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The Existence of Radial Solutions to the Schrödinger System Containing a Nonlinear Operator

Guotao Wang [1,*], Zhuobin Zhang [1] and Zedong Yang [2]

[1] School of Mathematics and Computer Science, Shanxi Normal University, Taiyuan 030031, China; zhangzhuobin89757@163.com
[2] College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, China; yangzd1229@163.com
* Correspondence: wgt2512@163.com

Abstract: In this paper, we investigate a class of nonlinear Schrödinger systems containing a nonlinear operator under Osgood-type conditions. By employing the iterative technique, the existence conditions for entire positive radial solutions of the above problem are given under the cases where components μ and ν are bounded, μ and ν are blow-up, and one of the components is bounded, while the other is blow-up. Finally, we present two examples to verify our results.

Keywords: Osgood-type condition; Schrödinger system; monotone iterative method; nonlinear operator; radial solution

MSC: 35B08; 35B09; 35J10

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1. Introduction

Osgood-type condition is of great significance in the field of mathematics and has been widely applied to different equations or systems by many authors. In 1898, under the Osgood type condition

$$\int_0^U \frac{ds}{\psi(s)} = \infty, \ \forall U > 0,$$

Osgood [1] presented the existence result of solutions for the following equation without the Cauchy–Lipschitz condition

$$\frac{dy}{d\chi} = \psi(\chi, y),$$

where $\psi(s)$ is a continuous function satisfying $|\psi(\chi, y) - \psi(\chi, y')| \le \varphi(|y - y'|)$. Then, lots of authors began to consider applying the Osgood-type condition to other problems and gained many excellent results such as the comparison result of viscosity upper and lower solutions for fully nonlinear parabolic equations [2], the existence result of solutions for backward stochastic differential equations (BSDEs) [3], and the nonexistence result of the local solution for semilinear fractional heat Equation [4]. For more results, see [5–9].

The Schrödinger equation was derived from mathematical physics and closely related to several physical phenomena. In [10], Kurihura used it to model the superfluid film equation in plasma physics. In [11,12], it was used to model the phenomena of the self-channeling of a high-power ultrashort laser in matter. More examples and details of applications can be found in [13–16].

In 2017, by employing the analysis technique and weighted norm method, Sun [17] established the existence result of solutions to the following Schrödinger equation

$$\Delta\mu + \psi(|\chi|)b(\mu) = 0,$$

where $|\chi| \in E_D$, $\psi(|\chi|) \in C^{\lambda}_{loc}(E_D, R)$, $\lambda \in (0,1)$, $b(\mu) \in C^{\lambda}_{loc}(R, R)$ (locally Hölder continuous), $E_D = \{\chi \in R^2 : |\chi| > D\}$, $S_D = \{\chi \in R^2 : |\chi| = D\}$, for $D > 0$.

In 2018, by introducing a growth condition and employing the iterative technique, Zhang, Wu and Cui [18] established the nonexistence and existence results of the entire blow-up solutions to the following Schrödinger equation

$$div(\Lambda(|\nabla\mu|)\nabla\mu) = b(|\chi|)\psi(\mu), \quad \chi \in \mathbb{R}^n,$$

where $n \geq 2$, Λ is a nonlinear operator belonging to the set $\{\Lambda \in C^2([0,\infty),(0,\infty)) | \exists\beta \in (0,\infty) : \Lambda(ms) \leq m^{\beta}\Lambda(s), 0 < m < 1\}$.

In 2020, by employing the iterative technique, Wang et al. [19] established the existence result of the entire radial solutions for the following Schrödinger system

$$\begin{cases} div\big(\Lambda(|\nabla\mu|^{p-2})\nabla\mu\big) = b(|\chi|)\psi(\nu), & \chi \in \mathbb{R}^n, \\ div\big(\Lambda(|\nabla\nu|^{p-2})\nabla\nu\big) = h(|\chi|)\varphi(\mu), & \chi \in \mathbb{R}^n, \end{cases}$$

where $n \geq 3$, $b, h, \psi, \varphi \in C([0,\infty),[0,\infty))$ and Λ is a nonlinear operator belonging to $\theta = \{\Lambda \in C^2([0,\infty),(0,\infty)) | \exists p \in (2,\infty) : \Lambda(ms) \leq m^{p-2}\Lambda(s), 0 < m < 1\}$.

Motivated by the above work, we studied the existence of entire positive radial solutions to the following Schrödinger system

$$\begin{cases} div\big(\Lambda(|\nabla\mu|^{p-2})\nabla\mu\big) = b(|\chi|)\psi(\mu,\nu), & \chi \in \mathbb{R}^n, \\ div\big(\Lambda(|\nabla\nu|^{p-2})\nabla\nu\big) = h(|\chi|)\varphi(\mu,\nu), & \chi \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $n \geq 3$, b, h are continuous functions, Λ is a nonlinear operator belonging to θ and ψ, φ are continuous functions satisfying Osgood-type conditions

$$\int_i^{\infty} \frac{1}{\left(\psi\left(t, (\varphi(t,t))^{\frac{1}{p-1}}\right) + 1\right)^{\frac{1}{p-1}}} dt = \infty, \quad \forall i > 0$$

and

$$\int_j^{\infty} \frac{1}{\left(\varphi\left((\psi(t,t))^{\frac{1}{p-1}}, t\right) + 1\right)^{\frac{1}{p-1}}} dt = \infty, \quad \forall j > 0.$$

By employing the monotone iterative method, we give the existence results of positive entire radial solutions to the Schrödinger system (1) under the cases where the components μ and ν are bounded, μ and ν are blow-up, and one of the components is bounded while the other is blow-up. The monotone iterative method plays a significant role in the study of nonlinear problem, as can be seen in [18–28] and the references therein. To the best of our knowledge, there is no work about the existence of the positive radial solutions of the Schrödinger system (1) under the Osgood-type conditions. In addition, our results extended the work of authors in [18,28–33].

2. Preliminaries

In this section, we give a definition, some notations, assumptions and Lemmas that are subsequently needed in the proof.

Firstly, we present the definition about the classification of solutions.

Definition 1 ([34]). *A solution $(\mu,\nu) \in C^2[0,\infty) \times C^2[0,\infty)$ of system (1) is called an entire bounded solution if condition (2) is established; it is called an entire blow-up solution if condition (3) is established; it is called a semifinite entire blow-up solution if condition (4) or (5) is established.*

Finite case: both components μ and ν are bounded, that is

$$\lim_{|\chi|\to\infty} \mu(|\chi|) < \infty \quad and \quad \lim_{|\chi|\to\infty} \nu(|\chi|) < \infty. \tag{2}$$

Infinite case: both components μ, ν are blow-up, that is

$$\lim_{|\chi|\to\infty} \mu(|\chi|) = \infty \quad and \quad \lim_{|\chi|\to\infty} \nu(|\chi|) = \infty. \tag{3}$$

Semifinite Case: one of the components is bounded, while the other is blow-up, that is

$$\lim_{|\chi|\to\infty} \mu(|\chi|) < \infty \quad and \quad \lim_{|\chi|\to\infty} \nu(|\chi|) = \infty \tag{4}$$

or

$$\lim_{|\chi|\to\infty} \mu(|\chi|) = \infty \quad and \quad \lim_{|\chi|\to\infty} \nu(|\chi|) < \infty. \tag{5}$$

We then present the notations as follows: $\tau = |\chi|, i, j, c_1, c_2 \in (0,\infty)$ are suitably chosen,

$$\begin{aligned}
G_1(\tau) &= \int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)ds\Big)dt,\\
G_2(\tau) &= \int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)ds\Big)dt,\\
L(\tau) &= \int_{i+j}^\tau \frac{dt}{[(\psi+\varphi)(t,t)+1]^{\frac{1}{p-1}}}, \qquad L(\infty) := \lim_{\tau\to\infty} L(\tau),\\
G(\tau) &= G_1(\tau)+G_2(\tau), \qquad G_k(\infty) := \lim_{\tau\to\infty} G_k(\tau), \quad k=1,2,\\
\omega_1(\tau) &= \psi\Big(1, \Big(\frac{j}{\varphi(i,i)^{p-1}} + \big(c_2\varphi(1, 1+\frac{L^{-1}(G(\tau))}{i}) + \frac{1}{\varphi(i,i)}\big)^{\frac{1}{p-1}} G_2(\tau)\Big)\Big),\\
\omega_2(\tau) &= \varphi\Big(\Big(\frac{i}{\psi(j,j)^{p-1}} + \big(c_1\psi(1+\frac{L^{-1}(G(\tau))}{j}, 1) + \frac{1}{\psi(j,j)}\big)^{\frac{1}{p-1}} G_1(\tau)\Big), 1\Big),\\
U_1(\tau) &= \int_0^\tau \Im^{-1}\Big(\frac{1}{\varrho^{n-1}}\int_0^\varrho t^{n-1}b(t)\psi\Big(i, j+\big(\frac{1}{\varphi(i,j)+1}\big)^{\frac{1}{p-1}} G_2(t)\Big)dt\Big)d\varrho,\\
V_1(\tau) &= \int_0^\tau \Im^{-1}\Big(\frac{1}{\varrho^{n-1}}\int_0^\varrho t^{n-1}h(t)\varphi\Big(i+\big(\frac{1}{\psi(i,j)+1}\big)^{\frac{1}{p-1}} G_1(t), j\Big)dt\Big)d\varrho,\\
U_2(\tau) &= \int_0^\tau \Big(c_1\omega_1(t)+1\Big)^{\frac{1}{p-1}} \Im^{-1}\Big(\int_0^t b(s)ds\Big)dt,\\
V_2(\tau) &= \int_0^\tau \Big(c_2\omega_2(t)+1\Big)^{\frac{1}{p-1}} \Im^{-1}\Big(\int_0^t h(s)ds\Big)dt,\\
U_k(\infty) &:= \lim_{\tau\to\infty} U_k(\tau), \quad V_k(\infty) := \lim_{\tau\to\infty} V_k(\tau), \quad for\ k=1,2,\\
F_1(\tau) &= \int_i^\tau \frac{1}{\Big(\psi\big(t,(\varphi(t,t))^{\frac{1}{p-1}}\big)+1\Big)^{\frac{1}{p-1}}}dt, \quad F_1(\infty) := \lim_{r\to\infty} F_1(\tau),\\
F_2(\tau) &= \int_j^\tau \frac{1}{\Big(\varphi\big((\psi(t,t))^{\frac{1}{p-1}}, t\big)+1\Big)^{\frac{1}{p-1}}}dt, \quad F_2(\infty) := \lim_{r\to\infty} F_2(\tau).
\end{aligned}$$

Assume that ψ and φ satisfy the following assumptions.

$(N1)$ $\psi, \varphi \in C([0,\infty)\times[0,\infty), [0,\infty))$ are increasing for every variable and $\psi(\mu,\nu) > 0, \varphi(\mu,\nu) > 0$ for all $\mu, \nu > 0$;

$(N2)$ for fixed constants $i, j \in (0,\infty)$, there exist $c_1, c_2 \in (0,\infty)$ such that

$$\psi(t_1 s_1, t_2 s_2) \le c_1 \psi(t_1, t_2)\psi(s_1, s_2),$$

$$\varphi(t_1s_1, t_3s_3) \le c_2\varphi(t_1, t_3)\varphi(s_1, s_3),$$

$$\psi(i,j) \ge \frac{\sqrt{5}-1}{2} \quad and \quad \varphi(i,j) \ge \frac{\sqrt{5}-1}{2},$$

where $t_1 \ge \min\{i, j, \psi^{\frac{1}{p-1}}(j,j)\}$, $s_1 \ge \min\{1, i\psi^{\frac{1}{1-p}}(j,j)\}$, $t_2 \ge \min\{j, \varphi^{\frac{1}{p-1}}(i,i)\}$, $s_2 \ge \min\{1, j\varphi^{\frac{1}{1-p}}(i,i)\}$, $t_3 \ge \min\{i,j\}$, $s_3 \ge 1$;

$(S1)$ $U_2(\infty) < F_1(\infty) < \infty, V_2(\infty) < F_2(\infty) < \infty$;
$(S2)$ $U_1(\infty) < \infty, V_1(\infty) < \infty$;
$(S3)$ $F_1(\infty) = F_2(\infty) = \infty, U_2(\infty) = V_2(\infty) = \infty$;
$(S4)$ $U_1(\infty) = V_1(\infty) = \infty$;
$(S5)$ $F_1(\infty) = \infty, U_1(\infty) = \infty, U_2(\infty) = \infty$;
$(S6)$ $V_1(\infty) < \infty, V_2(\infty) < F_2(\infty) < \infty$;
$(S7)$ $F_2(\infty) = \infty, V_1(\infty) = \infty, V_2(\infty) = \infty$;
$(S8)$ $U_1(\infty) < \infty, U_2(\infty) < F_1(\infty) < \infty$.

Lemma 1 ([18]). *If $\Lambda \in \theta$, let $\Im(s) = s\Lambda(s^{p-2})$. We have*
(1) : *$\Im(s)$ has a nonnegative increasing inverse mapping $\Im^{-1}(s)$;*
(2) : *If $0 < q < 1$, we have*

$$\Im^{-1}(qs) \ge q^{\frac{1}{p-1}}\Im^{-1}(s);$$

(3) : *If $q \ge 1$, we have*

$$\Im^{-1}(qs) \le q^{\frac{1}{p-1}}\Im^{-1}(s).$$

Through the similar proof as in [19], we can obtain the following Lemma.

Lemma 2. *$(\mu,\nu) \in C^2[0,\infty) \times C^2[0,\infty)$ is a radial solution of the Schrödinger system (1) if and only if it is a solution of the following ordinary differential system*

$$\begin{cases} (\Lambda(|\mu'|^{p-2})\mu')' + \dfrac{n-1}{\tau}\Lambda(|\mu'|^{p-2})\mu' = b(\tau)\psi(\mu,\nu), & \tau > 0, \\ (\Lambda(|\nu'|^{p-2})\nu')' + \dfrac{n-1}{\tau}\Lambda(|\nu'|^{p-2})\nu' = h(\tau)\varphi(\mu,\nu), & \tau > 0. \end{cases} \tag{6}$$

3. The Entire Positive Bounded Radial Solutions

In this section, we prove Theorems 1 and 2.

Theorem 1. *Assume that $(N1)$, $(N2)$ hold, then the system (1) has an entire positive radial solution $(\mu,\nu) \in C^2[0,\infty) \times C^2[0,\infty)$.*

Proof. Through an operation on system (6), we obtain

$$\begin{cases} (\Im(\mu'))' + \dfrac{n-1}{\tau}\Im(\mu') = b(\tau)\psi(\mu(\tau),\nu(\tau)), & \tau > 0, \\ (\Im(\nu'))' + \dfrac{n-1}{\tau}\Im(\nu') = h(\tau)\varphi(\mu(\tau),\nu(\tau)), & \tau > 0. \end{cases}$$

Obviously, the above system can be transformed into the following system

$$\begin{cases} \mu(\tau) = \mu(0) + \displaystyle\int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\psi(\mu(s),\nu(s))ds\Big)dt, & \tau \ge 0, \\ \nu(\tau) = \nu(0) + \displaystyle\int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\varphi(\mu(s),\nu(s))ds\Big)dt, & \tau \ge 0. \end{cases}$$

Define the sequences $\{\mu_m(\tau)\}_{m\geq 0}$ and $\{\nu_m(\tau)\}_{m\geq 0}$ on $[0,\infty)$ by

$$\begin{cases} \mu_0(\tau) = \mu(0) = i, \ \ \nu_0(\tau) = \nu(0) = j, \ \ \tau \geq 0, \\ \mu_m(\tau) = \mu(0) + \int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\psi\big(\mu_{m-1}(s),\nu_{m-1}(s)\big)ds\Big)dt, \ \ \tau \geq 0, \\ \nu_m(\tau) = \nu(0) + \int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\varphi\big(\mu_{m-1}(s),\nu_{m-1}(s)\big)ds\Big)dt, \ \ \tau \geq 0. \end{cases} \tag{7}$$

Using the similar arguments as in [19], we obtain the sequences $\{\mu_m(\tau)\}_{m\geq 0}$ and $\{\nu_m(\tau)\}_{m\geq 0}$ are increasing and

$$\frac{(\mu_m(\tau)+\nu_m(\tau))'}{[(\psi+\varphi)(\mu_m(\tau)+\nu_m(\tau),\mu_m(\tau)+\nu_m(\tau))+1]^{\frac{1}{p-1}}} \leq G_1'(\tau)+G_2'(\tau).$$

We then arrive at

$$\int_{i+j}^{\mu_m(\tau)+\nu_m(\tau)} \frac{dt}{[(\psi+\varphi)(t,t)+1]^{\frac{1}{p-1}}} \leq G(\tau).$$

Therefore,

$$L(\mu_m(\tau)+\nu_m(\tau)) \leq G(\tau).$$

By $(N1)$, we can obtain that $L'(\tau) > 0$ and $L(\tau)$ is a bijection. Clearly, the inverse function L^{-1} is strictly increasing on $[0, L(\infty))$ and

$$\mu_m(\tau)+\nu_m(\tau) \leq L^{-1}(G(\tau)). \tag{8}$$

By Lemma 1, $(N1)$, $(N2)$, (7) and (8), the monotonicity of $\{\mu_m(\tau)\}_{m\geq 0}$ and $\{\nu_m(\tau)\}_{m\geq 0}$, we obtain

$$\begin{aligned} \mu_m(\tau) \leq & i + \int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)\psi\big(\mu_m(s),\nu_m(s)\big)ds\Big)dt \\ \leq & i + \Big(\psi\big(\mu_m(\tau),\nu_m(\tau)\big)+1\Big)^{\frac{1}{p-1}}\int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}b(s)ds\Big)dt \\ \leq & i + \Big(\psi\big(\nu_m(\tau)+L^{-1}(G(\tau)),\nu_m(\tau)\big)+1\Big)^{\frac{1}{p-1}}G_1(\tau) \\ = & i + \Big(\psi\big(\nu_m(\tau)(1+\frac{L^{-1}(G(\tau))}{\nu_m(\tau)}),\nu_m(\tau)\big)+1\Big)^{\frac{1}{p-1}}G_1(\tau) \\ \leq & i + \Big(\psi\big(\nu_m(\tau)(1+\frac{L^{-1}(G(\tau))}{j}),\nu_m(\tau)\big)+1\Big)^{\frac{1}{p-1}}G_1(\tau) \\ \leq & i + \Big(c_1\psi\big(\nu_m(\tau),\nu_m(\tau)\big)\psi\big(1+\frac{L^{-1}(G(\tau))}{j},1\big)+1\Big)^{\frac{1}{p-1}}G_1(\tau) \\ = & \Big(\psi\big(\nu_m(\tau),\nu_m(\tau)\big)\Big)^{\frac{1}{p-1}}\Big(\frac{i}{\psi(\nu_m(\tau),\nu_m(\tau))^{\frac{1}{p-1}}}+\big(c_1\psi(1+\frac{L^{-1}(G(\tau))}{j},1) \\ & +\frac{1}{\psi(\nu_m(\tau),\nu_m(\tau))}\big)^{\frac{1}{p-1}}G_1(\tau)\Big) \\ \leq & \Big(\psi\big(\nu_m(\tau),\nu_m(\tau)\big)\Big)^{\frac{1}{p-1}}\Big(\frac{i}{\psi(j,j)^{\frac{1}{p-1}}}+\big(c_1\psi(1+\frac{L^{-1}(G(\tau))}{j},1)+\frac{1}{\psi(j,j)}\big)^{\frac{1}{p-1}}G_1(\tau)\Big) \end{aligned} \tag{9}$$

and

$$
\begin{aligned}
\nu_m(\tau) \leq & j + \int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)\varphi\big(\mu_m(s),\nu_m(s)\big)ds\Big)dt \\
\leq & j + \Big(\varphi\big(\mu_m(\tau),\nu_m(\tau)\big)+1\Big)^{\frac{1}{p-1}}\int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}}\int_0^t s^{n-1}h(s)ds\Big)dt \\
\leq & j + \Big(\varphi\big(\mu_m(\tau),\mu_m(\tau)+L^{-1}(G(\tau))\big)+1\Big)^{\frac{1}{p-1}}G_2(\tau) \\
= & j + \Big(\varphi\big(\mu_m(\tau),\mu_m(\tau)(1+\frac{L^{-1}(G(\tau))}{\mu_m(\tau)})\big)+1\Big)^{\frac{1}{p-1}}G_2(\tau) \\
\leq & j + \Big(\varphi\big(\mu_m(\tau),\mu_m(\tau)(1+\frac{L^{-1}(G(\tau))}{i})\big)+1\Big)^{\frac{1}{p-1}}G_2(\tau) \\
\leq & j + \Big(c_2\varphi\big(\mu_m(\tau),\mu_m(\tau)\big)\varphi\big(1,1+\frac{L^{-1}(G(\tau))}{i}\big)+1\Big)^{\frac{1}{p-1}}G_2(\tau) \\
= & \Big(\varphi\big(\mu_m(\tau),\mu_m(\tau)\big)\Big)^{\frac{1}{p-1}}\Big(\frac{j}{\varphi(\mu_m(\tau),\mu_m(\tau))^{\frac{1}{p-1}}}+(c_2\varphi(1,1+\frac{L^{-1}(G(\tau))}{i}) \\
& +\frac{1}{\varphi(\mu_m(\tau),\mu_m(\tau))})^{\frac{1}{p-1}}G_2(\tau)\Big) \\
\leq & \Big(\varphi\big(\mu_m(\tau),\mu_m(\tau)\big)\Big)^{\frac{1}{p-1}}\Big(\frac{j}{\varphi(i,i)^{\frac{1}{p-1}}}+(c_2\varphi(1,1+\frac{L^{-1}(G(\tau))}{i})+\frac{1}{\varphi(i,i)})^{\frac{1}{p-1}}G_2(\tau)\Big).
\end{aligned} \tag{10}
$$

By $(N1)$, $(N2)$, (9) and (10) and the monotonicity of $\{\mu_m(\tau)\}_{m\geq 0}$ and $\{\nu_m(\tau)\}_{m\geq 0}$, we obtain

$$
\begin{aligned}
\Big(\Im\big((\mu_m(\tau))'\big)\Big)'+\frac{n-1}{\tau}\Im\Big((\mu_m(\tau))'\Big) = & b(\tau)\psi\big(\mu_{m-1}(\tau),\nu_{m-1}(\tau)\big) \\
\leq & b(\tau)\psi\big(\mu_m(\tau),\nu_m(\tau)\big) \\
\leq & b(\tau)\psi\Big(\mu_m(\tau),\Big(\varphi\big(\mu_m(\tau),\mu_m(\tau)\big)\Big)^{\frac{1}{p-1}}\Big(\frac{j}{\varphi(i,i)^{\frac{1}{p-1}}} \\
& +(c_2\varphi(1,1+\frac{L^{-1}(G(\tau))}{i})+\frac{1}{\varphi(i,i)})^{\frac{1}{p-1}}G_2(\tau)\Big)\Big) \\
\leq & b(\tau)c_1\psi\Big(\mu_m(\tau),\Big(\varphi\big(\mu_m(\tau),\mu_m(\tau)\big)\Big)^{\frac{1}{p-1}}\Big)\omega_1(\tau)
\end{aligned}
$$

and

$$
\begin{aligned}
\Big(\Im\big((\nu_m(\tau))'\big)\Big)'+\frac{n-1}{\tau}\Im\Big((\nu_m(\tau))'\Big) = & h(\tau)\varphi\big(\mu_{m-1}(\tau),\nu_{m-1}(\tau)\big) \\
\leq & h(\tau)\varphi\big(\mu_m(\tau),\nu_m(\tau)\big) \\
\leq & h(\tau)\varphi\Big(\Big(\psi\big(\nu_m(\tau),\nu_m(\tau)\big)\Big)^{\frac{1}{p-1}}\Big(\frac{i}{\psi(j,j)^{\frac{1}{p-1}}} \\
& +(c_1\psi(1+\frac{L^{-1}(G(\tau))}{j},1)+\frac{1}{\psi(j,j)})^{\frac{1}{p-1}}G_1(\tau)\Big),\nu_m(\tau)\Big) \\
\leq & h(\tau)c_2\varphi\Big(\Big(\psi\big(\nu_m(\tau),\nu_m(\tau)\big)\Big)^{\frac{1}{p-1}},\nu_m(\tau)\Big)\omega_2(\tau).
\end{aligned}
$$

From the above inequalities, we obtain

$$\begin{aligned}\left(\Im\left((\mu_m(\tau))'\right)\right)' \leq & \left(\Im\left((\mu_m(\tau))'\right)\right)' + \frac{n-1}{\tau}\Im^{-1}\left((\mu_m(\tau))'\right) \\ \leq & b(\tau)c_1\psi\left(\mu_m(\tau), \left(\varphi(\mu_m(\tau), \mu_m(\tau))\right)^{\frac{1}{p-1}}\right)\omega_1(\tau)\end{aligned} \tag{11}$$

and

$$\begin{aligned}\left(\Im\left((\nu_m(\tau))'\right)\right)' \leq & \left(\Im\left((\nu_m(\tau))'\right)\right)' + \frac{n-1}{\tau}\Im^{-1}\left((\nu_m(\tau))'\right) \\ \leq & h(\tau)c_2\varphi\left(\left(\psi(\nu_m(\tau), \nu_m(\tau))\right)^{\frac{1}{p-1}}, \nu_m(\tau)\right)\omega_2(\tau).\end{aligned} \tag{12}$$

We then arrive at

$$\Im\left((\mu_m(\tau))'\right) \leq \int_0^\tau b(s)c_1\psi\left(\mu_m(s), (\varphi(\mu_m(s), \mu_m(s)))^{\frac{1}{p-1}}\right)\omega_1(s)ds \tag{13}$$

and

$$\Im\left((\nu_m(\tau))'\right) \leq \int_0^\tau h(s)c_2\varphi\left((\psi(\nu_m(s), \nu_m(s)))^{\frac{1}{p-1}}, \nu_m(s)\right)\omega_2(s)ds. \tag{14}$$

By Lemma 1, $(N1)$, (13) and (14), we obtain

$$\begin{aligned}(\mu_m(\tau))' \leq & \Im^{-1}\left(\int_0^\tau b(s)c_1\psi\left(\mu_m(s), (\varphi(\mu_m(s), \mu_m(s)))^{\frac{1}{p-1}}\right)\omega_1(s)ds\right) \\ \leq & \Im^{-1}\left(c_1\omega_1(\tau)\int_0^\tau b(s)\psi\left(\mu_m(s), (\varphi(\mu_m(s), \mu_m(s)))^{\frac{1}{p-1}}\right)ds\right) \\ \leq & \left(c_1\omega_1(\tau)+1\right)^{\frac{1}{p-1}}\Im^{-1}\left(\int_0^\tau b(s)\psi\left(\mu_m(s), (\varphi(\mu_m(s), \mu_m(s)))^{\frac{1}{p-1}}\right)ds\right) \\ \leq & \left(c_1\omega_1(\tau)+1\right)^{\frac{1}{p-1}}\Im^{-1}\left(\psi\left(\mu_m(\tau), (\varphi(\mu_m(\tau), \mu_m(\tau)))^{\frac{1}{p-1}}\right)\int_0^\tau b(s)ds\right) \\ \leq & \left(c_1\omega_1(\tau)+1\right)^{\frac{1}{p-1}}\left(\psi(\mu_m(\tau), (\varphi(\mu_m(\tau), \mu_m(\tau)))^{\frac{1}{p-1}})+1\right)^{\frac{1}{p-1}}\Im^{-1}\left(\int_0^\tau b(s)ds\right)\end{aligned} \tag{15}$$

and

$$\begin{aligned}(\nu_m(\tau))' \leq & \Im^{-1}\left(\int_0^\tau h(s)c_2\varphi\left((\psi(\nu_m(s), \nu_m(s)))^{\frac{1}{p-1}}, \nu_m(s)\right)\omega_2(s)ds\right) \\ \leq & \Im^{-1}\left(c_2\omega_2(\tau)\int_0^\tau h(s)\varphi\left((\psi(\nu_m(s), \nu_m(s)))^{\frac{1}{p-1}}, \nu_m(s)\right)ds\right) \\ \leq & \left(c_2\omega_2(\tau)+1\right)^{\frac{1}{p-1}}\Im^{-1}\left(\int_0^\tau h(s)\varphi\left((\psi(\nu_m(s), \nu_m(s)))^{\frac{1}{p-1}}, \nu_m(s)\right)ds\right) \\ \leq & \left(c_2\omega_2(\tau)+1\right)^{\frac{1}{p-1}}\Im^{-1}\left(\varphi\left((\psi(\nu_m(\tau), \nu_m(\tau)))^{\frac{1}{p-1}}, \nu_m(\tau)\right)\int_0^\tau h(s)ds\right) \\ \leq & \left(c_2\omega_2(\tau)+1\right)^{\frac{1}{p-1}}\left(\varphi((\psi(\nu_m(\tau), \nu_m(\tau)))^{\frac{1}{p-1}}, \nu_m(\tau))+1\right)^{\frac{1}{p-1}}\Im^{-1}\left(\int_0^\tau h(s)ds\right).\end{aligned} \tag{16}$$

From the above two inequalities, we easily deduce that

$$\frac{(\mu_m(\tau))'}{\left(\psi(\mu_m(\tau), (\varphi(\mu_m(\tau), \mu_m(\tau)))^{\frac{1}{p-1}})+1\right)^{\frac{1}{p-1}}} \leq \left(c_1\omega_1(\tau)+1\right)^{\frac{1}{p-1}}\Im^{-1}\left(\int_0^\tau b(s)ds\right) \tag{17}$$

and

$$\frac{(\nu_m(\tau))'}{\Big(\varphi\big((\psi(\nu_m(\tau),\nu_m(\tau)))^{\frac{1}{p-1}},\nu_m(\tau)\big)+1\Big)^{\frac{1}{p-1}}}\leq\Big(c_2\omega_2(\tau)+1\Big)^{\frac{1}{p-1}}\Im^{-1}\Big(\int_0^\tau h(s)ds\Big). \tag{18}$$

We then arrive at

$$\int_i^{\mu_m(\tau)}\frac{1}{\Big(\psi\big(t,(\varphi(t,t))^{\frac{1}{p-1}}\big)+1\Big)^{\frac{1}{p-1}}}dt$$
$$\leq\int_0^\tau\Big(c_1\omega_1(t)+1\Big)^{\frac{1}{p-1}}\Im^{-1}\Big(\int_0^t b(s)ds\Big)dt$$

and

$$\int_j^{\nu_m(\tau)}\frac{1}{\Big(\varphi\big((\psi(t,t))^{\frac{1}{p-1}},t\big)+1\Big)^{\frac{1}{p-1}}}dt$$
$$\leq\int_0^\tau\Big(c_2\omega_2(t)+1\Big)^{\frac{1}{p-1}}\Im^{-1}\Big(\int_0^t h(s)ds\Big)dt.$$

Now the above two inequalities can be expressed as

$$F_1(\mu_m(\tau))\leq U_2(\tau),\quad\forall\tau\geq 0 \tag{19}$$

and

$$F_2(\nu_m(\tau))\leq V_2(\tau),\quad\forall\tau\geq 0. \tag{20}$$

It follows from the $(N1)$ that F_1^{-1} and F_2^{-1} are strictly increasing on $[0,F_1(\infty))$ and $[0,F_2(\infty))$ separately, we obtain

$$\mu_m(\tau)\leq F_1^{-1}(U_2(\tau)),\quad\forall\tau\geq 0$$

and

$$\nu_m(\tau)\leq F_2^{-1}(V_2(\tau)),\quad\forall\tau\geq 0.$$

Since

$$(\mu_m(\tau))'\geq 0\quad and\quad(\nu_m(\tau))'\geq 0,\quad\forall\tau\geq 0,$$

we obtain

$$\mu_m(\tau)\leq\mu_m(c_0)\leq W_1\quad and\quad\nu_m(\tau)\leq\nu_m(c_0)\leq W_2,\quad on\ [0,c_0],$$

where $W_1=F_1^{-1}(U_2(c_0))$ and $W_2=F_2^{-1}(V_2(c_0))$ are positive constants. Moreover, from (15) and (16), we can deduce that $\{(\mu_m(\tau))'\}$ and $\{(\nu_m(\tau))'\}$ are bounded on $[0,c_0]$ for arbitrary $c_0>0$. Therefore, the monotone sequences $\{\mu_m(\tau)\}$ and $\{\nu_m(\tau)\}$ are bounded and equicontinuous on $[0,c_0]$. By employing the Arzela–Ascoli theorem, we obtain the subsequences of $\{\mu_m(\tau)\}$ and $\{\nu_m(\tau)\}$ uniformly converging towards $\mu(r)$ and $\nu(r)$ on $[0,c_0]$. According to the arbitrariness of c_0, we obtain that (μ,ν) is an entire positive solution of the system (6). Thus, from Lemma 2, we obtain that (μ,ν) is an entire positive radial solution of the system (1). □

Theorem 2. *Assuming that* $(N1)$, $(N2)$, $(S1)$ *and* $(S2)$ *hold, then the system* (1) *has an entire positive bounded radial solution* (μ, ν) *such that*

$$\begin{cases} i + U_1(\tau) \le \mu(\tau) \le F_1^{-1}(U_2(\tau)), \\ j + V_1(\tau) \le \nu(\tau) \le F_2^{-1}(V_2(\tau)). \end{cases}$$

Proof. On the basis of $(N1)$ and $(N2)$, by Theorem 1, we see that the system (1) has an entire positive radial solution (μ, ν). Moreover, it follows from (19), (20) and $(S1)$ that

$$F_1(\mu_m(\tau)) \le U_2(\infty) < F_1(\infty) < \infty, \quad \forall \tau \ge 0$$

and

$$F_2(\nu_m(\tau)) \le V_2(\infty) < F_2(\infty) < \infty, \quad \forall \tau \ge 0.$$

Since F_1^{-1} and F_2^{-1} are strictly increasing on $[0, F_1(\infty))$ and $[0, F_2(\infty))$ separately, we obtain

$$\mu_m(\tau) \le F_1^{-1}\big(U_2(\infty)\big) < \infty, \quad \forall \tau \ge 0$$

and

$$\nu_m(\tau) \le F_2^{-1}\big(V_2(\infty)\big) < \infty, \quad \forall \tau \ge 0.$$

Letting $m \to \infty$ into the above two inequalities, we obtain

$$\mu(\tau) \le F_1^{-1}\big(U_2(\infty)\big) < \infty, \quad \forall \tau \ge 0 \tag{21}$$

and

$$\nu(\tau) \le F_2^{-1}\big(V_2(\infty)\big) < \infty, \quad \forall \tau \ge 0. \tag{22}$$

Letting $m \to \infty$ in (7), we obtain

$$\mu(\tau) = i + \int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}} \int_0^t s^{n-1} b(s) \psi\big(\mu(s), \nu(s)\big) ds\Big) dt$$

and

$$\nu(\tau) = j + \int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}} \int_0^t s^{n-1} h(s) \varphi\big(\mu(s), \nu(s)\big) ds\Big) dt.$$

Then, it follows from Lemma 1, $(N1)$, $(N2)$ and $(S2)$ that

$$\begin{aligned} \mu(r) =& i + \int_0^\tau \Im^{-1}\Big(\frac{1}{t^{n-1}} \int_0^t s^{n-1} b(s) \psi\big(\mu(s), \nu(s)\big) ds\Big) dt \\ \ge& i + \int_0^\tau \Im^{-1}\bigg(\frac{1}{\varrho^{n-1}} \int_0^\varrho t^{n-1} b(t) \psi\Big(i, j + \\ & \int_0^t \Im^{-1}\Big(\frac{1}{\sigma^{n-1}} \int_0^\sigma s^{n-1} h(s) \varphi(\mu(s), \nu(s)) ds\Big) d\sigma\Big) dt\bigg) d\varrho \\ \ge& i + \int_0^\tau \Im^{-1}\bigg(\frac{1}{\varrho^{n-1}} \int_0^\varrho t^{n-1} b(t) \psi\Big(i, j + (\frac{1}{\varphi(i,j)+1})^{\frac{1}{p-1}} G_2(t)\Big) dt\bigg) d\varrho \\ =& i + U_1(\tau). \end{aligned} \tag{23}$$

As with the above proof, we can prove that

$$\nu(\tau) \ge j + V_1(\tau). \tag{24}$$

□

4. The Entire Positive Blow-Up Radial Solutions

In this section, we prove Theorem 3.

Theorem 3. *Assume that $(N1)$, $(N2)$, $(S3)$ and $(S4)$ hold, then the system (1) has an entire positive blow-up radial solution $(\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)$.*

Proof. On the basis of $(N1)$, $(N2)$, by Theorem 1, we see that the system (1) has an entire positive radial solution $(\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)$. Moreover, it follows from (19) and (20) that

$$F_1(\mu_m(\tau)) \leq U_2(\infty), \quad \forall \tau \geq 0$$

and

$$F_2(\nu_m(\tau)) \leq V_2(\infty), \quad \forall \tau \geq 0.$$

Since F_1^{-1} and F_2^{-1} are strictly increasing on $[0, F_1(\infty))$ and $[0, F_2(\infty))$ separately, we arrive at

$$\mu_m(\tau) \leq F_1^{-1}(U_2(\infty)), \quad \forall \tau \geq 0$$

and

$$\nu_m(\tau) \leq F_2^{-1}(V_2(\infty)), \quad \forall \tau \geq 0.$$

When $(S3)$ holds, we see that $F_1^{-1}(\infty) = F_2^{-1}(\infty) = \infty$. Letting $m \to \infty$ into the above two inequalities, we have

$$\mu(\tau) \leq F_1^{-1}(U_2(\infty)), \quad \forall \tau \geq 0$$

and

$$\nu(\tau) \leq F_2^{-1}(V_2(\infty)), \quad \forall \tau \geq 0.$$

By condition $(S3)$, letting $\tau \to \infty$ into the above two inequalities, we obtain

$$\lim_{\tau \to \infty} \mu(\tau) \leq F_1^{-1}(U_2(\infty)) = \infty, \quad \forall \tau \geq 0 \tag{25}$$

and

$$\lim_{\tau \to \infty} \nu(\tau) \leq F_2^{-1}(V_2(\infty)) = \infty, \quad \forall \tau \geq 0. \tag{26}$$

Then, it follows from $(S4)$, (23) and (24) that

$$\lim_{\tau \to \infty} \mu(\tau) \geq i + \lim_{\tau \to \infty} U_1(\tau) > U_1(\infty) = \infty \tag{27}$$

and

$$\lim_{\tau \to \infty} \nu(\tau) \geq j + \lim_{\tau \to \infty} V_1(\tau) > V_1(\infty) = \infty. \tag{28}$$

Consequently,

$$\lim_{\tau \to \infty} \mu(\tau) = \infty \quad \textit{and} \quad \lim_{\tau \to \infty} \nu(\tau) = \infty,$$

which imply that the system (6) has an entire positive blow-up solution $(\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)$. From Lemma 2, the system (1) has an entire positive blow-up radial solution $(\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)$. □

5. The Semifinite Entire Positive Blow-Up Radial Solutions

In this section, we prove Theorems 4 and 5.

Theorem 4. *Assuming that* $(N1)$, $(N2)$, $(S5)$ *and* $(S6)$ *hold, the system (1) then has a semifinite entire positive blow-up radial solution* $(\mu, \nu) \in C^2[0,\infty) \times C^2[0,\infty)$.

Proof. In view of $(N1)$, $(N2)$, by Theorem 1, we see that system (1) has an entire positive radial solution $(\mu, \nu) \in C^2[0,\infty) \times C^2[0,\infty)$. By $(S5)$, (25) and (27), we obtain

$$\lim_{\tau\to\infty} \mu(\tau) \le F_1^{-1}(U_2(\infty)) = \infty$$

and

$$\lim_{\tau\to\infty} \mu(\tau) \ge i + \lim_{\tau\to\infty} U_1(\tau) > U_1(\infty) = \infty,$$

which imply that

$$\lim_{\tau\to\infty} \mu(\tau) = \infty.$$

Moreover, by $(S6)$, (22) and (24), we obtain

$$\lim_{\tau\to\infty} \nu(\tau) \le F_2^{-1}(V_2(\infty)) < \infty$$

and

$$\lim_{\tau\to\infty} \nu(\tau) \ge j + \lim_{\tau\to\infty} V_1(\tau) > V_1(\infty), \quad V_1(\infty) < \infty,$$

which imply that

$$\lim_{\nu\to\infty} \nu(\tau) < \infty.$$

Therefore, system (6) has a semifinite entire positive blow-up solution $(\mu, \nu) \in C^2[0,\infty) \times C^2[0,\infty)$. From Lemma 2, the system (1) has a semifinite entire positive blow-up radial solution $(\mu, \nu) \in C^2[0,\infty) \times C^2[0,\infty)$. □

Theorem 5. *Assume that* $(N1)$, $(N2)$, $(S7)$ *and* $(S8)$ *hold, then the system (1) has a semifinite entire positive blow-up radial solution* $(\mu, \nu) \in C^2[0,\infty) \times C^2[0,\infty)$.

Proof. In view of $(N1)$, $(N2)$, by Theorem 1, we see that system (1) has an entire positive radial solution $(\mu, \nu) \in C^2[0,\infty) \times C^2[0,\infty)$. By $(S7)$, (26) and (28), we obtain

$$\lim_{\tau\to\infty} \nu(\tau) \le F_2^{-1}(V_2(\infty)) = \infty$$

and

$$\lim_{\tau\to\infty} \nu(\tau) \ge j + \lim_{\tau\to\infty} V_1(\tau) > V_1(\infty) = \infty,$$

which imply that

$$\lim_{\tau\to\infty} \nu(\tau) = \infty.$$

Moreover, by $(S8)$, (21) and (23), we obtain

$$\lim_{\tau\to\infty} \mu(\tau) \le F_1^{-1}(U_2(\infty)) < \infty$$

and

$$\lim_{\tau\to\infty} \mu(\tau) \ge i + \lim_{\tau\to\infty} U_1(\tau) > U_1(\infty), \quad U_1(\infty) < \infty,$$

which imply that

$$\lim_{\tau\to\infty} \mu(\tau) < \infty.$$

Therefore, system (6) has a semifinite entire positive blow-up solution $(\mu,\nu) \in C^2[0,\infty) \times C^2[0,\infty)$. From Lemma 2, system (1) has a semifinite entire positive blow-up radial solution $(\mu,\nu) \in C^2[0,\infty) \times C^2[0,\infty)$. $\square$

6. Example

Example 1. *Consider the following Schrödinger system*

$$\begin{cases} div\big(\Lambda(|\nabla\mu|^5)\nabla\mu\big) = \dfrac{3}{4}\dfrac{3-|\chi|}{|\chi|e^{|\chi|}}\mu^{\frac{1}{2}}\nu^{\frac{1}{2}}, \quad \chi \in \mathbb{R}^6, \\ div\big(\Lambda(|\nabla\nu|^5)\nabla\nu\big) = \dfrac{1}{4}\dfrac{3-2|\chi|}{|\chi|e^{2|\chi|}}\mu^{\frac{1}{3}}\nu^{\frac{2}{3}}, \quad \chi \in \mathbb{R}^6. \end{cases} \tag{29}$$

Let $\Lambda(s) = s^5$, $p = 7$, then $\Lambda \in \theta$. Here $b(s) = \frac{3-s}{se^s}$, $h(s) = \frac{3-2s}{se^{2s}}$, $\psi(\mu,\nu) = \frac{3}{4}\mu^{\frac{1}{2}}\nu^{\frac{1}{2}}$, $\varphi(\mu,\nu) = \frac{1}{4}\mu^{\frac{1}{3}}\nu^{\frac{2}{3}}$, then ψ and φ are increasing for each variable and $(N1)$ holds. Obviously, when $i = j = 4$, we have $t_1 \geq \sqrt[6]{3}$, $s_1 \geq 1$, $t_2 \geq 1$, $s_2 \geq 1$, $t_3 \geq 4$, $s_3 \geq 1$,

$$\psi(t_1s_1, t_2s_2) = \frac{3}{4}t_1^{\frac{1}{2}}s_1^{\frac{1}{2}}t_2^{\frac{1}{2}}s_2^{\frac{1}{2}} \leq c_1\frac{3}{4}t_1^{\frac{1}{2}}t_2^{\frac{1}{2}}\frac{3}{4}s_1^{\frac{1}{2}}s_2^{\frac{1}{2}} = c_1\psi(t_1,t_2)\psi(s_1,s_2), \quad \forall c_1 \geq \frac{4}{3},$$

$$\varphi(t_1s_1, t_3s_3) = \frac{1}{4}t_1^{\frac{1}{3}}s_1^{\frac{1}{3}}t_3^{\frac{2}{3}}s_3^{\frac{2}{3}} \leq c_2\frac{1}{4}t_1^{\frac{1}{3}}t_3^{\frac{2}{3}}\frac{1}{4}s_1^{\frac{1}{3}}s_3^{\frac{2}{3}} = c_2\varphi(t_1,t_3)\varphi(s_1,s_3), \quad \forall c_2 \geq 4,$$

$$\psi(i,j) \geq \frac{\sqrt{5}-1}{2} \quad and \quad \varphi(i,j) \geq \frac{\sqrt{5}-1}{2},$$

meaning that $(N2)$ is established. From Theorem 1, the Schrödinger system (29) has an entire positive radial solution $(\mu,\nu) \in C^2[0,\infty) \times C^2[0,\infty)$.

Example 2. *Consider the following Schrödinger system*

$$\begin{cases} div\big(\Lambda(|\nabla\mu|^3)\nabla\mu\big) = |\chi|^3(\mu^4+\nu^3), \quad \chi \in \mathbb{R}^4, \\ div\big(\Lambda(|\nabla\nu|^3)\nabla\nu\big) = (3|\chi|^{-1}e^{|\chi|} + e^{|\chi|})\mu\nu^3, \quad \chi \in \mathbb{R}^4. \end{cases} \tag{30}$$

Let $\Lambda(s) = s^3$, $p = 5$, then $\Lambda \in \theta$. Here, $b(s) = s^3$, $h(s) = 3s^{-1}e^s + e^s$, $\psi(\mu,\nu) = \mu^4 + \nu^3$, $\varphi(\mu,\nu) = \mu\nu^3$, then φ and ψ are increasing for each variable and $(N1)$ holds. Obviously, when $i = j = 1$, we have $t_1 \geq 1$, $s_1 \geq \frac{1}{\sqrt[4]{2}}$, $t_2 \geq 1$, $s_2 \geq 1$, $t_3 \geq 1$, $s_3 \geq 1$,

$$\psi(t_1s_1, t_2s_2) = t_1^4s_1^4 + t_2^3s_2^3 \leq c_1(t_1^4+t_2^3)(s_1^4+s_2^3) = c_1\psi(t_1,t_2)\psi(s_1,s_2), \quad \forall c_1 \geq 1,$$

$$\varphi(t_1s_1, t_3s_3) = t_1s_1t_3^3s_3^3 \leq c_2t_1t_3^3s_1s_3^3 = c_2\varphi(t_1,t_3)\varphi(s_1,s_3), \quad \forall c_2 \geq 1,$$

$$\psi(i,j) \geq \frac{\sqrt{5}-1}{2} \quad and \quad \varphi(i,j) \geq \frac{\sqrt{5}-1}{2},$$

meaning that $(N2)$ is established. After a simple calculation, one has

$$\begin{aligned} U_2(\infty) &= \int_0^\infty \Big(c_1\omega_1(t)+1\Big)^{\frac{1}{4}}\Im^{-1}\Big(\int_0^t b(s)ds\Big)dt > \int_0^\infty \sqrt[10]{\frac{1}{4}t^4}dt \\ &= \sqrt[10]{\frac{1}{4}}\int_0^\infty t^{\frac{2}{5}}dt = \infty, \end{aligned}$$

$$\begin{aligned} V_2(\infty) &= \int_0^\infty \Big(c_2\omega_2(t)+1\Big)^{\frac{1}{4}}\Im^{-1}\Big(\int_0^t h(s)ds\Big)dt > \int_0^\infty \sqrt[10]{e^t}dt \\ &= \int_0^\infty e^{\frac{t}{10}}dt = \infty, \end{aligned}$$

$$F_1(\infty) = \int_i^\infty \frac{1}{\Big(\psi(t,(\varphi(t,t))^{\frac{1}{4}})+1\Big)^{\frac{1}{4}}}dt = \int_i^\infty \frac{1}{\sqrt[4]{t^4+t^3+1}}dt = \infty$$

and

$$F_2(\infty) = \int_j^{\infty} \frac{1}{\left(\varphi((\psi(t,t))^{\frac{1}{4}}, t) + 1\right)^{\frac{1}{4}}} dt = \int_j^{\infty} \frac{1}{\sqrt[4]{(t^4+t^3)^{\frac{1}{4}} t^3 + 1}} dt = \infty,$$

meaning that $(S3)$ is established. We then have

$$G_1(\tau) = \int_0^{\tau} \Im^{-1}\Big(\frac{1}{t^3}\int_0^t s^3 b(s)ds\Big)dt = \int_0^{\tau} \Big(\frac{1}{t^3}\int_0^t s^6 ds\Big)^{\frac{1}{10}} dt = \sqrt[10]{\frac{1}{7}} \int_0^{\tau} t^{\frac{2}{5}} dt = \frac{5}{7}\sqrt[10]{\frac{1}{7}}\tau^{\frac{7}{5}},$$

$$G_2(\tau) = \int_0^{\tau} \Im^{-1}\Big(\frac{1}{t^3}\int_0^t s^3 h(s)ds\Big)dt = \int_0^{\tau} \Big(\frac{1}{t^3}\int_0^t s^2 e^s(3+s)ds\Big)^{\frac{1}{10}} dt = \int_0^{\tau} e^{\frac{t}{10}} dt = 10e^{\frac{\tau}{10}},$$

$$\begin{aligned} U_1(\infty) &= \int_0^{\infty} \Im^{-1}\Bigg(\frac{1}{\varrho^3}\int_0^{\varrho} t^3 b(t)\psi\Big(i, j + (\frac{1}{\varphi(i,j)+1})^{\frac{1}{p-1}} G_2(t)\Big)dt\Bigg)d\varrho \\ &> \int_0^{\infty} \Im^{-1}\Big(\frac{1}{\varrho^3}\int_0^{\varrho} t^3 b(t)\psi(i,j)dt\Big)d\varrho \\ &> \int_0^{\infty} \Im^{-1}\Big(\frac{1}{\varrho^3}\int_0^{\varrho} t^3 b(t)(\frac{1}{\psi(i,j)+1})dt\Big)d\varrho \\ &> \Big(\frac{1}{\psi(i,j)+1}\Big)^{\frac{1}{p-1}} \int_0^{\infty} \Im^{-1}\Big(\frac{1}{\varrho^3}\int_0^{\varrho} t^3 b(t)dt\Big)d\varrho \\ &= \Big(\frac{1}{2+1}\Big)^{\frac{1}{4}} G_1(\infty) = \infty \end{aligned}$$

and

$$\begin{aligned} V_1(\infty) &= \int_0^{\infty} \Im^{-1}\Bigg(\frac{1}{\varrho^3}\int_0^{\varrho} t^3 h(t)\varphi\Big(i + (\frac{1}{\psi(i,j)+1})^{\frac{1}{p-1}} G_1(t), j\Big)dt\Bigg)d\varrho \\ &> \int_0^{\infty} \Im^{-1}\Big(\frac{1}{\varrho^3}\int_0^{\varrho} t^3 h(t)\varphi(i,j)dt\Big)d\varrho \\ &> \int_0^{\infty} \Im^{-1}\Big(\frac{1}{\varrho^3}\int_0^{\varrho} t^3 h(t)(\frac{1}{\varphi(i,j)+1})dt\Big)d\varrho \\ &> \Big(\frac{1}{\varphi(i,j)+1}\Big)^{\frac{1}{p-1}} \int_0^{\infty} \Im^{-1}\Big(\frac{1}{\varrho^3}\int_0^{\varrho} t^3 h(t)dt\Big)d\varrho \\ &= \Big(\frac{1}{1+1}\Big)^{\frac{1}{4}} G_2(\infty) = \infty, \end{aligned}$$

meaning that $(S4)$ is established. From Theorem 3, the Schrödinger system (30) has an entire positive blow-up radial solution $(\mu,\nu) \in C^2[0,\infty) \times C^2[0,\infty)$.

Example 3. *Consider the following Schrödinger system*

$$\begin{cases} div(\Lambda(|\nabla\mu|^4)\nabla\mu) = |\chi|^5(\mu^4+\nu), & \chi \in \mathbb{R}^5, \\ div(\Lambda(|\nabla\nu|^4)\nabla\nu) = (6|\chi|^{-1}e^{|\chi|} + 2e^{|\chi|})\mu^2\nu^2, & \chi \in \mathbb{R}^5. \end{cases} \quad (31)$$

Let $\Lambda(s) = s^4$, $p = 6$, then $\Lambda \in \theta$. Here, $b(s) = s^5$, $h(s) = 6s^{-1}e^s + 2e^s$, $\psi(\mu,\nu) = \mu^4 + \nu$, $\varphi(\mu,\nu) = \mu^2\nu^2$, then φ and ψ are increasing for each variable and $(N1)$ holds. Obviously, when $i = j = 1$, we have $t_1 \geq 1$, $s_1 \geq \frac{1}{\sqrt[5]{2}}$, $t_2 \geq 1$, $s_2 \geq 1$, $t_3 \geq 1$, $s_3 \geq 1$,

$$\psi(t_1s_1, t_2s_2) = t_1^4s_1^4 + t_2s_2 \leq c_1(t_1^4+t_2)(s_1^4+s_2) = c_1\psi(t_1,t_2)\psi(s_1,s_2), \quad \forall c_1 \geq 1,$$

$$\varphi(t_1s_1, t_3s_3) = t_1^2s_1^2t_3^2s_3^2 \leq c_2t_1^2t_3^2s_1^2s_3^2 = c_2\varphi(t_1,t_3)\varphi(s_1,s_3), \quad \forall c_2 \geq 1,$$

$$\psi(i,j) \geq \frac{\sqrt{5}-1}{2} \quad and \quad \varphi(i,j) \geq \frac{\sqrt{5}-1}{2},$$

meaning that $(N2)$ is established. After a simple calculation, one has

$$\begin{aligned} U_2(\infty) &= \int_0^\infty \Big(c_1\omega_1(t)+1\Big)^{\frac{1}{4}} \Im^{-1}\Big(\int_0^t b(s)ds\Big)dt > \int_0^\infty \sqrt[17]{\frac{1}{6}t^6}dt \\ &= \sqrt[17]{\frac{1}{6}} \int_0^\infty t^{\frac{6}{17}} dt = \infty, \end{aligned}$$

$$\begin{aligned} V_2(\infty) &= \int_0^\infty \Big(c_2\omega_2(t)+1\Big)^{\frac{1}{4}} \Im^{-1}\Big(\int_0^t h(s)ds\Big)dt > \int_0^\infty \sqrt[17]{e^t}dt \\ &= \int_0^\infty e^{\frac{t}{17}} dt = \infty, \end{aligned}$$

$$F_1(\infty) = \int_i^\infty \frac{1}{\Big(\psi\big(t,(\varphi(t,t))^{\frac{1}{4}}\big)+1\Big)^{\frac{1}{4}}}dt = \int_i^\infty \frac{1}{\sqrt[4]{t^4+t+1}}dt = \infty$$

and

$$F_2(\infty) = \int_j^\infty \frac{1}{\Big(\varphi\big((\psi(t,t))^{\frac{1}{4}},t\big)+1\Big)^{\frac{1}{4}}}dt = \int_j^\infty \frac{1}{\sqrt[4]{t^2(t^4+t)^{\frac{1}{2}}+1}}dt = \infty,$$

meaning that $(S3)$ is established. We then have

$$G_1(\tau) = \int_0^\tau \Im^{-1}\Big(\frac{1}{t^3}\int_0^t s^3 b(s)ds\Big)dt = \int_0^\tau \Big(\frac{1}{t^3}\int_0^t s^8 ds\Big)^{\frac{1}{17}} dt = \sqrt[17]{\frac{1}{9}}\int_0^\tau t^{\frac{6}{17}}dt = \frac{17}{23}\sqrt[10]{\frac{1}{9}}\tau^{\frac{23}{17}},$$

$$G_2(\tau) = \int_0^\tau \Im^{-1}\Big(\frac{1}{t^3}\int_0^t s^3 h(s)ds\Big)dt = \int_0^\tau \Big(\frac{1}{t^3}\int_0^t s^2 e^s(6+2s)ds\Big)^{\frac{1}{17}} dt = \sqrt[17]{2}\int_0^\tau e^{\frac{t}{17}}dt = 17\sqrt[17]{2}e^{\frac{\tau}{17}},$$

$$\begin{aligned} U_1(\infty) &= \int_0^\infty \Im^{-1}\Bigg(\frac{1}{\varrho^4}\int_0^\varrho t^4 b(t)\psi\Big(i, j+(\frac{1}{\varphi(i,j)+1})^{\frac{1}{p-1}}G_2(t)\Big)dt\Bigg)d\varrho \\ &> \int_0^\infty \Im^{-1}\Big(\frac{1}{\varrho^4}\int_0^\varrho t^4 b(t)\psi(i,j)dt\Big)d\varrho \\ &> \int_0^\infty \Im^{-1}\Big(\frac{1}{\varrho^4}\int_0^\varrho t^4 b(t)(\frac{1}{\psi(i,j)+1})dt\Big)d\varrho \\ &> \Big(\frac{1}{\psi(i,j)+1}\Big)^{\frac{1}{p-1}} \int_0^\infty \Im^{-1}\Big(\frac{1}{\varrho^4}\int_0^\varrho t^4 b(t)dt\Big)d\varrho \\ &= \Big(\frac{1}{2+1}\Big)^{\frac{1}{5}} G_1(\infty) = \infty \end{aligned}$$

and

$$\begin{aligned} V_1(\infty) &= \int_0^\infty \Im^{-1}\Bigg(\frac{1}{\varrho^4}\int_0^\varrho t^4 h(t)\varphi\Big(i+(\frac{1}{\psi(i,j)+1})^{\frac{1}{p-1}}G_1(t), j\Big)dt\Bigg)d\varrho \\ &> \int_0^\infty \Im^{-1}\Big(\frac{1}{\varrho^4}\int_0^\varrho t^4 h(t)\varphi(i,j)dt\Big)d\varrho \\ &> \int_0^\infty \Im^{-1}\Big(\frac{1}{\varrho^4}\int_0^\varrho t^4 h(t)(\frac{1}{\varphi(i,j)+1})dt\Big)d\varrho \\ &> \Big(\frac{1}{\varphi(i,j)+1}\Big)^{\frac{1}{p-1}} \int_0^\infty \Im^{-1}\Big(\frac{1}{\varrho^4}\int_0^\varrho t^4 h(t)dt\Big)d\varrho \\ &= \Big(\frac{1}{1+1}\Big)^{\frac{1}{5}} G_2(\infty) = \infty, \end{aligned}$$

which mean that $(S4)$ is established. From Theorem 3, the Schrödinger system (31) has an entire positive blow-up radial solution $(\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)$.

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