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Super Connected Direct Product of Graphs and Cycles

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Abstract: The topology of an interconnection network can be modeled by a graph $G = (V(G), E(G))$. The connectivity of graph G is a parameter used to measure the reliability of a corresponding network. The direct product is an important graph product. This paper mainly focuses on the super connectedness of the direct product of graphs and cycles. The connectivity of G, denoted by $\kappa(G)$, is the size of a minimum vertex set $S \subseteq V(G)$ such that $G - S$ is not connected or has only one vertex. The graph G is said to be super connected, simply super-κ, if every minimum vertex cut is the neighborhood of a vertex with minimum degree. The direct product of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{(u_1, v_1)(u_2, v_2) | u_1 u_2 \in E(G), v_1 v_2 \in E(H)\}$. In this paper, we give some sufficient conditions for the direct product $G \times C_n$ to be super connected, where C_n is the cycle on n vertices. Furthermore, those sufficient conditions are the best possible.

Keywords: connectivity; super connected graphs; direct product; cycles

MSC: 05C40

1. Introduction

For a simple graph G with vertex set $V(G)$ and edge set $E(G)$, $u, v \in V(G)$ are adjacent if $uv \in E(G)$. The set of all vertices adjacent to u is called the *neighborhood* of u in G, denoted by $N_G(u)$. The *degree* of u, denoted by $d_G(v)$, is $|N_G(u)|$. The *minimum degree* of G is $\delta(G) = min\{d_G(v)|v \in V(G)\}$. For a vertex set $S \subseteq V(G)$, if $G - S$ is not connected, then S is a vertex cut of G. We know that only complete graphs do not have vertex cuts. If G is not a complete graph, then the *connectivity* of G, denoted by $\kappa(G)$, is the size of a minimum vertex cut of G. Otherwise, $\kappa(G) = |V(G)| - 1$. The *edge connectivity* of G, denoted by $\kappa'(G)$, is the the size of a minimum edge set $F \subseteq G$ such that $G - F$ is not connected. K_n, $K_{m,n}$ and C_n are used to denote a complete graph, complete bipartite graph and cycle, respectively. We follow Bondy and Murty [1] for undefined notation and terminology.

The topology of an interconnection network can be modeled by a graph $G = (V(G), E(G))$, where $V(G)$ represents the set of nodes and $E(G)$ represents the set of communication links in the network. The connectivity $\kappa(G)$ of G can be used to measure the reliability and fault tolerance of the network. In general, the larger $\kappa(G)$ is, the more reliable the network is. It is well known that $\kappa(G) \le \delta(G)$. The graph G with $\kappa(G) = \delta(G)$ is called *maximally connected*, simply *max-κ*. For the maximally connected graphs, it is believed that the graphs with the smallest number of minimum vertex cuts are more reliable than the others. Boesch in [2] proposed the concept of super connected graph. If every minimum vertex cut is a neighborhood of some vertex of G with minimum degree, then the graph G is said to be *super connected*, simply *super-κ*. By definition, a super connected graph is also maximally connected. The converse is not always true. For example, C_n ($n \ge 6$) is maximally connected but not super connected.

The direct product of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{(u_1, v_1)(u_2, v_2) | u_1 u_2 \in$

$E(G), v_1v_2 \in E(H)$. Weichsel [3] proved that the direct product $G \times H$ of two nontrivial connected graphs G and H is connected if and only if at least one of G and H is not bipartite.

We list some results on the edge connectivity of the direct product of graphs as follows. Some bounds on the edge connectivity of the direct product of graphs were given by Špacapan in [4]. Cao, Brglez, Špacapan and Vumar [5] determined the edge connectivity of the direct product of a nontrivial graph and a complete graph. In [6], Špacapan not only obtained the edge connectivity of the direct product of two graphs but also characterized the structure of a minimum edge cut in the direct product of two graphs.

This paragraph lists some results on the connectivity of direct product of graphs. Some bounds on the connectivity of the direct product of graphs were also given by Špacapan in [4]. Mamut and Vumar [7] proved that the connectivity of the direct product of two complete graphs K_m and K_n is $(m-1)(n-1)$, where $m \geq n \geq 2$. In [8], Guji and Vumar proved that the connectivity of the direct product of a complete graph K_n $(n \geq 3)$ and a bipartite graph G is $min\{n\kappa(G), (n-1)\delta(G)\}$, and furthermore, the authors also conjectured that this is true for all nontrivial graph G. Later, Wang and Wu [9] and Wang and Xue [10] independently confirmed this conjecture. Wang and Yan [11] determined the connectivity of $G \times K_2$. Recently, Sonawane and Borse [12] determined the connectivity of the direct product of graphs and cycles.

The results for the super connected direct product graphs are presented in the following. Guo, Qin and Guo [13] proved that for a maximally connected bipartite graph G, $G \times K_n$ $(n \geq 3)$ is super connected. In [14], the authors generalized this result by showing that for a maximally connected nonbipartite graph G, $G \times K_n$ $(n \geq 3)$ is super connected. In [15], Zhou completely characterized the super connected direct product of a nontrivial graphs G and a complete graph K_n $(n \geq 3)$; that is, $G \times K_n$ is not super connected if and only if either $\kappa(G \times K_n) = n\kappa(G)$ or $\kappa(G \times K_n) \cong K_{m,m} \times K_3$ $(m \geq 1)$. Wu and Tian [16] studied the super connected direct product of paths, cycles and cycles.

Motivated by the results above, especially those in [12], we study the super connected direct product of graphs and cycles. In the next section, we present a key lemma, which is used in the proof of our main results in Section 3. Conclusions are given in the last section.

2. A Key Lemma

In [12], Sonawane and Borse constructed a graph $\widetilde{G}_n$ (see Figure 1) from a connected bipartite graph G as follows. Let (X, Y) be a bipartition of G and $n \geq 2$ be an integer and $X_i = \{(x, i) : x \in X\}$ and $Y_i = \{(y, i) : y \in Y\}$ for $i = 1, 2, \ldots, n$. Let H_i and H_i' be graphs isomorphic to G with bipartitions (X_i, Y_i) and (X_{i+1}, Y_i), respectively, for each $i \in \{1, 2, \ldots, n\}$. Let
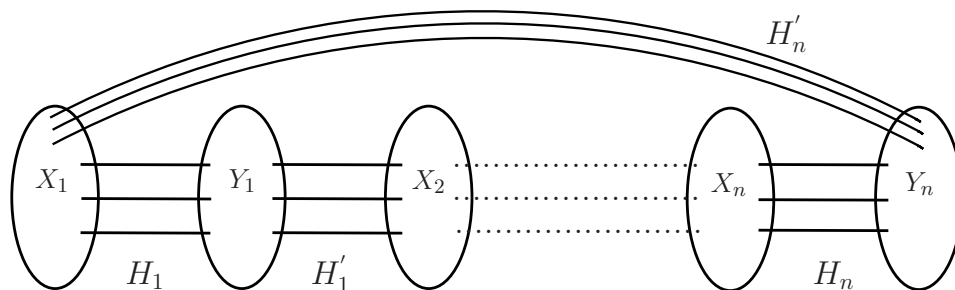
$$\widetilde{G}_n = \bigcup_{i=1}^{n} \left(H_i \cup H_i' \right).$$

Figure 1. The graph $\widetilde{G}_n$.

Sonawane and Borse [12] determined the connectivity of $\widetilde{G}_n$ as follows.

Theorem 1 ([12])**.** $\kappa(\widetilde{G}_n) = min\{n\kappa(G), 2\delta(G)\}$, where $n \geq 2$ is an integer.

In the following, we present a key lemma in this paper, which is used to prove the main results in the next section. Furthermore, the sufficient condition for $\widetilde{G}_n$ to be super connected in this lemma is the best possible.

Lemma 1. (**A key lemma**) *Let G be a connected bipartite graph with bipartition X and Y, and let $n \geq 3$ be an integer. If $|X| \geq \delta(G) + 1$, $|Y| \geq \delta(G) + 1$ and $\kappa(G) > \frac{2}{n}\delta(G)$, then $\widetilde{G}_n$ is super-κ.*

Proof. By Theorem 1 and $\kappa(G) > \frac{2}{n}\delta(G)$, we have $\kappa(\widetilde{G}_n) = 2\delta(G)$. By contradiction, assume that $\widetilde{G}_n$ is not super-κ; then, there is a vertex cut S with $|S| = 2\delta(G)$ such that $\widetilde{G}_n - S$ is not connected and has no isolated vertices. Let $D_1, D_2, \ldots, D_r$ $(r \geq 2)$ be the components of $\widetilde{G}_n - S$. Then, $|D_i| \geq 2$ holds for each $i \in \{1, \ldots, r\}$. Denote $S_{X_i} = S \cap X_i$, $S_{Y_i} = S \cap Y_i$, $X'_i = X_i - S_{X_i}$ and $Y'_i = Y_i - S_{Y_i}$ for $i = 1, 2, \ldots, n$.

Since each vertex x_i in X_i has at least $\delta(G)$ neighbors in both Y_{i-1} and Y_i, and each vertex y_j in Y_j has at least $\delta(G)$ neighbors in both X_j and X_{j+1}, we may make the following claim.

Claim 1. If $D_k \cap X'_i \neq \emptyset$, $|S_{Y_{i-1}}| < \delta(G)$ and $|S_{Y_i}| < \delta(G)$, then both $D_k \cap Y'_{i-1} \neq \emptyset$ and $D_k \cap Y'_i \neq \emptyset$ hold. Similarly, if $D_k \cap Y'_j \neq \emptyset$, $|S_{X_j}| < \delta(G)$ and $|S_{X_{j+1}}| < \delta(G)$, then both $D_k \cap X'_j \neq \emptyset$ and $D_k \cap X'_{j+1} \neq \emptyset$ hold.

In the following, we consider two cases.

Case 1. $|S_{X_i}| < \delta(G)$ and $|S_{Y_j}| < \delta(G)$ for any $i, j \in \{1, 2, \ldots, n\}$. By Claim 1, we have $D_k \cap X'_i \neq \emptyset$ and $D_k \cap Y'_j \neq \emptyset$ for $k \in \{1, \ldots, r\}$ and $i, j \in \{1, \ldots, n\}$. Since $|S| = (|S_{X_1}| + |S_{Y_1}|) + (|S_{X_2}| + |S_{Y_2}|) + \cdots + (|S_{X_n}| + |S_{Y_n}|) < n\kappa(G)$, we have $|S_{X_i}| + |S_{Y_i}| < \kappa(G)$ for some $i \in \{1, 2, \ldots, n\}$. Note that $H_i = (X_i, Y_i)$ is isomorphic to G. Then $\widetilde{G}_n[X'_i \cup Y'_i] = H_i - S_{X_i} \cup S_{Y_i}$ is connected. Thus $\widetilde{G}_n - S$ is connected, contradicting to the assumption.

Case 2. There is an $i \in \{1, 2, \ldots, n\}$ such that $|S_{X_i}| \geq \delta(G)$ or there is a $j \in \{1, 2, \ldots, n\}$ such that $|S_{Y_j}| \geq \delta(G)$.

Assume, without loss of generality, that $|S_{X_1}| \geq \delta(G)$.

Subcase 2.1. $|S_{X_i}| < \delta(G)$ for any $i \in \{2, 3, \ldots, n\}$ and $|S_{Y_j}| < \delta(G)$ for any $j \in \{1, 2, \ldots, n\}$.

If $D_k \cap X'_1 \neq \emptyset$ for some $k \in \{1, \ldots, r\}$, then by Claim 1, $D_k \cap X'_i \neq \emptyset$ and $D_k \cap Y'_j \neq \emptyset$ for all $i, j \in \{1, \ldots, n\}$. Otherwise, $D_k \cap X'_i \neq \emptyset$ and $D_k \cap Y'_j \neq \emptyset$ for all $i \in \{2, \ldots, n\}$ and $j \in \{1, \ldots, n\}$.

Since $|S_{Y_1}| + (|S_{X_2}| + |S_{Y_2}|) + \cdots + (|S_{X_n}| + |S_{Y_n}|) < n\kappa(G) - |S_{X_1}| \leq (n-1)\kappa(G)$, we have $|S_{X_i}| + |S_{Y_i}| < \kappa(G)$ for some $i \in \{2, 3, \ldots, n\}$. Then, $\widetilde{G}_n[X'_i \cup Y'_i] = H_i - S_{X_i} \cup S_{Y_i}$ is connected. Thus, $\widetilde{G}_n - S$ is connected, which contradicts the assumption.

Subcase 2.2. There is an $i' \in \{2, 3, \ldots, n\}$ such that $|S_{X_{i'}}| \geq \delta(G)$.

Since $|S| = 2\delta(G)$, we have $|S_{X_1}| = \delta(G)$ and $|S_{X_{i'}}| = \delta(G)$.

If $i' = 2$ or n, by symmetry, assume $i' = 2$; then, $\widetilde{G}_n[Y'_2 \cup X'_3 \cup \cdots \cup X'_n \cup Y'_n] = \widetilde{G}_n[Y_2 \cup X_3 \cup \cdots \cup X_n \cup Y_n]$ is connected. Furthermore, $\widetilde{G}_n[X'_2 \cup Y'_2 \cup X'_3 \cup \cdots \cup X'_n \cup Y'_n \cup X'_1]$ is connected. Since $\widetilde{G}_n - S$ has no isolated vertices, each vertex in Y_1 has at least one neighbor in X'_1 or X'_2. Therefore, $\widetilde{G}_n - S$ is connected, which is a contradiction.

If $i' \neq 2$ and $i' \neq n$, then by $\widetilde{G}_n[Y_1 \cup X_2 \cup \cdots \cup Y_{i'-1}]$ and $\widetilde{G}_n[Y_{i'} \cup X_{i'+1} \cup \cdots \cup X_n \cup Y_n]$ are connected, we find that $\widetilde{G}_n - S$ is connected, contradicting the assumption.

Subcase 2.3. There is a $j' \in \{1, 2, \ldots, n\}$ such that $|S_{Y_{j'}}| \geq \delta(G)$.

Since $|S| = 2\delta(G)$, we have $|S_{X_1}| = \delta(G)$ and $|S_{Y_{j'}}| = \delta(G)$.

If $j' = 1$ or n, by symmetry, assume $j' = 1$; then, $\widetilde{G}_n[X'_2 \cup Y'_2 \cup \cdots \cup X'_n \cup Y'_n] = \widetilde{G}_n[X_2 \cup Y_2 \cup \cdots \cup X_n \cup Y_n]$ is connected. Thus, $\widetilde{G}_n - S = \widetilde{G}_n[Y'_1 \cup X'_2 \cup Y'_2 \cup \cdots \cup X'_n \cup Y'_n \cup X'_1]$ is connected, which is a contradiction.

If $j' \neq 1$ and $j' \neq n$, then by $\widetilde{G}_n[Y_1 \cup X_2 \cup \cdots \cup X_{j'}]$ and $\widetilde{G}_n[X_{j'+1} \cup Y_{j'+1} \cup \cdots \cup X_n \cup Y_n]$ are connected, we find that $\widetilde{G}_n - S$ is connected, which contradicts the assumption.

Since all cases lead to contradiction, the proof is thus complete. □

3. Main Results

Motivated by the connectivity of the direct product of graphs and cycles, we obtain some sufficient conditions for the direct product of graphs and cycles to be super connected. Considering four cases arising from whether G is bipartite or not and n is even or odd, Sonawane and Borse [12] obtained the connectivity of the direct product of graphs and cycles in the following four theorems.

Theorem 2 ([12]). *Let G be a connected bipartite graph and $n \geq 3$ be an odd integer. Then $\kappa(G \times C_n) = min\{n\kappa(G), 2\delta(G)\}$.*

Theorem 3 ([12]). *Let G be a connected bipartite graph and $n \geq 4$ be an even integer. Then the graph $G \times C_n$ has two isomorphic components each with connectivity $min\{\frac{n}{2}\kappa(G), 2\delta(G)\}$.*

Theorem 4 ([12]). *Let G be a connected non-bipartite graph and $n \geq 4$ be an even integer. Then $\kappa(G \times C_n) = min\{\frac{n}{2}\kappa(G \times K_2), 2\delta(G)\}$.*

Theorem 5 ([12]). *Let G be a connected non-bipartite graph and $n \geq 5$ be an odd integer. Then $min\{\frac{n-1}{2}\kappa(G \times K_2), 2\delta(G)\} \leq \kappa(G \times C_n) \leq min\{\frac{n+1}{2}\kappa(G \times K_2), 2\delta(G)\}$.*

Similarly, we consider four cases to study the super connectedness of the direct product of graphs and cycles in the following. The cycle C_n of length n (≥ 3) is denoted by $C_n = \langle 1, 2, \ldots, n \rangle$.

Theorem 6. *Let G be a connected bipartite graph with bipartition X and Y, and let $n \geq 3$ be an odd integer. If $|X| \geq \delta(G) + 1$, $|Y| \geq \delta(G) + 1$ and $\kappa(G) > \frac{2}{n}\delta(G)$, then $G \times C_n$ is super-κ.*

Proof. Let $V_i = \{(v, i) : v \in V(G)\}$ for $i = 1, 2, \ldots, n$. Then $V(G \times C_n) = \cup_{i=1}^{n} V_i$. Let $X_{\frac{i+1}{2}} = \{(x, i) : x \in X\}$ and $Y_{\frac{n+i}{2}} = \{(y, i) : y \in Y\}$ for $i = 1, 3, \ldots, n$. Let $X_{\frac{n+i+1}{2}} = \{(x, i) : x \in X\}$ and $Y_{\frac{i}{2}} = \{(y, i) : y \in Y\}$ for $i = 2, 4, \ldots, n-1$. Then $V_i = X_{\frac{i+1}{2}} \cup Y_{\frac{n+i}{2}}$ for $i = 1, 3, \ldots, n$ and $V_i = X_{\frac{n+i+1}{2}} \cup Y_{\frac{i}{2}}$ for $i = 2, 4, \ldots, n-1$. Let H_i and H_i' be subgraphs of $G \times C_n$ induced by $X_i \cup Y_i$ and $X_{i+1} \cup Y_i$, respectively for $i = 1, 2, \ldots, n$. Then $G \times C_n = \cup_{i=1}^{n}\left(H_i \cup H_i'\right)$. The graph $G \times C_n$ is shown in Figure 2. For each i, H_i and H_i' isomorphic to G. By Lemma 1, $G \times C_n$ is super-κ. □
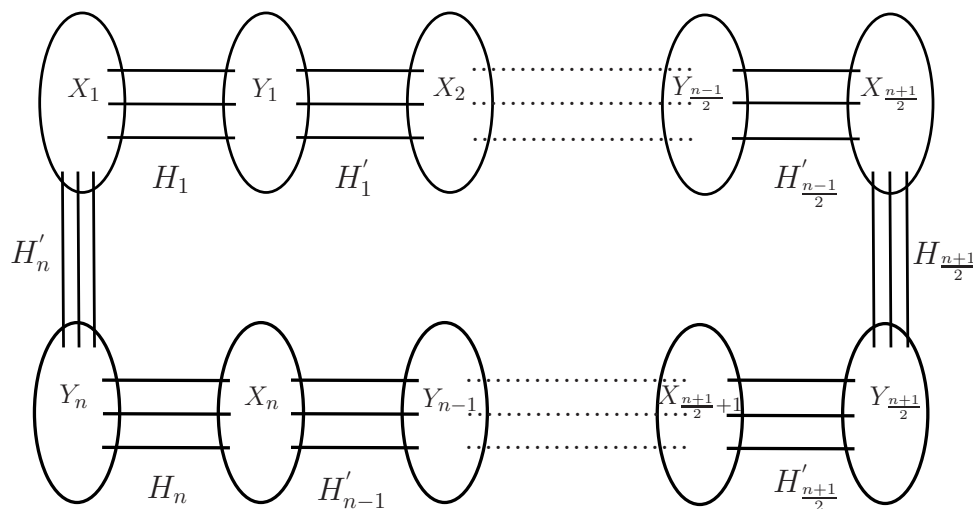
Figure 2. The graph $G \times C_n$ when G is bipartite and n is odd.

Theorem 7. *Let G be a connected bipartite graph with bipartition X and Y, and let $n \geq 6$ be an even integer. If $|X| \geq \delta(G) + 1$, $|Y| \geq \delta(G) + 1$ and $\kappa(G) > \frac{4}{n}\delta(G)$, then the two isomorphic components G_1 and G_2 of $G \times C_n$ are both super-κ.*

Proof. Let $V_i = \{(v, i) : v \in V(G)\}$ for $i = 1, 2, \ldots, n$. Then $V(G \times C_n) = \cup_{i=1}^{n} V_i$. Let $X_{\frac{i+1}{2}} = \{(x, i) : x \in X\}$ and $Y_{\frac{n+i+1}{2}} = \{(y, i) : y \in Y\}$ for $i = 1, 3, \ldots, n - 1$. Let $X_{\frac{n+i}{2}} = \{(x, i) : x \in X\}$ and $Y_{\frac{i}{2}} = \{(y, i) : y \in Y\}$ for $i = 2, 4, \ldots, n$. Then $V_i = X_{\frac{i+1}{2}} \cup Y_{\frac{n+i+1}{2}}$ for $i = 1, 3, \ldots, n - 1$ and $V_i = X_{\frac{n+i}{2}} \cup Y_{\frac{i}{2}}$ for $i = 2, 4, \ldots, n$. Let H_i and H_i' be subgraphs of $G \times C_n$ induced by $X_i \cup Y_i$ and $X_{i+1} \cup Y_i$, respectively, for $i = 1, 2, \ldots, n$. For each i, H_i and H_i' are isomorphic to G. Note that the two isomorphic components G_1 and G_2 are $\cup_{i=1}^{\frac{n}{2}} \left(H_i \cup H_i'\right)$ and $\cup_{i=\frac{n}{2}+1}^{n} \left(H_i \cup H_i'\right)$. Thus, by Lemma 1, both G_1 and G_2 (shown in Figure 3) are super-κ. $\square$
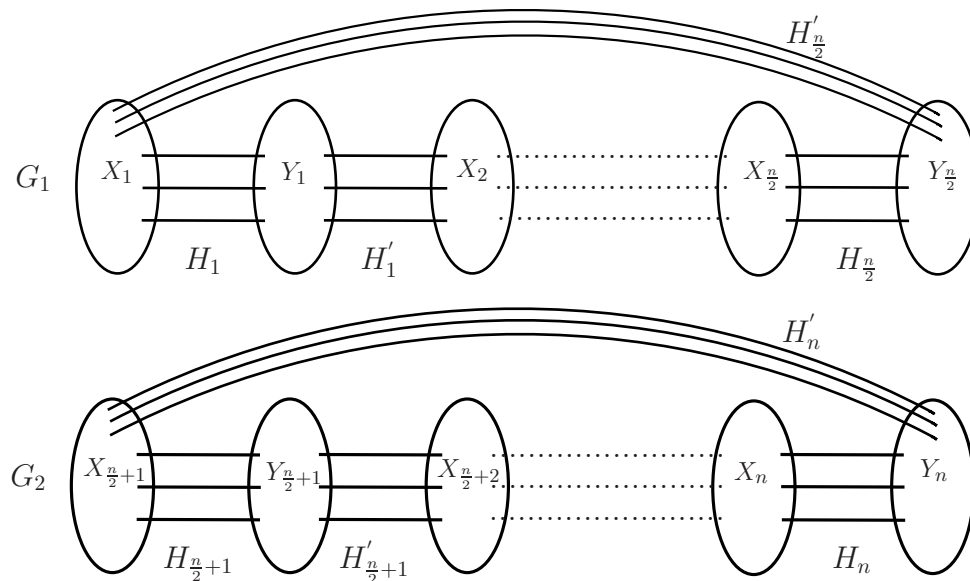

Figure 3. The graphs G_1 and G_2.

Theorem 8. *Let G be a connected non-bipartite graph and $n \geq 6$ be an even integer. If $\kappa(G \times K_2) > \frac{4}{n}\delta(G)$, then $G \times C_n$ is super-κ.*

Proof. Let $V_i = \{(v,i) : v \in V(G)\}$ for $i = 1, 2, \ldots, n$. Then $V(G \times C_n) = \cup_{i=1}^{n} V_i$. Let $H_{\frac{i+1}{2}}$ be the subgraph of $G \times C_n$ induced by $V_i \cup V_{i+1}$ for $i = 1, 3, \ldots, n-1$. Let $H'_{\frac{i}{2}}$ be the subgraph of $G \times C_n$ induced by $V_i \cup V_{i+1}$ for $i = 2, 4, \ldots, n$. Then H_i and H'_i are isomorphic to the bipartite graph $G \times K_2$ with bipartitions (V_{2i-1}, V_{2i}) and (V_{2i}, V_{2i+1}), respectively for $i = 1, 2, \ldots, \frac{n}{2}$. Note that $G \times C_n = \cup_{i=1}^{\frac{n}{2}} \left(H_i \cup H'_i \right)$ (see Figure 4) and $\delta(G \times K_2) = \delta(G)$. By Lemma 1, $G \times C_n$ is super-κ. $\square$
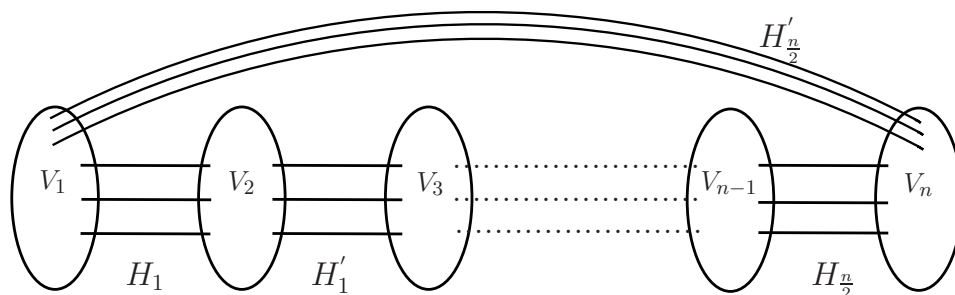

Figure 4. The graph $G \times C_n$ when G is non-bipartite and n is an even.

Theorem 9. *Let G be a connected non-bipartite graph and $n \geq 7$ be an odd integer. If $\kappa(G \times K_2) > \frac{4}{n-1}\delta(G)$, then $G \times C_n$ is super-κ.*

Proof. Let $V_i = \{(v,i) : v \in V(G)\}$ for $i = 1, 2, \ldots, n$. Then $V(G \times C_n) = \cup_{i=1}^{n} V_i$. Let $H_{\frac{i+1}{2}}$ be the subgraph of $G \times C_n$ induced by $V_i \cup V_{i+1}$ for $i = 1, 3, \ldots, n$. Let $H'_{\frac{i}{2}}$ be the subgraph of $G \times C_n$ induced by $V_i \cup V_{i+1}$ for $i = 2, 4, \ldots, n-1$. Then, H_i is isomorphic to the bipartite graph $G \times K_2$ with bipartition (V_{2i-1}, V_{2i}) for $i = 1, 2, \ldots, \frac{n+1}{2}$ and H'_i is isomorphic to the bipartite graph $G \times K_2$ with bipartition (V_{2i}, V_{2i+1}) for $i = 1, 2, \ldots, \frac{n-1}{2}$. Note that $G \times C_n = (\cup_{i=1}^{\frac{n-1}{2}} \left(H_i \cup H'_i \right)) \cup H_{\frac{n+1}{2}}$ (see Figure 5) and $\delta(G \times K_2) = \delta(G)$. By similar arguments as the proof in Lemma 1, we can prove that $G \times C_n$ is super-κ. $\square$
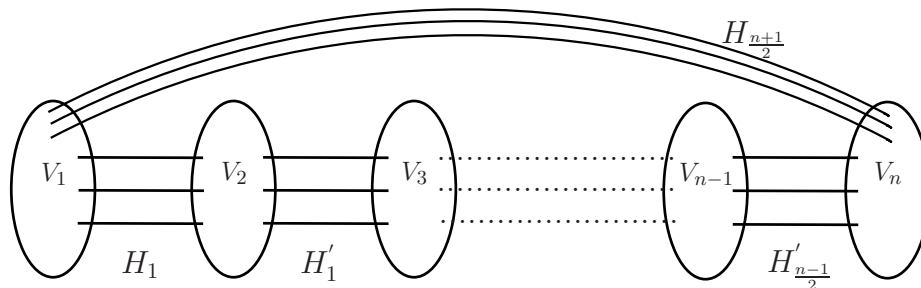

Figure 5. The graph $G \times C_n$ when G is non-bipartite and n is odd.

Theorem 10 ([12]). *If H is the direct product of $k \geq 1$ odd cycles, then $\kappa(H \times K_2) = 2^k$.*

Combing Theorem 10 with Theorems 8 and 9, respectively, we have the following two corollaries.

Corollary 1. *Let G be the direct product of $k \geq 1$ odd cycles and $n \geq 6$ be an even integer. Then, $G \times C_n$ is super-κ.*

Proof. Let $G = C_{l_1} \times C_{l_2} \times \cdots \times C_{l_k}$. By Theorem 10, $\kappa(G \times K_2) = 2^k = \delta(G)$. Then, $G \times C_n$ is super-κ by Theorem 8. $\square$

Corollary 2. *Let G be the direct product of $k \geq 1$ odd cycles and $n \geq 7$ be an odd integer. Then, $G \times C_n$ is super-κ.*

Proof. Let $G = C_{l_1} \times C_{l_2} \times \cdots \times C_{l_k}$. By Theorem 10, $\kappa(G \times K_2) = 2^k = \delta(G)$. Then, by Theorem 9, $G \times C_n$ is super-κ. □

4. Concluding Remarks

Motivated by the results on the connectivity of the direct product of graphs and cycles in [12], we focus on studying the super connectedness of the direct product of graphs and cycles in this paper. By using a key lemma obtained in Section 2, we give some sufficient conditions for the direct product of a graph and a cycle to be super connected. However, there are few results on the connectivity of the direct product of two general graphs. Consequently, we will explore the connectivity and super connectedness of the direct product of two general graphs in the future.

Conflicts of Interest: The authors declare no conflict of interest.

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