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Certain Subclasses of Bi-Starlike Function of Complex Order Defined by Erdély–Kober-Type Integral Operator

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Abstract: In the present paper, we introduce new subclasses of bi-starlike and bi-convex functions of complex order associated with Erdély–Kober-type integral operator in the open unit disc and find the estimates of initial coefficients in these classes. Moreover, we obtain Fekete-Szegő inequalities for functions in these classes. Some of the significances of our results are pointed out as corollaries.

Keywords: univalent functions; analytic functions; bi-univalent functions; coefficient bounds; bi-starlike and bi-convex functions of complex order; fractional calculus; Erdély–Kober-type integral operator

MSC: 30C45; 30C50; 30C55

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1. Introduction and Preliminaries

Let $\mathfrak{A}$ signify the class of functions of the following form:

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n \tag{1}$$

which are analytic in the open unit disc $\mathfrak{U} = \{\xi : |\xi| < 1\}$ and normalized as $f(0) = 0$ and $f'(0) = 1$. Furthermore, let $\mathfrak{S}$ represent the class of all functions in $\mathfrak{A}$ that are univalent in $\mathfrak{U}$. Some of the imperative and well-investigated subclasses of the univalent function class $\mathfrak{S}$ include (for example) the class $\mathfrak{S}^*(\delta)$ of starlike functions of order δ in $\mathfrak{U}$ and the class $\mathfrak{K}(\delta)$ of convex functions of order δ $(0 \le \delta < 1)$ in $\mathfrak{U}$. It is known that if $f \in \mathfrak{S}$, then there exists inverse function f^{-1} because normalization is defined in some neighborhood of the origin. In some cases, f^{-1}can be defined in the entire $\mathfrak{U}$. Clearly, f^{-1} is also univalent. For this reason, class Σ is defined as follows.

It is well known that every function $f \in \mathfrak{S}$ has an inverse f^{-1} defined by the following:

$$f^{-1}(f(\xi)) = \xi \ \ (\xi \in \mathfrak{U})$$
$$\text{and} \ \ f(f^{-1}(w)) = w \ \ (|w| < r_0(f); r_0(f) \ge 1/4)$$

where the following is the case.

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f(\xi) \in \mathfrak{A}$ is said to be bi-univalent in $\mathfrak{U}$ if both $f(\xi)$ and $f^{-1}(\xi)$ are univalent in $\mathfrak{U}$. Let Σ denote the class of bi-univalent functions in $\mathfrak{U}$ given by (1). Note that the following functions:

$$f_1(\xi) = \frac{\xi}{1-\xi}, \qquad f_2(\xi) = \frac{1}{2}\log\frac{1+\xi}{1-\xi}, \qquad f_3(\xi) = -\log(1-\xi)$$

with their corresponding inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \qquad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}, \qquad f_3^{-1}(w) = \frac{e^{w}-1}{e^{w}}$$

are elements of Σ (see [1–3]). Certain subclasses of Σ are explicitly bi-starlike functions of order $\delta(0 < \delta \leq 1)$ denoted by $\mathfrak{S}^*_\Sigma(\delta)$ and bi-convex function of order δ designated by $\mathfrak{K}_\Sigma(\delta)$ familiarized by Brannan and Taha [1]. For each $f \in \mathfrak{S}^*_\Sigma(\delta)$ and $f \in \mathfrak{K}_\Sigma(\delta)$, non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ were established [1,2], but the problem to find the general coefficient bounds on the following Taylor–Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\}; \;\; \mathbb{N} := \{1,2,3,\cdots\})$$

is still an open problem (see [1–5]). Several researchers (see [6–11]) have introduced and explored some inspiring subclasses Σ and they have initiated non-sharp estimates $|a_2|$ and $|a_3|$. For two functions f_1 and $f_2 \in \mathfrak{A}$, we say that function f_1 is subordinate to f_2 if there exists a Schwarz function ω that is holomorphic in $\mathfrak{U}$ with property $w(0) = 0; |\omega(\xi)| < 1$ and satisfying $f_1(\xi) = f_2(w(\xi))$ This subordination is symbolically written as $f_1(\xi) \prec f_2(\xi)$. Lately, Ma and Minda [12]-unified subclasses of starlike and convex functions are subordinate to a general superordinate function. For this purpose, they considered an analytic function $\mathfrak{W}$ with positive real parts in the unit disk $\mathfrak{U}, \mathfrak{W}(0) = 1, \mathfrak{W}'(0) > 0$, and $\mathfrak{W}$ maps $\mathfrak{U}$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. In the consequence, it is assumed that $\mathfrak{W}$ is an analytic function with positive real part in the unit disk $\mathfrak{U}$, with $\mathfrak{W}(0) = 1, \mathfrak{W}'(0) > 0$, and $\mathfrak{W}(\mathfrak{U})$ is symmetric with respect to the real axis. Such functions are of the following form.

$$\mathfrak{W}(\xi) = 1 + \mathbf{m}_1\xi + \mathbf{m}_2\xi^2 + \mathbf{m}_3\xi^3 + \cdots, \quad (\mathbf{m}_1 > 0). \tag{3}$$

The study of operators plays a central role in geometric function theory and its correlated fields. In the recent years, there has been an collective importance in problems concerning the evaluations of various differential and integral operators. For our study, we recall the Erdély–Kober type ([13] Ch. 5; also see [14–17]) for the integral operator definition, which shall be used throughout the paper as stated below.

Erdély–Kober Fractional-Order Derivative

Let $\kappa > 0, \varsigma, \tau \in \mathbb{C}$ be such that $\Re(\tau - \varsigma) \geq 0$, an Erdély–Kober type integral operator:

$$\mathfrak{I}_\kappa^{\varsigma,\tau} : \mathfrak{A} \to \mathfrak{A}$$

be defined for $\Re(\tau - \varsigma) > 0$ and $\Re(\varsigma) > -\kappa$ by the following.

$$\mathfrak{I}_\kappa^{\varsigma,\tau} f(\xi) = \frac{\Gamma(\tau+\kappa)}{\Gamma(\varsigma+\kappa)} \frac{1}{\Gamma(\tau-\varsigma)} \int_0^1 (1-t)^{\tau-\varsigma-1} t^{\varsigma-1} f(\xi t^\kappa) dt, \kappa > 0. \tag{4}$$

For $\kappa > 0, \Re(\tau - \varsigma) \geq 0, \Re(\vartheta) > -\kappa$ and $f \in \mathfrak{A}$ of the form (1), we have the following:

$$\begin{aligned} \mathfrak{I}_{\kappa}^{\varsigma,\tau} f(\xi) &= \xi + \sum_{n=2}^{\infty} \frac{\Gamma(\tau+\kappa)\Gamma(\varsigma+n\kappa)}{\Gamma(\varsigma+\kappa)\Gamma(\tau+n\kappa)} a_n \xi^n \quad (\xi \in \mathfrak{U}) \qquad (5) \\ &= \xi + \sum_{n=2}^{\infty} \Upsilon_{\kappa}^{\varsigma,\tau}(n) a_n \xi^n \quad (\xi \in \mathfrak{U}) \qquad (6) \end{aligned}$$

where the following is the case.

$$\Upsilon_{\kappa}^{\varsigma,\tau}(n) = \frac{\Gamma(\tau+\kappa)\Gamma(\varsigma+n\kappa)}{\Gamma(\varsigma+\kappa)\Gamma(\tau+n\kappa)} \qquad (7)$$

and $\Gamma(n+1) = n!$.

Note that the following is the case.

$$\mathfrak{I}_{\kappa}^{\varsigma,\varsigma} f(\xi) = f(\xi)$$

Remark 1. *By fixing the parameters $\varsigma, \tau, \vartheta$ as mentioned below, the operator $\mathfrak{I}_{\kappa}^{\varsigma,\tau}$ includes various operators studied in the literature as cited below:*

1. *For $\varsigma = \beta; \tau = \alpha + \beta$ and $\kappa = 1$, we obtain the operator $\mathfrak{Q}_{\beta}^{\alpha} f(\xi)(\alpha \geq 0; \beta > 1)$ studied by Jung et al. [18];*
2. *For $\varsigma = \alpha - 1; \tau = \beta - 1$ and $\kappa = 1$, we obtain the operator $\mathfrak{L}_{\alpha,\beta} f(\xi)(\alpha; \beta \in \mathbb{C} \in \mathbb{Z}_0; \mathbb{Z}_0 = \{0; -1; -2; \cdots\}$ studied by Carlson and Shafer [19];*
3. *For $\varsigma = \rho - 1; \tau = \mathfrak{l}$ and $\kappa = 1$, we obtain the operator $\mathfrak{I}_{\rho,\mathfrak{l}}(\rho > 0; \mathfrak{l} > 1)$ studied by Choi et al. [20];*
4. *For $\varsigma = \alpha; \tau = 0$ and $\kappa = 1$, we obtain the operator $\mathfrak{D}^{\alpha}(\alpha > 1)$ studied by Ruscheweyh [21];*
5. *For $\varsigma = 1; \tau = n$ and $\mu = 1$, we obtain the operator $\mathfrak{I}_n(n > \mathbb{N}_0)$ studied in [22,23];*
6. *For $\varsigma = \beta; \tau = \beta + 1$ and $\kappa = 1$; we obtain the integral operator $\mathfrak{I}_{\beta,1}$ which studied by Bernardi [24];*
7. *For $\varsigma = 1; \tau = 2$ and $\kappa = 1$, we obtain the integral operator $\mathfrak{I}_{1,1} = \mathfrak{I}$ studied by Libera [25] and Livingston [26].*

The motivation of our present investigation stems from (by Silverman and Silvia [27] (also see [28])) the seminal paper on bi-univalent functions by Srivastava et al. [8] and by the recent works by many authors (for example Deniz [7], Huo Tang et al. [6], El-Deeb et al. [29–31], and Murugusundaramoorthy and Janani [32]). In the present paper, we introduce two new subclasses of the function class Σ of complex order $\vartheta \in \mathbb{C}\backslash\{0\}$, involving the linear operator $\mathfrak{I}_{\kappa}^{\varsigma,\tau}$ given in Definition 1. We find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions $f \in \mathfrak{S}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta, \ell)$. Several related classes are also considered, and connections to earlier known results are provided. Moreover we obtain the Fekete-Szegő inequalities for $f \in \mathfrak{S}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta, \ell)$ and $f \in \mathfrak{K}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta, \ell)$.

Definition 1. *Let $f \in \Sigma$ be assumed by (1) and $f \in \mathfrak{S}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta, \ell)$, if the subsequent conditions holds:*

$$1 + \frac{1}{\vartheta}\left(\frac{\xi(\mathfrak{I}_{\kappa}^{\varsigma,\tau} f(\xi))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau} f(\xi)} + \left(\frac{1+e^{i\ell}}{2}\right)\frac{\xi^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau} f(\xi))''}{\mathfrak{I}_{\kappa}^{\varsigma,\tau} f(\xi)} - 1\right) \prec \mathfrak{W}(\xi) \qquad (8)$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{w(\mathfrak{I}_{\kappa}^{\varsigma,\tau} g(w))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau} g(w)} + \left(\frac{1+e^{i\ell}}{2}\right)\frac{w^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau} g(w))''}{\mathfrak{I}_{\kappa}^{\varsigma,\tau} g(w)} - 1\right) \prec \mathfrak{W}(w), \qquad (9)$$

where $\vartheta \in \mathbb{C}\backslash\{0\};\ \ell \in (-\pi, \pi];\ \xi, w \in \mathfrak{U}$ and g is given by (2).

Definition 2. *Let $f \in \Sigma$ be assumed by (1) and $f \in \mathfrak{K}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta, \ell)$, if the subsequent conditions are satisfied:*

$$1 + \frac{1}{\vartheta}\left(\frac{[\xi(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))' + \left(\frac{1+e^{i\ell}}{2}\right)\xi^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'']'}{(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'} - 1\right) \prec \mathfrak{W}(\xi) \tag{10}$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{[w(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))' + \left(\frac{1+e^{i\ell}}{2}\right)w^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'']'}{(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'} - 1\right) \prec \mathfrak{W}(w), \tag{11}$$

where $\vartheta \in \mathbb{C}\backslash\{0\}; \ell \in (-\pi, \pi]; \xi, w \in \mathfrak{U}$ *and* g *is given by* (2).

Remark 2. *For a function* $f(\xi) \in \Sigma$ *specified by* (1) *and for* $\ell = \pi$, *interpret that* $\mathfrak{S}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta, \ell) \equiv \mathfrak{S}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta)$ *satisfies the ensuing conditions :*

$$\left[1 + \frac{1}{\vartheta}\left(\frac{\xi(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi)} - 1\right)\right] \prec \mathfrak{W}(\xi) \text{ and } \left[1 + \frac{1}{\vartheta}\left(\frac{w(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w)} - 1\right)\right] \prec \mathfrak{W}(w)$$

where $\vartheta \in \mathbb{C}\backslash\{0\};\ \xi, w \in \mathfrak{U}$ *and* g *is given by* (2).

Remark 3. *A function* $f(\xi) \in \Sigma$ *specified by* (1) *and for* $\ell = \pi$, *we interpret that* $\mathfrak{K}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta, \ell) \equiv \mathfrak{K}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta)$ *satisfies the ensuing conditions correspondingly:*

$$\left[1 + \frac{1}{\vartheta}\left(\frac{\xi(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))''}{(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'}\right)\right] \prec \mathfrak{W}(\xi) \text{ and } \left[1 + \frac{1}{\vartheta}\left(\frac{w(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))''}{(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'}\right)\right] \prec \mathfrak{W}(w),$$

where $\vartheta \in \mathbb{C}\backslash\{0\};\ \xi, w \in \mathfrak{U}$ *and* g *is given by* (2).

Remark 4. *For a function* $f(\xi) \in \Sigma$ *given by* (1) *and for* $\vartheta = 1$, *we note that* $\mathfrak{S}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta, \ell) \equiv \mathfrak{S}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\ell)$ *and satisfies the following conditions, respectively:*

$$\left(\frac{\xi(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi)} + \left(\frac{1+e^{i\ell}}{2}\right)\frac{\xi^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))''}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi)}\right) \prec \mathfrak{W}(\xi)$$

and the following is the case.

$$\left(\frac{w(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w)} + \left(\frac{1+e^{i\ell}}{2}\right)\frac{w^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))''}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w)}\right) \prec \mathfrak{W}(w).$$

Moreover, $\mathfrak{K}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta, \ell) \equiv \mathfrak{K}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\ell)$ *and it satisfies the following conditions:*

$$\left(\frac{[\xi(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))' + \left(\frac{1+e^{i\ell}}{2}\right)\xi^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'']'}{(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'}\right) \prec \mathfrak{W}(\xi)$$

and the following is the case:

$$\left(\frac{[w(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))' + \left(\frac{1+e^{i\ell}}{2}\right)w^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'']'}{(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'}\right) \prec \mathfrak{W}(w),$$

where $\ell \in (-\pi, \pi]; \xi, w \in \mathfrak{U}$ *and* g *is given by* (2).

2. Coefficient Estimates for $f \in \mathfrak{S}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta,\ell)$ and $f \in \mathfrak{K}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta,\ell)$

For notational simplicity, in the sequel we let the following be the case:

$$\kappa > 0, \Re(\tau-\varsigma) \geq 0, \quad \Re(\varsigma) > -\kappa \quad \textit{and} \quad \mathfrak{I}^{\varsigma,\tau}_{\kappa} f(\xi)$$

and it is provided by (5):

$$\Upsilon_2 = \Upsilon^{\varsigma,\tau}_{\kappa}(2) = \frac{\Gamma(\tau+\kappa)\Gamma(\varsigma+2\kappa)}{\Gamma(\varsigma+\kappa)\Gamma(\tau+2\kappa)}, \tag{12}$$

$$\Upsilon_3 = \Upsilon^{\varsigma,\tau}_{\kappa}(3) = \frac{\Gamma(\tau+\kappa)\Gamma(\varsigma+3\kappa)}{\Gamma(\varsigma+\kappa)\Gamma(\tau+3\kappa)} \tag{13}$$

and the following.

$$\ell \in (-\pi, \pi].$$

For deriving our main results, we need the following lemma.

Lemma 1. *Ref. [33] states that if $h \in \mathfrak{P}$, then $|c_k| \leq 2$ for each k, where $\mathfrak{P}$ is the family of all functions h analytic in $\mathfrak{U}$ for which $\Re(h(\xi)) > 0$ and the following is the case.*

$$h(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots \quad \textit{for } \xi \in \mathfrak{U}.$$

Define the functions $p(\xi)$ and $q(\xi)$ by the following:

$$p(\xi) := \frac{1+u(\xi)}{1-u(\xi)} = 1 + \wp_1\xi + \wp_2\xi^2 + \cdots$$

and the following.

$$q(w) := \frac{1+v(w)}{1-v(w)} = 1 + \mathfrak{q}_1 w + \mathfrak{q}_2 w^2 + \cdots.$$

It follows that the following is the case:

$$u(\xi) := \frac{p(\xi)-1}{p(\xi)+1} = \frac{1}{2}\left[\wp_1\xi + \left(\wp_2 - \frac{\wp_1^2}{2}\right)\xi^2 + \cdots\right]$$

and

$$v(w) := \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left[\mathfrak{q}_1 w + \left(\mathfrak{q}_2 - \frac{\mathfrak{q}_1^2}{2}\right)w^2 + \cdots\right].$$

Then, $p(\xi)$ and $q(w)$ are analytic in $\mathfrak{U}$ with $p(0) = 1 = q(0)$.

Since $u, v : \mathfrak{U} \to \mathfrak{U}$, the functions $p(\xi)$ and $q(w)$ have a positive real part in $\mathfrak{U}$, and $|\wp_i| \leq 2$ and $|\mathfrak{q}_i| \leq 2$ for each i.

Theorem 1. *Let f given by (1) be in the class $\mathfrak{S}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta,\ell)$, $\vartheta \in \mathbb{C}\backslash\{0\}$ and $\ell \in (-\pi,\pi]$. Then, we have the following:*

$$|a_2| \leq \frac{|\vartheta|\mathbf{m}_1\sqrt{\mathbf{m}_1}}{\sqrt{|\vartheta[(5+3e^{i\ell})\Upsilon_3 - (2+e^{i\ell})\Upsilon_2^2]\mathbf{m}_1^2 + (2+e^{i\ell})^2(\mathbf{m}_1-\mathbf{m}_2)\Upsilon_2^2|}} \tag{14}$$

and the following.

$$|a_3| \leq \frac{|\vartheta|^2\mathbf{m}_1^2}{|2+e^{i\ell}|^2\Upsilon_2^2} + \frac{|\vartheta|\mathbf{m}_1}{|5+3e^{i\ell}|\Upsilon_3}. \tag{15}$$

Proof. It follows from (8) and (9) that we have the following:

$$1+\frac{1}{\vartheta}\left(\frac{\xi(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi)}+\left(\frac{1+e^{i\ell}}{2}\right)\frac{\xi^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))''}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi)}-1\right)=\mathfrak{W}(u(\xi)) \tag{16}$$

and

$$1+\frac{1}{\vartheta}\left(\frac{w(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w)}+\left(\frac{1+e^{i\ell}}{2}\right)\frac{w^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))''}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w)}-1\right)=\mathfrak{W}(v(w)), \tag{17}$$

where

$$\mathfrak{W}(u(\xi))=\frac{1}{2}\mathbf{m}_1\wp_1\xi+\left(\frac{1}{2}\mathfrak{m}_1(\wp_2-\frac{\wp_1^2}{2})+\frac{1}{4}\mathbf{m}_2\wp_1^2\right)\xi^2+\cdots. \tag{18}$$

and

$$\mathfrak{W}(v(w))=\frac{1}{2}\mathbf{m}_1\mathfrak{q}_1w+\left(\frac{1}{2}\mathfrak{m}_1(\mathfrak{q}_2-\frac{\mathfrak{q}_1^2}{2})+\frac{1}{4}\mathbf{m}_2\mathfrak{q}_1^2\right)w^2+\cdots. \tag{19}$$

For a given $f(z)$ of form (1), a computation shows the following:

$$\frac{zf'(z)}{f(z)}=1+a_2\Upsilon_2z+(2\Upsilon_3a_3-a_2^2\Upsilon_2^2)z^2+(3a_4\Upsilon_4+a_2^3\Upsilon_2^3-3a_3a_2\Upsilon_2\Upsilon_3)z^3+\cdots$$

and

$$\frac{zf''(z)}{f'(z)}=2a_2\Upsilon_2^2z+(6a_3\Upsilon_3-4a_2^2\Upsilon_2^2)z^2+\cdots.$$

Using these in the left hand side of (16) and (17), a simple computation produces the following:

$$\begin{aligned}1+\frac{1}{\vartheta}\left(\frac{\xi(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi)}+\left(\frac{1+e^{i\ell}}{2}\right)\frac{\xi^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))''}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi)}-1\right)&=1+\frac{1}{\vartheta}(2+e^{i\ell})\Upsilon_2a_2\varsigma\\&+\frac{1}{\vartheta}\left[(5+3e^{i\ell})\Upsilon_3a_3-(2+e^{i\ell})\Upsilon_2^2a_2^2\right]\varsigma^2+\cdots\end{aligned}$$

and

$$\begin{aligned}1+\frac{1}{\vartheta}\left(\frac{w(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w)}+\left(\frac{1+e^{i\ell}}{2}\right)\frac{w^2(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))''}{\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w)}-1\right)&=1-\frac{1}{\vartheta}(2+e^{i\ell})\Upsilon_2a_2w\\&+\frac{1}{\vartheta}\left([2(5+3e^{i\ell})\Upsilon_3-(2+e^{i\ell})\Upsilon_2^2]a_2^2-(5+3e^{i\ell})\Upsilon_3a_3\right)w^2=\cdots.\end{aligned}$$

Thus, by equating the coefficients of ς and ς^2 in (16) and (17), we obtain the following:

$$\frac{1}{\vartheta}(2+e^{i\ell})\Upsilon_2a_2=\frac{1}{2}\mathbf{m}_1\wp_1, \tag{20}$$

$$\frac{1}{\vartheta}\left[(5+3e^{i\ell})\Upsilon_3a_3-(2+e^{i\ell})\Upsilon_2^2a_2^2\right]=\frac{1}{2}\mathbf{m}_1(\wp_2-\frac{\wp_1^2}{2})+\frac{1}{4}\mathbf{m}_2\wp_1^2, \tag{21}$$

$$-\frac{1}{\vartheta}(2+e^{i\ell})\Upsilon_2a_2=\frac{1}{2}\mathbf{m}_1\mathfrak{q}_1, \tag{22}$$

and

$$\frac{1}{\vartheta}\left([2(5+3e^{i\ell})\Upsilon_3-(2+e^{i\ell})\Upsilon_2^2]a_2^2-(5+3e^{i\ell})\Upsilon_3a_3\right)=\frac{1}{2}\mathbf{m}_1(\mathfrak{q}_2-\frac{\mathfrak{q}_1^2}{2})+\frac{1}{4}\mathbf{m}_2\mathfrak{q}_1^2. \tag{23}$$

From (20) and (22), we obtain the following:

$$\wp_1=-\mathfrak{q}_1 \tag{24}$$

and

$$\begin{aligned} 8(2+e^{i\ell})^2 \Upsilon_2^2 a_2^2 &= \vartheta^2 \mathbf{m}_1^2 (\wp_1^2 + \mathfrak{q}_1^2) \\ a_2^2 &= \frac{\vartheta^2 \mathbf{m}_1^2 (\wp_1^2 + \mathfrak{q}_1^2)}{8(2+e^{i\ell})^2 \Upsilon_2^2}. \end{aligned} \tag{25}$$

Now, by adding (21) and (23) and then using (25), we obtain the following.

$$a_2^2 = \frac{\vartheta^2 \mathbf{m}_1^3 (\wp_2 + \mathfrak{q}_2)}{(4\{\vartheta[(5+3e^{i\ell})\Upsilon_3 - (2+e^{i\ell})\Upsilon_2^2]\mathbf{m}_1^2 + (2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\Upsilon_2^2\})}. \tag{26}$$

Applying Lemma (1) to the coefficients $\wp_2$ and $\mathfrak{q}_2$, we have the following.

$$|a_2| \le \frac{|\vartheta| \mathbf{m}_1 \sqrt{\mathbf{m}_1}}{\sqrt{|\vartheta[(5+3e^{i\ell})\Upsilon_3 - (2+e^{i\ell})\Upsilon_2^2]\mathbf{m}_1^2 + (2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\Upsilon_2^2|}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (21) from (23) and using (24), we obtain the following.

$$\begin{aligned} \frac{4}{\vartheta}\frac{(5+3e^{i\ell})}{2}\Upsilon_3(a_3 - a_2^2) &= \frac{\mathbf{m}_1}{2}(\wp_2 - \mathfrak{q}_2) \\ a_3 &= a_2^2 + \frac{\vartheta \mathbf{m}_1 (\wp_2 - \mathfrak{q}_2)}{4(5+3e^{i\ell})\Upsilon_3}. \end{aligned} \tag{27}$$

Substituting the value of a_2^2 given by (25), we obtain the following.

$$a_3 = \frac{\vartheta^2 \mathbf{m}_1^2 (\wp_1^2 + \mathfrak{q}_1^2)}{8(2+e^{i\ell})^2 \Upsilon_2^2} + \frac{\vartheta \mathbf{m}_1 (\wp_2 - \mathfrak{q}_2)}{4(5+3e^{i\ell})\Upsilon_3}.$$

Applying Lemma 1 once again to the coefficients $\wp_1, \wp_2, \mathfrak{q}_1$ and $\mathfrak{q}_2$, we obtain the following.

$$|a_3| \le \frac{|\vartheta|^2 \mathbf{m}_1^2}{|2+e^{i\ell}|^2 \Upsilon_2^2} + \frac{|\vartheta| \mathbf{m}_1}{|5+3e^{i\ell}|\Upsilon_3}.$$

□

Theorem 2. *Let f given by (1) be in the following class: $\mathfrak{K}_{\Sigma,\mathfrak{W}}^{\varsigma,\tau}(\vartheta,\ell)$, $\vartheta \in \mathbb{C}\backslash\{0\}$ and $\ell \in (-\pi,\pi]$. Then, we have the following:*

$$|a_2| \le \frac{|\vartheta| \mathbf{m}_1 \sqrt{\mathbf{m}_1}}{\sqrt{|\vartheta[3(5+3e^{i\ell})\Upsilon_3 - 4(2+e^{i\ell})\Upsilon_2^2]\mathrm{m}_1^2 + 4(2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\Upsilon_2^2|}} \tag{28}$$

and

$$|a_3| \le \frac{|\vartheta|^2 \mathbf{m}_1^2}{4|2+e^{i\ell}|^2 \Upsilon_2^2} + \frac{|\vartheta| \mathbf{m}_1}{3|5+3e^{i\ell}|\Upsilon_3}. \tag{29}$$

Proof. By Definition 2,the argument inequalities in (10) and (11) can be equivalently written as follows:

$$1 + \frac{1}{\vartheta}\left(\frac{[\xi(\mathfrak{I}_\kappa^{\varsigma,\tau} f(\xi))' + \left(\frac{1+e^{i\ell}}{2}\right)\xi^2(\mathfrak{I}_\kappa^{\varsigma,\tau} f(\xi))'']'}{(\mathfrak{I}_\kappa^{\varsigma,\tau} f(\xi))'} - 1\right) = \mathfrak{W}(u(\xi)) \tag{30}$$

and

$$1+\frac{1}{\vartheta}\left(\frac{[w(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'+\left(\frac{1+e^{i\ell}}{2}\right)w^{2}(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'']'}{(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'}-1\right)=\mathfrak{W}(v(w)), \tag{31}$$

and proceeding as in the proof of Theorem 1, we can arrive at the following relations:

$$1+\frac{1}{\vartheta}\left(\frac{[\xi(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'+\left(\frac{1+e^{i\ell}}{2}\right)\xi^{2}(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'']'}{(\mathfrak{I}_{\kappa}^{\varsigma,\tau}f(\xi))'}-1\right) = 1+\tfrac{2}{\vartheta}(2+e^{i\ell})\Upsilon_2 a_2\varsigma$$
$$+\frac{1}{\vartheta}[3(5+3e^{i\ell})\Upsilon_3 a_3-4(2+e^{i\ell})\Upsilon_2^2 a_2^2]\varsigma^2+\cdots$$

and

$$1+\frac{1}{\vartheta}\left(\frac{[w(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'+\left(\frac{1+e^{i\ell}}{2}\right)w^{2}(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'']'}{(\mathfrak{I}_{\kappa}^{\varsigma,\tau}g(w))'}-1\right)=1-\frac{2}{\vartheta}(2+e^{i\ell})\Upsilon_2 a_2 w$$
$$+\frac{1}{\vartheta}[3(5+3e^{i\ell})(2a_2^2-a_3)\Upsilon_3-4(2+e^{i\ell})\Upsilon_2^2 a_2^2]w^2+\cdots.$$

From (30) and (31), equating the coefficients of ς and ς^2, we obtain the following:

$$\frac{2}{\vartheta}(2+e^{i\ell})\Upsilon_2 a_2=\frac{1}{2}\mathbf{m}_1\wp_1, \tag{32}$$

$$\frac{1}{\vartheta}[3(5+3e^{i\ell})\Upsilon_3 a_3-4(2+e^{i\ell})\Upsilon_2^2 a_2^2]=\frac{1}{2}\mathbf{m}_1(\wp_2-\frac{\wp_1^2}{2})+\frac{1}{4}\mathbf{m}_2\wp_1^2, \tag{33}$$

and

$$-\frac{2}{\vartheta}(2+e^{i\ell})\Upsilon_2 a_2=\frac{1}{2}\mathbf{m}_1\mathfrak{q}_1, \tag{34}$$

$$\frac{1}{\vartheta}[3(5+3e^{i\ell})(2a_2^2-a_3)\Upsilon_3-4(2+e^{i\ell})\Upsilon_2^2 a_2^2]=\frac{1}{2}\mathbf{m}_1(\mathfrak{q}_2-\frac{\mathfrak{q}_1^2}{2})+\frac{1}{4}\mathbf{m}_2\mathfrak{q}_1^2. \tag{35}$$

From (32) and (34), we obtain the following:

$$\wp_1=-\mathfrak{q}_1 \tag{36}$$

and

$$32(2+e^{i\ell})^2\Upsilon_2^2 a_2^2=\vartheta^2\mathbf{m}_1^2(\wp_1^2+\mathfrak{q}_1^2). \tag{37}$$

If we add (33) and (35) and substitute value $\wp_1^2+\mathfrak{q}_1^2$, we obtain the following.

$$a_2^2=\frac{\vartheta^2\mathbf{m}_1^3(\wp_2+\mathfrak{q}_2)}{4[\vartheta[3(5+3e^{i\ell})\Upsilon_3-4(2+e^{i\ell})\Upsilon_2^2]\mathbf{m}_1^2+4(2+e^{i\ell})^2(\mathbf{m}_1-\mathbf{m}_2)\Upsilon_2^2]}. \tag{38}$$

Applying Lemma 1 to the coefficients $\wp_2$ and $\mathfrak{q}_2$, we have the desired inequality given in (28).

Next, if we subtract (33) from (35), we easily observe the following.

$$\begin{aligned}\frac{12}{\vartheta}\frac{(5+3e^{i\ell})}{2}(a_3-a_2^2)\Upsilon_3 &= \frac{\mathbf{m}_1}{2}(\wp_2-\mathfrak{q}_2)\\ a_3 &= \frac{\vartheta\mathbf{m}_1(\wp_2-\mathfrak{q}_2)}{12(5+3e^{i\ell})\Upsilon_3}+a_2^2\end{aligned}$$

Upon relieving the value of a_2^2 given in (37), the above equation leads to the following.

$$a_3 = \frac{\vartheta \mathbf{m}_1(\wp_2 - \mathfrak{q}_2)}{12(5+3e^{i\ell})\Upsilon_3} + \frac{\vartheta^2 \mathbf{m}_1^2(\wp_1^2 + \mathfrak{q}_1^2)}{32(2+e^{i\ell})^2\Upsilon_2^2}.$$

Applying Lemma (1) once again to the coefficients $\wp_1$, $\wp_2, \mathfrak{q}_1$, and $\mathfrak{q}_2$, we obtain the preferred coefficient provided in (29). □

Fixing $\ell = \pi$ in Theorems (1) and (2), we can state the coefficient estimates for the functions in subclasses $\mathfrak{S}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta)$ and $\mathfrak{K}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta)$, defined in Remark (2).

Corollary 1. *Let f assumed as (1) be in the class $\mathfrak{S}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta)$. Then, the following is the case.*

$$|a_2| \leq \frac{|\vartheta|\mathbf{m}_1\sqrt{\mathbf{m}_1}}{\sqrt{|\vartheta|(2\Upsilon_3 - \Upsilon_2^2)\mathbf{m}_1^2 + (\mathbf{m}_1 - \mathbf{m}_2)\Upsilon_2^2}} \quad \text{and} \quad |a_3| \leq \frac{|\vartheta|^2\mathbf{m}_1^2}{\Upsilon_2^2} + \frac{|\vartheta|\mathbf{m}_1}{2\Upsilon_3}.$$

Corollary 2. *Let f assumed as (1) be in class $\mathfrak{K}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta)$. Then, we have the following.*

$$|a_2| \leq \frac{|\vartheta|\mathbf{m}_1\sqrt{\mathbf{m}_1}}{\sqrt{2|\vartheta|(3\Upsilon_3 - 2\Upsilon_2^2)\mathbf{m}_1^2 + 4(\mathbf{m}_1 - \mathbf{m}_2)\Upsilon_2^2}} \quad \text{and} \quad |a_3| \leq \frac{|\vartheta|^2\mathbf{m}_1^2}{4\Upsilon_2^2} + \frac{|\vartheta|\mathbf{m}_1}{6\Upsilon_3}.$$

Fixing $\vartheta = 1$ in Theorems (1) and (2), we can state the coefficient estimates for the functions in the subclasses $\mathfrak{S}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\ell)$ and $\mathfrak{K}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\ell)$ defined in Remark (4).

Corollary 3. *Let f supposed by (1) be in class $\mathfrak{S}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\ell)$. Then, we have the following:*

$$|a_2| \leq \frac{\mathbf{m}_1\sqrt{\mathbf{m}_1}}{\sqrt{|[(5+3e^{i\ell})\Upsilon_3 - (2+e^{i\ell})\Upsilon_2^2]\mathbf{m}_1^2 + (2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\Upsilon_2^2|}}$$

and the following is the case.

$$|a_3| \leq \frac{\mathbf{m}_1^2}{|2+e^{i\ell}|^2\Upsilon_2^2} + \frac{\mathbf{m}_1}{|5+3e^{i\ell}|\Upsilon_3}.$$

Corollary 4. *Let f supposed by (1) be in class $\mathfrak{K}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\ell)$. Then, we have the following:*

$$|a_2| \leq \frac{\mathbf{m}_1\sqrt{\mathbf{m}_1}}{\sqrt{|[3(5+3e^{i\ell})\Upsilon_3 - 4(2+e^{i\ell})\Upsilon_2^2]\mathbf{m}_1^2 + 4(2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\Upsilon_2^2|}}$$

and

$$|a_3| \leq \frac{\mathbf{m}_1^2}{4|2+e^{i\ell}|^2\Upsilon_2^2} + \frac{\mathbf{m}_1}{3|5+3e^{i\ell}|\Upsilon_3}.$$

3. Fekete-Szegő Inequality

In this section, we discuss the Fekete-Szegő results [34] due toZaprawa [35] for functions $f \in \mathfrak{S}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta,\ell)$ and $f \in \mathfrak{K}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta,\ell)$.

Theorem 3. *Let f assumed by (1) be in class $\mathfrak{S}^{\varsigma,\tau}_{\Sigma,\mathfrak{W}}(\vartheta,\ell)$ and $\varrho \in \mathbb{R}$. Then, we have the following:*

$$| a_3 - \varrho a_2^2 | \leq \begin{cases} \frac{\vartheta\mathbf{m}_1}{|5+3e^{i\ell}|\Upsilon_3}, & 0 \leq | \phi(\varrho) | \leq \frac{\vartheta\mathbf{m}_1}{4|5+3e^{i\ell}|\Upsilon_3} \\ 4|\phi(\varrho)|, & |\phi(\varrho)| \geq \frac{\vartheta\mathbf{m}_1}{4|5+3e^{i\ell}|\Upsilon_3}. \end{cases}.$$

where the following is obtained.

$$\phi(\varrho) = \frac{(1-\varrho)\vartheta^2 \mathbf{m}_1^3}{4\{\vartheta[(5+3e^{i\ell})\mathrm{Y}_3 - (2+e^{i\ell})\mathrm{Y}_2^2]\mathbf{m}_1^2 + (2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\mathrm{Y}_2^2\}}.$$

Proof. From (26) and (27), we have the following:

$$\begin{aligned} a_3 - \varrho a_2^2 &= \frac{(1-\varrho)\vartheta^2 \mathbf{m}_1^3(\wp_2 + \mathfrak{q}_2)}{(4\{\vartheta[(5+3e^{i\ell})\mathrm{Y}_3 - (2+e^{i\ell})\mathrm{Y}_2^2]\mathbf{m}_1^2 + (2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\mathrm{Y}_2^2\})} + \frac{\vartheta \mathbf{m}_1(\wp_2 - \mathfrak{q}_2)}{4(5+3e^{i\ell})\mathrm{Y}_3} \\ &= \left[\phi(\varrho) + \frac{\vartheta \mathbf{m}_1}{4(5+3e^{i\ell})\mathrm{Y}_3}\right]\wp_2 + \left[\phi(\varrho) - \frac{\vartheta \mathbf{m}_1}{4(5+3e^{i\ell})\mathrm{Y}_3}\right]\mathfrak{q}_2 \end{aligned}$$

where the following is the case.

$$\phi(\varrho) = \frac{(1-\varrho)\vartheta^2 \mathbf{m}_1^3}{4\{\vartheta[(5+3e^{i\ell})\mathrm{Y}_3 - (2+e^{i\ell})\mathrm{Y}_2^2]\mathbf{m}_1^2 + (2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\mathrm{Y}_2^2\}}$$

Thus, by applying Lemma 1, we obtain the following.

$$|\, a_3 - \varrho a_2^2 \,| \leq \begin{cases} \frac{\vartheta \mathbf{m}_1}{|5+3e^{i\ell}|\mathrm{Y}_3}, & 0 \leq |\, \phi(\varrho) \,| \leq \frac{\vartheta \mathbf{m}_1}{4|5+3e^{i\ell}|\mathrm{Y}_3} \\ 4|\phi(\varrho)|, & |\phi(\varrho)| \geq \frac{\vartheta \mathbf{m}_1}{4|5+3e^{i\ell}|\mathrm{Y}_3}. \end{cases}$$

In particular, by fixing $\varrho = 1$, we obtain the following.

$$|\, a_3 - a_2^2 \,| \leq \frac{\vartheta \mathbf{m}_1}{|5+3e^{i\ell}|\mathrm{Y}_3}.$$

□

Theorem 4. *Let f given by (1) be in class $\mathfrak{K}_{\Sigma,\mathfrak{W}}^{\mathcal{C},\tau}(\vartheta,\ell)$ and $\aleph \in \mathbb{R}$. Then, we have the following:*

$$|\, a_3 - \aleph a_2^2 \,| \leq \begin{cases} \frac{\vartheta \mathbf{m}_1}{3|5+3e^{i\ell}|\mathrm{Y}_3}, & 0 \leq |\, \phi(\aleph) \,| \leq \frac{\vartheta \mathbf{m}_1}{12|5+3e^{i\ell}|\mathrm{Y}_3} \\ 4|\phi(\aleph)|, & |\phi(\aleph)| \geq \frac{\vartheta \mathbf{m}_1}{12|5+3e^{i\ell}|\mathrm{Y}_3}. \end{cases}.$$

where

$$\phi(\aleph) = \frac{(1-\aleph)\vartheta^2 \mathbf{m}_1^3}{4[\vartheta[3(5+3e^{i\ell})\mathrm{Y}_3 - 4(2+e^{i\ell})\mathrm{Y}_2^2]\mathbf{m}_1^2 + 4(2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\mathrm{Y}_2^2]}.$$

Proof. From (27) and (38), we have the following.

$$\begin{aligned} a_3 - \aleph a_2^2 &= \frac{(1-\aleph)\vartheta^2 \mathbf{m}_1^3(\wp_2 + \mathfrak{q}_2)}{4[\vartheta[3(5+3e^{i\ell})\mathrm{Y}_3 - 4(2+e^{i\ell})\mathrm{Y}_2^2]\mathbf{m}_1^2 + 4(2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\mathrm{Y}_2^2]} + \frac{\vartheta \mathbf{m}_1(\wp_2 - \mathfrak{q}_2)}{12(5+3e^{i\ell})\mathrm{Y}_3} \\ &= \left[\phi(\aleph) + \frac{\vartheta \mathbf{m}_1}{12(5+3e^{i\ell})\mathrm{Y}_3}\right]\wp_2 + \left[\phi(\aleph) - \frac{\vartheta \mathbf{m}_1}{12(5+3e^{i\ell})\mathrm{Y}_3}\right]\mathfrak{q}_2 \end{aligned}$$

where the following is the case.

$$\phi(\aleph) = \frac{(1-\aleph)\vartheta^2 \mathbf{m}_1^3}{4[\vartheta[3(5+3e^{i\ell})\mathrm{Y}_3 - 4(2+e^{i\ell})\mathrm{Y}_2^2]\mathbf{m}_1^2 + 4(2+e^{i\ell})^2(\mathbf{m}_1 - \mathbf{m}_2)\mathrm{Y}_2^2]}.$$

Thus, by Lemma 1, we obtain the following.

$$|\, a_3 - \aleph a_2^2 \,| \leq \begin{cases} \frac{\vartheta \mathbf{m}_1}{3|5+3e^{i\ell}|\mathrm{Y}_3}, & 0 \leq |\, \phi(\aleph) \,| \leq \frac{\vartheta \mathbf{m}_1}{12|5+3e^{i\ell}|\mathrm{Y}_3} \\ 4|\phi(\aleph)|, & |\phi(\aleph)| \geq \frac{\vartheta \mathbf{m}_1}{12|5+3e^{i\ell}|\mathrm{Y}_3}. \end{cases}$$

In particular, by taking $\aleph = 1$, we obtain the following.

$$| a_3 - a_2^2 | \leq \frac{\vartheta \mathbf{m}_1}{3|5 + 3e^{i\ell}|\Upsilon_3}.$$

□

4. Conclusions

By fixing $\mathfrak{W}(\xi)$ as listed below, one can determine new results as in Theorems 1–4 for the subclasses introduced in this paper by suitably fixing $\mathbf{m}_1$ and $\mathbf{m}_2$:

1. For the class of strongly starlike functions, function $\mathfrak{W}$ is given by $\mathfrak{W}(\xi) = \left(\frac{1+\xi}{1-\xi}\right)^\alpha = 1 + 2\alpha\xi + 2\alpha^2\xi^2 + \cdots \quad (0 < \alpha \leq 1)$, which gives $\mathbf{m}_1 = 2\alpha$ and $\mathbf{m}_2 = 2\alpha^2$, (see [36]);
2. On the other hand, if we take $\mathfrak{W}(\xi) = \frac{1+(1-2\beta)\xi}{1-\xi} = 1 + 2(1-\beta)\xi + 2(1-\beta)\xi^2 + \cdots \quad (0 \leq \beta < 1)$, then $\mathbf{m}_1 = \mathbf{m}_2 = 2(1-\beta)$, (see [36]);
3. For $\mathfrak{W}(\xi) = \frac{1+A\xi}{1+B\xi}$ $(-1 \leq B < A \leq 1)$, we obtain class $\mathfrak{S}^*(A, B)$ (see [37]);
4. For $\mathfrak{W}(\xi) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\right)^2$, which was considered and studied in [38];
5. For $\mathfrak{W}(\xi) = \sqrt{1+\xi}$, the class is denoted by $\mathfrak{S}^*_L$, , which was considered and studied in [39] further in discussed [40];
6. For $\mathfrak{W}(\xi) = \xi + \sqrt{1+\xi^2}$, the class is denoted by $\mathfrak{S}^*_l$ (see [41]);
7. If $\mathfrak{W}(\xi) = 1 + \frac{4}{3}\xi + \frac{2}{3}\xi^2$, then such class denoted by $\mathfrak{S}^*_C$ was introduced in [42] and further studied by [43];
8. For $\mathfrak{W}(\xi) = e^\xi$, class $\mathfrak{S}^*_e$ was defined and studied in [44,45];
9. For $\mathfrak{W}(\xi) = \cosh(\xi)$, the class is denoted by $\mathfrak{S}^*_{\cosh}$ (see [46]);
10. For $\mathfrak{W}(\xi) = 1 + \sin(\xi)$, the class is denoted by $\mathfrak{S}^*_{\sin}$ (see [47]); for details and further investigation, (see [48]).

In the current paper, we mainly obtain the upper bounds of the initial Taylors coefficients of bi-starlike and bi-convex functions of complex order involving Erdély–Kober-type integral operators in the open unit. Furthermore, we find the Fekete-Szegő inequalities for the function in these classes. Several consequences of the results are also pointed out as examples. Moreover, we note that by assuming $\mathfrak{W}$ with some particular functions as illustrated above, one can determine new results for the subclasses introduced in this paper. Moreover, by fixing $\ell = 0$ and $\ell = \pi$ in the above Theorems, we can easily state the results for various subclasses of Σ illustrated in Remarks 2–4. By appropriately fixing the parameters in Theorems 3 and 4, we can deduce the Fekete-Szegő functional for these function classes. Moreover, motivating further research on the subject-matter of this, we have chosen to draw the attention of the concerned readers toward a significantly large number of interrelated publications(see [49–52]) and developments in the area of Geometric Function Theory of Complex Analysis. In conclusion, we choose to reiterate an important observation, which was offered in the recently published survey-cum-expository article by Srivastava ([49], p. 340), who pointed out the fact that the results for the above-mentioned or new $q-$ analogues can easily (and possibly or unimportantly) be interpreted into the equivalent results for the so-called $(p;q)-$ analogues (with $0 < |q| < p \leq 1$) by smearing some recognizable parametric and argument variations with the additional parameter p being redundant.

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References

1. Brannan, D.A.; Taha, T.S. On some classes of bi-univalent functions. *Studia Univ. Babeş-Bolyai Math.* **1986**, *31*, 70–77.
2. Taha, T.S. Topics in Univalent Function Theory. Ph.D. Thesis, University of London, London, UK, 1981.
3. Brannan, D.A.; Clunie, J. *Aspects of Contemporary Complex Analysis*; Academic Press: New York, NY, USA; London, UK, 1980.
4. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **1967**, *18*, 63–68. [CrossRef]
5. Netanyahu, E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Arch. Rational. Mech. Anal.* **1969**, *32*, 100–112.
6. Tang, H.; Deng, G.T.; Li, S.H. Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions. *J. Inequal. Appl.* **2013**, *2013*, 317. [CrossRef]
7. Deniz, E. Certain subclasses of bi-univalent functions satisfying subordinate conditions. *J. Class. Anal.* **2013**, *2*, 49–60. [CrossRef]
8. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23* 1188–1192. [CrossRef]
9. Çaglar, M.; Orhan, H.; Yagmur, N. Coefficient bounds for new subclasses of bi-univalent functions. *Filomat* **2013**, *27*, 1165–1171. [CrossRef]
10. Xu, Q.-H.; Gui, Y.-C.; Srivastava, H.M. Coefficient estimates for a certain subclass of analytic and bi-univalent functions. *Appl. Math. Lett.* **2012**, *25*, 990–994. [CrossRef]
11. Xu, Q.-H.; Xiao, H.-G.; Srivastava, H.M. A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. *Appl. Math. Comput.* **2012**, *218*, 11461–11465. [CrossRef]
12. Ma, W.C.; Minda, D. A unified treatment of some special classes of functions. In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; pp. 157–169.
13. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Pitman Research Notes in Mathematics Series, 301; John Willey & Sons, Inc.: New York, NY, USA, 1994.
14. Erdelyi, A.; Magnus,W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill: New York, NY, USA; Toronto, ON, Canada; London, UK, 1953; Volume I.
15. Akdemir, A.O.; Karaoblan, A.; Ragusa, M.A.; Set, E. Fractional integral inequalities via Atangana-Baleanu operators for convex and concave functions. *J. Funct.* **2021**, *2021*, 1055434. [CrossRef]
16. Luchko, Y. The four-parameters Wright function of the second kind and its applications in FC. *Mathematics* **2020**, *8*, 970. [CrossRef]
17. Tang, J.H.; Yin, C.T. Analysis of the generalized fractional diferential system. *Aims Math.* **2022**, *7*, 8654–8684. [CrossRef]
18. Jung, I.B.; Kim Y.C.; Srivastava, H.M. The Hardy space of analytic functions associated with certain one-parameter families of integral operators. *J. Math. Anal. Appl.* **1993**, *176*, 138–147. [CrossRef]
19. Carlson, B.C.; Shafer, D.B. Starlike and prestarlike Hypergeometric functions. *J. Math. Anal.* **1984**, *15*, 737–745. [CrossRef]
20. Choi, J.H.; Saigo, M.; Srivastava, H.M. Some inclusion properties of a certain family of integral operators. *J. Math. Anal. Appl.* **2002**, *276*, 432–445. [CrossRef]
21. Ruscheweyh, S. New criteria for univalent functions. *Proc. Am. Math. Soc.* **1975**, *49*, 109–115. [CrossRef]
22. Noor, K.I. On new classes of integral operators. *J. Nat. Geom.* **1999**, *16*, 71–80.
23. Noor, K.I.; Noor, M.A. On integral operators. *J. Math. Anal. Appl.* **1999**, *238*, 341–352. [CrossRef]
24. Bernardi, S.D. Convex and starlike univalent functions. *Trans. Am. Math. Soc.* **1969**, *135*, 429–446. [CrossRef]
25. Libera, R.J. Some classes of regular univalent functions. *Proc. Am. Math. Soc.* **1965**, *16*, 755–758. [CrossRef]
26. Livingston, A.E. On the radius of univalence of certain analytic functions. *Proc. Am. Math. Soc.* **1966**, *17*, 352–357. [CrossRef]
27. Silverman, H.; Silvia, E.M. Characterizations for subclasses of univalent functions. *Math. Jpn.* **1999**, *50*, 103–109.
28. Silverman, H. A class of bounded starlike functions. *Int. J. Math. Math. Sci.* **1994**, *17*, 249–252. [CrossRef]
29. El-Deeb, S.M. Maclaurin Coefficient estimates for new subclasses of bi-univalent functions connected with a q-analogue of Bessel function. *Abstr. Appl. Anal.* **2020**, *2020*, 1–7. [CrossRef]
30. El-Deeb, S.M.; Bulboacă, T.; El-Matary, B.M. Maclaurin coefficient estimates of bi-univalent functions connected with the q-derivative. *Mathematics* **2020**, *8*, 418. [CrossRef]
31. El-Deeb, S.M.; El-Matary, B.M. Subclasses of bi-univalent functions associated with q-confluent hypergeometric distribution based upon the Horadam polynomials. *Adv. Theory Nonlinear Anal. Appl.* **2021**, *5*, 82–93. [CrossRef]

32. Murugusundaramoorthy, G.; Janani, T. Bi-starlike function of complex order associated with double Zeta functions. *Afr. Mat.* **2015**, *26*, 1025–1036. [CrossRef]
33. Pommerenke, C. *Univalent Functions*; Vandenhoeck & Ruprecht, Göttingen: Göttingen, Germany, 1975.
34. Fekete, M.; Szegő, G. Eine Bemerkungüber ungerade schlichte Functionen. *J. Lond. Math. Soc.* **1933**, *8*, 85–89. [CrossRef]
35. Zaprawa, P. On the Fekete-Szegö problem for classes of bi-univalent functions. *Bull. Belg. Math. Soc. Simon Stevin* **2014**, *21*, 169–178. [CrossRef]
36. Robertson, M.S. Certain classes of starlike functions. *Mich. Math. J.* **1985**, *32*, 135–140. [CrossRef]
37. Janowski, W. Extremal problems for a family of functions with positive real part and for some related families. *Ann. Pol. Math.* **1970**, *23*, 159–177. [CrossRef]
38. Ronning, F. Uniformly convex functions and a corresponding class of starlike functions. *Proc. Am. Math. Soc.* **1993**, *118*, 189–196. [CrossRef]
39. Sokół, J. Radius problem in the class $\mathfrak{SL}^*$. *Appl. Math. Comput.* **2009**, *214*, 569–573. [CrossRef]
40. Mohsin, M.; Malik, S.N. Upper bound of third hankel determinant for class of analytic functions related with lemniscate of bernoulli. *J. Inequal. Appl.* **2013**, *2013*, 412. [CrossRef]
41. Raina, R.K.; Sokół, J. On coefficient estimates for a class of starlike functions. *Hacet. J. Math. Statist.* **2015**, *44*, 1427–1433. [CrossRef]
42. Sharma, K.; Jain, N.K.; Ravichandran, V. Starlike functions associated with a cardioid. *Afr. Math.* **2016**, *27*, 923–939. [CrossRef]
43. Shi, L.; Izaz, A.; Arif, M.; Cho, N.E.; Hussain, S.; Hassan, K.A. Study of third hankel determinant problem for certain subfamilies of analytic functions involving cardioid domain. *Mathematics* **2019**, *7*, 418. [CrossRef]
44. Mendiratta, R.; Nagpal, S.; Ravichandran, V. On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 365–386. [CrossRef]
45. Shi, L.; Srivastava, H.M.; Arif, M.; Hussain, S.; Khan, H. An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function. *Symmetry* **2019**, *11*, 598. [CrossRef]
46. Alotaibi, A.; Arif, M.; Alghamdi, M.A.; Hussain, S. Starlikeness associated with cosine hyperbolic function. *Mathematics* **2020**, *8*, 16. [CrossRef]
47. Cho, N.E.; Kumar, V.; Kumar, S.S.; Ravichandran, V. Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc.* **2019**, *45*, 213–232. [CrossRef]
48. Arif, M.; Raza, M.; Huo, T.; Hussain, S.; Khan, H. Hankel determinant of order three for familiar subsets of analytic functions related with sine function. *Open Math.* **2019**, *17*, 1615–1630. [CrossRef]
49. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in Geometric Function theory of Complex Analysis. *Iran. J. Sci. Technol. Trans. Sci.* **2020**, *44*, 327–344. [CrossRef]
50. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Wiley: New York, NY, USA, 1985.
51. Srivastava, H.M.; Raducanu, D.; Zaprawa, P.A. Certain subclass of analytic functions defined by means of differential subordination. *Filomat* **2016**, *30*, 3743–3757. [CrossRef]
52. Srivastava, H.M. Certain q-polynomial expansions for functions of several variables. I and II. *IMA J. Appl. Math.* **1983**, *30*, 205–209. [CrossRef]