axioms

Article

Nonoscillation and Oscillation Criteria for a Class of Second-Order Nonlinear Neutral Delay Differential Equations with Positive and Negative Coefficients

Rongrong Guo †, Qingdao Huang *,† and Haifeng Tian †

School of Mathematics, Jilin University, Changchun 130012, China; guorr16@mails.jlu.edu.cn (R.G.); tianhf14@mails.jlu.edu.cn (H.T.)
* Correspondence: huangqd@jlu.edu.cn
† These authors contributed equally to this work.

Abstract: In this paper, we investigate some nonoscillatory and oscillatory solutions for a class of second-order nonlinear neutral delay differential equations with positive and negative coefficients. By means of the method of contraction mapping principle and some integral inequality techniques, we extend the recent results provided in the literature.

Keywords: delayed argument; differential equation; second-order; neutral; nonlinear; oscillation; integral inequality technique

check for updates

Citation: Guo, R.; Huang, Q.; Tian, H. Nonoscillation and Oscillation Criteria for a Class of Second-Order Nonlinear Neutral Delay Differential Equations with Positive and Negative Coefficients. *Axioms* **2022**, *11*, 281. https://doi.org/10.3390/axioms11060281

Academic Editor: Serkan Araci

Received: 13 May 2022
Accepted: 6 June 2022
Published: 10 June 2022

1. Introduction

Only recently, some scholars ([1,2]) studied the oscillation of the following mixed-type second-order equation:

$$(r(x)(y'(x))^{\alpha})' = p(x)y^{\alpha}(\tau(x)), \quad x \geq 0,$$

where $p, r \in C([x_0, \infty), (0, \infty))$, α is the ratio of two positive odd integers, $\lim_{x\to\infty} \tau(x) = \infty$, $\tau(x) \in C^1([x_0, \infty), \Re)$ and $\tau'(x) > 0$.

The researchers ([3,4]) studied the oscillation of the following second-order half-linear neutral delay differential equation:

$$(r(x)(y(x) + p(x)y(\tau(x)))^{\alpha})')' + q(x)y^{\alpha}(\sigma(x)) = 0, \quad x \geq x_0 > 0,$$

where α is the ratio of two positive odd integers, $r, p \in C^1([x_0, \infty), (0, \infty))$, $q \in C([x_0, \infty), \Re)$, $\tau, \sigma \in C([x_0, \infty), \Re)$, $\tau(x) \leq x$, $\sigma(x) \leq x$ and $\lim_{x\to\infty} \tau(x) = \lim_{x\to\infty} \sigma(x) = \infty$.

Baculíková et al. [5] considered the oscillation of the following second-order delay differential equation:

$$(a(x)(y(x) - p(x)y^{\alpha}(\tau(x)))')' + q(x)y^{\beta}(\sigma(x)) = 0, \quad x \geq x_0 > 0,$$

where $0 < \alpha \leq 1$, α and β are the ratio of two positive odd integers, $a \in C^1([x_0, \infty), (0, \infty))$, $p, q \in C([x_0, \infty), (0, \infty))$, $0 < p(x) \leq p < 1$, $\tau, \sigma \in C^1([x_0, \infty), (0, \infty))$, $\tau(x) \leq x$, $\sigma(x) \leq x$, $\tau'(x) > 0$, $\sigma'(x) > 0$ and $\lim_{x\to\infty} \tau(x) = \lim_{x\to\infty} \sigma(x) = \infty$.

Oscillation phenomena take part in delay differential equations from real world applications. We refer the reader to [6–23] (where oscillation and/or delay situations take part in models from mathematical biology and physics when their formulation includes cross-diffusion terms) and the references cited therein.

Thus, many scholars were concerned about the second-order equation with positive and negative coefficients. In [24], Lin et al. studied the following equation:

$$[a(x)(y(x)+p(x)y(x-\tau))']' + q(x)G(y(x-\delta)) - r(x)H(y(x-\sigma)) = 0, \quad (1)$$

where $x \geq x_0$, $\tau \in (0,\infty)$, $\delta, \sigma \in [0,\infty)$, $p, q, r \in C([x_0,\infty), \Re)$ and $G, H \in C(\Re, \Re)$, and $a(x), p(x), q(x), r(x), G(x)$ and $H(x)$ satisfy some of the following assumptions.

Assumption 1 (c1). *G and H satisfy local Lipschitz condition, and $uG(u) > 0$, $uH(u) > 0$ for $u \neq 0$.*

Assumption 2 (c2). $a(x) > 0$, $q(x), r(x) \geq 0$, $\int^{\infty} A(x)q(x)dx < \infty$, $\int^{\infty} A(x)r(x)dx < \infty$, *where* $A(x) = \int_{x_0}^{x} \frac{1}{a(s)} ds$.

Assumption 3 (c3). *$mq(x) - r(x)$ is eventually non-negative for every $m > 0$.*

Assumption 4 (c4). *$|p(x)| \leq p_0 < \frac{1}{2}$ eventually.*

Assumption 5 (c5). *$p(x) \geq 0$ eventually, and $0 < p_1 < 1$; $p(x) \leq 0$ eventually, and $-1 < p_2 < 0$, where $p_1 = \limsup_{x\to\infty} p(x)$, $p_2 = \liminf_{x\to\infty} p(x)$.*

Assumption 6 (c6). *$p(x) > 1$ eventually, and $1 < p_2 < p_1 < p_2^2 < +\infty$; $p(x) < -1$ eventually, and $-\infty < p_2 < p_1 < -1$, where p_1 and p_2 are the same as that in (c5).*

By using the contraction mapping principle, they obtained the existence of nonoscillatory solutions of (1) when (c1)–(c3), (c4) (or (c1)–(c3), (c5) or (c1)–(c3) and (c6)) hold.

In [25], Zhang et al. extended the results of [24] to the case $p(x) = 1$ and indicated that the condition (c3) is redundant.

When $a(x) \equiv 1$, we extend the number of neutral terms and positive and negative coefficient terms from single to multiple, and then we obtain the following equation:

$$\left[y(x) + \eta \sum_{i=1}^{l} p_i(x)y(x-\tau_i)\right]'' + \sum_{j=1}^{m} q_j(x)G(y(x-\delta_j)) - \sum_{k=1}^{n} r_k(x)H(y(x-\sigma_k)) = 0, \quad (2)$$

where $x \geq x_0$, $\eta = \pm 1$, $l, m, n \in N$, $p_i(x)$ $(i = 1, \cdots, l) \in C^2([x_0,\infty), \Re)$, $q_j(x)$ $(j = 1, \cdots, m)$ and $r_k(x)$ $(k = 1, \cdots, n) \in C([x_0,\infty), \Re)$, $G, H \in C([x_0,\infty), \Re)$ and $G(v) = H(v) = 0$ for $v = 0$.

When we consider (2), some of the following five assumptions are satisfied.

Assumption 7 (H1). $0 < h_1 \leq q_j(x) \leq h_2$, $j = 1, 2, \cdots, m$;

Assumption 8 (H2). *Set the following values:*

$$A := \{y \in X : M_2 \leq y(x) \leq M_1,\ x \geq x_0\},$$

where X denotes the set which includes all continuous and bounded functions on $[x_0,\infty)$ with the sup norm, $M_1 > 1$ and $M_2 > 0$. Let G and H satisfy Lipschitz conditions in A; that is, for any $y_1, y_2 \in A$, there exist $L_1, L_2 > 0$ such that the following is the case.

$$|G(y_1) - G(y_2)| \leq L_1|y_1 - y_2|,$$

$$|H(y_1) - H(y_2)| \leq L_2|y_1 - y_2|;$$

Assumption 9 (H3). $0 < m_1 \leq \frac{G(u)}{u} \leq m_2$ *and* $0 < N_1 \leq \frac{H(u)}{u} \leq N_2$ *for* $u \neq 0$.

Assumption 10 (H4). $\sum_{k=1}^{n} \int_{x_0}^{+\infty} x r_k(x)dx < \infty$.

Assumption 11 (H5). $\sum_{j=1}^{m} \int_{x_0}^{+\infty} x q_j(x)dx < \infty$.

Let $\kappa := \max\{\tau_1, \tau_2, \cdots, \tau_l, \delta_1, \delta_2, \cdots, \delta_m, \sigma_1, \sigma_2, \cdots, \sigma_n\}$.

Definition 1. *A function* y *is called a solution of* (2) *on the interval* $I = [x_0, \infty)$, *if* y *is continuous,* $y(x) + \eta \sum_{i=1}^{l} p_i(x)y(x - \tau_i)$ *is continuously differentiable and* y *satisfies* (2) *on* $x \in I$.

We only consider the nontrivial solution of (2), which satisfies $\sup\{|y(x)| : x \geq X\} > 0$ for all $X \geq x_0$.

Definition 2. *A nontrivial solution of* (2) *is nonoscillatory if it is eventually positive or eventually negative. Otherwise, it is oscillatory.*

Motivated by the useful work of Lin et al. and Zhang et al., in this paper, we obtain some new conditions of the existence of nonoscillatory solutions of the Equation (2).

Recently, the scholars ([26–33]) investigated the oscillatory properties of Equations (1) and (2). When $a(x) \equiv 1$, in [31], Thandapani et al. obtained that every solution of (1) is oscillatory if the following assumptions are satisfied.

Assumption 12 (B0)**.** τ, δ *and* σ *are nonnegative constants with* $\delta \geq \sigma \geq \tau$;

Assumption 13 (B1)**.** *There exist* $\alpha \geq 1$ *and a positive constant* M_1 *such that* $\frac{G(v)}{v^\alpha} \leq M_1$ *for* $v \neq 0$.

Assumption 14 (B2)**.** *There exist* $M_2, M > 0$ *such that* $0 \leq \frac{H(v)}{v} \leq M_2$ *and* $0 \leq \frac{G(v)}{H(v)} \leq M$ *for* $v \neq 0$.

Assumption 15 (B3)**.** $p(x)$ *is bounded.*

Assumption 16 (B4)**.** $\int_{x_0}^{\infty} \int_{u-\delta+\sigma}^{u} r(v)dvdu < \infty$.

Assumption 17 (B5)**.** *There exists a constant* k *such that* $q(x) - Mr(x - \delta + \sigma) \geq k > 0$ *for all* $x \geq x_0$.

In [28,29,33], the authors established some criteria that ensured that every solution of (1) with $G \equiv H$ is oscillatory. In particular, some authors ([26,27,32]) considered the oscillatory and asymptotic behavior of Equation (2) with $G(v) = H(v) = v$.

The above research has greatly stimulated our interest. Thus, in this article, we investigate the oscillatory behavior of the Equation (2) under some assumptions that are different from the previous ones.

Under some new assumptions (i.e., (c3) is not needed and we replace assumption (c2) with (H4) and (H5); (B0), (B1) and (B5) are not required and we provide assumption (H4) instead of (B4)), we study the second-order nonlinear delay differential equation with multiple neutral terms and positive and negative coefficients terms. Motivated by the above research, we obtain some new conditions of the existence of nonoscillatory solution of (2) by using the contraction mapping principle, and we obtain some criteria that ensure the oscillation of bounded solutions of Equation (2) by utilizing the integral inequality technique. Our results extend the research work in this field.

2. Nonoscillatory Solution

When $\eta = 1$, (2) becomes the following.

$$\left[y(x) + \sum_{i=1}^{l} p_i(x)y(x - \tau_i)\right]'' + \sum_{j=1}^{m} q_j(x)G(y(x - \delta_j)) - \sum_{k=1}^{n} r_k(x)H(y(x - \sigma_k)) = 0, \quad x \geq x_0. \tag{3}$$

When $\eta = -1$, (2) becomes the following.

$$\left[y(x)-\sum_{i=1}^{l}p_i(x)y(x-\tau_i)\right]''+\sum_{j=1}^{m}q_j(x)G(y(x-\delta_j))-\sum_{k=1}^{n}r_k(x)H(y(x-\sigma_k))=0,\quad x\geq x_0. \tag{4}$$

In this section, we investigate the existence of the nonoscillatory solution of Equations (3) and (4).

Lemma 1. *Suppose* $(H2)$, $(H4)$ *and* $(H5)$ *hold. If* $p_i(x)$ *satisfies the following:*

$$0<\sum_{i=1}^{l}p_i(x)\leq p<1, \tag{5}$$

then (3) has a nonoscillatory solution.

Proof. It is easy to verify that if (H2) holds, then (H3) holds. Let $L=\max\{L_1,L_2\}$, $\alpha_1=\max\{G(y):y\in A\}$ and $\alpha_2=\max\{H(y):y\in A\}$. According to (H5) and (H4), we have the following:

$$0<\sum_{j=1}^{m}\int_{x_0}^{+\infty}(s-x_0)q_j(s)ds<\sum_{j=1}^{m}\int_{x_0}^{+\infty}sq_j(s)ds<\infty \tag{6}$$

and the following is obtained.

$$0<\sum_{k=1}^{n}\int_{x_0}^{+\infty}(s-x_0)r_k(s)ds<\sum_{k=1}^{n}\int_{x_0}^{+\infty}sr_k(s)ds<\infty. \tag{7}$$

By (6) and (7), we obtain the following.

$$\lim_{x_0\to\infty}\sum_{j=1}^{m}\int_{x_0}^{+\infty}(s-x_0)q_j(s)ds=0$$

$$\lim_{x_0\to\infty}\sum_{k=1}^{n}\int_{x_0}^{+\infty}(s-x_0)r_k(s)ds=0.$$

Thus, we can choose a sufficiently large x such that the following is the case.

$$\sum_{j=1}^{m}\int_{x}^{+\infty}(s-x)q_j(s)ds+\sum_{k=1}^{n}\int_{x}^{+\infty}(s-x)r_k(s)ds<\frac{1-p}{2L}, \tag{8}$$

$$\sum_{j=1}^{m}\int_{x}^{+\infty}\alpha_1(s-x)q_j(s)ds<1-pM_1-M_2, \tag{9}$$

$$\sum_{k=1}^{n}\int_{x}^{+\infty}\alpha_2(s-x)r_k(s)ds<M_1-1. \tag{10}$$

Set the following.

$$z(x):=y(x)+\sum_{i=1}^{l}p_i(x)y(x-\tau_i), \tag{11}$$

$$w(x):=-\sum_{j=1}^{m}\int_{x}^{+\infty}(s-x)q_j(s)G(y(s-\delta_j))ds$$

$$u(x):=\sum_{k=1}^{n}\int_{x}^{+\infty}(s-x)r_k(s)H(y(s-\sigma_k))ds. \tag{12}$$

Define a mapping $T:A\to X$ with the following.

$$
\begin{aligned}
(Ty)(x) :=& 1 - \sum_{i=1}^{l} p_i(x)y(x-\tau_i) - \sum_{j=1}^{m} \int_x^{+\infty} (s-x)q_j(s)G(y(s-\delta_j))ds \\
&+ \sum_{k=1}^{n} \int_x^{+\infty} (s-x)r_k(s)H(y(s-\sigma_k))ds.
\end{aligned}
$$

Clearly, Ty is continuous. For every $y \in A$ and $x \geq x_1$, from (10), we obtain the following.

$$
\begin{aligned}
(Ty)(x) &\leq 1 + \sum_{k=1}^{n} \int_x^{+\infty} (s-x)r_k(s)H(y(s-\sigma_k))ds \\
&\leq 1 + \alpha_2 \sum_{k=1}^{n} \int_x^{+\infty} (s-x)r_k(s)ds \leq M_1.
\end{aligned}
$$

From (9), we have the following.

$$
\begin{aligned}
(Ty)(x) &\geq 1 - pM_1 - \sum_{j=1}^{m} \int_x^{+\infty} (s-x)q_j(s)G(y(s-\delta_j))ds \\
&\geq 1 - pM_1 - \alpha_1 \sum_{j=1}^{m} \int_x^{+\infty} (s-x)q_j(s)ds \geq M_2.
\end{aligned}
$$

Thus, $TA \subset A$. We claim that T is a contraction mapping on A. Indeed, for any $y_1, y_2 \in A$ and $x \geq x_1$, by (8), we have the following.

$$
\begin{aligned}
|(Ty_1)(x) - (Ty_2)(x)| \leq& |p(y_1(x-\tau_i) - y_2(x-\tau_i))| \\
&+ L_1 \sum_{j=1}^{m} \int_x^{\infty} (s-x)q_j(s)|(y_1(s-\delta_j) - y_2(s-\delta_j))|ds \\
&+ L_2 \sum_{k=1}^{n} \int_x^{\infty} (s-x)r_k(s)|(y_1(s-\sigma_k) - y_2(s-\sigma_k))|ds \\
&\leq \|y_1 - y_2\| \left\{ p + L\left(\sum_{j=1}^{m} \int_x^{\infty} (s-x)q_j(s)ds + \sum_{k=1}^{n} \int_x^{\infty} (s-x)r_k(s)ds \right) \right\} \\
&< \frac{p+1}{2}\|y_1 - y_2\|.
\end{aligned}
$$

By taking the sup norm of the above inequality, we have the following.

$$
\|Ty_1 - Ty_2\| < \frac{p+1}{2}\|y_1 - y_2\|.
$$

Because of (5), we obtain $\frac{p+1}{2} < 1$. Refer to a similar proof of ([24], Theorem 2), and we know that T has a fixed point y^*. □

Lemma 2. *If the assumptions of Lemma 1 are satisfied, then Equation (4) has a nonoscillatory solution.*

Proof. Similarly to the proof of Lemma 1, according to (H4) and (H5), we have (6), and (7) holds. Thus, we choose a sufficiently large x such that (8) the following is the case:

$$
\sum_{j=1}^{m} \int_x^{+\infty} \alpha_1 (s-x)q_j(s)ds < 1 - M_2, \tag{13}
$$

and

$$\sum_{k=1}^{n}\int_{x}^{+\infty}\alpha_2(s-x)r_k(s)ds<(1-p)M_1-1, \tag{14}$$

hold. Similarly, we define the mapping $T_1 : A \to X$ as follows.

$$\begin{aligned}(T_1y)(x) :=& 1+\sum_{i=1}^{l}p_i(x)y(x-\tau_i)-\sum_{j=1}^{m}\int_{x}^{+\infty}(s-x)q_j(s)G(y(s-\delta_j))ds\\ &+\sum_{k=1}^{n}\int_{x}^{+\infty}(s-x)r_k(s)H(y(s-\sigma_k))ds.\end{aligned}$$

Obviously, T_1y is continuous. For any $y \in A$ and $x \geq x_2$, by (14), we have the following.

$$\begin{aligned}(T_1y)(x) &\leq 1+\sum_{i=1}^{l}p_i(x)y(x-\tau_i)+\sum_{k=1}^{n}\int_{x}^{+\infty}(s-x)r_k(s)H(y(s-\sigma_k))ds\\ &\leq 1+pM_1+\alpha_2\sum_{k=1}^{n}\int_{x}^{+\infty}(s-x)r_k(s)ds\leq M_1.\end{aligned}$$

By (13), we have the following.

$$\begin{aligned}(T_1y)(x) &\geq 1-\sum_{j=1}^{m}\int_{x}^{+\infty}(s-x)q_j(s)G(y(s-\delta_j))ds\\ &\geq 1-\alpha_1\sum_{j=1}^{m}\int_{x}^{+\infty}(s-x)q_j(s)ds\geq M_2.\end{aligned}$$

Thus, $T_1A \subset A$. Next, we prove that T_1 is a contraction mapping on A. For any $y_1, y_2 \in A$ and $x \geq x_2$, by (8), we have the following.

$$\begin{aligned}|(T_1y_1)(x)-(T_1y_2)(x)| &\leq |p(y_1(x-\tau_i)-y_2(x-\tau_i))|\\ &+L_1\sum_{j=1}^{m}\int_{x}^{\infty}(s-x)q_j(s)|(y_1(s-\delta_j)-y_2(s-\delta_j))|ds\\ &+L_2\sum_{k=1}^{n}\int_{x}^{\infty}(s-x)r_k(s)|(y_1(s-\sigma_k)-y_2(s-\sigma_k))|ds\\ &\leq \|y_1-y_2\|\left\{p+L\left(\sum_{j=1}^{m}\int_{x}^{\infty}(s-x)q_j(s)ds+\sum_{k=1}^{n}\int_{x}^{\infty}(s-x)r_k(s)ds\right)\right\}\\ &<\frac{p+1}{2}\|y_1-y_2\|.\end{aligned}$$

Refer to a similar proof of ([24], Theorem 2), we obtain that T_1 has a fixed point y^*. □

Theorem 1. *Suppose* $(H2)$, $(H4)$ *and* $(H5)$ *hold. If* $p_i(x)$ *satisfies* (5), *then* (2) *has a nonoscillatory solution.*

Proof. According to Lemmas 1 and 2, we obtain (2), which has a nonoscillatory solution. □

Example 1. *Consider the following equation.*

$$\left[y(x)+\frac{1}{8}y(x-1)\right]''-e^{-\frac{x}{2}}\left(1-\frac{1}{e^{-x}+2}\right)\frac{y(\frac{x}{2})[y^2(\frac{x}{2})+2]}{y^2(\frac{x}{2})+1} \\ -\frac{1}{8}e^{-\frac{x-1}{2}}\left(1-\frac{2}{e^{-x+1}+3}\right)\frac{y(\frac{x-1}{2})[y^2(\frac{x-1}{2})+3]}{y^2(\frac{x-1}{2})+1}=0. \tag{15}$$

Here, we have the following.

$$l=1,\ p_1(x)=\frac{1}{8},$$
$$q(x)=0,$$
$$n=2,\ r_1(x)=e^{-\frac{x}{2}}\left(1-\frac{1}{e^{-x}+2}\right),\ r_2(x)=\frac{1}{8}e^{-\frac{x-1}{2}}\left(1-\frac{2}{e^{-x+1}+3}\right).$$

It is easy to verify that $p_1(x)$ and $q(x)$ satisfy (5) and (H5), respectively, and $r_1(x)$ and $r_2(x)$ satisfy (H4). Therefore, by Theorem 1, $y(x)=e^{-x}$ is a nonoscillatory solution of (15).

Example 2. *Consider the following equation:*

$$\left[y(x)-\frac{3}{4}y(x-2)\right]''+\frac{3}{4}e^{-\frac{x-2}{2}}\left(1-\frac{2}{e^{-x+2}+3}\right)\frac{y(\frac{x-2}{2})[y^2(\frac{x-2}{2})+3]}{y^2(\frac{x-2}{2})+1} \\ -e^{-\frac{x}{2}}\left(1-\frac{1}{e^{-x}+2}\right)\frac{y(\frac{x}{2})[y^2(\frac{x}{2})+2]}{y^2(\frac{x}{2})+1}=0, \tag{16}$$

and we have the following.

$$l=1,\ p_1(x)=\frac{3}{4},$$
$$m=1,\ q_1(x)=\frac{3}{4}e^{-\frac{x-2}{2}}\left(1-\frac{2}{e^{-x+2}+3}\right),$$
$$n=1,\ r_1(x)=e^{-\frac{x}{2}}\left(1-\frac{1}{e^{-x}+2}\right).$$

It is easily verified that $p_1(x)$, $q_1(x)$ and $r_1(x)$ satisfy (5), (H5) and (H4), respectively. By Theorem 1, (16) has a nonoscillatory $y(x)=e^{-x}$.

3. Oscillatory Criteria

In this section, the oscillation criteria of (2) will be given, and some examples will be illustrated to demonstrate the results.

Lemma 3. *Suppose that (H1), (H3) and (H4) hold and $p_i(x)\in C([x_0,\infty),\Re^+)$ is bounded. If the bounded solution $y(x)$ of (3) satisfies $\lim_{x\to\infty}y(x)\neq 0$, then $y(x)$ is oscillatory.*

Proof. Suppose toward a contradiction, there is no loss of generality in assuming that y is an eventually a positive-bounded solution of (3). Thus, there exists $x_1\geq x_0+\kappa$ such that $y(x-\kappa)>0$ for $x\geq x_1$. Furthermore, there exists $K>0$ such that $y(x)\leq K$ for $x\geq x_1$. From (7), we may choose a sufficiently large $x>x_2\geq x_1+\kappa$, such that the following is the case.

$$\sum_{k=1}^{n}\int_{x}^{+\infty}(s-x)r_k(s)ds<\frac{1}{2N_2}. \tag{17}$$

Let the following be the case:

$$w(x)=z(x)-u(x), \tag{18}$$

where $z(x)$ and $u(x)$ are defined in (11) and (12), respectively. Then, we have the following.

$$w''(x) = -\sum_{j=1}^{m} q_j(x)G(y(x-\delta_j)). \tag{19}$$

From (H1) and (H3), we obtain $w''(x) < 0$. Hence, $w'(x) > 0$ or $w'(x) < 0$ for all $x \geq x_3 \geq x_2 + \kappa$, and x_3 is sufficiently large.

If $w'(x) < 0$ for all $x \geq x_3$, then the following is the case.

$$\lim_{x\to\infty} w(x) = -\infty. \tag{20}$$

According to (3), (17), (18) and (H3), we obtain the following:

$$w(x) \geq -N_2 K \sum_{k=1}^{n} \int_x^{+\infty} (s-x) r_k(s) ds \geq -\frac{K}{2} > -\infty,$$

which contradicts (20). Thus, $w(x)$ is increasing for all $x \geq x_3$. From (H1), $q_j(x) \geq h_1$, $j = 1, \cdots, m$ for all $x \geq x_3$. Integrating (19) from x_3 to $+\infty$, we obtain the following.

$$\infty > w'(x_3) \geq \sum_{j=1}^{m} \int_{x_3}^{+\infty} q_j(x)G(y(x-\delta_j))dx \geq h_1 m_1 \sum_{j=1}^{m} \int_{x_3}^{+\infty} y(x-\delta_j)dx.$$

Therefore, $y \in L^1([x_3, \infty))$, which contradicts $\lim_{x\to\infty} y(x) \neq 0$. The proof is complete. □

Lemma 4. *If the assumptions of Lemma 3 are satisfied and if the bounded solution $y(x)$ of Equation (4) satisfies $\lim_{x\to\infty} y(x) \neq 0$, then $y(x)$ is oscillatory.*

Proof. Just as in the proof of Lemma 3, assume that y is an eventually positive-bounded solution of (4). Since $\lim_{x\to\infty} y(x) \neq 0$, $0 < y(x-\kappa) < K$ for $x \geq x_1 \geq x_0 + \kappa$, where $K > 0$.

Define the following.

$$Y(x) = y(x) - \sum_{i=1}^{l} p_i(x) y(x-\tau_i).$$

$$w_1(x) = Y(x) - u(x). \tag{21}$$

Then, the following is the case.

$$\begin{aligned} w_1''(x) &= Y''(x) - u''(x) \\ &= -\sum_{j=1}^{m} q_j(x)G(y(x-\delta_j)) < 0, \qquad x \geq x_2 \geq x_1 + \kappa. \end{aligned} \tag{22}$$

Hence, $w_1'(x) > 0$ or $w_1'(x) < 0$ for all $x \geq x_3$, where $x_3 \geq x_2 + \kappa$ is sufficiently large. If $w_1'(x) < 0$ for all $x \geq x_3$, we have the following.

$$\lim_{x\to\infty} w_1(x) = -\infty. \tag{23}$$

Because $p_i(x)$ is bounded, then we have the following:

$$p_i(x) \leq p_i,\ i = 1, \cdots, l,\ \sum_{i=1}^{l} p_i < P \tag{24}$$

where p_i and P are non-negative constants. By means of (17), (21), (24) and (H3), we have the following:

$$w_1(x) \geq -\left[\sum_{i=1}^{l} p_i(x) + N_2 \sum_{k=1}^{n} \int_{x_2}^{+\infty} (s-x) r_k(s) ds\right] K \geq -\left(P + \frac{1}{2}\right) K > -\infty,$$

which contradicts (23). Hence, $w_1(x)$ is increasing for all $x \geq x_3$. Following the same method as in Lemma 3, from (H1) and inequality (22), we obtain $y \in L^1([x_3, \infty))$, which contradicts $\lim_{x\to\infty} y(x) \neq 0$. The proof is complete. □

Theorem 2. *Suppose that (H1), (H3) and (H4) hold and $p_i(x) \in C([x_0,\infty), \Re^+)$ is bounded. If the bounded solution $y(x)$ of (2) satisfies $\lim_{x\to\infty} y(x) \neq 0$, then $y(x)$ is oscillatory.*

Proof. According to Lemmas 3 and 4, we obtain that if the bounded solution $y(x)$ of (2) satisfies $\lim_{x\to\infty} y(x) \neq 0$, then $y(x)$ is oscillatory. □

Example 3. *Consider the following equation.*

$$\begin{aligned} &\left[y(x) + y(x - \frac{3\pi}{2})\right]'' + (e^{-x}+1)\left(1 - \frac{1}{\sin^2 x + 2}\right)\frac{y(x-2\pi)[y^2(x-2\pi)+2]}{y^2(x-2\pi)+1} \\ &+ \left(1 - \frac{1}{\cos^2 x + 2}\right)\frac{y(x-\frac{3\pi}{2})[y^2(x-\frac{3\pi}{2})+2]}{y^2(x-\frac{3\pi}{2})+1} \\ &- e^{-x}\left(1 - \frac{2}{\sin^4 x + 3}\right)\frac{y(x-2\pi)[y^4(x-2\pi)+3]}{y^4(x-2\pi)+1} = 0. \end{aligned} \tag{25}$$

Here, we have the following.

$$l = 1,\ p_1(x) = 1,$$

$$m = 2,\ q_1(x) = (e^{-x}+1)\left(1 - \frac{1}{\sin^2 x + 2}\right),\ q_2(x) = \left(1 - \frac{1}{\cos^2 x + 2}\right),$$

$$n = 1,\ r_1(x) = e^{-x}\left(1 - \frac{2}{\sin^4 x + 3}\right).$$

It is easy to verify that $\frac{1}{2} \leq q_1(x) \leq \frac{4}{3}$, $\frac{1}{2} \leq q_2(x) \leq \frac{2}{3}$ and $\int_{x_0}^{+\infty} (s-x) r_1(s) ds < \infty$. Therefore, according to Theorem 2, we know that every bounded solution of (2) that does not tend to zero is oscillatory. Indeed, $y(x) = \sin x$ is a bounded oscillatory solution of (25).

Example 4. *Consider the following equation.*

$$\begin{aligned} &\left[y(x) - \frac{1}{2^3} y(x-\pi) - \frac{1}{2^3} y(x-2\pi)\right]'' \\ &+ (e^{-x}+1)\left(1 - \frac{1}{\sin^2 x + 2}\right)\frac{y(x-2\pi)[y^2(x-2\pi)+2]}{y^2(x-2\pi)+1} \\ &- e^{-x}\left(1 - \frac{2}{sin^4 x + 3}\right)\frac{y(x-2\pi)[y^4(x-2\pi)+3]}{y^4(x-2\pi)+1} = 0. \end{aligned} \tag{26}$$

We have the following.

$$l = 2,\ p_1(x) = p_2(x) = \frac{1}{2^3},$$

$$m = 1,\ q_1(x) = (e^{-x}+1)\left(1 - \frac{1}{\sin^2 x + 2}\right),$$

$$n = 1,\ r_1(x) = e^{-x}\left(1 - \frac{2}{\sin^4 x + 3}\right).$$

It is clear that $\sum_{i=1}^{2} p_i(x) \le 1$, $\frac{1}{2} \le q_1(x) \le \frac{4}{3}$ *and* $\int_{x_0}^{+\infty}(s-x)r_1(s)ds < \infty$. *Therefore, according to Theorem 2, we know that every bounded solution of (2) that does not tend to zero is oscillatory. Indeed,* $y(x) = \sin x$ *is a bounded oscillatory solution of (26).*

4. Remark

Comparing with the results of [24–29,31–33], we increased the number of the positive and negative coefficient terms and the neutral terms of the second-order delay differential equation with positive and negative coefficients from single to multiple and generalized the equation from a linear case to a nonlinear case.

Motivated by the useful work of Lin et al. and Zhang et al. ([24,25]), we provide some new conditions under which Equation (2) has a nonoscillatory solution. More precisely, (c3) is not needed and we replace assumption (c2) with (H4) and (H5).

For the oscillation of Equation (2), we present some assumptions that are different from those in [31], i.e., (B0), (B1) and (B5) are not necessary and we provide assumption (H4) instead of (B4). Compared with the studies of Malojlović et al. ([27,32]), we generalize their work to the nonlinear situation and provide different assumptions. Firstly, we provide condition (H4) instead of the following condition:

$$\sum_{i=1}^{n} \int_0^{\infty} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi ds < 1$$

in [27] or

$$\sum_{i=1}^{n} \int_0^{\infty} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi ds < p_{j_1}(x)$$

in [32]. Secondly, assumptions (H1) and (H2) in [27] (or (H2) and (H3) in [32]) are not needed, which means that there is no relationship between the positive and negative coefficients.

We obtain not only the oscillation criteria but also the existence of the nonoscillation solution of (2); thus, our results are an extension of theirs.

Author Contributions: All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of Jilin province (20180101221JC).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data available.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Baculíková, B. Oscillation of second order half-linear differential equations with deviating arguments of mixed type. *Appl. Math. Lett.* **2021**, *119*, 107228. [CrossRef]
2. Baculíková, B.; Dzurina, J. New asymptotic results for half-linear differential equations with deviating argument. *Carpathian J. Math.* **2022**, *38*, 327–335. [CrossRef]
3. Agarwal, R.P.; Zhang, C.; Li, T. Some remarks on oscillation of second order neutral differential equations. *Appl. Math. Comput.* **2016**, *274*, 178–181. [CrossRef]
4. Jadlovská, I. New criteria for sharp oscillation of second-order neutral delay differential equations. *Mathematics* **2021**, *9*, 2089. [CrossRef]
5. Baculíková, B.; Sudha, B.; Thangavelu, K.; Thandapani, E. Oscillation of second order delay differential equations with nonlinear nonpositive neutral term. *Math. Slovaca* **2022**, *72*, 103–110. [CrossRef]
6. Agarwal, R.P.; Bazighifan, O.; Ragusa, M.A. Nonlinear neutral delay differential equations of fourth-order: Oscillation of solutions. *Entropy* **2021**, *23*, 129. [CrossRef]
7. Agarwal, R.P.; Grace, S.R.; Regan, D.O. *Oscillation Theory for Difference and Functional Differential Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2000.

8. Abbas, M.I.; Ragusa, M.A. On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function. *Symmetry* **2021**, *13*, 264. [CrossRef]
9. Bazighifan, O.; Cesarano, C. Some new oscillation criteria for second order neutral differential equations with delayed arguments. *Mathematics* **2019**, *7*, 619. [CrossRef]
10. Bohner, M.; Hassan, T.S.; Li, T. Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments. *Indag. Math.* **2018**, *29*, 548–560. [CrossRef]
11. Bohner, M.; Li, T. Oscillation of second-order p-Laplace dynamic equations with a nonpositive neutral coefficient. *Appl. Math. Lett.* **2014**, *37*, 72–76. [CrossRef]
12. Bohner, M.; Li, T. Kamenev-type criteria for nonlinear damped dynamic equations. *Sci. China Math.* **2015**, *58*, 1445–1452. [CrossRef]
13. Chiu, K.-S.; Li, T. Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments. *Math. Nachr.* **2019**, *292*, 2153–2164. [CrossRef]
14. Džurina, J.; Grace, S.R.; Jadlovská, I.; Li, T. Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term. *Math. Nachr.* **2020**, *293*, 910–922. [CrossRef]
15. Frassu, S.; Viglialoro, G. Boundedness in a chemotaxis system with consumed chemoattractant and produced chemorepellent. *Nonlinear Anal.* **2021**, *213*, 112505. [CrossRef]
16. Li, T.; Pintus, N.; Viglialoro, G. Properties of solutions to porous medium problems with different sources and boundary conditions. *Z. Angew. Math. Phys.* **2019**, *70*, 86. [CrossRef]
17. Li, T.; Rogovchenko, Y.V. Oscillation of second-order neutral differential equations. *Math. Nachr.* **2015**, *288*, 1150–1162. [CrossRef]
18. Li, T.; Rogovchenko, Y.V. Oscillation criteria for even-order neutral differential equations. *Appl. Math. Lett.* **2016**, *61*, 35–41. [CrossRef]
19. Li, T.; Rogovchenko, Y.V. Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations. *Monatsh. Math.* **2017**, *184*, 489–500. [CrossRef]
20. Li, T.; Rogovchenko, Y.V. On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations. *Appl. Math. Lett.* **2017**, *67*, 53–59. [CrossRef]
21. Li, T.; Rogovchenko, Y.V. On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations. *Appl. Math. Lett.* **2020**, *105*, 106293. [CrossRef]
22. Li, T.; Viglialoro, G. Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime. *Differ. Integral. Equ.* **2021**, *34*, 315–336.
23. Zhang, C.; Agarwal, R.P.; Bohner, M.; Li, T. Oscillation of fourth-order delay dynamic equations. *Sci. China Math.* **2015**, *58*, 143–160. [CrossRef]
24. Lin, S.-Z.; Qu, Y.; Yu, Y.-H. Existence of nonoscillatory solution of second order nonlinear neutral delay equations. *Kyungpook Math. J.* **2006**, *46*, 273–284.
25. Zhang, J.; Jin, Z.; Zhang, H. Existence of nonoscillatory solution for second order nonlinear neutral delay differential equation. In Proceedings of the International Conference on Software Engineering, Qingdao, China, 30 July–1 August 2007; pp. 339–342.
26. Karpuz, B.; Manojlović, J.V.; Öcalan, Ö; Shoukaku, Y. Oscillation criteria for a class of second-order neutral delay differential equations. *Appl. Math. Comput.* **2009**, *210*, 303–312. [CrossRef]
27. Manojlović, J.; Shoukaku, Y.; Tanigawa, T.; Yoshida, N. Oscillation criteria for second order differential equations with positive and negative coefficients. *Appl. Math. Comput.* **2006**, *181*, 853–863. [CrossRef]
28. Padhi, S. Oscillation and asymptotic behaviour of solutions of second order homogeneous neutral differential equations with positive and negative coefficients. *Funct. Differ. Equ.* **2007**, *14*, 363–371.
29. Padhi, S. Oscillation and asymptotic behaviour of solutions of second order neutral differential equations with positive and negative coefficients. *Fasc. Math.* **2007**, *38*, 105–114.
30. Shoukaku, Y. Oscillation theory of second order differential equations with positive and negative coefficients. *Hacet. J. Math. Stat.* **2022**, *in press*. [CrossRef]
31. Thandapani, E.; Muthulakshmi, V.; Graef, J.R. Oscillation criteria for second order nonlinear neutral delay differential equations with positive and negative coefficients. *Int. J. Pure Appl. Math.* **2011**, *70*, 261–274.
32. Weng, A.; Sun, J. Oscillation of second order delay differential equations. *Appl. Math. Comput.* **2008**, *198*, 930–935. [CrossRef]
33. Yildiz, M.K.; Karpuz, B.; Öcalan, Ö. Oscillation of nonlinear neutral delay differential equations of second-order with positive and negative coefficients. *Turk. J. Math.* **2009**, *33*, 341–350.