partial U$, relation (14) becomes:

$$\varphi\big(p(\xi_0,\zeta),\xi_0\cdot p'_z(\xi_0,\zeta);\xi_0,\zeta\ \big)\notin\phi\big(U\times\overline{U}\big),\zeta\in\ \overline{U}. \tag{15}$$

Using relation (6) we get:

$$\varphi\big(B(a(\zeta),c(\zeta);z,\zeta),z\cdot B'_z(a(\zeta),c(\zeta);z,\zeta);z,\zeta\big)\in\phi\big(U\times\overline{U}\big),z\in U,\zeta\in\ \overline{U}. \tag{16}$$

For $z=z_0\in U$, (16) is written as:

$$\varphi\big(B(a(\zeta),c(\zeta);z_0,\zeta),z_0\cdot B'_z(a(\zeta),c(\zeta);z_0,\zeta);z_0,\zeta\big)\in\phi\big(U\times\overline{U}\big),z_0\in U,\zeta\in\ \overline{U}. \tag{17}$$

In order to finalize the proof, Lemma 1 and admissibility condition (A') will be applied.

Suppose that $q(z,\zeta)=B(a(\zeta),c(\zeta);z,\zeta)$ is not subordinate to $p(z,\zeta)$ for $z\in U,\zeta\in\ \overline{U}$. Then, using Lemma 1, we know that there are points $z_0=r_0e^{i\theta_0}\in U$ and $\xi_0\in\partial U\backslash E(p(z,\zeta))$ and an $m\geq n\geq 1$such that

$$(z_0,\zeta)=B(a(\zeta),c(\zeta);z_0,\zeta)=p(\xi_0,\zeta)\text{ and}$$

$$z_0\cdot q'_z(z_0,\zeta)=z_0\cdot B'_z(a(\zeta),c(\zeta);z_0,\zeta)=m\xi_0p'_z(\xi_0,\zeta).$$

Using those conditions with $r=q(z_0,\zeta)$ and $s=\frac{z_0\cdot q'_z(z_0,\zeta)}{m}$ for $\xi=\xi_0$ in Definition 3 and taking into consideration the admissibility condition (A'), we obtain:

$$\begin{aligned}\varphi(p(\xi_0,\zeta),\xi_0\mathrm{p}'_z(\xi_0,\zeta);\xi_0,\zeta)&=\varphi\Big(B(a(\zeta),c(\zeta);z_0,\zeta),\tfrac{z_0\cdot B'_z(a(\zeta),c(\zeta);z_0,\zeta)}{m};z_0,\zeta\Big)\\&\in\phi\big(U\times\overline{U}\big).\end{aligned}$$

Using $m=1$ in the previous relation, we get

$$\begin{aligned}\varphi(p(\xi_0,\zeta),\xi_0p'_z(\xi_0,\zeta);\xi_0,\zeta)&=\varphi(B(a(\zeta),c(\zeta);z_0,\zeta),z_0\cdot B'_z(a(\zeta),c(\zeta);z_0,\zeta);z_0,\zeta)\\&\in\phi\big(U\times\overline{U}\big)\end{aligned}$$

and using (17) we write

$$\varphi\big(p(\xi_0,\zeta),\xi_0\mathrm{p}'_z(\xi_0,\zeta);\xi_0,\zeta\big)\in\phi\big(U\times\overline{U}\big),z\in U,\zeta\in\ \overline{U},$$

which contradicts the result obtained in relation (15). Hence, the assumption made is false and we must have:

$$B(a(\zeta),c(\zeta);z,\zeta)\ll p(z,\zeta)\text{ for }z\in U,\zeta\in\ \overline{U}.$$

Since $q(z,\zeta) = B(a(\zeta), c(\zeta); z, \zeta)$ satisfies the differential Equation (6), we conclude that $q(z,\zeta) = B(a(\zeta), c(\zeta); z, \zeta)$ is the best subordinant. □

Remark 2. *For $\gamma = 1$, instead of Kummer–Bernardi integral operator, Kummer–Libera integral operator defined in (4) is used in Theorem 1 and the following corollary can be written:*

Corollary 1. *Consider the confluent hypergeometric function $\phi(a(\zeta), c(\zeta); z, \zeta)$ defined by (5) and Kummer–Libera integral operator $L(a(\zeta), c(\zeta); z, \zeta)$ given by (4). Let $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ be an admissible function with the properties seen in Definition 3. Suppose that $\phi(a(\zeta), c(\zeta); z, \zeta)$ is a univalent solution of the equation*

$$\phi(a(\zeta), c(\zeta); z, \zeta) = \varphi\big(L(a(\zeta), c(\zeta); z, \zeta), z \cdot L'_z(a(\zeta), c(\zeta); z, \zeta); z, \zeta\big).$$

If $\varphi \in \Phi_n\big[h\big(U \times \overline{U}\big), q(z,\zeta)\big]$, $p(z,\zeta) \in Q_\zeta(1)$ and $\varphi(p(z,\zeta), z \cdot p'_z(z,\zeta); z, \zeta)$ are univalent in U for all$\zeta \in \overline{U}$, then strong superordination

$$\phi(a(\zeta), c(\zeta); z, \zeta) \lll \varphi\big(p(z,\zeta), z \cdot p'_z(z,\zeta); z, \zeta\big)$$

implies

$$L(a(\zeta), c(\zeta); z, \zeta) \lll p(z,\zeta),\ z \in U,\ \zeta \in \overline{U}.$$

The function $q(z,\zeta) = L(a(\zeta), c(\zeta); z, \zeta)$ is the best subordinant.

Theorem 2. *Let $q(z,\zeta)$ be a convex function in the unit disc for all $\zeta \in \overline{U}$, consider the confluent hypergeometric function $\phi(a(\zeta), c(\zeta); z, \zeta)$ defined by (5) and Kummer–Bernardi integral operator $B(a(\zeta), c(\zeta); z, \zeta)$ given by (3). Let $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ be an admissible function with the properties seen in Definition 3 and define the analytic function*

$$h(z,\zeta) = \left(1 + \frac{1}{\gamma}\right) q(z,\zeta) + \frac{1}{\gamma} z \cdot q'_z(z,\zeta), z \in U,\ \zeta \in \overline{U}.$$

If $\phi'_z(a(\zeta), c(\zeta); z, \zeta)$ and $B'_z(a(\zeta), c(\zeta); z, \zeta) \in H_\zeta[1,1] \cap Q_\zeta(1)$ are univalent functions in U for all $\zeta \in \overline{U}$, then strong differential superordination

$$h(z,\zeta) \lll \phi'_z(a(\zeta), c(\zeta); z, \zeta) \tag{18}$$

implies

$$q(z,\zeta) \lll B'_z(a(\zeta), c(\zeta); z, \zeta),\ z \in U,\ \zeta \in \overline{U}.$$

Proof. Using relation (9) from the proof of Theorem 1 and differentiating it with respect to z, we obtain:

$$\phi'_z(a(\zeta), c(\zeta); z, \zeta) = \left(1 + \frac{1}{\gamma}\right) B'_z(a(\zeta), c(\zeta); z, \zeta) + \frac{1}{\gamma} z \cdot B''_{z^2}(a(\zeta), c(\zeta); z, \zeta), z \in U,\ \zeta \in \overline{U}. \tag{19}$$

Using (19), strong differential superordination (18) becomes:

$$h(z,\zeta) \lll \left(1 + \frac{1}{\gamma}\right) B'_z(a(\zeta), c(\zeta); z, \zeta) + \frac{1}{\gamma} z \cdot B''_{z^2}(a(\zeta), c(\zeta); z, \zeta). \tag{20}$$

For the proof of this theorem to be complete, Lemma 1 and the admissibility condition (A′) will be applied.

In order to do that, we define the admissible function $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, $\varphi(r, s; z, \zeta) \in \Phi_n\big[h\big(U \times \overline{U}\big), q(z,\zeta)\big]$, given by:

$$\varphi(r, s; z, \zeta) = \left(1 + \frac{1}{\gamma}\right) r + \frac{1}{\gamma} s,\ \ r, s \in \mathbb{C},\ \gamma > 0. \tag{21}$$

Taking $r = B'_z(a(\zeta), c(\zeta); z, \zeta)$, $s = z \cdot B''_{z^2}(a(\zeta), c(\zeta); z, \zeta)$ relation (21) becomes:

$$\varphi\Big(B'_z(a(\zeta), c(\zeta); z, \zeta), z \cdot B''_{z^2}(a(\zeta), c(\zeta); z, \zeta); z, \zeta\Big) = \Big(1 + \frac{1}{\gamma}\Big)B'_z(a(\zeta), c(\zeta); z, \zeta) + \frac{1}{\gamma} z \cdot B''_{z^2}(a(\zeta), c(\zeta); z, \zeta); z, \zeta). \tag{22}$$

Using relation (22) in (20) we get:

$$h(z, \zeta) \ll \varphi\Big(B'_z(a(\zeta), c(\zeta); z, \zeta), z \cdot B''_{z^2}(a(\zeta), c(\zeta); z, \zeta); z, \zeta\Big).$$

Using Definition 1 and Remark 1, a) for this strong differential superordination, we get:

$$h(0, \zeta) = \varphi(B'_z(a(\zeta), c(\zeta); 0, \zeta), 0; 0, \zeta)$$

and

$$h(U \times \overline{U}) \subset \varphi(U \times \overline{U}). \tag{23}$$

Interpreting relation (23) we conclude that

$$\varphi\Big(B'_z(a(\zeta), c(\zeta); \xi, \zeta), \xi \cdot B''_{z^2}(a(\zeta), c(\zeta); \xi, \zeta); \xi, \zeta\Big) \notin h(U \times \overline{U}),\ \xi \in \partial U, \zeta \in \overline{U}. \tag{24}$$

For $\xi = \xi_0 \in \partial U$, relation (24) becomes:

$$\varphi\Big(B'_z(a(\zeta), c(\zeta); \xi_0, \zeta), \xi_0 \cdot B''_{z^2}(a(\zeta), c(\zeta); \xi_0, \zeta); \xi_0, \zeta\Big) \notin h(U \times \overline{U}), \zeta \in \overline{U}. \tag{25}$$

Suppose that $q(z, \zeta)$ is not subordinate to $B'_z(a(\zeta), c(\zeta); z, \zeta)$ for $z \in U, \zeta \in \overline{U}$. Then, using Lemma 1, we know that there are points $z_0 = r_0 e^{i\theta_0} \in U$ and $\xi_0 \in \partial U \backslash E(B'_z(a(\zeta), c(\zeta); z, \zeta))$ and an $m \geq n \geq 1$ such that

$$q(z_0, \zeta) = B'_z(a(\zeta), c(\zeta); z_0, \zeta) = p(\xi_0, \zeta) \text{ and}$$

$$z_0 q'_z(z_0, \zeta) = m\xi_0 B''_{z^2}(a(\zeta), c(\zeta); z_0, \zeta) = m\xi_0 p'_z(\xi_0, \zeta).$$

Using those conditions with $r = B'_z(a(\zeta), c(\zeta); z_0, \zeta)$ and $s = \xi_0 B''_{z^2}(a(\zeta), c(\zeta); z_0, \zeta)$ for $\xi = \xi_0$ in Definition 3 and taking into consideration the admissibility condition (A'), we obtain:

$$\varphi(q(z_0, \zeta), z_0 q'_z(z_0, \zeta); z_0, \zeta) = \varphi\left(B'_z(a(\zeta), c(\zeta); z_0, \zeta), \frac{\xi_0 B''_{z^2}(a(\zeta), c(\zeta); \xi_0, \zeta)}{m}; z_0, \zeta\right) \in h(U \times \overline{U}).$$

Using $m = 1$ in the previous relation, we get

$$\varphi\Big(B'_z(a(\zeta), c(\zeta); z_0, \zeta), \xi_0 B''_{z^2}(a(\zeta), c(\zeta); z_0, \zeta); z_0, \zeta\Big) \in h(U \times \overline{U}), \zeta \in \overline{U},$$

which contradicts the result obtained in relation (25). Hence, the assumption made is false and we must have:

$$q(z, \zeta) \ll B'_z(a(\zeta), c(\zeta); z, \zeta) \text{ for } z \in U, \zeta \in \overline{U}.$$

□

Remark 3. *For $\gamma = 1$, instead of Kummer–Bernardi integral operator, Kummer–Libera integral operator defined in (4) is used in Theorem 2 and the following corollary can be written:*

Corollary 2. *Let $q(z, \zeta)$ be a convex function in the unit disc for all $\zeta \in \overline{U}$, consider the confluent hypergeometric function $\phi(a(\zeta), c(\zeta); z, \zeta)$ defined by (5) and Kummer–Libera integral operator*

$L(a(\zeta), c(\zeta); z, \zeta)$ given by (4). Let $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ be an admissible function with the properties seen in Definition 3 and define the analytic function:

$$h(z,\zeta) = \left(1+\frac{1}{\gamma}\right)q(z,\zeta) + \frac{1}{\gamma} z \cdot q'_z(z,\zeta), z \in U,\ \zeta \in \overline{U}.$$

If $\phi'_z(a(\zeta), c(\zeta); z, \zeta)$ and $L'_z(a(\zeta), c(\zeta); z, \zeta) \in H_\zeta[1,1] \cap Q_\zeta(1)$ are univalent functions in U for all $\zeta \in \overline{U}$, then strong differential superordination

$$h(z,\zeta) \lll \phi'_z(a(\zeta), c(\zeta); z, \zeta)$$

implies

$$q(z,\zeta) \lll L'_z(a(\zeta), c(\zeta); z, \zeta),\ z \in U,\ \zeta \in \overline{U}.$$

In Theorems 1 and 2, parameter γ is a real number, $\gamma > 0$. In the next theorem, a necessary and sufficient condition is determined such that Kummer–Bernardi integral operator is the best subordinant for a certain strong differential superordination considering γ a complex number with $Re\ \gamma > 0$.

Theorem 3. *Let $h(z,\zeta)$ with $h(0,\zeta) = a$ be a convex function in the unit disc for all $\zeta \in \overline{U}$ and let γ be a complex number with $Re\ \gamma > 0$. Consider the confluent hypergeometric function $\phi(a(\zeta), c(\zeta); z, \zeta)$ defined by (5) and Kummer–Bernardi integral operator $B(a(\zeta), c(\zeta); z, \zeta)$ given by (3). Let $p(z,\zeta) \in H_\zeta[a,1] \cap Q_\zeta(a)$.*
If $p(z,\zeta) + \frac{z \cdot p'_z(z,\zeta)}{\gamma}$ is univalent in U for all $\zeta \in \overline{U}$ and the following strong differential superordination is satisfied

$$B(a(\zeta), c(\zeta); z, \zeta) + \frac{z \cdot B'_z(a(\zeta), c(\zeta); z, \zeta)}{\gamma} \lll p(z,\zeta) + \frac{z \cdot p'_z(z,\zeta)}{\gamma}, \tag{26}$$

then

$$q(z,\zeta) = B(a(\zeta), c(\zeta); z, \zeta) \lll p(z,\zeta), z \in U,\ \zeta \in \overline{U}.$$

Function $q(z,\zeta) = B(a(\zeta), c(\zeta); z, \zeta)$ is convex and is the best subordinant.

Proof. Lemma 2 will be used for the proof of this theorem. Using the definition of Kummer–Bernardi operator given by (3) and differentiating this relation with respect to z, we obtain:

$$\gamma \cdot z^{\gamma-1} \cdot B(a(\zeta), c(\zeta); z, \zeta) + z^{\gamma} \cdot B'_z(a(\zeta), c(\zeta); z, \zeta) = \gamma \cdot h(z,\zeta) \cdot z^{\gamma-1},\ z \in U,\ \zeta \in \overline{U}.$$

After a simple calculation, we get:

$$B(a(\zeta), c(\zeta); z, \zeta) + \frac{z \cdot B'_z(a(\zeta), c(\zeta); z, \zeta)}{\gamma} = h(z,\zeta), z \in U,\ \zeta \in \overline{U}. \tag{27}$$

Using (27), the strong differential subordination (26) becomes

$$h(z,\zeta) \lll p(z,\zeta) + \frac{z \cdot p'_z(z,\zeta)}{\gamma}, z \in U,\ \zeta \in \overline{U}.$$

Since $h(z,\zeta)$ is a convex function and $p(z,\zeta) + \frac{z \cdot p'_z(z,\zeta)}{\gamma}$ is univalent in U for all $\zeta \in \overline{U}$, by applying Lemma 2 we obtain:

$$q(z,\zeta) = B(a(\zeta), c(\zeta); z, \zeta) \lll p(z,\zeta), z \in U,\ \zeta \in \overline{U}.$$

Since function $q(z,\zeta) = B(a(\zeta), c(\zeta); z, \zeta)$ satisfies Equation (27) and is analytic in U for all $\zeta \in \overline{U}$, we conclude that $q(z,\zeta) = B(a(\zeta), c(\zeta); z, \zeta)$ is the best subordinant. □

Example 1. *Let $a = -1$, $c = \frac{i}{2\zeta}$, $\frac{i}{2\zeta} \neq 0, -1, -2, \ldots$, $\zeta \neq 0$, $\gamma \in \mathbb{C}$, $Re\ \gamma > 0$. We evaluate:*

$$\phi\left(-1, \frac{i}{2\zeta}; z, \zeta\right) = 1 + \frac{-1}{\frac{i}{2\zeta}} \cdot \frac{z}{1!} = 1 - \frac{2\zeta \cdot z}{i} = 1 + 2i\zeta z.$$

Further, we use this expression to obtain Kummer–Bernardi integral operator's expression:

$$\begin{aligned} B\left(\phi\left(-1, \tfrac{i}{2\zeta}; z, \zeta\right)\right) &= \tfrac{\gamma}{z^\gamma} \int_0^z \phi\left(-1, \tfrac{i}{2\zeta}; t, \zeta\right) t^{\gamma-1} dt = \tfrac{\gamma}{z^\gamma} \int_0^z (1 + 2i\zeta t) t^{\gamma-1} dt \\ &= \tfrac{\gamma}{z^\gamma} \left(\tfrac{z^\gamma}{\gamma} + 2i\zeta \tfrac{z^{\gamma+1}}{\gamma+1} \right) = 1 + 2i\zeta \tfrac{\gamma}{\gamma+1} \cdot z. \end{aligned}$$

Functions $p(z, \zeta) = 1 + z\zeta$ *and* $p(z, \zeta) + \frac{z \cdot p'_z(z,\zeta)}{\gamma} = 1 + z\left(\zeta + \frac{\zeta}{\gamma}\right)$ *are univalent in* U *for all* $\zeta \in \overline{U}$.

Using Theorem 3, we get:

If the following strong differential superordination is satisfied

$$1 + 2i\zeta \frac{\gamma}{\gamma + 1} \cdot z + \frac{2i\zeta \cdot z}{\gamma + 1} \ll 1 + z\left(\zeta + \frac{\zeta}{\gamma}\right),$$

then

$$1 + 2i\zeta \frac{\gamma}{\gamma + 1} \cdot z \ll 1 + z\zeta, z \in U, \ \zeta \in \overline{U}.$$

Function $q(z, \zeta) = 1 + 2i\zeta \frac{\gamma}{\gamma+1} \cdot z$ *is convex and is the best subordinant.*

Example 2. *Let* $a = -1, \ c = \frac{i}{2\zeta}, \ \frac{i}{2\zeta} \neq 0, -1, -2, \ \ldots, \ \zeta \neq 0, \ \gamma = 1 + i \in \mathbb{C}, \ Re\,\gamma = 1 > 0.$ *We evaluate:*

$$\phi\left(-1, \frac{i}{2\zeta}; z, \zeta\right) = 1 + \frac{-1}{\frac{i}{2\zeta}} \cdot \frac{z}{1!} = 1 - \frac{2\zeta \cdot z}{i} = 1 + 2i\zeta z.$$

Further, we use this expression to obtain Kummer–Bernardi integral operator's expression:

$$\begin{aligned} B\left(\phi\left(-1, \tfrac{i}{2\zeta}; z, \zeta\right)\right) &= \tfrac{\gamma}{z^\gamma} \int_0^z \phi\left(-1, \tfrac{i}{2\zeta}; t, \zeta\right) t^{\gamma-1} dt = \tfrac{1+i}{z^{1+i}} \int_0^z (1 + 2i\zeta t) t^{\gamma-1} dt \\ &= \tfrac{1+i}{z^{1+i}} \left(\tfrac{z^{1+i}}{1+i} + 2i\zeta \tfrac{z^{1+i+1}}{1+i+1} \right) = 1 + 2i\zeta \tfrac{z(i+1)}{i+2} = 1 + \tfrac{2}{5}(-1 + 3i) z\zeta. \end{aligned}$$

Functions $p(z, \zeta) = 1 + z\zeta$ *and* $p(z, \zeta) + \frac{z \cdot p'_z(z,\zeta)}{1+i} = 1 + \frac{3}{2} z\zeta(3 - i)$ *are univalent in* U *for all* $\zeta \in \overline{U}$.

Using Theorem 3, we get:

If $1 + \frac{3}{2} z\zeta(3 - i)$ *is univalent in* U *for all* $\zeta \in \overline{U}$ *and the following strong differential superordination is satisfied*

$$1 + 2i\zeta t \ll 1 + \frac{3}{2} z\zeta(3 - i),$$

then

$$1 + \frac{2}{5}(-1 + 3i) z\zeta \ll 1 + z\zeta, z \in U, \ \zeta \in \overline{U}.$$

Function $q(z, \zeta) = 1 + \frac{2}{5}(-1 + 3i) z\zeta$ *is convex and is the best subordinant.*

3. Discussion

The study presented in this paper is inspired by the nice results published which involve confluent hypergeometric function and certain operators defined by using this interesting function. For this research, the environment of the theory of strong differential superordination is considered. Confluent hypergeometric function and Kummer–Bernardi and Kummer–Libera operators defined in [38] are used in order to obtain certain strong differential superordinations. Their best subordinants are given in the three theorems proved in the main results part. Theorems 1 and 2 use the convexity of confluent hypergeometric function $\phi(a(\zeta), c(\zeta); z, \zeta)$ given in (5) where it is adapted to certain classes of analytic functions specific for the theory of strong differential superordination. The methods related to strong differential superordination theory are applied in order to find necessary conditions for Kummer–Bernardi integral operator presented in Definition 5, relation (3), to be the best subordinant of a certain strong differential superordination involving confluent hypergeometric function $\phi(a(\zeta), c(\zeta); z, \zeta)$. As corollary, the similar result is given for Kummer–Libera operator. For those two theorems, the parameter γ is a real number, $\gamma > 0$. In Theorem 3, $\gamma \in \mathbb{C}$, with $Re\,\gamma > 0$ is considered and a necessary and sufficient condition is determined such that Kummer–Bernardi integral operator to be the best subordinant for a certain strong differential superordination. Two examples are constructed for the case when $\gamma \in \mathbb{C}$, with $Re\,\gamma > 0$.

4. Conclusions

In this paper, new strong differential superordinations are investigated using a special form of confluent hypergeometric function given in (5) and two operators previously introduced in [38]. In the three theorems proved as a result of the study, the two operators called Kummer–Bernardi and Kummer–Libera integral operators are the best subordinants of the strong differential superordinations.

The novelty of the study resides in the forms of the confluent hypergeometric function and of the two operators considered by adaptation to the new classes depending on the extra parameter ζ introduced in the theory of strong differential subordination in [27].

As future studies, the dual notion of strong differential subordination can be considered for investigations concerning confluent hypergeometric function and the two operators used in the present study. Sandwich-type results could be obtained as seen in recent papers [13,39,40].

New subclasses of univalent functions could be introduced in the context of strong differential subordination and superordination theories using the operators presented in this paper as seen in [41].

It might also be interesting to consider other hypergeometric functions and operators defined with them following the ideas presented in this paper.

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