axioms

MDPI

Article

Two Invariants for Geometric Mappings

Nenad O. Vesić [1,†], Vladislava M. Milenković [2,†] and Mića S. Stanković [3,*,†]

[1] Mathematical Institute of Serbian Academy of Sciences and Arts, 11000 Belgrade, Serbia; n.o.vesic@outlook.com
[2] Faculty of Technology, University of Niš, 16000 Leskovac, Serbia; vanja.dunja91@gmail.com
[3] Faculty of Science and Mathematics, University of Niš, 18000 Niš, Serbia
* Correspondence: mica.stankovic@pmf.edu.rs
† These authors contributed equally to this work.

Abstract: Two invariants for mappings of affine connection spaces with a special form of deformation tensors are obtained in this paper. We used the methodology of Vesić to obtain the form of these invariants. At the end of this paper, we used these forms to obtain two invariants for third-type almost-geodesic mappings of symmetric affine connection.

Keywords: affine connection space; Riemannian space; geometric mapping; deformation tensor; invariants of geometric mapping

check for updates

Citation: Vesić, N.O.; Milenković, V.M.; Stanković, M.S. Two Invariants for Geometric Mappings. *Axioms* **2022**, *11*, 239. https://doi.org/10.3390/axioms11050239

Academic Editor: Anna Maria Fino

Received: 11 May 2022
Accepted: 19 May 2022
Published: 20 May 2022

1. Introduction

Invariants for different mappings of symmetric and non-symmetric affine connection spaces have been obtained by different authors. The generalizations of the Weyl conformal and the Weyl projective tensor and the Thomas projective parameters are objects that have been generalized in different papers about invariants for geometric mappings.

Vesić [1] developed the methodology of obtaining invariants for mappings defined on symmetric and non-symmetric affine connection spaces. We develop one result obtained in [1] below.

1.1. Affine Connection Spaces

An N-dimensional manifold $\mathcal{M}_N$ equipped with an affine connection ∇ is the affine connection space. If this affine connection is torsion-free, i.e., if

$$\nabla_X Y \equiv \overset{0}{\nabla}_X Y, \quad \nabla_X Y - \nabla_Y X = [X, Y],$$

the pair $(\mathcal{M}_N, \overset{0}{\nabla})$ is symmetric affine connection space $\mathbb{A}_N$ (see [2,3]).

The affine connection coefficients of the space $\mathbb{A}_N$ are $L^i_{\underline{jk}}$, $L^i_{\underline{jk}} = L^i_{\underline{kj}}$.

The partial derivative of a tensor a^i_j of the type $(1,1)$ by x^k, $\partial a^i_j/\partial x^k = a^i_{j,k}$, is not a tensor. the covariant derivative $a^i_{j|k}$ of the tensor a^i_j by x^k is the tensor of the type $(1,2)$, whose components are

$$a^i_{j|k} = a^i_{j,k} + L^i_{\underline{\alpha k}} a^\alpha_j - L^\alpha_{\underline{jk}} a^i_\alpha. \tag{1}$$

Remark 1. *For a tensor $A^{i_1\ldots i_p}_{j_1\ldots j_q}$ of the type (p,q), the partial derivative $A^{i_1\ldots i_p}_{j_1\ldots j_q,k}$ is not a tensor, but the tensor is the corresponding covariant derivative:*

$$A^{i_1\ldots i_p}_{j_1\ldots j_q|k} = A^{i_1\ldots i_p}_{j_1\ldots j_q,k} + \sum_{u=1}^{p} L^{i_u}_{\underline{\alpha k}} a^{i_1\ldots i_{u_1}\alpha i_{u+1}\ldots i_p}_{j_1\ldots j_q} - \sum_{v=1}^{q} L^{\alpha}_{\underline{j_v k}} a^{i_1\ldots i_p}_{j_1\ldots j_{v-1}\alpha j_{v+1}\ldots j_q}. \tag{2}$$

With respect to symmetric affine connection $\overset{0}{\nabla}_X Y$ and for the tensor a^i_j of type $(1,1)$, one Ricci identity exists [2,3]:

$$a^i_{j|m|n} - a^i_{j|n|m} = a^\alpha_j \overset{0}{R}{}^i_{\alpha mn} - a^i_\alpha \overset{0}{R}{}^\alpha_{jmn}, \tag{3}$$

for the curvature tensor $\overset{0}{R}{}^i_{jmn}$ of the space $\mathbb{A}_N$ given as

$$\overset{0}{R}{}^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^\alpha_{\underline{jm}} L^i_{\underline{\alpha n}} - L^\alpha_{\underline{jn}} L^i_{\underline{\alpha m}}. \tag{4}$$

The Ricci tensor of space $\mathbb{A}_N$ is

$$\overset{0}{R}_{ij} = \overset{0}{R}{}^\alpha_{ij\alpha} = L^\alpha_{\underline{ij},\alpha} - L^\alpha_{\underline{i\alpha},j} + L^\alpha_{\underline{ij}} L^\beta_{\underline{\alpha\beta}} - L^\alpha_{\underline{i\beta}} L^\beta_{\underline{j\alpha}}. \tag{5}$$

By the anti-symmetrization of the Ricci tensor $\overset{0}{R}_{ij}$ without division, the next geometrical object is obtained:

$$\overset{0}{R}_{[ij]} = \overset{0}{R}_{ij} - \overset{0}{R}_{ji} = -L^\alpha_{\underline{i\alpha},j} + L^\alpha_{\underline{j\alpha},i} = -L^\alpha_{[\underline{i\alpha},j]}. \tag{6}$$

1.2. Riemannian Spaces

Special symmetric affine connection spaces are *the Riemannian spaces* [2–4].

Let a symmetric metric tensor $\hat{g}$ of the type $(0,2)$, whose components are $g_{\underline{ij}}$, $g_{\underline{ij}} = g_{\underline{ji}}$, be defined at any point of the manifold $\mathcal{M}_N$. The pair $(\mathcal{M}_N, \hat{g})$ is Riemannian space $\mathbb{R}_N$ (see [2–4]).

We assume that the matrix $[g_{\underline{ij}}]$ is non-degenerate, i.e., $g = \det[g_{\underline{ij}}] \neq 0$. The components of the contravariant metric tensor are $g^{\underline{ij}}$, determined by $[g^{\underline{ij}}] = [g_{\underline{ij}}]^{-1}$.

The Christoffel symbols $\Gamma^i_{\underline{jk}}$ uniquely determine the affine connection ∇^g of the space $\mathbb{R}_N$. The affine connection coefficients of $\mathbb{R}_N$ are $\Gamma^i_{\underline{jk}}$.

The next equation holds:

$$\Gamma^\alpha_{\underline{i\alpha}} = \Gamma^\alpha_{\underline{\alpha i}} = \frac{1}{2\sqrt{|g|}} \frac{\partial |g|}{\partial x^k} = \frac{1}{2}|g|^{-1/2}|g|_{,i}. \tag{7}$$

Analogously to the case of space $\mathbb{A}_N$, covariant derivative of the tensor a^i_j by x^k with respect to the affine connection ∇^g is defined as [2,3]

$$a^i_{j|^g k} = a^i_{j,k} + \Gamma^i_{\underline{\alpha k}} a^\alpha_j - \Gamma^\alpha_{\underline{jk}} a^i_\alpha. \tag{8}$$

The corresponding Ricci identity is [2,3]

$$a^i_{j|^g m|^g n} - a^i_{j|^g n|^g m} = a^\alpha_j R^{g\,i}{}_{\alpha mn} - a^i_\alpha R^{g\,\alpha}{}_{jmn}, \tag{9}$$

where

$$R^{g\,i}{}_{jmn} = \Gamma^i_{\underline{jm},n} - \Gamma^i_{\underline{jn},m} + \Gamma^\alpha_{\underline{jm}} \Gamma^i_{\underline{\alpha n}} - \Gamma^\alpha_{\underline{jn}} \Gamma^i_{\underline{\alpha m}}, \tag{10}$$

is the curvature tensor of space $\mathbb{R}_N$.

The Ricci tensor of space $\mathbb{R}_N$ is

$$R^g_{ij} = R^{g\,\alpha}{}_{ij\alpha} = \Gamma^\alpha_{\underline{ij},\alpha} - \Gamma^\alpha_{\underline{i\alpha},j} + \Gamma^\alpha_{\underline{ij}} \Gamma^\beta_{\underline{\alpha\beta}} - \Gamma^\alpha_{\underline{i\beta}} \Gamma^\beta_{\underline{j\alpha}}. \tag{11}$$

The scalar curvature of space $\mathbb{R}_N$ is

$$R^g = g^{\underline{\gamma\delta}} R^g{}_{\gamma\delta} = g^{\underline{\gamma\delta}} \big(\Gamma^{\alpha}_{\underline{\gamma\delta},\alpha} - \Gamma^{\alpha}_{\underline{\gamma\alpha},\delta} + \Gamma^{\alpha}_{\underline{\gamma\delta}} \Gamma^{\beta}_{\underline{\alpha\beta}} - \Gamma^{\alpha}_{\underline{\gamma\beta}} \Gamma^{\beta}_{\underline{\delta\alpha}} \big). \tag{12}$$

1.3. Geodesic Mappings

The affine connection coefficients $L^i_{\underline{jk}}$ and the Christoffel symbols $\Gamma^i_{\underline{jk}}$ are not tensors. With respect to transformation of coordinate systems $(O, x^1, \dots, x^N) \to (O', x'^1, \dots, x'^N)$, the corresponding transformation rules are [2,3]

$$\begin{aligned} L^{i'}_{\underline{j'k'}} &= x^{i'}_{\alpha} x^{\beta}_{j'} x^{\gamma}_{k'} L^{\alpha}_{\underline{\beta\gamma}} + x^{i'}_{\alpha} x^{\alpha}_{j'k'}, \\ \Gamma^{i'}_{\underline{j'k'}} &= x^{i'}_{\alpha} x^{\beta}_{j'} x^{\gamma}_{k'} \Gamma^{\alpha}_{\underline{\beta\gamma}} + x^{i'}_{\alpha} x^{\alpha}_{j'k'}, \end{aligned} \tag{13}$$

for $x^{i'}_{\alpha} = \dfrac{\partial x'^{i'}}{\partial x^{\alpha}}$, $x^{\alpha}_{j'} = \dfrac{\partial x^{\alpha}}{\partial x^{j'}}$, $x^{\alpha}_{j'k'} = \dfrac{\partial^2 x^{\alpha}}{\partial x^{j'} \partial x^{k'}}$.

The differences $\overset{0}{P}{}^i_{\underline{jk}} = \overline{L}^i_{\underline{jk}} - L^i_{\underline{jk}}$ and $P^g{}^i_{\underline{jk}} = \overline{\Gamma}^i_{\underline{jk}} - \Gamma^i_{\underline{jk}}$ are tensors. These tensors are named *the deformation tensors*.

It was found [2,3] that after adding a tensor of the type $(1,2)$, symmetric by covariant indices, to any of affine connection coefficients, $L^i_{\underline{jk}}$ or $\Gamma^i_{\underline{jk}}$, the resulting sums are affine connection coefficients. That is the motivation for studying the transformation rules of curvature tensors $\overset{0}{R}{}^i_{jmn} \to \overset{0}{\overline{R}}{}^i_{jmn}$ or $R^g{}^i_{jmn} \to \overline{R}^g{}^i_{jmn}$ caused by transformations of affine connection coefficients $L^i_{\underline{jk}} \to \overline{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \overset{0}{P}{}^i_{\underline{jk}}$ or $\Gamma^i_{\underline{jk}} \to \overline{\Gamma}^i_{\underline{jk}} = \Gamma^i_{\underline{jk}} + P^g{}^i_{\underline{jk}}$. Transformations like that are called *the mappings*.

Before we present the motivational results for our current research, we need to define the geodesic lines of manifolds [2,3].

A curve $\ell = \big(\ell^1, \dots, \ell^N\big)$ that satisfies the corresponding system of the following differential equations:

$$\frac{d^2\ell^i}{dt^2} + L^i_{\underline{\alpha\beta}} \frac{d\ell^{\alpha}}{dt} \frac{d\ell^{\beta}}{dt} = \overset{0}{\rho} \ell^i, \tag{14}$$

$$\frac{d^2\ell^i}{dt^2} + \Gamma^i_{\underline{\alpha\beta}} \frac{d\ell^{\alpha}}{dt} \frac{d\ell^{\beta}}{dt} = \rho^g \ell^i, \tag{15}$$

where $\overset{0}{\rho}$ and ρ^g are scalar functions and t is a scalar parameter, is the geodesic line of the corresponding spaces $\mathbb{A}_N$ and $\mathbb{R}_N$, respectively.

The mappings $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ and $f : \mathbb{R}_N \to \overline{\mathbb{R}}_N$, which any geodesic line of spaces $\mathbb{A}_N$ or $\mathbb{R}_N$ transform to a geodesic line of the corresponding space $\overline{\mathbb{A}}_N$ or $\overline{\mathbb{R}}_N$, are called the geodesic mappings of symmetric affine connection space $\mathbb{A}_N$ or Riemannian space $\mathbb{R}_N$, respectively.

The basic equations of geodesic mappings $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ and $f : \mathbb{R}_N \to \overline{\mathbb{R}}_N$ are [2,3]

$$\begin{aligned} \overline{L}^i_{\underline{jk}} &= L^i_{\underline{jk}} + \overset{0}{\psi}_j \delta^i_k + \overset{0}{\psi}_k \delta^i_j, \\ \overline{\Gamma}^i_{\underline{jk}} &= \Gamma^i_{\underline{jk}} + \psi^g{}_j \delta^i_k + \psi^g{}_k \delta^i_j, \end{aligned} \tag{16}$$

for the 1-forms $\overset{0}{\psi}$ and ψ^g.

Invariant geometrical structures under transformation (16) of the corresponding affine connection coefficients are the Thomas projective parameters [2,3,5]:

$$\overset{0}{T}{}^i_{jk} = L^i_{\underline{jk}} - \frac{1}{N+1} \big(L^{\alpha}_{\underline{j\alpha}} \delta^i_k + L^{\alpha}_{\underline{k\alpha}} \delta^i_j \big) \quad \text{and} \quad T^g{}^i_{jk} = \Gamma^i_{\underline{jk}} - \frac{1}{N+1} \big(\Gamma^{\alpha}_{\underline{j\alpha}} \delta^i_k + \Gamma^{\alpha}_{\underline{k\alpha}} \delta^i_j \big). \tag{17}$$

The geometrical objects that are invariant under the transformation of curvature tensors $\overset{0}{R}{}^i_{jmn}$ and $R^g{}^i_{jmn}$ caused by Equation (14) are the corresponding Weyl projective tensors [2,3,6]:

$$\overset{0}{W}{}^i_{jmn} = \overset{0}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j \overset{0}{R}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\overset{0}{R}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\overset{0}{R}_{n]j}, \tag{18}$$

$$W^g{}^i_{jmn} = R^g{}^i_{jmn} + \frac{1}{N-1}\delta^i_{[m}R^g{}_{jn]}. \tag{19}$$

The Thomas projective parameters (17) and the Weyl projective tensors (18) and (19) are invariants for the corresponding geodesic mappings.

Because geodesic mappings are not only transformations of affine connections, different authors have been motivated to obtain invariants for mappings of affine connection and Riemannian spaces.

Many authors have obtained invariants for different mappings of symmetric and non-symmetric affine connection spaces. Some of them are J. Mikeš with his research group [2,7–15], V. E. Berezovski [13–15], M.S. Stanković [16], M.Lj. Zlatanović [17,18], and many others.

These invariants are used as the motivation for obtaining invariants for mappings of non-symmetric affine connection spaces. Some interesting invariants were obtained in [17–19].

N. O. Vesić was motivated to develop the methodology for obtaining invariants for geometric mappings of symmetric and non-symmetric affine connections spaces. The corresponding results were presented in [1].

The formulas presented in [1] were applied in [19] for obtaining invariants of the corresponding geometric mappings. We were motivated by the results presented in [1] to obtain invariants for mappings determined with a deformation tensor of a special form in this paper.

1.4. Motivation from Physics and Two Kinds of Invariants

When stating the Theory of General Relativity, A. Einstein stated the corresponding principles. The most important of these principles in this paper is [20] *the Principle of General Covariance*. This principle states that the laws of physics maintain the same form under a specified set of transformations.

If we make them parallel with invariants for different geometric mappings, we may see that they have the same forms before and after transformations.

In an attempt to generalize this mathematical property of invariants for mappings, Vesić and Simjanović defined different kinds of invariance for geometrical objects.

Definition 1 (see [19])**.** *Let $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ be a mapping, and let $U^{i_1\ldots i_p}_{j_1\ldots j_q}$ be a geometrical object of the type (p,q):*

- *If the transformation f preserves the value of the object $U^{i_1\ldots i_p}_{j_1\ldots j_q}$, but changes its form to $\overline{V}^{i_1\ldots i_p}_{j_1\ldots j_q}$, then the invariance for geometrical object $U^{i_1\ldots i_p}_{j_1\ldots j_q}$ under transformation f is* valued.
- *If the transformation f preserves both the value and the form of the geometrical object $U^{i_1\ldots i_p}_{j_1\ldots j_q}$, then the invariance for the geometrical object $U^{i_1\ldots i_p}_{j_1\ldots j_q}$ under transformation f is* total.

Valued invariants for the third-type almost-geodesic mappings of a non-symmetric affine connection space and the basic condition for them to be total were obtained in [19].

1.5. Motivation

In [1], the methodology for obtaining invariants for mappings of affine connection spaces is presented. As basics for these invariants, the author used transformation rule:

$$\overline{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \overline{\omega}^i_{jk} - \omega^i_{jk}, \tag{20}$$

for geometrical objects $\overline{\omega}^i_{jk}$, ω^i_{jk} of the type $(1,2)$, such that $\overline{\omega}^i_{jk} = \omega^i_{kj}$ and $\omega^i_{jk} = \omega^i_{kj}$. Based on Equation (20), the associated basic invariants of the Thomas and Weyl type for this mapping are obtained [1]:

$$\overset{0}{\mathcal{T}}{}^i_{jk} = L^i_{\underline{jk}} - \omega^i_{jk}, \tag{21}$$

$$\overset{0}{\mathcal{W}}{}^i_{jmn} = \overset{0}{R}{}^i_{jmn} - \omega^i_{jm|n} + \omega^i_{jn|m} + \omega^\alpha_{jm}\omega^i_{\alpha n} - \omega^\alpha_{jn}\omega^i_{\alpha m}. \tag{22}$$

Moreover, Vesić considered [1] the case of difference $\overline{\omega}^i_{jk} - \omega^i_{jk}$ expressed as the sum of $\psi_j\delta^i_k + \psi_k\delta^i_j$, for 1-form ψ_j, and tensor σ^i_{jk} symmetric by j and k and obtained the single invariant of the Thomas type and two invariants of the Weyl type for a mapping. In this paper, we will develop this research with respect to expression $\sigma^i_{jk} = \overline{F}^i_{\underline{jk}} - F^i_{\underline{jk}}$ for the tensors $F^i_{\underline{jk}}$ and $\overline{F}^i_{\underline{jk}}$ of the type $(1,2)$, which are symmetric by j and k.

The main purpose of this paper is to obtain invariants for mappings whose deformation tensor is of the form $P^i_{\underline{jk}} = \psi_j\delta^i_k + \psi_k\delta^i_j + \overline{F}^i_{\underline{jk}} - F^i_{\underline{jk}}$, $\overline{F}^i_{\underline{jk}} = \overline{F}^i_{\underline{kj}}$, $F^i_{\underline{jk}} = F^i_{\underline{kj}}$. We obtained these results for mappings of symmetric affine connection spaces and point out the corresponding results of mappings defined on Riemannian spaces of Eisenhart's sense.

Sinyukov used the covariant vector q_i such that $\varphi^\alpha q_\alpha = e$, $e = \pm 1$, to obtain invariants for the third-type almost-geodesic mappings. Our next aim in this paper is to obtain the geometrical object ω^i_{jk} from the invariant [3]:

$$\begin{aligned} \underset{3}{T}{}^i_{jk} &= L^i_{\underline{jk}} + e\varphi^i q_{j|k} \\ &\quad - \frac{1}{N}\big(\delta^i_j - e\varphi^i q_j\big)\Big[L^\alpha_{\underline{k\alpha}} + e\varphi^\alpha q_{\alpha|k} + \frac{1}{N-1} q_k\big(e\varphi^\beta L^\alpha_{\underline{\beta\alpha}} + \varphi^\alpha\varphi^\beta q_{\alpha|\beta}\big)\Big] \\ &\quad - \frac{1}{N}\big(\delta^i_k - e\varphi^i q_k\big)\Big[L^\alpha_{\underline{j\alpha}} + e\varphi^\alpha q_{\alpha|j} + \frac{1}{N-1} q_j\big(e\varphi^\beta L^\alpha_{\underline{\beta\alpha}} + \varphi^\alpha\varphi^\beta q_{\alpha|\beta}\big)\Big]. \end{aligned}$$

2. Review of Basic and Derived Invariants

Let us consider a mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ whose deformation tensor is [1]

$$P^i_{\underline{jk}} = \overline{L}^i_{\underline{jk}} - L^i_{\underline{jk}} = \psi_j\delta^i_k + \psi_k\delta^i_j + \overline{F}^i_{\underline{jk}} - F^i_{\underline{kj}}, \tag{23}$$

for geometrical objects $\overline{F}^i_{\underline{jk}}$, $F^i_{\underline{jk}}$ of the type $(1,2)$ symmetric by j and k.

After contracting Equation (23) by i and k, one obtains [1]

$$\psi_j = \frac{1}{N+1}\big(\overline{L}^\alpha_{\underline{j\alpha}} - \overline{F}^\alpha_{\underline{j\alpha}}\big) - \frac{1}{N+1}\big(L^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big) \tag{24}$$

If substituting Equation (24) in (23), one obtains [1]:

$$\begin{aligned} \overline{L}^i_{\underline{jk}} - L^i_{\underline{jk}} = \overline{F}^i_{\underline{jk}} &+ \frac{1}{N+1}\Big[\delta^i_k\big(\overline{L}^\alpha_{\underline{j\alpha}} - \overline{F}^\alpha_{\underline{j\alpha}}\big) + \delta^i_j\big(\overline{L}^\alpha_{\underline{k\alpha}} - \overline{F}^\alpha_{\underline{k\alpha}}\big)\Big] \\ &- F^i_{\underline{jk}} - \frac{1}{N+1}\Big[\delta^i_k\big(L^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big) + \delta^i_j\big(L^\alpha_{\underline{k\alpha}} - F^\alpha_{\underline{k\alpha}}\big)\Big]. \end{aligned} \tag{25}$$

If we compare Equation (25) with (20), we obtain

$$\omega^i_{jk} = F^i_{\underline{jk}} + \frac{1}{N+1}\Big[\delta^i_k\big(L^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big) + \delta^i_j\big(L^\alpha_{\underline{k\alpha}} - F^\alpha_{\underline{k\alpha}}\big)\Big]. \tag{26}$$

Therefore, the corresponding basic invariants are

$$\overset{0}{\mathcal{T}}{}^i_{jk} = L^i_{\underline{jk}} - F^i_{\underline{jk}} - \frac{1}{N+1}\Big[\delta^i_k\big(L^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big) + \delta^i_j\big(L^\alpha_{\underline{k\alpha}} - F^\alpha_{\underline{k\alpha}}\big)\Big], \tag{27}$$

$$\begin{aligned}\overset{0}{\mathcal{W}}{}^i_{jmn} = {} & R^i_{jmn} + \frac{1}{N+1}\delta^i_j R_{[mn]} - F^i_{\underline{jm}|n} + F^i_{\underline{jn}|m} + F^\alpha_{\underline{jm}} F^i_{\underline{\alpha n}} - F^\alpha_{\underline{jn}} F^i_{\underline{\alpha m}} \\ & - \frac{1}{N+1}\Big[\delta^i_m\big(L^\alpha_{\underline{j\alpha}|n} - F^\alpha_{\underline{j\alpha}|n}\big) - \delta^i_n\big(L^\alpha_{\underline{j\alpha}|m} - F^\alpha_{\underline{j\alpha}|m}\big) - \delta^i_j F^\alpha_{[\underline{m\alpha}|n]}\Big] \\ & + \frac{1}{N+1}\Big[F^i_{\underline{jm}}\big(L^\alpha_{\underline{n\alpha}} - F^\alpha_{\underline{n\alpha}}\big) - F^i_{\underline{jn}}\big(L^\alpha_{\underline{m\alpha}} - F^\alpha_{\underline{m\alpha}}\big)\Big] \\ & - \frac{1}{N+1}\delta^i_{[m} F^\alpha_{\underline{jn}]}\big(L^\beta_{\underline{\alpha\beta}} - F^\beta_{\underline{\alpha\beta}}\big) - \frac{1}{(N+1)^2}\delta^i_m\big(L^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big)\big(L^\alpha_{\underline{n\alpha}} - F^\alpha_{\underline{n\alpha}}\big) \\ & + \frac{1}{(N+1)^2}\delta^i_n\big(L^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big)\big(L^\alpha_{\underline{m\alpha}} - F^\alpha_{\underline{m\alpha}}\big),\end{aligned} \tag{28}$$

for $L^i_{\underline{jm}|n} = L^i_{\underline{jm},n} + L^i_{\underline{\alpha n}} L^\alpha_{\underline{jm}} - L^\alpha_{\underline{jn}} L^i_{\underline{\alpha m}} - L^\alpha_{\underline{mn}} L^i_{\underline{j\alpha}}$, i.e., $L^\alpha_{\underline{i\alpha}|j} = L^\alpha_{\underline{i\alpha},j} - L^\beta_{\underline{\alpha\beta}} L^\alpha_{\underline{ij}}$.

The basic invariant $\overset{0}{\mathcal{W}}{}^i_{jmn}$ may be expressed as

$$\begin{aligned}\overset{0}{\mathcal{W}}{}^i_{jmn} = {} & R^i_{jmn} + \frac{1}{N+1}\delta^i_j R_{[mn]} - F^i_{\underline{jm}|n} + F^i_{\underline{jn}|m} + F^\alpha_{\underline{jm}} F^i_{\underline{\alpha n}} - F^\alpha_{\underline{jn}} F^i_{\underline{\alpha m}} \\ & + \frac{1}{N+1}\Big[F^i_{\underline{jm}}\big(L^\alpha_{\underline{n\alpha}} - F^\alpha_{\underline{n\alpha}}\big) - F^i_{\underline{jn}}\big(L^\alpha_{\underline{m\alpha}} - F^\alpha_{\underline{m\alpha}}\big)\Big] + \frac{1}{N+1}\delta^i_j F^\alpha_{[\underline{m\alpha}|n]} \\ & + \delta^i_m \mathcal{Q}_{jn} - \delta^i_n \mathcal{Q}_{jm},\end{aligned} \tag{29}$$

for

$$\mathcal{Q}_{ij} = -\frac{1}{N+1}\Big[L^\alpha_{\underline{i\alpha}|j} - F^\alpha_{\underline{i\alpha}|j} + F^\alpha_{\underline{ij}}\big(L^\beta_{\underline{\alpha\beta}} - F^\beta_{\underline{\alpha\beta}}\big)\Big] - \frac{1}{(N+1)^2}\big(L^\alpha_{\underline{i\alpha}} - F^\alpha_{\underline{i\alpha}}\big)\big(L^\beta_{\underline{j\beta}} - F^\beta_{\underline{j\beta}}\big). \tag{30}$$

The transformed invariant $\overset{0}{\overline{\mathcal{W}}}{}^i_{jmn}$ is

$$\begin{aligned}\overset{0}{\overline{\mathcal{W}}}{}^i_{jmn} = {} & \overline{R}^i_{jmn} + \frac{1}{N+1}\delta^i_j \overline{R}_{[mn]} - \overline{F}^i_{\underline{jm}\|n} + \overline{F}^i_{\underline{jn}\|m} + \overline{F}^\alpha_{\underline{jm}} \overline{F}^i_{\underline{\alpha n}} - \overline{F}^\alpha_{\underline{jn}} \overline{F}^i_{\underline{\alpha m}} \\ & + \frac{1}{N+1}\Big[\overline{F}^i_{\underline{jm}}\big(\overline{L}^\alpha_{\underline{n\alpha}} - \overline{F}^\alpha_{\underline{n\alpha}}\big) - \overline{F}^i_{\underline{jn}}\big(\overline{L}^\alpha_{\underline{m\alpha}} - \overline{F}^\alpha_{\underline{m\alpha}}\big)\Big] + \frac{1}{N+1}\delta^i_j \overline{F}^\alpha_{[\underline{m\alpha}\|n]} \\ & + \delta^i_m \overline{\mathcal{Q}}_{jn} - \delta^i_n \overline{\mathcal{Q}}_{jm},\end{aligned} \tag{31}$$

for

$$\overline{\mathcal{Q}}_{ij} = -\frac{1}{N+1}\Big[\overline{L}^\alpha_{\underline{i\alpha}\|j} - \overline{F}^\alpha_{\underline{i\alpha}\|j} + \overline{F}^\alpha_{\underline{ij}}\big(\overline{L}^\beta_{\underline{\alpha\beta}} - \overline{F}^\beta_{\underline{\alpha\beta}}\big)\Big] - \frac{1}{(N+1)^2}\big(\overline{L}^\alpha_{\underline{i\alpha}} - \overline{F}^\alpha_{\underline{i\alpha}}\big)\big(\overline{L}^\beta_{\underline{j\beta}} - \overline{F}^\beta_{\underline{j\beta}}\big). \tag{32}$$

The equality $0 = \overset{0}{\overline{\mathcal{W}}}{}^i_{jmn} - \overset{0}{\mathcal{W}}{}^i_{jmn}$, i.e.,

$$\begin{aligned}0 = {}&\overline{R}^i_{jmn} - R^i_{jmn} + \frac{1}{N+1}\delta^i_j\big(\overline{R}_{[mn]} - R_{[mn]}\big)\\ &- \overline{F}^i_{\underline{jm}\|n} + \overline{F}^i_{\underline{jn}\|m} + \overline{F}^\alpha_{\underline{jm}}\overline{F}^i_{\underline{\alpha n}} - \overline{F}^\alpha_{\underline{jn}}\overline{F}^i_{\underline{\alpha m}} + F^i_{\underline{jm}|n} - F^i_{\underline{jn}|m} - F^\alpha_{\underline{jm}}F^i_{\underline{\alpha n}} + F^\alpha_{\underline{jn}}F^i_{\underline{\alpha m}}\\ &+ \frac{1}{N+1}\Big[\overline{F}^i_{\underline{jm}}\big(\overline{L}^\alpha_{\underline{n\alpha}} - \overline{F}^\alpha_{\underline{n\alpha}}\big) - \overline{F}^i_{\underline{jn}}\big(\overline{L}^\alpha_{\underline{m\alpha}} - \overline{F}^\alpha_{\underline{m\alpha}}\big)\Big]\\ &- \frac{1}{N+1}\Big[F^i_{\underline{jm}}\big(L^\alpha_{\underline{n\alpha}} - F^\alpha_{\underline{n\alpha}}\big) - F^i_{\underline{jn}}\big(L^\alpha_{\underline{m\alpha}} - F^\alpha_{\underline{m\alpha}}\big)\Big]\\ &+ \frac{1}{N+1}\delta^i_j\big(\overline{F}^\alpha_{[\underline{m\alpha}\|n]} - F^\alpha_{[\underline{m\alpha}|n]}\big) + \delta^i_m\big(\overline{\mathcal{Q}}_{jn} - \mathcal{Q}_{jn}\big) - \delta^i_n\big(\overline{\mathcal{Q}}_{jm} - \mathcal{Q}_{jm}\big).\end{aligned} \tag{33}$$

After contracting Equation (33) by i and n, we obtain

$$\begin{aligned}\overline{\mathcal{Q}}_{jm} - \mathcal{Q}_{jm} = {}&\frac{N}{N^2-1}\big(\overline{R}_{jm} - R_{jm}\big) + \frac{1}{N^2-1}\big(\overline{R}_{mj} - R_{mj}\big)\\ &- \frac{1}{N-1}\big(\overline{F}^\alpha_{\underline{jm}\|\alpha} - \overline{F}^\alpha_{\underline{j\alpha}\|m} - \overline{F}^\alpha_{\underline{jm}}\overline{F}^\beta_{\underline{\alpha\beta}} + \overline{F}^\alpha_{\underline{j\beta}}\overline{F}^\beta_{\underline{\alpha m}}\big)\\ &+ \frac{1}{N-1}\big(F^\alpha_{\underline{jm}|\alpha} - F^\alpha_{\underline{j\alpha}|m} - F^\alpha_{\underline{jm}}F^\beta_{\underline{\alpha\beta}} + F^\alpha_{\underline{j\beta}}F^\beta_{\underline{\alpha m}}\big)\\ &+ \frac{1}{N^2-1}\Big[\overline{F}^\beta_{\underline{jm}}\big(\overline{L}^\alpha_{\underline{\alpha\beta}} - \overline{F}^\alpha_{\underline{\alpha\beta}}\big) - \overline{F}^\beta_{\underline{j\beta}}\big(\overline{L}^\alpha_{\underline{m\alpha}} - \overline{F}^\alpha_{\underline{m\alpha}}\big)\Big]\\ &- \frac{1}{N^2-1}\Big[F^\beta_{\underline{jm}}\big(L^\alpha_{\underline{\alpha\beta}} - F^\alpha_{\underline{\alpha\beta}}\big) - F^\beta_{\underline{j\beta}}\big(L^\alpha_{\underline{m\alpha}} - F^\alpha_{\underline{m\alpha}}\big)\Big]\\ &- \frac{1}{N^2-1}\big(\overline{F}^\alpha_{[\underline{j\alpha}\|m]} - F^\alpha_{[\underline{j\alpha}|m]}\big).\end{aligned} \tag{34}$$

Equation (34) should be rewritten in the more suitable form:

$$\overline{\mathcal{Q}}_{ij} - \mathcal{Q}_{ij} = \frac{N}{N^2-1}\big(\overline{R}_{ij} - R_{ij}\big) + \frac{1}{N^2-1}\big(\overline{R}_{ji} - R_{ji}\big) + \overset{0}{\overline{\mathcal{S}}}_{ij} - \overset{0}{\mathcal{S}}_{ij}, \tag{35}$$

for

$$\begin{aligned}\overset{0}{\mathcal{S}}_{ij} = {}&-\frac{1}{N-1}\big(F^\alpha_{\underline{ij}|\alpha} - F^\alpha_{\underline{i\alpha}|j} - F^\alpha_{\underline{ij}}F^\beta_{\underline{\alpha\beta}} + F^\alpha_{\underline{i\beta}}F^\beta_{\underline{\alpha j}}\big)\\ &+ \frac{1}{N^2-1}\Big[F^\beta_{\underline{ij}}\big(L^\alpha_{\underline{\alpha\beta}} - F^\alpha_{\underline{\alpha\beta}}\big) - F^\beta_{\underline{i\beta}}\big(L^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big) - F^\alpha_{[\underline{i\alpha}|j]}\Big],\end{aligned} \tag{36}$$

and the corresponding $\overset{0}{\overline{\mathcal{S}}}_{ij}$.

If we substitute the expression (35) in Equation (33), we obtain

$$\overset{0}{\overline{W}}{}^i_{jmn} = \overset{0}{W}{}^i_{jmn},$$

for

$$\begin{aligned}\overset{0}{W}{}^i_{jmn} = {}&R^i_{jmn} + \frac{1}{N+1}\delta^i_j\big(R_{[mn]} + F^\alpha_{[\underline{m\alpha}|n]}\big) + \frac{N}{N^2-1}\delta^i_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}R_{n]j}\\ &+ \delta^i_{[m}\overset{0}{\mathcal{S}}_{jn]} - F^i_{\underline{jm}|n} + F^i_{\underline{jn}|m} + F^\alpha_{\underline{jm}}F^i_{\underline{\alpha n}} - F^\alpha_{\underline{jn}}F^i_{\underline{\alpha m}}\\ &+ \frac{1}{N+1}\Big[F^i_{\underline{jm}}\big(L^\alpha_{\underline{n\alpha}} - F^\alpha_{\underline{n\alpha}}\big) - F^i_{\underline{jn}}\big(L^\alpha_{\underline{m\alpha}} - F^\alpha_{\underline{m\alpha}}\big)\Big],\end{aligned} \tag{37}$$

and the corresponding $\overset{0}{\overline{W}}{}^i_{jmn}$.

The next theorem is proven in this way.

Theorem 1. *Let $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ be a mapping determined by deformation tensor $P^i_{\underline{jk}} = \psi_j\delta^i_k + \psi_k\delta^i_j + \overline{F}^i_{\underline{jk}} - F^i_{\underline{jk}}$ for the one-form ψ_j and the tensors $\overline{F}^i_{\underline{jk}}$, $F^i_{\underline{jk}}$ of the type* (1,2) *symmetric by covariant indices.*

The geometrical object $\overset{0}{\mathcal{T}}{}^i_{jk}$ given by (27) *is the associated basic invariant of the Thomas type for the mapping f.*

The geometrical object $\overset{0}{\mathcal{W}}{}^i_{jmn}$ equivalently given by Equations (28) and (29) *is the associated basic invariant of the Weyl type for the mapping f.*

The geometrical object $\overset{0}{W}{}^i_{jmn}$ given by (37) *is the derived associative invariant of the Weyl type for the mapping f.*

Because the forms of invariants $\overset{0}{\mathcal{T}}{}^i_{jk}$, $\overset{0}{\mathcal{W}}{}^i_{jmn}$, $\overset{0}{W}{}^i_{jmn}$ coincide with the forms of their images, $\overline{\overset{0}{\mathcal{T}}}{}^i_{jk}$, $\overline{\overset{0}{\mathcal{W}}}{}^i_{jmn}$, $\overline{\overset{0}{W}}{}^i_{jmn}$, these invariants are total.

Invariants for Mappings of Riemannian Space

In Riemannian space $\mathbb{R}_N$, the affine connection coefficients are Christoffel symbols $\Gamma^i_{\underline{jk}}$. After changing $L^i_{\underline{jk}}$ with $\Gamma^i_{\underline{jk}}$ and $L^\alpha_{\underline{i\alpha}}$ with $\Gamma^\alpha_{\underline{i\alpha}} = \dfrac{1}{2}|g|^{-1}|g|_{,i}$ in Equations (27)–(29), and (37), we obtain the corresponding invariants for the mapping $f : \mathbb{R}_N \to \overline{\mathbb{R}}_N$, whose components are

$$\mathcal{T}^{g}{}^i_{jk} = \Gamma^i_{\underline{jk}} - F^i_{\underline{jk}} - \frac{1}{2(N+1)|g|}\Big[\delta^i_k\big(|g|_{,j} - 2|g|F^\alpha_{\underline{j\alpha}}\big) + \delta^i_j\big(|g|_{,k} - 2|g|F^\alpha_{\underline{k\alpha}}\big)\Big], \tag{38}$$

$$\begin{aligned}
\mathcal{W}^{g}{}^i_{jmn} = {}& R^{g}{}^i_{jmn} - F^i_{\underline{jm}|^g n} + F^i_{\underline{jn}|^g m} + F^\alpha_{\underline{jm}}F^i_{\underline{\alpha n}} - F^\alpha_{\underline{jn}}F^i_{\underline{\alpha m}} \\
&- \frac{1}{N+1}\Big[\delta^i_m\big(\Gamma^\alpha_{\underline{j\alpha}|^g n} - F^\alpha_{\underline{j\alpha}|^g n}\big) - \delta^i_n\big(\Gamma^\alpha_{\underline{j\alpha}|^g m} - F^\alpha_{\underline{j\alpha}|^g m}\big) - \delta^i_j F^\alpha_{[\underline{m\alpha}|^g n]}\Big] \\
&+ \frac{1}{N+1}\Big[F^i_{\underline{jm}}\big(\Gamma^\alpha_{\underline{n\alpha}} - F^\alpha_{\underline{n\alpha}}\big) - F^i_{\underline{jn}}\big(\Gamma^\alpha_{\underline{m\alpha}} - F^\alpha_{\underline{m\alpha}}\big)\Big] \\
&- \frac{1}{N+1}\delta^i_{[m}F^\alpha_{\underline{jn}]}\big(\Gamma^\beta_{\underline{\alpha\beta}} - F^\beta_{\underline{\alpha\beta}}\big) - \frac{1}{(N+1)^2}\delta^i_m\big(\Gamma^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big)\big(\Gamma^\alpha_{\underline{n\alpha}} - F^\alpha_{\underline{n\alpha}}\big) \\
&+ \frac{1}{(N+1)^2}\delta^i_n\big(\Gamma^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big)\big(\Gamma^\alpha_{\underline{m\alpha}} - F^\alpha_{\underline{m\alpha}}\big),
\end{aligned} \tag{39}$$

$$\begin{aligned}
W^{g}{}^i_{jmn} = {}& R^{g}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j F^\alpha_{[\underline{m\alpha}|^g n]} + \frac{1}{N-1}\delta^i_{[m}R^{g}{}_{jn]} \\
&+ \delta^i_{[m}\mathcal{S}^{g}{}_{jn]} - F^i_{\underline{jm}|^g n} + F^i_{\underline{jn}|^g m} + F^\alpha_{\underline{jm}}F^i_{\underline{\alpha n}} - F^\alpha_{\underline{jn}}F^i_{\underline{\alpha m}} \\
&+ \frac{1}{N+1}\Big[F^i_{\underline{jm}}\big(\Gamma^\alpha_{\underline{n\alpha}} - F^\alpha_{\underline{n\alpha}}\big) - F^i_{\underline{jn}}\big(\Gamma^\alpha_{\underline{m\alpha}} - F^\alpha_{\underline{m\alpha}}\big)\Big],
\end{aligned} \tag{40}$$

for

$$\Gamma^\alpha_{\underline{i\alpha}|^g j} = \frac{1}{2}\big(|g|^{-1}|g|_{,ij} - |g|^{-2}|g|_{,i}|g|_{,j}\big) - \frac{1}{2}\Gamma^\alpha_{\underline{ij}}|g|^{-1}|g|_{,\alpha}, \tag{41}$$

$$\begin{aligned}
\mathcal{S}^{g}{}_{ij} = {}& -\frac{1}{N-1}\big(F^\alpha_{\underline{ij}|^g \alpha} - F^\alpha_{\underline{i\alpha}|^g j} + F^\alpha_{\underline{ij}}F^\beta_{\underline{\alpha\beta}} + F^\alpha_{\underline{i\beta}}F^\beta_{\underline{\alpha j}}\big) \\
&+ \frac{1}{N^2-1}\Big[F^\beta_{\underline{ij}}\big(\Gamma^\alpha_{\underline{\alpha\beta}} - F^\alpha_{\underline{\alpha\beta}}\big) - F^\beta_{\underline{i\beta}}\big(\Gamma^\alpha_{\underline{j\alpha}} - F^\alpha_{\underline{j\alpha}}\big) - F^\alpha_{[\underline{i\alpha}|^g j]}\Big],
\end{aligned} \tag{42}$$

where $|^g$ denotes the covariant derivative in $\mathbb{R}_N$.

By denoting

$$\mathcal{Q}^{g}{}_{ij} = -\frac{1}{N+1}\Big[\Gamma^\alpha_{\underline{i\alpha}|^g j} - F^\alpha_{\underline{i\alpha}|^g j} + F^\alpha_{\underline{ij}}\big(\Gamma^\beta_{\underline{\alpha\beta}} - F^\beta_{\underline{\alpha\beta}}\big)\Big] - \frac{1}{(N+1)^2}\big(\Gamma^\alpha_{\underline{i\alpha}} - F^\alpha_{\underline{i\alpha}}\big)\big(\Gamma^\beta_{\underline{j\beta}} - F^\beta_{\underline{j\beta}}\big), \tag{43}$$

we can represent (39) in the form

$$\begin{aligned}\mathcal{W}^{g}{}^{i}_{jmn} &= R^{g}{}^{i}_{jmn} - F^{i}_{\underline{jm}|^{g}n} + F^{i}_{\underline{jn}|^{g}m} + F^{\alpha}_{\underline{jm}} F^{i}_{\underline{\alpha n}} - F^{\alpha}_{\underline{jn}} F^{i}_{\underline{\alpha m}} \\ &+ \frac{1}{N+1}\left[F^{i}_{\underline{jm}}(\Gamma^{\alpha}_{\underline{n\alpha}} - F^{\alpha}_{\underline{n\alpha}}) - F^{i}_{\underline{jn}}(\Gamma^{\alpha}_{\underline{m\alpha}} - F^{\alpha}_{\underline{m\alpha}})\right] + \frac{1}{N+1}\delta^{i}_{j} F^{\alpha}_{[\underline{m\alpha}|^{g}n]} \\ &+ \delta^{i}_{m}\mathcal{Q}^{g}{}_{jn} - \delta^{i}_{n}\mathcal{Q}^{g}{}_{jm}.\end{aligned} \tag{44}$$

The next theorem holds.

Theorem 2. *Let $f : \mathbb{R}_N \to \overline{\mathbb{R}}_N$ be a mapping determined by deformation tensor $P^{g}{}^{i}_{\underline{jk}} = \psi_j\delta^i_k + \psi_k\delta^i_j + \overline{F}^i_{\underline{jk}} - F^i_{\underline{jk}}$ for the one-form ψ_j and the tensors $\overline{F}^i_{\underline{jk}}$, $F^i_{\underline{jk}}$ of the type* (1,2) *symmetric by covariant indices.*

The geometrical object $\mathcal{T}^g$ given by (38) *is the associated basic invariant of the Thomas type for the mapping f.*

The geometrical object $\mathcal{W}^g$ equivalently given by Equations (39) and (44) *is the associated basic invariant of the Weyl type for the mapping f.*

The geometrical object W^g given by (40) *is the derived associative invariant of the Weyl type for the mapping f.*

Because the forms of invariants $\mathcal{T}^g$, $\mathcal{W}^g$, W^g coincide with the forms of their images, $\overline{\mathcal{T}}^g$, $\overline{\mathcal{W}}^g$, $\overline{W}^g$, these invariants are total.

3. Invariants for Third-Type almost-Geodesic Mappings

In an attempt to generalize the concept of geodesic lines, Sinyukov started the research about almost-geodesic lines.

Definition 2 (see [3,21]). *A curve $\overset{0}{\ell}$ on manifold $\mathcal{M}_N$, equipped with the affine connections $\overset{0}{\nabla}$ and $\overset{0}{\overline{\nabla}}$ whose coefficients are $L^i_{\underline{jk}}$ and $\overline{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \overset{0}{P}{}^i_{\underline{jk}}$,* is the almost-geodesic line *with respect to the affine connection $\overset{0}{\nabla}$ if the next equation holds:*

$$(\overset{0}{P}{}^{i}_{\underline{\beta\gamma}|\delta} + \overset{0}{P}{}^{\alpha}_{\underline{\gamma\delta}}\overset{0}{P}{}^{i}_{\underline{\alpha\beta}})\frac{d\overset{0}{\ell}{}^{\beta}}{dt}\frac{d\overset{0}{\ell}{}^{\gamma}}{dt}\frac{d\overset{0}{\ell}{}^{\delta}}{dt} = \overset{0}{b}\overset{0}{P}{}^{i}_{\underline{\alpha\beta}}\frac{d\overset{0}{\ell}{}^{\alpha}}{dt}\frac{d\overset{0}{\ell}{}^{\beta}}{dt} + \overset{0}{a}\frac{d\overset{0}{\ell}{}^{i}}{dt}, \tag{45}$$

where $\overset{0}{a}$ and $\overset{0}{b}$ are scalar functions.

A mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$, which any geodesic line of the space $\mathbb{A}_N$ transforms to an almost geodesic line of the space $\overline{\mathbb{A}}_N$, is the almost-geodesic mapping of symmetric affine connection space $\mathbb{A}_N$.

Sinyukov recognized three types of almost-geodesic mappings [2,3] π_1, π_2, π_3. The almost-geodesic mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ of a type π_k, $k = 1, 2, 3$, has the property of reciprocity if its inverse mapping is the almost-geodesic mapping of the type π_k.

In the literature, different authors obtained invariants for almost-geodesic mappings, which have the property of reciprocity.

The basic equations of almost-geodesic mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ are [2,3]

$$\begin{cases} \overline{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \overset{0}{\psi}_j\delta^i_k + \overset{0}{\psi}_k\delta^i_j + \overset{0}{\sigma}_{jk}\overset{0}{\varphi}{}^i \\ \overset{0}{\varphi}{}^i_{|j} = \overset{0}{\nu}\delta^i_j + \overset{0}{\mu}_j\varphi^i, \end{cases} \tag{46}$$

for the scalar function $\overset{0}{\nu}$, 1-forms $\overset{0}{\psi}$, $\overset{0}{\mu}$, and symmetric tensor $\overset{0}{\sigma}_{ij}$ of the type $(0,2)$.

Let us prove the following proposition.

Proposition 1. *The tensor $\overset{0}{\mu}_i$ and the vector $\overset{0}{\varphi}{}^i$ from the basic Equation (46) satisfy the following equation:*

$$\overset{0}{\mu}_{\alpha|i}\overset{0}{\varphi}{}^{\alpha} = \overset{0}{\varphi}{}^{\alpha}_{|\alpha i} - N\overset{0}{\nu}_i - \overset{0}{\nu}\overset{0}{\mu}_i - \overset{0}{\mu}_i\overset{0}{\mu}_{\alpha}\overset{0}{\varphi}{}^{\alpha}, \tag{47}$$

for $\overset{0}{\nu}_i = \overset{0}{\nu}_{|i}$.

Proof. After contracting the second of basic Equation (46), we obtain the equation

$$\overset{0}{\varphi}{}^{\alpha}_{|\alpha} = N\overset{0}{\nu} + \overset{0}{\mu}_{\alpha}\overset{0}{\varphi}{}^{\alpha}. \tag{48}$$

The covariant derivatives of the left and right sides of Equation (48) in the direction of x^i are equal to

$$\overset{0}{\varphi}{}^{\alpha}_{|\alpha i} = N\overset{0}{\nu}_i + \overset{0}{\mu}_{\alpha|i}\overset{0}{\varphi}{}^{\alpha} + \overset{0}{\mu}_{\alpha}\big(\overset{0}{\nu}\delta^{\alpha}_i + \overset{0}{\mu}_i\overset{0}{\varphi}{}^{\alpha}\big),$$

which completes the proof for this proposition. □

Let us combine Sinyukov's methodology for obtaining invariants for almost-geodesic mappings of the third type and the corresponding formulas from [1], in this paper listed in Equations (27)–(29), to obtain invariants for almost-geodesic mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ of the type π_3.

We know that almost-geodesic mappings of the type π_3 have the property of reciprocity [3]. Sinyukov involved the covariant vector $\overset{0}{q}_i$ such that (see [3], p. 193)

$$\overset{0}{q}_{\alpha}\overset{0}{\varphi}{}^{\alpha} = e \quad (e = \pm 1). \tag{49}$$

Because the almost-geodesic mapping f has the property of reciprocity, we may involve the corresponding geometrical objects $\overset{0}{\overline{\varphi}}{}^i$ and $\overset{0}{\overline{q}}_i$ such that

$$\overset{0}{\overline{\varphi}}{}^{\alpha}\overset{0}{\overline{q}}_{\alpha} = \overline{e} \quad (\overline{e} = \pm 1). \tag{50}$$

After some computation, Sinyukov obtained the invariant $\underset{3}{T}{}^i_{jk}$ (with respect to transformation of affine connection coefficients $L^i_{\underline{jk}}$) for the almost-geodesic mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$.

The form of invariant $\underset{3}{T}{}^i_{jk}$ is

$$\begin{aligned}\underset{3}{T}{}^i_{jk} = {} & L^i_{\underline{jk}} + e\overset{0}{\varphi}{}^i\overset{0}{q}_{j|k} \\ & - \frac{1}{N}\big(\delta^i_j - e\overset{0}{\varphi}{}^i\overset{0}{q}_j\big)\Big[L^{\alpha}_{\underline{k\alpha}} + e\overset{0}{\varphi}{}^{\alpha}\overset{0}{q}_{\alpha|k} + \frac{1}{N-1}q_k\big(e\overset{0}{\varphi}{}^{\beta}L^{\alpha}_{\underline{\beta\alpha}} + \overset{0}{\varphi}{}^{\alpha}\overset{0}{\varphi}{}^{\beta}\overset{0}{q}_{\alpha|\beta}\big)\Big] \\ & - \frac{1}{N}\big(\delta^i_k - e\overset{0}{\varphi}{}^i\overset{0}{q}_k\big)\Big[L^{\alpha}_{\underline{j\alpha}} + e\overset{0}{\varphi}{}^{\alpha}\overset{0}{q}_{\alpha|j} + \frac{1}{N-1}\overset{0}{q}_j\big(e\overset{0}{\varphi}{}^{\beta}L^{\alpha}_{\underline{\beta\alpha}} + \overset{0}{\varphi}{}^{\alpha}\overset{0}{\varphi}{}^{\beta}\overset{0}{q}_{\alpha|\beta}\big)\Big].\end{aligned} \tag{51}$$

Let

$$\overset{0}{\zeta}_i = e\overset{0}{\varphi}{}^{\alpha}\overset{0}{q}_{\alpha|i} + \frac{1}{N-1}\overset{0}{q}_i\big(e\overset{0}{\varphi}{}^{\beta}L^{\alpha}_{\underline{\beta\alpha}} + \overset{0}{\varphi}{}^{\alpha}\overset{0}{\varphi}{}^{\beta}\overset{0}{q}_{\alpha|\beta}\big), \tag{52}$$

$$\overset{0}{\overline{\zeta}}_i = e\overset{0}{\overline{\varphi}}{}^{\alpha}\overset{0}{\overline{q}}_{\alpha\|i} + \frac{1}{N-1}\overset{0}{\overline{q}}_i\big(e\overset{0}{\overline{\varphi}}{}^{\beta}\overline{L}^{\alpha}_{\underline{\beta\alpha}} + \overset{0}{\overline{\varphi}}{}^{\alpha}\overset{0}{\overline{\varphi}}{}^{\beta}\overset{0}{\overline{q}}_{\alpha\|\beta}\big). \tag{53}$$

In this case, the invariant $\underset{3}{T}{}^i_{jk}$ given by (51) takes the form

$$\underset{3}{T}{}^i_{jk} = L^i_{\underline{jk}} + e\overset{0}{\varphi}{}^i\overset{0}{q}_{j|k} - \frac{1}{N}\big(\delta^i_j - e\overset{0}{\varphi}{}^i\overset{0}{q}_j\big)\big(L^{\alpha}_{\underline{k\alpha}} + \overset{0}{\zeta}_k\big) - \frac{1}{N}\big(\delta^i_k - e\overset{0}{\varphi}{}^i\overset{0}{q}_k\big)\big(L^{\alpha}_{\underline{j\alpha}} + \overset{0}{\zeta}_j\big). \tag{54}$$

After comparing Equations (54) and (21), we obtain

$$\omega^i_{jk} = -e\overset{0}{\varphi}{}^i\overset{0}{q}_{j|k} + \frac{1}{N}\big(\delta^i_j - e\overset{0}{\varphi}{}^i\overset{0}{q}_j\big)\big(L^\alpha_{\underline{k\alpha}} + \overset{0}{\zeta}_k\big) + \frac{1}{N}\big(\delta^i_k - e\overset{0}{\varphi}{}^i\overset{0}{q}_k\big)\big(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j\big), \tag{55}$$

i.e.,

$$\begin{aligned}\omega^i_{jk} &= \frac{1}{N}\big(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j\big)\delta^i_k + \frac{1}{N}\big(L^\alpha_{\underline{k\alpha}} + \overset{0}{\zeta}_k\big)\delta^i_j \\ &- e\overset{0}{\varphi}{}^i\overset{0}{q}_{j|k} - \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\big(L^\alpha_{\underline{k\alpha}} + \overset{0}{\zeta}_k\big) - \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_k\big(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j\big).\end{aligned} \tag{56}$$

After some computing and with respect to the Ricci identity (3), one obtains that the geometrical object

$$\begin{aligned}\overset{0}{\underset{3}{\mathcal{W}}}{}^i_{jmn} &= \overset{0}{R}{}^i_{jmn} + \frac{1}{N}\delta^i_j\big(\overset{0}{R}_{[mn]} - \overset{0}{\zeta}_{[m|n]}\big) + \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\big(\overset{0}{R}_{[mn]} - 2\overset{0}{\zeta}_{[m|n]}\big) + e\overset{0}{\varphi}{}^i\overset{0}{q}_\alpha\overset{0}{R}{}^\alpha_{jmn} \\ &- \delta^i_m\big(\xi_{j|n} - \overset{0}{\varphi}{}^\alpha\xi_\alpha\overset{0}{q}_{j|n} + \xi_j\xi_n + (e\overset{0}{\nu} + \overset{0}{\varphi}{}^\alpha\xi_\alpha)(\overset{0}{q}_j\xi_n + \overset{0}{q}_n\xi_j)\big) \\ &+ \delta^i_n\big(\xi_{j|m} - \overset{0}{\varphi}{}^\alpha\xi_\alpha\overset{0}{q}_{j|m} + \xi_j\xi_m + (e\overset{0}{\nu} + \overset{0}{\varphi}{}^\alpha\xi_\alpha)(\overset{0}{q}_j\xi_m + \overset{0}{q}_m\xi_j)\big) \\ &+ e\big(\overset{0}{q}_{j|m}\overset{0}{\mu}_n - \overset{0}{q}_{j|[m}\xi_{n]} - \overset{0}{q}_{j|n}\overset{0}{\mu}_m + \overset{0}{q}_{[m}\xi_{j|n]}\big)\overset{0}{\varphi}{}^i \\ &+ e\overset{0}{\mu}_n\overset{0}{\varphi}{}^i\big(\overset{0}{q}_j\xi_m + \overset{0}{q}_m\xi_j\big) - e\overset{0}{\mu}_m\overset{0}{\varphi}{}^i\big(\overset{0}{q}_j\xi_n + \overset{0}{q}_n\xi_j\big),\end{aligned} \tag{57}$$

for $\xi_i = \dfrac{1}{N}\big(L^\alpha_{\underline{i\alpha}} + \overset{0}{\zeta}_i\big)$, is the basic invariant of the Thomas type for the almost-geodesic mapping f. This invariant is total.

After taking the image $\overline{\underset{3}{T}}{}^i_{jk}$ of the invariant $\underset{3}{T}{}^i_{jk}$ given by (56), we obtain

$$\begin{aligned}\omega^i_{jk} &= \xi_j\delta^i_k + \xi_k\delta^i_j - e\overset{0}{\varphi}{}^i\overset{0}{q}_{j|k} - e\overset{0}{\varphi}{}^i\overset{0}{q}_j\xi_k - e\overset{0}{\varphi}{}^i\overset{0}{q}_k\xi_j, \\ \overline{\omega}^i_{jk} &= \overline{\xi}_j\delta^i_k + \overline{\xi}_k\delta^i_j - e\overset{0}{\overline{\varphi}}{}^i\overset{0}{\overline{q}}_{j\|k} - e\overset{0}{\overline{\varphi}}{}^i\overset{0}{\overline{q}}_j\overline{\xi}_k - e\overset{0}{\overline{\varphi}}{}^i\overset{0}{\overline{q}}_k\overline{\xi}_j,\end{aligned} \tag{58}$$

for $\overline{\xi}_i = \dfrac{1}{N}\big(\overline{L}^\alpha_{\underline{i\alpha}} + \overset{0}{\overline{\zeta}}_i$, the image $\overset{0}{\overline{\varphi}}{}^i$ of vector $\overset{0}{\varphi}{}^i$ from the first of basic Equation (46), and the corresponding $\overset{0}{\overline{q}}_i$ such that $\overset{0}{\overline{\varphi}}{}^\alpha\overset{0}{\overline{q}}_\alpha = \overline{e}, \overline{e} = \pm 1$.

Because $\big(\overset{0}{\varphi}{}^i\overset{0}{q}_j\big)_{|k} = \overset{0}{\varphi}{}^i_{|k}\overset{0}{q}_j + \overset{0}{\varphi}{}^i\overset{0}{q}_{j|k}$, and with respect to the second of the basic Equation (46), the next equalities hold $0 = \big(\overset{0}{\varphi}{}^\alpha\overset{0}{q}_\alpha\big)_{|k} = \big(\overset{0}{\nu}\delta^\alpha_k + \overset{0}{\mu}_k\overset{0}{\varphi}{}^\alpha\big)\overset{0}{q}_\alpha + \overset{0}{\varphi}{}^\alpha\overset{0}{q}_{\alpha|k}$, i.e.,

$$\begin{cases}\overset{0}{\varphi}{}^\alpha\overset{0}{q}_{\alpha|k} = -\overset{0}{\nu}\,\overset{0}{q}_k - e\,\overset{0}{\mu}_k, \\ \overset{0}{\overline{\varphi}}{}^\alpha\overset{0}{\overline{q}}_{\alpha\|k} = -\overset{0}{\overline{\nu}}\,\overset{0}{\overline{q}}_k - \overline{e}\,\overset{0}{\overline{\mu}}_k.\end{cases} \tag{59}$$

The next equation also holds.

$$\begin{aligned}\overset{0}{P}{}^i_{\underline{jk}} &= \overset{0}{\overline{\omega}}{}^i_{jk} - \overset{0}{\omega}{}^i_{jk} \\ &= \big(\overline{\xi}_j - \xi_j\big)\delta^i_k + \big(\overline{\xi}_k - \xi_k\big)\delta^i_j - e\overset{0}{\overline{\varphi}}{}^i\big(\overset{0}{\overline{q}}_{j\|k} + \overset{0}{\overline{q}}_j\,\overline{\xi}_k + \overset{0}{\overline{q}}_k\,\overline{\xi}_j\big) + e\overset{0}{\varphi}{}^i\big(\overset{0}{q}_{j|k} + \overset{0}{q}_j\,\xi_k + \overset{0}{q}_k\,\xi_j\big).\end{aligned} \tag{60}$$

With respect to Equation (20) and after comparing this equation with Equation (23), we obtain

$$\psi_i = \overline{\xi}_i - \xi_i, \quad \overline{F}{}^i_{\underline{jk}} = -e\overset{0}{\overline{\varphi}}{}^i\big(\overset{0}{\overline{q}}_{j\|k} + \overset{0}{\overline{q}}_j\,\overline{\xi}_k + \overset{0}{\overline{q}}_k\,\overline{\xi}_j\big), \quad F^i_{\underline{jk}} = -e\overset{0}{\varphi}{}^i\big(\overset{0}{q}_{j|k} + \overset{0}{q}_j\,\xi_k + \overset{0}{q}_k\,\xi_j\big). \tag{61}$$

For the reason of $F^i_{\underline{jk}} = F^i_{\underline{kj}}$, we obtain that $q_{j|k} = q_{k|j}$.

To present the corresponding invariants, we need the next expressions.

$$\overset{0}{\varphi}{}^{\alpha}\overset{0}{\varphi}{}^{\beta}\overset{0}{q}_{\alpha|\beta i} = -e\overset{0}{\varphi}{}^{\alpha}_{|\alpha i} + (N-1)e\overset{0}{\nu}_i + 2(\overset{0}{\nu})^2\overset{0}{q}_i + 4e\overset{0}{\nu}\overset{0}{\mu}_i + 2e\overset{0}{\mu}_i\overset{0}{\varphi}{}^{\alpha}\overset{0}{\mu}_\alpha \tag{62}$$

$$\begin{aligned}\overset{0}{\zeta}_{[m|n]} &= -e\overset{0}{\mu}_{[m}\overset{0}{\varphi}{}^\alpha\overset{0}{q}_{\alpha|n]} - e\varphi^\alpha\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha mn} + \frac{1}{N-1}e\big(\overset{0}{\nu}\overset{0}{q}_{[m}L^\alpha_{\underline{n]\alpha}} + \overset{0}{\varphi}{}^\alpha\overset{0}{q}_{[m}L^\beta_{\underline{\alpha\beta}|n]}\big)\\
&\quad + \frac{1}{N-1}q_m\big(e(\overset{0}{\mu}_n\overset{0}{\varphi}{}^\beta)L^\alpha_{\underline{\beta\alpha}} + 2\overset{0}{\nu}\overset{0}{\varphi}{}^\beta\overset{0}{q}_{\beta|n} + 2\overset{0}{\mu}_n\overset{0}{\varphi}{}^\alpha\overset{0}{\varphi}{}^\beta\overset{0}{q}_{\alpha|\beta} + \varphi^\alpha\varphi^\beta\overset{0}{q}_{\alpha|\beta n}\big)\\
&= -e\overset{0}{\varphi}{}^\alpha\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha mn} + \overset{0}{\tilde{\zeta}}_{mn},\end{aligned} \tag{63}$$

$$\begin{aligned}\zeta_{[m|n]} &= -\frac{1}{N}\overset{0}{R}_{[mn]} + \frac{1}{N}\overset{0}{\zeta}_{[m|n]}\\
&= -\frac{1}{N}\overset{0}{R}_{[mn]} - \frac{1}{N}e\overset{0}{\varphi}{}^\alpha\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha mn} + \frac{1}{N}\overset{0}{\tilde{\zeta}}_{mn},\end{aligned} \tag{64}$$

$$F^\alpha_{\underline{i\alpha}} = \overset{0}{\mu}_i + e\overset{0}{\nu}\overset{0}{q}_i - \frac{1}{N}\big(L^\alpha_{\underline{i\alpha}} + \overset{0}{\zeta}_i\big) - \frac{1}{N}e\overset{0}{q}_i\overset{0}{\varphi}{}^\alpha\big(L^\beta_{\underline{\alpha\beta}} + \overset{0}{\zeta}_\alpha\big) = \overset{0}{\widetilde{\mathcal{F}}}_i, \tag{65}$$

$$\begin{aligned}F^\alpha_{\underline{ij}|\alpha} &= -e\big(N\overset{0}{\nu} + \overset{0}{\mu}_\beta\overset{0}{\varphi}{}^\beta\big)\Big(\overset{0}{q}_{i|j} + \frac{1}{N}\overset{0}{q}_i\big(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j\big) + \frac{1}{N}\overset{0}{q}_j\big(L^\alpha_{\underline{i\alpha}} + \overset{0}{\zeta}_i\big)\Big)\\
&\quad + \frac{1}{N}\big(e\overset{0}{\nu}\overset{0}{q}_i + \overset{0}{\mu}_i\big) + \frac{1}{N}\big(e\overset{0}{\nu}\overset{0}{q}_j + \overset{0}{\mu}_j\big)\big(L^\beta_{\underline{i\beta}} + \overset{0}{\zeta}_i\big)\\
&\quad - \frac{1}{N}e\overset{0}{\varphi}{}^\alpha\overset{0}{q}_i\big(L^\beta_{\underline{j\beta}|\alpha} + \overset{0}{\zeta}_{j|\alpha}\big) - \frac{1}{N}e\overset{0}{\varphi}{}^\alpha\overset{0}{q}_j\big(L^\beta_{\underline{i\beta}|\alpha} + \overset{0}{\zeta}_{i|\alpha}\big) = \overset{0}{\widetilde{\mathcal{F}}}_{ij},\end{aligned} \tag{66}$$

$$\begin{aligned}F^\alpha_{[\underline{i\alpha}|j]} &= \overset{0}{\mu}_{[i|j]} - e\overset{0}{\nu}_{[i}\overset{0}{q}_{j]} + \frac{1}{N}\overset{0}{R}_{[ij]} - \frac{1}{N}\overset{0}{\zeta}_{[i|j]} - \frac{1}{N}e\overset{0}{q}_{[i|j]}\big[\overset{0}{\varphi}{}^\alpha\big(L^\beta_{\underline{\alpha\beta}} + \overset{0}{\zeta}_\alpha\big) - N\overset{0}{\nu}\big]\\
&\quad - \frac{1}{N}\overset{0}{\nu}\big(\overset{0}{q}_{[i}L^\beta_{\underline{j]\beta}} + \overset{0}{q}_{[i}\overset{0}{\zeta}_{j]}\big) - \frac{1}{N}e\overset{0}{q}_{[i}\overset{0}{\mu}_{j]}\overset{0}{\varphi}{}^\alpha\big(L^\beta_{\underline{\alpha\beta}} + \overset{0}{\zeta}_\alpha\big) - \frac{1}{N}e\overset{0}{q}_{[i}\overset{0}{\varphi}{}^\alpha\big(L^\beta_{\underline{\alpha\beta}} + \overset{0}{\zeta}_\alpha\big)_{|j]}\\
&= \frac{1}{N}\overset{0}{R}_{[ij]} + \frac{1}{N}e\overset{0}{\varphi}{}^\alpha\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha ij} - \frac{1}{N}\overset{0}{\tilde{\zeta}}_{ij} + \overset{0}{\widetilde{\mathcal{G}}}_{ij},\end{aligned} \tag{67}$$

$$\begin{aligned}-F^i_{\underline{jm}|n} + F^i_{\underline{jn}|m} &= -e\overset{0}{\nu}\delta^i_m\Big(\overset{0}{q}_{j|n} + \frac{1}{N}\overset{0}{q}_j\big(L^\alpha_{\underline{n\alpha}} + \overset{0}{\zeta}_n\big) + \frac{1}{N}\overset{0}{q}_m\big(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j\big)\Big)\\
&\quad + e\overset{0}{\nu}\delta^i_n\Big(\overset{0}{q}_{j|m} + \frac{1}{N}\overset{0}{q}_j\big(L^\alpha_{\underline{m\alpha}} + \overset{0}{\zeta}_m\big) + \frac{1}{N}\overset{0}{q}_m\big(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j\big)\Big)\\
&\quad - e\varphi^i\overset{0}{\mu}_{[m}\overset{0}{q}_{j|n]} - \frac{1}{N}e\varphi^i\big(\overset{0}{\mu}_{[m}\overset{0}{q}_jL^\alpha_{\underline{n]\alpha}} + \overset{0}{\mu}_{[m}\overset{0}{q}_{n]}L^\alpha_{\underline{j\alpha}} + \overset{0}{q}_{j|[m}L^\alpha_{\underline{n]\alpha}} - \overset{0}{q}_{[m}L^\alpha_{\underline{j\alpha}|n]}\big)\\
&\quad + \frac{1}{N}e\overset{0}{\varphi}{}^i\big(\overset{0}{q}_{[m}\overset{0}{\zeta}_{j|n]} - \overset{0}{\mu}_{[m}\overset{0}{q}_j\overset{0}{\zeta}_{n]} - \overset{0}{\mu}_{[m}\overset{0}{q}_{n]}\overset{0}{\zeta}_j - \overset{0}{q}_{j|[m}\overset{0}{\zeta}_{n]}\big)\\
&\quad - e\varphi^i\overset{0}{q}_\alpha\overset{0}{R}{}^\alpha_{jmn} - \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\overset{0}{R}_{[mn]} + \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\overset{0}{\zeta}_{[m|n]}\\
&= -e\overset{0}{\varphi}{}^i\overset{0}{q}_\alpha\overset{0}{R}{}^\alpha_{jmn} - \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\overset{0}{R}_{[mn]} - \frac{1}{N}\overset{0}{\varphi}{}^i\overset{0}{q}_je\overset{0}{\varphi}{}^\alpha\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha mn}\\
&\quad + \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\overset{0}{\tilde{\zeta}}_{mn} + \overset{0}{\widetilde{\mathcal{H}}}{}^i_{jmn}\end{aligned} \tag{68}$$

$$\begin{aligned}F^\alpha_{\underline{jm}}F^i_{\underline{\alpha n}} - F^\alpha_{\underline{jn}}F^i_{\underline{\alpha m}} &= -e\delta^\alpha_{[m}\delta^\beta_{n]}\varphi^i\big(\overset{0}{\nu}\overset{0}{q}_\alpha + e\overset{0}{\mu}_\alpha\big)\big(\overset{0}{q}_{j|\beta} + \overset{0}{q}_j\xi_\beta + \overset{0}{q}_\beta\xi_j\big)\\
&\quad - e\varphi^\alpha\varphi^i\big(\overset{0}{q}_{j|[m}\overset{0}{q}_\alpha\xi_{n]} + \overset{0}{q}_{j|[m}\overset{0}{q}_{n]}\xi_\alpha - \overset{0}{q}_j\overset{0}{q}_{[m}\xi_\alpha\xi_{n]} + \overset{0}{q}_\alpha\overset{0}{q}_{[m}\xi_j\xi_{n]}\big) = \overset{0}{\widetilde{\mathcal{K}}}{}^i_{jmn},\end{aligned} \tag{69}$$

for the corresponding geometrical objects $\overset{0}{\tilde{\zeta}}_{ij}$, $\overset{0}{\widetilde{\mathcal{F}}}_i$, $\overset{0}{\widetilde{\mathcal{F}}}_{ij}$, $\overset{0}{\widetilde{\mathcal{G}}}_{ij}$, $\overset{0}{\widetilde{\mathcal{H}}}{}^i_{jmn}$, and $\overset{0}{\widetilde{\mathcal{K}}}{}^i_{jmn}$ uniquely determined by Equations (63) and (65)–(69).

After substituting the expression (65) in (21), we obtain the associated basic invariant for the almost-geodesic mapping f, whose components are

$$\overset{0}{\widetilde{\mathcal{T}}}{}^i_{jk} = L^i_{\underline{jk}} + e\overset{0}{\varphi}{}^i\Big(\overset{0}{q}_{j|k} + \frac{1}{N}\overset{0}{q}_j(L^\alpha_{\underline{k\alpha}} + \overset{0}{\zeta}_k) + \frac{1}{N}\overset{0}{q}_k(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j)\Big) - \frac{1}{N+1}\Big(\delta^i_k(L^\alpha_{\underline{j\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_j) + \delta^i_j(L^\alpha_{\underline{k\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_k)\Big), \tag{70}$$

for $\overset{0}{\zeta}_i$ given by (52) and $\overset{0}{\widetilde{\mathcal{F}}}_i$ expressed with Equation (65).

If substituting the expressions (64) and (65), (67)–(69) in Equation (22), one obtains the basic invariant for the almost-geodesic mapping f, whose components are

$$\begin{aligned}\overset{0}{\widetilde{\mathcal{W}}}{}^i_{jmn} &= \overset{0}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j(\overset{0}{R}_{[mn]} + \frac{1}{N}\overset{0}{R}_{[mn]} + \frac{1}{N}e\overset{0}{\varphi}{}^\alpha\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha mn} - \frac{1}{N}\overset{0}{\tilde{\zeta}}_{mn} + \overset{0}{\widetilde{\mathcal{G}}}_{mn})\\
&- \frac{1}{(N+1)^2}\delta^i_m\Big((N+1)(L^\alpha_{\underline{j\alpha}|n} - \overset{0}{\widetilde{\mathcal{F}}}_{j|n}) + (L^\alpha_{\underline{j\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_j)(L^\beta_{\underline{n\beta}} - \overset{0}{\widetilde{\mathcal{F}}}_n)\Big)\\
&+ \frac{1}{(N+1)^2}\delta^i_n\Big((N+1)(L^\alpha_{\underline{j\alpha}|m} - \overset{0}{\widetilde{\mathcal{F}}}_{j|m}) + (L^\alpha_{\underline{j\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_j)(L^\beta_{\underline{m\beta}} - \overset{0}{\widetilde{\mathcal{F}}}_m)\Big)\\
&- e\overset{0}{\varphi}{}^i\overset{0}{q}_\alpha\overset{0}{R}{}^\alpha_{jmn} - \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\overset{0}{R}_{[mn]} - \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{\varphi}{}^\alpha\overset{0}{q}_j\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha mn} + \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\overset{0}{\tilde{\zeta}}_{mn}\\
&- \frac{1}{N+1}e\overset{0}{\varphi}{}^i\Big(\overset{0}{q}_{j|m} + \frac{1}{N}\overset{0}{q}_j(L^\alpha_{\underline{m\alpha}} + \overset{0}{\zeta}_m) + \frac{1}{N}\overset{0}{q}_m(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j)\Big)(L^\alpha_{\underline{n\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_n)\\
&+ \frac{1}{N+1}e\overset{0}{\varphi}{}^i\Big(\overset{0}{q}_{j|n} + \frac{1}{N}\overset{0}{q}_j(L^\alpha_{\underline{n\alpha}} + \overset{0}{\zeta}_n) + \frac{1}{N}\overset{0}{q}_n(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j)\Big)(L^\alpha_{\underline{m\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_m)\\
&+ \overset{0}{\widetilde{\mathcal{H}}}{}^i_{jmn} + \overset{0}{\widetilde{\mathcal{K}}}{}^i_{jmn}\end{aligned} \tag{71}$$

Analogously as above, with respect to Equation (37) and the expressions (36), (65), (66), (68), and (69), we obtain the derived invariant of the Weyl type for mapping f whose components are

$$\begin{aligned}\overset{0}{\widetilde{W}}{}^i_{jmn} &= \overset{0}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j(\overset{0}{R}_{[mn]} + \frac{1}{N}\overset{0}{R}_{[mn]} + \frac{1}{N}e\overset{0}{\varphi}{}^\alpha\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha mn} - \frac{1}{N}\overset{0}{\tilde{\zeta}}_{mn} + \overset{0}{\widetilde{\mathcal{G}}}_{mn})\\
&+ \frac{N}{N^2-1}\delta^i_{[m}\overset{0}{R}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\overset{0}{R}_{n]j} + \delta^i_{[m}\overset{0}{\widetilde{\mathcal{S}}}_{jn]} - e\overset{0}{\varphi}{}^i\overset{0}{q}_\alpha\overset{0}{R}{}^\alpha_{jmn} - \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\overset{0}{R}_{[mn]}\\
&- \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{\varphi}{}^\alpha\overset{0}{q}_j\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha mn} + \frac{1}{N}e\overset{0}{\varphi}{}^i\overset{0}{q}_j\overset{0}{\tilde{\zeta}}_{mn} + \overset{0}{\widetilde{\mathcal{H}}}{}^i_{jmn} + \overset{0}{\widetilde{\mathcal{K}}}{}^i_{jmn}\\
&- \frac{1}{N+1}e\overset{0}{\varphi}{}^i(\overset{0}{q}_{j|m} + \frac{1}{N}\overset{0}{q}_j(L^\alpha_{\underline{m\alpha}} + \overset{0}{\zeta}_m) + \overset{0}{q}_m(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j))(L^\alpha_{\underline{n\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_n)\\
&+ \frac{1}{N+1}e\overset{0}{\varphi}{}^i(\overset{0}{q}_{j|n} + \frac{1}{N}\overset{0}{q}_j(L^\alpha_{\underline{n\alpha}} + \overset{0}{\zeta}_n) + \overset{0}{q}_n(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j))(L^\alpha_{\underline{m\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_m),\end{aligned} \tag{72}$$

for

$$\begin{aligned}\overset{0}{\widetilde{\mathcal{S}}}_{ij} &= -\frac{1}{N(N^2-1)}(\overset{0}{R}_{[ij]} + e\overset{0}{\varphi}{}^\alpha\overset{0}{q}_\beta\overset{0}{R}{}^\beta_{\alpha ij} - \overset{0}{\tilde{\zeta}}_{ij}) - \frac{1}{N^2-1}\overset{0}{\widetilde{\mathcal{G}}}_{ij}\\
&- \frac{1}{N-1}(\overset{0}{\widetilde{\mathcal{F}}}{}^\alpha_{\underline{ij}|\alpha} - \overset{0}{\widetilde{\mathcal{F}}}_{i|j} - \overset{0}{\widetilde{\mathcal{K}}}{}^\alpha_{ij\alpha}) - \frac{1}{N^2-1}\overset{0}{\widetilde{\mathcal{F}}}_i(L^\alpha_{\underline{j\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_j)\\
&- \frac{1}{N^2-1}e\overset{0}{\varphi}{}^\beta\Big(\overset{0}{q}_{j|m} + \frac{1}{N}\overset{0}{q}_j(L^\alpha_{\underline{m\alpha}} + \overset{0}{\zeta}_m) + \overset{0}{q}_m(L^\alpha_{\underline{j\alpha}} + \overset{0}{\zeta}_j)\Big)(L^\alpha_{\underline{\beta\alpha}} - \overset{0}{\widetilde{\mathcal{F}}}_\beta).\end{aligned} \tag{73}$$

In this way, the following theorem was proven.

Theorem 3. *Let $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ be an almost geodesic mapping of the type π_3.*

The geometrical object $\overset{0}{\widetilde{\mathcal{T}}}{}^i_{jk}$ given by (70) is the associated basic invariant of the Thomas type for the mapping f.

The geometrical object $\overset{0}{\widetilde{\mathcal{W}}}{}^i_{jmn}$ given by (71) is the associated basic invariant of the Weyl type for the mapping f.

The geometrical object $\overset{0}{W}{}^i_{jmn}$ given by (72) is the associated derived invariant of the Weyl type for the mapping f.

The invariants (70)–(72) for mapping f are total.

4. Discussion

In this paper, we continued the idea presented in [1] about obtaining invariants for geometric mappings in a universal way. In most of the previous research, the authors obtained just one invariant with respect to the transformation of curvature tensor $\overset{0}{R}{}^i_{jmn}$. After the research in [1] was published, it became clear that at least one invariant for the studied mappings of symmetric affine connection space has been lost. In this paper, we obtained general formulas of invariants for mappings whose deformation tensors are sums of the object $\psi_k\delta^i_j + \psi_j\delta^i_k$ and some other symmetric tensor of the type $(1,2)$. We proved that there are two invariants for the studied mappings of a symmetric affine connection space with respect to the transformation of its curvature tensor. The findings of this paper motivate us to answer the following questions: (i) Are the two invariants obtained in this paper the only invariants for mappings of symmetric affine connection spaces with respect to the transformations of curvature tensors? (ii) What is the tensor character of the two mappings obtained in this paper? (iii) How many families of invariants for mappings of non-symmetric affine connection spaces may be obtained?

Author Contributions: All authors have equally contributed to this work. All authors wrote, read, and approved the final manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: N.O.V. was supported by the Serbian Ministry of Education, Science and Technological Development through the Mathematical Institute of the Serbian Academy of Sciences and Arts. M.S.S. acknowledges the grant of the Ministry of Education, Science and Technological Development of Serbia 451-03-68/2020-14/200124 for carrying out this research.

Acknowledgments: The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Vesić, N.O. Basic Invariants of Geometric Mappings. *Miskolc. Math. Notes* **2020**, *21*, 473–487. [CrossRef]
2. Mikeš, J. *Differential Geometry of Special Mappings*, 1st ed.; Palacký University Press: Olomouc, Czech Republic, 2015.
3. Sinyukov, N.S. *Geodesic Mappings of Riemannian Spaces*; Nauka: Moscow, Russia, 1979.
4. Eisenhart, L.P. *Riemannian Geometry*; Princeton University Press: London, UK, 1967.
5. Thomas, T.Y. On the projective and equi-projective geometries of paths. *Proc. Nat. Acad. Sci. USA* **1925**, *11*, 199–203. [CrossRef] [PubMed]
6. Weyl, H. *Zur Infinitesimal Geometrie: Einordnung der Projectiven und der Konformen Auffsung*; Gottingen Nachrichten: Göttingen, Germany, 1921; pp. 99–112.
7. Mikeš, J. Geodesic mappings of affine-connected and Riemannian spaces. *J. Math. Sci. N. Y.* **1996**, *78*, 311–333. [CrossRef]
8. Mikeš, J.; Pokorná, O.; Starko, G. On almost-geodesic mappings $\pi_2(e)$ onto Riemannian spaces. In *Proceedings of the 23rd Winter School "Geometry and Physics", Srni, Czech Republic, 18–25 January 2003*; Circolo Matematico di Palermo: Palermo, Italy, 2004; Volume 72, pp. 151–157.
9. Mikeš, J. Holomorphically Projective Mappings and Their Generalizations. *J. Math. Sci. N. Y.* **1998**, *89*, 1334–1353. [CrossRef]
10. Mikeš, J.; Berezovski, V.E.; Stepanova, E.; Chudá, H. Geodesic Mappings and Their Generalizations. *J. Math. Sci. N. Y.* **2016**, *217*, 607–623. [CrossRef]

11. Mikeš, J.; Shina, M.; Vanžurová, A. Invariant objects by holomorphically projective mappings of parabolically Kahler spaces. In Proceedings of the 8th International Conference on Applied Mathematics (APLIMAT 2009), Bratislava, Slovakia, 3–6 February 2009; pp. 439–444.
12. Mikeš, J.; Vanžurová, A.; Hinterleitner, I. *Geodesic Mappings and Some Generalizations*; Palacky University: Olomouc, Czech Republic, 2009.
13. Berezovski, V.E.; Báscó, S.; Mikeš, J. Almost geodesic mappings of affinely connected spaces that preserve the Riemannian curvature. *Ann. Math. Inf.* **2015**, *45*, 3–10.
14. Berezovski, V.; Báscó, S.; Mikeš, J. Diffeomorphism of Affine Connected Spaces Which Preserved Riemannian and Ricci Curvature Tensors. *Miskolc Math. Notes* **2017**, *18*, 117–124. [CrossRef]
15. Berezovskij, V.; Mikeš, J. On special almost-geodesic mappings of type π_1 of spaces with affine connection. *Acta Univ. Palacki. Olomoucensis Math.* **2004**, *43*, 21–26.
16. Stanković, M.S.; Vesić, N.O. Some relations in non-symmetric affine connection spaces with regard to a special almost-geodesic mappings of the third type. *Filomat* **2015**, *29*, 1941–1951. [CrossRef]
17. Zlatanović, M.L. New projective tensors for equitorsion geodesic mappings. *Appl. Math. Lett.* **2012**, *25*, 890–897. [CrossRef]
18. Zlatanović, M.; Stanković, V. Some invariants of holomorphically projective mappings of generalized Kählerian spaces. *J. Math. Analys. Appl.* **2018**, *458*, 601–610. [CrossRef]
19. Simjanović, D.J.; Vesić, N.O. Novel Invariants for Almost Geodesic Mappings of the Third Type. *Miskolc Math. Notes* **2021**, *22*, 961–975. [CrossRef]
20. Einstein, A. The basis of the general theory of relativity. *Ann. Phys.* **1916**, *49*, 517–571. (In German)
21. Sinyukov, N.S. Almost geodesic mappings of affinely connected and Riemannian spaces. *Sov. Math. Dokl.* **1963**, *4*, 1086–1088.