axioms

Article

Controllability of a Class of Impulsive ψ-Caputo Fractional Evolution Equations of Sobolev Type

Qing Yang, Chuanzhi Bai and Dandan Yang *

Department of Mathematics, Huaiyin Normal University, Huaian 223300, China; yq511115@hytc.edu.cn (Q.Y.); czbai@hytc.edu.cn (C.B.)
* Correspondence: ydd@hytc.edu.cn

Abstract: In this paper, we investigate the controllability of a class of impulsive ψ-Caputo fractional evolution equations of Sobolev type in Banach spaces. Sufficient conditions are presented by two new characteristic solution operators, fractional calculus, and Schauder fixed point theorem. Our works are generalizations and continuations of the recent results about controllability of a class of impulsive ψ-Caputo fractional evolution equations. Finally, an example is given to illustrate the effectiveness of the main results.

Keywords: controllability; ψ-Caputo fractional derivative; impulsive; functional evolution equations; Sobolev; fixed point theorem

MSC: 47A10; 93B05

check for updates

Citation: Yang, Q.; Bai, C.; Yang, D. Controllability of a Class of Impulsive ψ-Caputo Fractional Evolution Equations of Sobolev Type. *Axioms* **2022**, *11*, 283. https://doi.org/10.3390/axioms11060283

Academic Editor: Adrian Petrusel

Received: 14 May 2022
Accepted: 8 June 2022
Published: 10 June 2022

1. Introduction

Fractional calculus sweeps the board in science and engineering, such as chemistry, economics, biology control of dynamical systems, financing viscoelastic materials, signal processing, and so on. For more details on the theory and applications in this filed, one may see the monographs [1–5], and the references cited therein. The fractional calculus from the physical community, for instance, fractional calculus and anomalous diffusion have been studied intensively [6–9].

Controllability of fractional semilinear evolution systems in Banach spaces has been paid much attention. Many researchers have focused on this topic. We refer the readers to El-Borai [10,11], Balachandran and Park [12], Wang et al. [13–15], Zhou and Jiao [16,17], Sakthivel et al. [18], Debbouchra and Baleanu [19], Li et al. [20], Kumar and Sukavanam [21], and Lord et al. [22] and the references therein. In 2013, Fečkan et al. [23] investigated the controllability of $q \in (0,1)$-order Caputo fractional functional evolution equations of Sobolev type in Banach space X :

$$\begin{cases} {}_0^C D_t^q Ex(t) + Ax(t) = f(t, x_t) + Bu(t), & t \in J, \\ x(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{1}$$

where $\phi \in C([-r,0],X)$, A and E are linear operators, A is closed, E is bijective, and E^{-1} is compact. By utilizing the Schauder fixed point theorem and the properties of two new characteristic solution operators, the authors presented the exact controllability of system (1).

Due to important and potential applications of impulse and delay, the study of dynamical systems with impulses and time delay has gained more and more attention. Impulsive fractional differential equations with delays have been widely applied to many fields, such as weather predicting, drug delivery processing, agricultural insect pests control, and some other optimization problems. We refer the readers to [24–28] and the references therein.

In 2021, Zhao [29] studied the exact controllability of a class of impulsive fractional nonlinear evolution equations with delay in Banach spaces:

$$
\begin{cases}
{}^{C}D^{\gamma}x(t) = Ax(t) + f(t, x(t), x_t) + Bu(t), & a.e.\ t \in I := [0, a], \\
\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), & i = 1, \dots, m, \\
x(t) = \phi(t), & t \in [-b, 0],
\end{cases} \tag{2}
$$

where $\gamma \in (0,1)$, $A : \mathcal{D} \subset X \to X$ is a closed the linear unbounded operator on X with dense domain $\mathcal{D}$. In Ref. [29], Zhao defined the mild solution of system (2) as follows:

$$
x(t) = \begin{cases}
\phi(0) + \frac{1}{\Gamma(\gamma)} A \left(\sum\limits_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\gamma-1} x(s) ds + \int_{t_k}^{t} (t-s)^{\gamma-1} x(s) ds \right) \\
\quad + \frac{1}{\Gamma(\gamma)} \left(\sum\limits_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\gamma-1} (f(s, x(s), x_s) + Bu(s)) ds \right) \\
\quad + \frac{1}{\Gamma(\gamma)} \int_{t_k}^{t} (t-s)^{\gamma-1} (f(s, x(s), x_s) + Bu(s)) ds + \sum\limits_{0<t_k<t} I_k(x(t_k)), \quad t \in [0, \tau], \\
\phi(t), \quad t \in [-b, 0].
\end{cases} \tag{3}
$$

Unfortunately, (3) is not correct. In fact, from [30] we know that the mild solution of system (2) should be defined as below.

$$
x(t) = \begin{cases}
\phi(0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} (Ax(s) + f(s, x(s), x_s) + Bu(s)) ds, \quad t \in [0, t_1), \\
\phi(0) + I_1(x(t_1)) \\
\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} (Ax(s) + f(s, x(s), x_s) + Bu(s)) ds, \quad t \in (t_1, t_2), \\
\phi(0) + I_1(x(t_1)) + I_2(x(t_2)) \\
\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} (Ax(s) + f(s, x(s), x_s) + Bu(s)) ds, \quad t \in (t_2, t_3), \\
\vdots \\
\phi(0) + \sum\limits_{i=1}^{m} I_i(x(t_i)) \\
\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} (Ax(s) + f(s, x(s), x_s) + Bu(s)) ds, \quad t \in (t_m, a], \\
\phi(t), \quad t \in [-b, 0].
\end{cases} \tag{4}
$$

In recent decades, the generalizations of the fractional calculus operators have been done [31–34], since they are more general operators that allow for the discussion and analysis of a wide class of particular cases. Considering the Caputo fractional derivative of a function with respect to another function ψ, Almeida [35] generalized the definition of Caputo fractional derivative, in which the advantage of this new definition of the fractional derivative is that by choosing a suitable function ψ, a higher accuracy of the model could be achieved. For recent relevant work on generalized fractional derivatives, one may see refs. [36–38]. In ref. [39], Suechori and Ngiamsunthorn studied the following semilinear ψ-Caputo fractional evolution equations:

$$
\begin{cases}
{}_{0}^{C}D_{\psi}^{\alpha} u(t) = \mathcal{A}u(t) + f(t, u(t)), & t \in (0, T], \\
u(0) = u_0,
\end{cases} \tag{5}
$$

where $0 < \alpha < 1$, $T < \infty$, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of uniformly bounded linear operators $\{T(t)\}_{t \geq 0}$. Existence results of mild solutions to (5) have been obtained. These results generalize the previous work in which the classical Caputo fractional derivative is studied.

Motivated by the above works, we consider the following impulsive ψ-Caputo fractional evolution equations of Sobolev type:

$$
\begin{cases}
{}_{0}^{C}D_{t}^{\alpha,\psi}(Ex)(t) = Ax(t) + f(t, x(t), x_t) + Bu(t), & a.e.\ t \in J', \\
\Delta x(t_k) = I_k(x(t_k)), & k = 1, \dots, m, \\
x(t) = \phi(t), & t \in [-r, 0],
\end{cases} \tag{6}
$$

where $0 < \alpha < 1$, $J = [0, b]$ $(b > 0)$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, the $\{t_k\}$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$, ${}_0^C D_t^{\alpha,\psi} x(t)$ is the Caputo fractional derivative of a function x with respect to another function ψ. The operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$, where X and Y are two real Banach spaces, $x(\cdot) \in X$ and the control function $u(\cdot) \in \mathcal{U}$. The Banach space of admissible control functions is denoted by $\mathcal{U}$ involving a Banach space U, in which we define either $\mathcal{U} := L^2(J, U)$ for $\frac{1}{2} < \alpha < 1$ or $\mathcal{U} := L^\infty(J, U)$ for $0 < \alpha < 1$. A bounded linear operator B is from U into Y, $x : J^* := [-r, b] \to X$, $x_t \in C := C([-r, 0], X)$ defined by $x_t(s) := x(t + s)$, $-r \le s \le 0$. $D(E)$ of E is a Banach space, $\|x\|_{D(E)} := \|Ex\|_Y$, $x \in D(E)$ and $\phi \in C(E) := C([-r, 0], D(E))$, $f : J \times X \times C(E) \to X$ and $I_k : PC(J^*, X) \to X$, $k = 1, \ldots, m$ are appropriate functions which will be specified later. $PC(J^*, X) = \{x : J^* \to X, x(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $x(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$. Obviously, $PC(J^*, X)$ is a Banach space with the norm $\|x\| = \sup_{t \in J^*}\{\|x(t)\| : x \in PC(J^*, X)\}$.

In this paper, by means of two new characteristic solution operators and Schauder fixed point theorem, we present the controllability of impulsive ψ-Caputo fractional evolution equations of Sobolev type in Banach spaces. This paper will be organized as follows. In Section 2, we will briefly recall some definitions and preliminaries. In Section 3, sufficient conditions ensuring exact controllability of the systems are provided. In Section 4, an example is given to illustrate our theoretical result. Finally, we give the conclusions in Section 5.

To the best of our knowledge, no such results in the literature studied theoretically the impulsive fractional evolution equations of Sobolev type containing the fractional derivative of a function with respect to another function. Our goal is to cover this gap in this paper. Our results extend the main results of Ref. [23].

2. Preliminaries

In this section, we recall some basic definitions and lemmas that will be used later.

Definition 1 ([40]). *Let $\alpha > 0$, f be an integrable function defined on $[a, b]$ and $\psi \in C^1([a, b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. The left ψ-Riemann–Liouville fractional integral operator of order α of a function f is defined by*

$$ {}_a I_t^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds. \tag{7}$$

Definition 2 ([35,40]). *Let $n - 1 < \alpha < n$, $f \in C^n([a, b])$ and $\psi \in C^n([a, b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. The left ψ-Caputo fractional derivative of order α of a function f is defined by*

$$\begin{aligned} {}_a^C D_t^{\alpha,\psi} f(t) &= ({}_a I_t^{n-\alpha,\psi} f^{[n]})(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t) - \psi(s))^{n-\alpha-1} f^{[n]}(s) \psi'(s) ds, \end{aligned} \tag{8}$$

where $n = [\alpha] + 1$ and $f^{[n]}(t) := \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n f(t)$ on $[a, b]$.

We will give some properties of the fractional integral and the fractional derivatives of a function with respect to another function.

Lemma 1 ([35]). *Let $f \in C^n([a, b])$ and $n - 1 < \alpha < n$. Then we have*

(1) ${}_a^C D_t^{\alpha,\psi} {}_a I_t^{\alpha,\psi} f(t) = f(t)$;

(2) $I_t^{\alpha,\psi} {}_a^C D_t^{\alpha,\psi} f(t) = f(t) - \displaystyle\sum_{k=0}^{n-1} \frac{f^{[k]}(a^+)}{\Gamma(k - \alpha)} (\psi(t) - \psi(a))^k.$

In special case, given $\alpha \in (0, 1)$, we have

$$I_t^{\alpha,\psi} {}_a^C D_t^{\alpha,\psi} = f(t) - f(a).$$

Definition 3 ([40]). *Let $u, \psi : [a, \infty) \to \mathbb{R}$ be real valued functions such that $\psi(t)$ is continuous and $\psi'(t) > 0$ on $[a, \infty)$. The generalized Laplace transform of u is denoted by*

$$\mathcal{L}_\psi\{u(t)\} = \int_a^\infty e^{-s(\psi(t)-\psi(a))} u(t)\psi'(t)dt \tag{9}$$

for all s.

From Ref. [40], we have the following property of the generalized Laplace transform of the Caputo fractional operators with respect to function ψ.

Lemma 2. *Assume that $0 < \alpha < 1$, h is continuous on $[a, \infty)$ and of ψ-exponential order, while ${}_a^C D_t^{\alpha,\psi} h(t)$ is piecewise continuous on $[a, \infty)$. Then*

$$\mathcal{L}_\psi\left\{ \left({}_a^C D_t^{\alpha,\psi} f \right)(t) \right\} = s^\alpha \mathcal{L}_\psi\{f(t)\} - s^{\alpha-1} f(a).$$

For problem (6), throughout this paper, the following assumptions on the operators A and E are satisfied.

(H1) A and E are linear operators, and A is closed.
(H2) $D(E) \subset D(A)$ and E is bijective.
(H3) Linear operator $E^{-1} : Y \to D(E) \subset X$ is compact (which implies that E^{-1} is bounded).

By (H3) we know that E is closed. In fact, E^{-1} is closed and injective, then the inverse is also closed. Note (H1)–(H3) and the closed graph theorem that the boundedness of the linear operator $-AE^{-1} : Y \to Y$. Consequently, $-AE^{-1}$ generates a semigroup $\{T(t), t \geq 0\}$, $T(t) := e^{-AE^{-1}t}$. We assume $M := \sup_{t \geq 0} \|T(t)\| < \infty$.

For convenience, denote $J_0 = [t_0, t_1]$, $J_i = (t_i, t_{i+1}]$, $i = 1, \ldots, m$.

By Definitions 1 and 2, and Lemma 1, the impulsive problem (6) could be written as the following fractional integral equation

$$Ex(t) = \begin{cases} E\phi(0) + \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (\psi(t) - \psi(s))^{\gamma-1} (Ax(s) + f(s, x(s), x_s) + Bu(s))ds, \quad t \in J_0, \\ E\phi(0) + \sum\limits_{i=1}^n EI_i(x(t_i)) \\ \quad + \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (\psi(t) - \psi(s))^{\gamma-1} (Ax(s) + f(s, x(s), x_s) + Bu(s))ds, \quad t \in J_n,\ n = 1, \ldots, m, \\ E\phi(t), \quad t \in [-r, 0], \end{cases} \tag{10}$$

if the integral in (10) exists.

Lemma 3. *Suppose that (H1)–(H3) hold, then*

(i) ${}_0^C D_t^{\alpha,\psi} [E S_E^{\alpha,\psi}(t, 0)w] = A S_E^{\alpha,\psi}(t, 0)w,$

(ii) ${}_0^C D_t^{\alpha,\psi} [E S_E^{\alpha,\psi}(t, t_i)w] = A S_E^{\alpha,\psi}(t, t_i)w, \quad i = 1, \ldots, m,$

(iii)
$$ {}_0^C D_t^{\alpha,\psi} \left[E \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T_E^{\alpha,\psi}(t, s) g(s) ds \right] = A \int_0^t (\psi(t) - \psi(s))^{\alpha-1} T_E^{\alpha,\psi}(t, s) g(s) ds + g(t),$$

where $S_E^{\alpha,\psi}(t, s)$ and $T_E^{\alpha,\psi}(t, s)$ are called characteristic solutions given by

$$S_E^{\alpha,\psi}(t, s)v := \int_0^\infty E^{-1} \xi_\alpha(\theta) T((\psi(t) - \psi(s))^\alpha \theta) v d\theta, \tag{11}$$

and

$$T_E^{\alpha,\psi}(t, s)v := \alpha \int_0^\infty E^{-1} \theta \xi_\alpha(\theta) T((\psi(t) - \psi(s))^\alpha \theta) v d\theta, \tag{12}$$

for $0 \le s \le t \le b$, *here* $T(t) := e^{-AE^{-1}t}$,

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\rho_\alpha(\theta^{-\frac{1}{\alpha}}),$$

and

$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\theta^{-\alpha k-1}\frac{\Gamma(\alpha k+1)}{k!}\sin(k\pi\alpha),$$

where ξ_α *is the probability density function defined on* $(0,\infty)$.

Proof. (i) For $t \ge 0$, by (11) and Definition 3, we get

$$\begin{aligned}
\mathcal{L}_\psi\{ES_E^{\alpha,\psi}(t,0)w\} &= \textstyle\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\left(\int_0^\infty \xi_\alpha(\theta)T((\psi(t)-\psi(0))^\alpha\theta)wd\theta\right)\psi'(t)dt \\
&= \textstyle\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\left(\int_0^\infty \rho_\alpha(\theta)T\left(\frac{(\psi(t)-\psi(0))^\alpha}{\theta^\alpha}\right)wd\theta\right)\psi'(t)dt \\
&= \textstyle\int_0^\infty\int_0^\infty \theta\rho_\alpha(\theta)e^{-(\lambda(\psi(t)-\psi(0))\theta}T((\psi(t)-\psi(0))^\alpha)w\psi'(t)d\theta dt \\
&= \textstyle\int_0^\infty -\frac{1}{\lambda}\frac{d}{dt}e^{-(\lambda(\psi(t)-\psi(0)))^\alpha}T((\psi(t)-\psi(0))^\alpha)wdt \\
&= \textstyle\alpha\int_{t_i}^\infty \lambda^{\alpha-1}(\psi(t)-\psi(0))^{\alpha-1}e^{-(\lambda(\psi(t)-\psi(0)))^\alpha}T((\psi(t)-\psi(0))^\alpha)w\psi'(t)dt \\
&= \textstyle\lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}T(s)wds \quad (s=(\psi(t)-\psi(0))^\alpha) \\
&= \lambda^{\alpha-1}(\lambda^\alpha I - AE^{-1})^{-1}w.
\end{aligned} \tag{13}$$

On the other hand, by Lemma 2, one has

$$\begin{aligned}
\mathcal{L}_\psi\{{}_0^CD_t^{\alpha,\psi}[ES_E^{\alpha,\psi}(t,0)w]\} &= \lambda^\alpha\mathcal{L}_\psi\{ES_E^{\alpha,\psi}(t,0)w\} - \lambda^{\alpha-1}ES_E^{\alpha,\psi}(0,0)w \\
&= \lambda^\alpha[\lambda^{\alpha-1}(\lambda^\alpha I - AE^{-1})^{-1}w] - \lambda^{\alpha-1}w \\
&= \lambda^{\alpha-1}(\lambda^\alpha I - AE^{-1})^{-1}[\lambda^\alpha - (\lambda^\alpha - AE^{-1})]w \\
&= AE^{-1}\lambda^{\alpha-1}(\lambda^\alpha I - AE^{-1})^{-1}w.
\end{aligned} \tag{14}$$

Combing (13) with (14), we obtain

$${}_0^CD_t^{\alpha,\psi}[ES_E^{\alpha,\psi}(t,0)w] = AE^{-1}ES_E^{\alpha,\psi}(t,0)w = AS_E^{\alpha,\psi}(t,0)w.$$

(ii) For $t \ge t_i$, similar to the proof of (i), we can prove that (ii) holds, so we omit it here.
(iii) For $t \ge 0$, by (12), we have

$$\begin{aligned}
&\mathcal{L}_\psi\left\{E\textstyle\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)g(s)ds\right\} \\
&= \textstyle\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\int_0^t\int_0^\infty \alpha\theta\xi_\alpha(\theta)(\psi(t)-\psi(s))^{\alpha-1}T((\psi(t)-\psi(s))^\alpha\theta)g(s)\psi'(s)d\theta ds \\
&= \textstyle\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\int_0^t\int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\psi(t)-\psi(s))^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\psi(t)-\psi(s))^\alpha}{\theta^\alpha}\right)g(s)\psi'(s)d\theta ds \\
&= \textstyle\int_0^\infty\int_0^\infty \alpha(\psi(s)-\psi(0))^{\alpha-1}e^{-(\lambda(\psi(s)-\psi(0)))^\alpha}T((\psi(s)-\psi(0))^\alpha)e^{-\lambda(\psi(t)-\psi(0))}g(t)\psi'(s)\psi'(t)dsdt \\
&= \textstyle\int_0^\infty e^{-\lambda^\alpha z}T(z)\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}g(t)\psi'(t)dtdz \\
&= (\lambda^\alpha I - AE^{-1})^{-1}\mathcal{L}_\psi\{g(t)\}.
\end{aligned} \tag{15}$$

From Lemma 2, we have

$$\begin{aligned}
&\mathcal{L}_\psi\Big\{{}_0^C D_t^{\alpha,\psi}\Big(E\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)g(s)ds\Big)\Big\}\\
&\quad=\lambda^\alpha\mathcal{L}_\psi\Big\{E\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)g(s)ds\Big\}-\lambda^{\alpha-1}\cdot 0\\
&\quad=\lambda^\alpha(\lambda^\alpha I-AE^{-1})^{-1}\mathcal{L}_\psi\{g(t)\}\\
&\quad=[(\lambda^\alpha I-AE^{-1})+AE^{-1}](\lambda^\alpha I-AE^{-1})^{-1}\mathcal{L}_\psi\{g(t)\}\\
&\quad=AE^{-1}(\lambda^\alpha I-AE^{-1})^{-1}\mathcal{L}_\psi\{g(t)\}+\mathcal{L}_\psi\{g(t)\}.
\end{aligned}\tag{16}$$

Thanks to (15) and (16), we obtain

$$\begin{aligned}
&{}_0^C D_t^{\alpha,\psi}\Big(E\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)g(s)ds\Big)\\
&\qquad=A\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)g(s)ds+g(t).
\end{aligned}$$

□

Lemma 4. *Assume that (H1)–(H3) hold, then problem (6) has a unique solution $x\in PC(J^*,X)$ and satisfies the following integral equation:*

$$x(t)=\begin{cases}S_E^{\alpha,\psi}(t,0)E\phi(0)+\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)(f(s,x(s),x_s)+Bu(s))\psi'(s)ds, & t\in J_0,\\
S_E^{\alpha,\psi}(t,0)E\phi(0)+\sum_{i=1}^n S_E^{\alpha,\psi}(t,t_i)EI_i(x(t_i)) & \\
\quad+\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)(f(s,x(s),x_s)+Bu(s))\psi'(s)ds, & t\in J_n,\ n=1,\dots,m,\\
\phi(t), & t\in[-r,0].\end{cases}\tag{17}$$

Here $S_E^{\alpha,\psi}(t,s)$ and $T_E^{\alpha,\psi}(t,s)$ are as in (11) and (12), respectively.

Proof. If $t\in J_0=[0,t_1]$, then we get by Lemma 3 that

$$\begin{aligned}
{}_0^C D_t^{\alpha,\psi}Ex(t)&={}_0^C D_t^{\alpha,\psi}\Big(ES_E^{\alpha,\psi}(t,0)E\phi(0)\\
&\qquad+E\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)(f(s,x(s),x_s)+Bu(s))\psi'(s)ds\Big)\\
&=AS_E^{\alpha,\psi}(t,0)E\phi(0)+A\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)(f(s,x(s),x_s)+Bu(s))\psi'(s)ds\\
&\quad+f(t,x(t),x_t)+Bu(t)\\
&=Ax(t)+f(t,x(t),x_t)+Bu(t).
\end{aligned}$$

If $t\in J_n=(t_n,t_{n+1}]$, then we obtain by Lemma 3 that

$$\begin{aligned}
{}_0^C D_t^{\alpha,\psi}Ex(t)&={}_0^C D_t^{\alpha,\psi}\Bigg(ES_E^{\alpha,\psi}(t,0)E\phi(0)+\sum_{i=1}^n ES_E^{\alpha,\psi}(t,t_i)EI_i(x(t_i))\\
&\qquad+E\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)(f(s,x(s),x_s)+Bu(s))\psi'(s)ds\Bigg)\\
&=AS_E^{\alpha,\psi}(t,0)E\phi(0)+A\sum_{i=1}^n S_E^{\alpha,\psi}(t,t_i)EI_i(x(t_i))
\end{aligned}$$

$$
\begin{aligned}
&+A\int_0^t(\psi(t)-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t,s)(f(s,x(s),x_s)+Bu(s))\psi'(s)ds\\
&+f(t,x(t),x_t)+Bu(t)\\
=\ &Ax(t)+f(t,x(t),x_t)+Bu(t).
\end{aligned}
$$

For $t=0$, one has

$$x(0)=S_E^{\alpha,\psi}(0,0)E\phi(0)=\int_0^\infty E^{-1}\xi_\alpha(\theta)T(0)d\theta E\phi(0)=E^{-1}E\phi(0)=\phi(0).$$

Moreover, we have

$$
\begin{aligned}
\Delta x(t_k) &= x(t_k^+)-x(t_k^-)\\
&=\left[\sum_{i=1}^{k}S_E^{\alpha,\psi}(t,t_i)EI_i(x(t_i))-\sum_{i=1}^{k-1}S_E^{\alpha,\psi}(t,t_i)EI_i(x(t_i))\right]_{t=t_k}\\
&=S_E^{\alpha,\psi}(t_k,t_k)EI_k(x(t_k))\\
&=\int_0^\infty E^{-1}\xi_\alpha(\theta)T(0)EI_k(x(t_k))d\theta\\
&=E^{-1}EI_k(x(t_k))=I_k(x(t_k)).
\end{aligned}
$$

Thus, expression (17) is a solution of problem (6). □

Definition 4. *For each $u\in\mathcal{U}$ and $\phi\in C(E)$, a function $x\in PC(J^*,X)$ is called a mild solution of (6) if (17) holds.*

From [35], we can obtain easily that the following properties of $S_E^{\alpha,\psi}(t,s)$ and $T_E^{\alpha,\psi}(t,s)$.

Lemma 5. *Suppose that conditions (H1)–(H3) hold. Then the operators $S_E^{\alpha,\psi}$ and $T_E^{\alpha,\psi}$ have the following properties:*

(i) *For any fixed $t\geq s\geq 0$, $S_E^{\alpha,\psi}(t,s)$ and $T_E^{\alpha,\psi}(t,s)$ are bounded linear operators with*

$$\|S_E^{\alpha,\psi}(t,s)(x)\|\leq M\|E^{-1}\|\|x\|\quad\text{and}\quad\|T_E^{\alpha,\psi}(t,s)(x)\|\leq\frac{M\|E^{-1}\|}{\Gamma(\alpha)}\|x\|,$$

for each $x\in X$.

(ii) *$S_E^{\alpha,\psi}(t,s)$ and $T_E^{\alpha,\psi}(t,s)$ are strongly continuous for all $t\geq s\geq 0$, that is, for each $x\in X$ and $0\leq s\leq t_1<t_2\leq b$ we have*

$$\|S_E^{\alpha,\psi}(t_2,s)-S_E^{\alpha,\psi}(t_1,s)\|\to 0\quad\text{and}\quad\|T_E^{\alpha,\psi}(t_2,s)-T_E^{\alpha,\psi}(t_1,s)\|\to 0$$

as $t_1\to t_2$.

(iii) *If $T(t)$ is compact operator for every $t>0$, then $S_E^{\alpha,\psi}(t,s)$ and $T_E^{\alpha,\psi}(t,s)$ are compact for all $t,s>0$.*

(iv) *If $S_E^{\alpha,\psi}(t,s)$ and $T_E^{\alpha,\psi}(t,s)$ are compact strongly continuous semigroup of bounded linear operator for $t,s>0$, then $S_E^{\alpha,\psi}(t,s)$ and $T_E^{\alpha,\psi}(t,s)$ are continuous in the uniform operator topology.*

3. Main Results

According to the exact controllability considered in Ref. [41], we give the following definition.

Definition 5. *The fractional system (6) is the exact controllability on $J = [0, b]$ if for any initial function $\phi \in C(E)$ and $x_1 \in D(E)$, there has a control $u \in \mathcal{U}$ such that the mild solution x of (6) on $[-r, b]$ satisfies $x(b) = x_1$.*

Besides (H1)–(H3), we need the following hypotheses.

(H4) If f fulfills the following two conditions:

(i) For each $x \in PC(J^*, D(E))$ and $\varphi \in C(E)$, the function $f(\cdot, x, \varphi) : J \to Y$ is strongly measurable and for each $t \in J$, the function $f(t, \cdot, \cdot) : PC(J^*, D(E)) \times C(E) \to Y$ is continuous;

(ii) For any $t \in J$ and $x \in C$, there are two continuous nondecreasing functions μ_1, μ_2 and constant L such that, for any $(t, x, \varphi) \in J \times PC(J^*, D(E)) \times C(E)$, such that

$$\|f(t, x, x_t)\| \le L(1 + \mu_1(\|x\|) + \mu_2(\|x_t\|), \quad \lim_{s\to\infty} \inf \frac{\mu_1(s) + \mu_2(s)}{s} = \Lambda < \infty, \quad (18)$$

(H5) For every $i = 1, 2, \dots, m$, $I_i : PC(J^*, D(E)) \to PC(J^*, D(E))$ is continuous, and there exists constant k_i such that

$$\|I_i(u)\| \le k_i\|u\|, \quad u \in PC(J^*, D(E)). \quad (19)$$

(H6) For $\psi \in C^1(J, \mathbb{R})$, and there exists a constant $\gamma > 0$ such that $0 < \psi'(t) \le \gamma$, $\forall t \in J$.

(H7) $B : U \to Y$ is a bounded linear operator and a linear operator $W : \mathcal{U} \to D(E)$ defined by

$$Wu := \int_0^b (\psi(b) - \psi(s))^{\alpha-1} T_E^{\alpha,\psi}(b, s)Bu(s)\psi'(s)ds. \quad (20)$$

The right inverse operator $W^{-1} : D(E) \to \mathcal{U}$ is bounded, i.e., $WW^{-1} = I_{D(E)}$, and thus there exist two constants $M_1, M_2 > 0$ such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$, then by determining M_2 we could define the norm $\|\cdot\|_{D(E)}$ on $D(E)$.

If $\alpha \in (0, 1)$, then we have

$$\begin{aligned}\int_0^t (\psi(t) - \psi(s))^{\alpha-1}\|u(s)\|\psi'(s)ds &\le \int_0^t (\psi(t) - \psi(s))^{\alpha-1}\psi'(s)ds\|u\|_\infty \\ &= \frac{(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha)}\|u\|_\infty \le \frac{(\psi(b) - \psi(0))^\alpha}{\Gamma(\alpha)}\|u\|_\infty := \frac{(\psi(b) - \psi(0))^\alpha}{\Gamma(\alpha)}\|u\|_{\mathcal{U}}.\end{aligned}$$

If $\alpha \in (\frac{1}{2}, 1)$, then one has by (H6) that

$$\begin{aligned}\int_0^t (\psi(t) - \psi(s))^{\alpha-1}\|u(s)\|\psi'(s)ds &\le \left(\int_0^t (\psi(t) - \psi(s))^{2\alpha-2}(\psi'(s))^2 ds\right)^{\frac{1}{2}}\|u\|_{L^2(J,U)} \\ &\le \sqrt{\gamma}\left(\int_0^t (\psi(t) - \psi(s))^{2\alpha-2}\psi'(s)ds\right)^{\frac{1}{2}}\|u\|_{L^2(J,U)} \\ &= \sqrt{\gamma}\frac{(\psi(t) - \psi(0))^{2\alpha-1}}{\Gamma(2\alpha - 1)}\|u\|_{L^2(J,U)} \le \sqrt{\gamma}\frac{(\psi(b) - \psi(0))^{2\alpha-1}}{\Gamma(2\alpha - 1)}\|u\|_{L^2(J,U)} \\ &:= \sqrt{\gamma}\frac{(\psi(b) - \psi(0))^{2\alpha-1}}{\Gamma(2\alpha - 1)}\|u\|_{\mathcal{U}}.\end{aligned}$$

Thus

$$\int_0^t (\psi(t) - \psi(s))^{\alpha-1}\|u(s)\|\psi'(s)ds \le K_{\alpha,\psi}\|u\|_{\mathcal{U}}, \quad (21)$$

for any $t \in J$, where

$$K_{\alpha,\psi} = \begin{cases} \frac{(\psi(b)-\psi(0))^{2\alpha-1}}{2\alpha-1}\gamma, & u \in L^2(J,U),\ \frac{1}{2} < \alpha < 1, \\ \frac{(\psi(b)-\psi(0))^{\alpha}}{\alpha}, & u \in L^\infty(J,U),\ 0 < \alpha < 1. \end{cases} \tag{22}$$

Obviously, $Wu \in D(E)$ and W is well defined. In fact, by Lemma 5 and (21), one has

$$\begin{aligned} \|EWu\| &= \left\| \int_0^b (\psi(b)-\psi(s))^{\alpha-1} T_I^{\alpha,\psi}(b,s)Bu(s)\psi'(s)ds \right\| \\ &\le \frac{M\|B\|}{\Gamma(\alpha)} \int_0^b (\psi(b)-\psi(s))^{\alpha-1}\psi'(s)\|u(s)\|ds \\ &\le \frac{M\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi}\|u\|_{\mathcal{U}}. \end{aligned}$$

For an arbitrary function $x(\cdot)$, by means of the above assumptions, it is suitable to define the following control formula:

$$\begin{aligned} u(t) := W^{-1}\Bigg[& x_1 - S_E^{\alpha,\psi}(b,0)E\phi(0) - \sum_{i=1}^{m} S_E^{\alpha,\psi}(b,t_i)EI_i(x(t_i)) \\ & - \int_0^b (\psi(b)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(b,s)f(s,x(s),x_s)\psi'(s)ds \Bigg]. \end{aligned} \tag{23}$$

In the following, we will prove that, in view of the control u in (23), the operator $\mathcal{P}$ defined by

$$(\mathcal{P}x)(t) := \begin{cases} S_E^{\alpha,\psi}(t,0)E\phi(0) + \int_0^t (\psi(t)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(t,s)f(s,x(s),x_s)\psi'(s)ds \\ \quad + \int_0^t (\psi(t)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(t,s)Bu(s)\psi'(s)ds, \quad t \in J_0, \\ S_E^{\alpha,\psi}(t,0)E\phi(0) + \sum_{i=1}^{n} S_E^{\alpha,\psi}(t,t_i)EI_i(x(t_i)) \\ \quad + \int_0^t (\psi(t)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(t,s)f(s,x(s),x_s)\psi'(s)ds \\ \quad + \int_0^t (\psi(t)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(t,s)Bu(s)\psi'(s)ds, \quad t \in J_n, \quad n = 1,2,\ldots,m, \\ \phi(t), \quad t \in [-r,0], \end{cases}$$

from $PC(J^*, D(E))$ into $PC(J^*, D(E))$, has a fixed point. It is obvious that this fixed point is just a solution of system (6). Moreover, we can check that

$$\begin{aligned} (\mathcal{P}x)(b) := {} & S_E^{\alpha,\psi}(b,0)E\phi(0) + \sum_{i=1}^{m} S_E^{\alpha,\psi}(b,t_i)EI_i(x(t_i)) \\ & + \int_0^b (\psi(b)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(b,s)f(s,x(s),x_s)\psi'(s)ds \\ & + \int_0^b (\psi(b)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(b,s)Bu(s)\psi'(s)ds \\ = {} & S_E^{\alpha,\psi}(b,0)E\phi(0) + \sum_{i=1}^{m} S_E^{\alpha,\psi}(b,t_i)EI_i(x(t_i)) \\ & + \int_0^b (\psi(b)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(b,s)f(s,x(s),x_s)\psi'(s)ds \\ & + \int_0^b (\psi(b)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(b,s)BW^{-1}\Bigg[x_1 - S_E^{\alpha,\psi}(b,0)E\phi(0) - \sum_{i=1}^{m} S_E^{\alpha,\psi}(b,t_i)EI_i(x(t_i)) \end{aligned}$$

$$-\int_0^b (\psi(b)-\psi(\tau))^{\alpha-1} T_E^{\alpha,\psi}(b,\tau) f(\tau, x(\tau), x_\tau)\psi'(\tau)d\tau\Big]\psi'(s)ds = x_1.$$

For each number $K > 0$, set

$$\mathcal{B}_K := \{x \in PC(J^*, D(E)) : \|x(t)\| \le K,\ t \in J^*\}.$$

Clearly, $\mathcal{B}_K$ is a bounded, closed, convex subset in $PC(J, D(E))$.

Lemma 6. *Assume that (H1)–(H7) are satisfied. Then there exists a* $K \ge \max\left\{\|\phi\|, \frac{N_3}{1-\rho}\right\}$ *where*

$$\rho := \begin{cases} M\sum\limits_{i=1}^m k_i + \frac{M\|E^{-1}\|L\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^\alpha \\ \quad + \frac{M^2\|W^{-1}\|\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi}\sqrt{b}\left(\|\sum\limits_{i=1}^m k_i + \frac{L\|E^{-1}\|\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^\alpha\right) < 1, \quad \mathcal{U} = L^2(J,U), \\ M\sum\limits_{i=1}^m k_i + \frac{M\|E^{-1}\|L\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^\alpha \\ \quad + \frac{M^2\|W^{-1}\|\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi}\left(\sum\limits_{i=1}^m k_i + \frac{L\|E^{-1}\|\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^\alpha\right) < 1, \quad \mathcal{U} = L^\infty(J,U), \end{cases} \tag{24}$$

and

$$N_3 = \begin{cases} M\|\phi(0)\| + \frac{M\|E^{-1}\|}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^\alpha(L+\sqrt{b}\|B\|N_1), \quad \mathcal{U} = L^2(J,U), \\ M\|\phi(0)\| + \frac{M\|E^{-1}\|}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^\alpha(L+\|B\|N_1), \quad \mathcal{U} = L^\infty(J,U), \end{cases}$$

here

$$N_1 = \|Ex_1\| + M\|E\phi(0)\| + \frac{ML}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^\alpha,$$

such that $P\mathcal{B}_K \subset \mathcal{B}_K$.

Proof. Let $x \in \mathcal{B}_K$. If $t \in [-r, 0]$ then $\|(\mathcal{P}x)(t)\| = \|\phi(t)\| \le \max_{t\in[-r,0]}\|\phi(t)\| = \|\phi\|$. If $t \in [0, b]$, then

$$\|x_t\| = \sup_{\tau\in[-r,0]}\|x(t+\tau)\| \le \max\{\|\phi\|, \|x\|\}. \tag{25}$$

Since $K \ge \|\phi\|$, by (25), we note that the control u defined in (23) satisfies

$$\begin{aligned} \|u(t)\| &\le \|W^{-1}\|\Bigg\| x_1 - S_E^{\alpha,\psi}(b,0)E\phi(0) - \sum_{i=1}^m S_E^{\alpha,\psi}(b,t_i)EI_i(x(t_i)) \\ &\qquad - \int_0^b (\psi(b)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(b,s) f(s,x(s),x_s)\psi'(s)ds\Bigg\|_{D(E)} \\ &= \|W^{-1}\|\Bigg\| E\Bigg(\Bigg[x_1 - S_E^{\alpha,\psi}(b,0)E\phi(0) - \sum_{i=1}^m S_E^{\alpha,\psi}(b,t_i)EI_i(x(t_i)) \\ &\qquad - \int_0^b (\psi(b)-\psi(s))^{\alpha-1} T_E^{\alpha,\psi}(b,s) f(s,x(s),x_s)\psi'(s)ds\Bigg]\Bigg)\Bigg\| \\ &\le \|W^{-1}\|\Bigg[\|Ex_1\| + M\left(\|E\phi(0)\| + K\|E\|\sum_{i=1}^m k_i\right) \\ &\qquad + \frac{M}{\Gamma(\alpha)}\int_0^b (\psi(b)-\psi(s))^{\alpha-1} L[1+\mu_1(K)+\mu_2(\max\{\|\phi\|,K\})]\psi'(s)ds\Bigg] \end{aligned}$$

$$
\begin{aligned}
&\leq \|W^{-1}\| \Bigg[\|Ex_1\| + M\Bigg(\|E\phi(0)\| + K\|E\| \sum_{i=1}^{m} k_i \Bigg) \\
&\qquad + L(1 + \mu_1(K) + \mu_2(K)) \frac{M}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha} \Bigg] \\
&:= N_1 + N_2(K)K,
\end{aligned}
$$

which implies that

$$
\|u\|_{\mathcal{U}} \leq \begin{cases} \sqrt{b}(N_1 + N_2(K)K), & \mathcal{U} = L^2(J,U), \\ N_1 + N_2(K)K, & \mathcal{U} = L^{\infty}(J,U), \end{cases} \tag{26}
$$

where

$$
N_1 = \left[\|Ex_1\| + M\|E\phi(0)\| + \frac{ML}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha} \right] \|W^{-1}\|,
$$

$$
N_2(K) = \left[M\|E\| \sum_{i=1}^{m} k_i + \frac{ML}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha} \frac{\mu_1(K) + \mu_2(K)}{K} \right] \|W^{-1}\|.
$$

Thus, for $t \in J_n$, we derive by (21) and (26) that

$$
\begin{aligned}
\|(\mathcal{P}x)(t)\| &\leq M\|E^{-1}\| \Bigg(\|E\phi(0)\| + \sum_{i=1}^{n} k_i \|E\| x(t_i)\| \Bigg) \\
&\quad + \frac{M\|E^{-1}\|}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} L[1 + \mu_1(\|x(s)\|) + \mu_2(\max\{\|x(s)\|, \|x_s\|\})] \psi'(s) ds \\
&\quad + \frac{M\|E^{-1}\|\|B\|}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) \|u(s)\| ds \\
&\leq M\Bigg(\|\phi(0)\| + K\sum_{i=1}^{m} k_i \Bigg) + \frac{M\|E^{-1}\|L}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha} (1 + \mu_1(K) + \mu_2(K)) \\
&\quad + \frac{M\|E^{-1}\|\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi} \|u\|_{\mathcal{U}} \\
&\leq M\Bigg(\|\phi(0)\| + K\sum_{i=1}^{m} k_i \Bigg) + \frac{M\|E^{-1}\|L}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha} (1 + \mu_1(K) + \mu_2(K)) \\
&\quad + \frac{M\|E^{-1}\|\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi} \begin{cases} \sqrt{b}(N_1 + N_2(K)K), & \mathcal{U} = L^2(J,U), \\ N_1 + N_2(K)K, & \mathcal{U} = L^{\infty}(J,U), \end{cases} \\
&:= N_3 + N_4(K)K,
\end{aligned}
$$

where

$$
N_3 = \begin{cases} M\|\phi(0)\| + \frac{M\|E^{-1}\|L}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha} + \frac{M\|E^{-1}\|\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi} \sqrt{b} N_1, & \mathcal{U} = L^2(J,U), \\ M\|\phi(0)\| + \frac{M\|E^{-1}\|L}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha} + \frac{M\|E^{-1}\|\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi} N_1, & \mathcal{U} = L^{\infty}(J,U), \end{cases}
$$

$$
N_4(K) = \begin{cases} M\sum_{i=1}^{m} k_i + \frac{M\|E^{-1}\|L}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha} \frac{\mu_1(K) + \mu_2(K)}{K} \\ \quad + \frac{M\|E^{-1}\|\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi} \sqrt{b} N_2(K), & \mathcal{U} = L^2(J,U), \\ M\sum_{i=1}^{m} k_i + \frac{M\|E^{-1}\|L}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha} \frac{\mu_1(K) + \mu_2(K)}{K} \\ \quad + \frac{M\|E^{-1}\|\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi} N_2(K), & \mathcal{U} = L^{\infty}(J,U). \end{cases}
$$

From (18), we have $\frac{\mu_1(K)+\mu_2(K)}{K} \geq \Lambda$ for $K > 0$. Thus

$$N_4(K) \geq \begin{cases} M\sum_{i=1}^{m} k_i + \frac{M\|E^{-1}\|L\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^{\alpha} \\ \quad + \frac{M^2\|W^{-1}\|B\|}{\Gamma(\alpha)}K_{\alpha,\psi}\sqrt{b}\left(\|\sum_{i=1}^{m} k_i + \frac{L\|E^{-1}\|\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^{\alpha}\right), & \mathcal{U} = L^2(J,U), \\ M\sum_{i=1}^{m} k_i + \frac{M\|E^{-1}\|L\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^{\alpha} \\ \quad + \frac{M^2\|W^{-1}\|B\|}{\Gamma(\alpha)}K_{\alpha,\psi}\left(\sum_{i=1}^{m} k_i + \frac{L\|E^{-1}\|\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^{\alpha}\right), & \mathcal{U} = L^{\infty}(J,U), \end{cases}$$

$$:= \rho.$$

So $N_3 + N_4(K)K \leq K$ for each $K \geq \max\left\{\|\phi\|, \frac{N_3}{1-\rho}\right\}$ sufficiently large, that is $\mathcal{P}\mathcal{B}_K \subset \mathcal{B}_K$. We complete the proof. □

Lemma 7. *Assume that (H1)–(H7) are satisfied. Then, for any fixed $t \in J$ the set $V_K(t) := \{(\mathcal{P}x)(t) : x \in \mathcal{B}_K\}$ is precompact in X.*

Proof. For $t \in [-r, 0]$, obviously, it holds. For $t \in J \setminus \{t_1, \ldots, t_m\}$ be fixed. Without loss of generality, let $t \in J_n$. Note that

$$(\mathcal{P}x)(t) = E^{-1}(\mathcal{P}_0 x)(t),$$

$$(\mathcal{P}_0 x)(t) = S_I^{\alpha,\psi}(t,0)\phi(0) + \sum_{i=1}^{n} S_I^{\alpha,\psi}(t,t_i)I_i(x(t_i))$$

$$+ \int_0^t (\psi(t)-\psi(s))^{\alpha-1} T_I^{\alpha,\psi}(t,s) f(s,x(s),x_s)\psi'(s)ds$$

$$+ \int_0^t (\psi(t)-\psi(s))^{\alpha-1} T_I^{\alpha,\psi}(t,s) Bu(s)\psi'(s)ds, \quad \text{for } t \in J_n.$$

For $x \in \mathcal{B}_K$, we can derive

$$\|(\mathcal{P}_0 x)(t)\| \leq M\left(\|\phi(0)\| + K\sum_{i=1}^{m} k_i\right)$$

$$+ \frac{ML}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^{\alpha}(1+\mu_1(K)+\mu_2(K)) + \frac{M}{\Gamma(\alpha)}K_{\alpha,\psi}\|B\|\|u\|_{\mathcal{U}}.$$

Thus, $\{(\mathcal{P}_0 x)(t) : x \in \mathcal{B}_K\}$ is bounded in Y by (28).

Since $E^{-1} : Y \to Y$ is compact, then $(\mathcal{P}x)(t) := E^{-1}(\{(\mathcal{P}_0 x)(t) : x \in \mathcal{B}_K\})$ is precompact in X. The proof is completed. □

Lemma 8. *Assume that (H1)–(H7) are satisfied. Then $\mathcal{P}\mathcal{B}_K := \{\mathcal{P}x : x \in \mathcal{B}_K\}$ is equicontinuous.*

Proof. Let $x \in \mathcal{B}_K$ and $t', t'' \in J_n$, $t' < t''$. One has

$$\|(\mathcal{P}x)(t'') - (\mathcal{P}x)(t')\|$$

$$\leq \left\| S_E^{\alpha,\psi}(t'',0)E\phi(0) + \sum_{i=1}^{n} S_E^{\alpha,\psi}(t'',t_i)EI_i(x(t_i)) \right.$$

$$\left. - S_E^{\alpha,\psi}(t',0)E\phi(0) - \sum_{i=1}^{n} S_E^{\alpha,\psi}(t',t_i)EI_i(x(t_i)) \right\|$$

$$
\begin{aligned}
&+\left\|\int_0^{t''}(\psi(t'')-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t'',s)f(s,x(s),x_s)\psi'(s)ds\right.\\
&\qquad\left.-\int_0^{t'}(\psi(t')-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t',s)f(s,x(s),x_s)\psi'(s)ds\right\|\\
&+\left\|\int_0^{t''}(\psi(t'')-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t'',s)Bu(s)\psi'(s)ds\right.\\
&\qquad\left.-\int_0^{t'}(\psi(t')-\psi(s))^{\alpha-1}T_E^{\alpha,\psi}(t',s)Bu(s)\psi'(s)ds\right\|\\
\le&\left\|[S_E^{\alpha,\psi}(t'',0)-S_E^{\alpha,\psi}(t',0)]E\phi(0)\right\|\\
&+\sum_{i=1}^m\left\|[S_E^{\alpha,\psi}(t'',t_i)-S_E^{\alpha,\psi}(t',t_i)]I_i(x(t_i))\right\|\\
&+\int_0^{t'}[(\psi(t'')-\psi(s))^{\alpha-1}-(\psi(t')-\psi(s))^{\alpha-1}]\|T_E^{\alpha,\psi}(t'',s)f(s,x(s),x_s)\|\psi'(s)ds\\
&+\int_0^{t'}(\psi(t')-\psi(s))^{\alpha-1}\|[T_E^{\alpha,\psi}(t'',s)-T_E^{\alpha,\psi}(t',s)]f(s,x(s),x_s)\|\psi'(s)ds\\
&+\int_0^{t'}[(\psi(t'')-\psi(s))^{\alpha-1}-(\psi(t')-\psi(s))^{\alpha-1}]\|T_E^{\alpha,\psi}(t'',s)Bu(s)\|\psi'(s)ds\\
&+\int_0^{t'}(\psi(t')-\psi(s))^{\alpha-1}\|[T_E^{\alpha,\psi}(t'',s)-T_E^{\alpha,\psi}(t',s)]Bu(s)\|\psi'(s)ds\\
&+\int_{t'}^{t''}(\psi(t'')-\psi(s))^{\alpha-1}\|[T_E^{\alpha,\psi}(t'',s)f(s,x(s),x_s)\|\psi'(s)ds\\
&+\int_{t'}^{t''}(\psi(t'')-\psi(s))^{\alpha-1}\|[T_E^{\alpha,\psi}(t'',s)Bu(s)\|\psi'(s)ds\\
\le& J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &:= \left\|S_E^{\alpha,\psi}(t'',0)-S_E^{\alpha,\psi}(t',0)\right\|\|E\phi(0)\|,\\
J_2 &:= \sum_{i=1}^m\left\|S_E^{\alpha,\psi}(t'',t_i)-S_E^{\alpha,\psi}(t',t_i)\right\|k_i\|x(t_i)\|,\\
J_3 &:= \frac{M\|E^{-1}\|L}{\Gamma(\alpha)}(1+\mu_1(K)+\mu_2(K))\\
&\qquad\cdot\int_0^{t'}[(\psi(t'')-\psi(s))^{\alpha-1}-(\psi(t')-\psi(s))^{\alpha-1}]\psi'(s)ds,\\
J_4 &:= \frac{L}{\alpha}(\psi(b)-\psi(0))^{\alpha}(1+\mu_1(K)+\mu_2(K))\sup_{s\in[0,t']}\|T_E^{\alpha,\psi}(t'',s)-T_E^{\alpha,\psi}(t',s)\|,\\
J_5 &:= \frac{M\|E^{-1}\|\|B\|}{\Gamma(\alpha)}\int_0^{t'}[(\psi(t'')-\psi(s))^{\alpha-1}-(\psi(t')-\psi(s))^{\alpha-1}]\|u(s)\|\psi'(s)ds,\\
J_6 &:= \sup_{s\in[0,t']}\|T_E^{\alpha,\psi}(t'',s)-T_E^{\alpha,\psi}(t',s)\|\|B\|\int_0^{t'}(\psi(t')-\psi(s))^{\alpha-1}\|u(s)\|\psi'(s)ds,
\end{aligned}
$$

$$J_7 := \frac{M\|E^{-1}\|L}{\Gamma(\alpha+1)}(1+\mu_1(K)+\mu_2(K))(\psi(t'')-\psi(t')^{\alpha},$$

$$J_8 := \frac{M\|E^{-1}\|\|B\|}{\Gamma(\alpha)}\int_{t'}^{t''}(\psi(t'')-\psi(s))^{\alpha-1}\|u(s)\|\psi'(s)ds.$$

Obviously, $J_7 \to 0$ as $t'' \to t'$. By Lemma 3(iv), $S_E^{\alpha,\psi}(t,s)$ and $T_E^{\alpha,\psi}(t,s)$ are continuous in the uniform operator topology for $t \geq s \geq 0$, and $u(\cdot)$ is bounded by (28). Then one can check the terms $J_1, J_2, J_4, J_6, J_8 \to 0$ as $t'' \to t'$. By virtue of Lebesgue's dominated convergence theorem, we obtain $J_3, J_5 \to 0$ as $t'' \to t'$. Hence, $\mathcal{PB}_K$ is equicontinuous and bounded. □

Theorem 1. *Suppose that (H1)–(H7) are satisfied. Then the system (6) is controllable on J provided that the condition (24) holds.*

Proof. From Lammas 6–8 and the Arzela–Ascoli theorem, we obtain that $\mathcal{PB}_K$ is precompact in $PC(J^*, X)$. Thus $\mathcal{P}$ is a completely continuous operator on $PC(J^*, X)$. By the Schauder fixed point theorem, $\mathcal{P}$ has a fixed point in $\mathcal{B}_K$. Each fixed point of $\mathcal{P}$ is a mild solution of the system (6) on J such that $(\mathcal{P}x)(t) = x(t) \in X$. Hence, the system (6) is controllable on J. □

Remark 1. *Let $\psi(t) \equiv t$, and $I_i(\cdot) \equiv 0$ $(i = 1, 2, \ldots, m)$, then, Theorem 1 reduces to Theorem 4.1 in Ref. [23]. That is, the classical Caputo fractional derivative and non-impulse cases in Ref. [23] are generalized to the ψ-Caputo fractional derivative and non-impulse cases.*

4. Example

An example is provided to demonstrate the controllability result for the proposed criteria.

Let $X = Y = U = L^2[0,\pi]$ equipped with the norm and inner product defined, respectively, for all $u, v \in L^2[0,\pi]$ by

$$\|u\| = \left(\int_0^{\pi}|u(x)|^2dx\right)^{\frac{1}{2}} \quad \text{and} \quad \langle u, v\rangle = \int_0^{\pi}u(x)\overline{v(x)}dx.$$

Consider the ψ-Caputo fractional differential control system of Sobolev type

$$\begin{cases} {}^CD_t^{\alpha,\psi}(x(t,z)-x_{zz}(t,z)) = x_{zz}(t,z)+f(t,x(t,z),x(t-r,z))+Bu(t,z), & a.e.\ t \in J',\ z \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0, \quad t \in J := [0,1], & \\ \Delta x(t_k,z) = I_k(x(t_k,z)), & k = 1,\ldots,m, \\ x(t,z) = \phi(t,z), & (t,z) \in [-\frac{1}{2},0]\times[0,\pi], \end{cases} \tag{27}$$

where $\alpha = \frac{4}{5}$, $\psi(t) = \sqrt{t+1}$,

$$f(t,x(t,z),x(t-r,z)) = e^{-t}+\frac{1}{3}x(t,z)+\frac{1}{6}\sin t\frac{x(t-r,z)}{1+|x(t-r,z)|}, \tag{28}$$

and

$$I_k(x(t_k,z)) = \frac{\sin(x(t_k,z))}{9m(1+|x(t_k,z)|)}, \quad k = 1,2,\ldots,m. \tag{29}$$

We define

$$D(A) = D(E) = \{x \in X : x, x_z \text{ are absolutely continuous, } x_{zz} \in X, x(t,0) = x(t,\pi) = 0\}.$$

$$Ax = x_{zz}, \quad Ex = x - x_{zz}.$$

It follows that A has eigenvalues $-n^2$, $n \in \mathbb{N}$ with corresponding orthogonal eigenvectors $e_n(z) = \sqrt{\frac{2}{\pi}}\sin(nz)$. From Ref. [42], A and E can be written as

$$Ax := \sum_{n=1}^{\infty} n^2 \langle x, e_n \rangle e_n, \quad x \in D(A),$$

$$Ex := \sum_{n=1}^{\infty} (1+n^2) \langle x, e_n \rangle e_n, \quad x \in D(A).$$

Furthermore, for each $x \in X$ one has

$$E^{-1}x := \sum_{n=1}^{\infty} \frac{1}{1+n^2} \langle x, e_n \rangle e_n, \quad -AE^{-1}x := \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2} \langle x, e_n \rangle e_n,$$

and

$$T(t)x = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t} \langle x, e_n \rangle e_n.$$

Obviously, E^{-1} is compact, $\|E^{-1}\| \leq 1$. We also have $-AE^{-1}$, which generates the above strongly continuous semigroup $T(t)$ on Z with $\|T(t)\| \leq e^{-t} \leq 1$. So, the two characterized operators $S_E^{\alpha,\psi}(\cdot,\cdot)$ and $T_E^{\alpha,\psi}(\cdot,\cdot)$ have the following formulas

$$S_E^{\alpha,\psi}(t,s) := \int_0^{\infty} E^{-1} \xi_{\frac{4}{5}}(\theta) T((\sqrt{t+1} - \sqrt{s+1})^{\frac{4}{5}} \theta) d\theta,$$

and

$$T_E^{\alpha,\psi}(t,s) := \frac{4}{5} \int_0^{\infty} E^{-1} \theta \xi_{\frac{4}{5}}(\theta) T((\sqrt{t+1} - \sqrt{s+1})^{\frac{4}{5}} \theta) d\theta.$$

Clearly,

$$\|S_E^{\alpha,\psi}(t,s)\| \leq 1, \quad \|T_E^{\alpha,\psi}(t,s)\| \leq \frac{1}{\Gamma(\frac{4}{5})}, \quad 0 \leq s \leq t \leq 1.$$

Let $B = \omega I : U \to Z$ ($\omega > 0$), then defined $W : \mathcal{U} \to D(E)$ by

$$Wu := \omega \int_0^1 \frac{(\psi(1) - \psi(s))^{-\frac{1}{5}}}{2\sqrt{s+1}} T_E^{\alpha,\psi}(1,s) u(s,z) ds.$$

Since $\alpha = \frac{4}{5} \in \left(\frac{1}{2}, 1\right)$, we take $\mathcal{U} := L^2(J, U)$, and

$$K_{\alpha,\psi} = \frac{(\psi(1) - \psi(0))^{2\alpha-1}}{2\alpha - 1} \gamma = \frac{5}{6} (\sqrt{2} - 1)^{\frac{3}{5}} = 0.4911.$$

Next, let $u(t,z) := x(z) \in U$. Thus,

$$\begin{aligned}
Wu &= \omega \int_0^1 \frac{(\psi(1) - \psi(s))^{-\frac{1}{5}}}{2\sqrt{s+1}} \frac{4}{5} \int_0^{\infty} E^{-1} \theta \xi_{\frac{4}{5}}(\theta) T((\sqrt{2} - \sqrt{s+1})^{\frac{4}{5}} \theta) d\theta x ds \\
&= \omega \int_0^1 \frac{(\psi(1) - \psi(s))^{-\frac{1}{5}}}{2\sqrt{s+1}} \frac{4}{5} \int_0^{\infty} E^{-1} \theta \xi_{\frac{4}{5}}(\theta) \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}(\sqrt{2} - \sqrt{s+1})^{\frac{4}{5}} \theta} \langle x, e_n \rangle e_n d\theta ds \\
&= \omega \int_0^{\infty} E^{-1} \xi_{\frac{4}{5}}(\theta) \sum_{n=1}^{\infty} \int_0^1 \frac{4}{5} \theta \frac{(\sqrt{2} - \sqrt{s+1})^{-\frac{1}{5}}}{2\sqrt{s+1}} e^{\frac{-n^2}{1+n^2}(\sqrt{2} - \sqrt{s+1})^{\frac{4}{5}} \theta} ds \langle x, e_n \rangle e_n d\theta \\
&= \omega \int_0^{\infty} \xi_{\frac{4}{5}}(\theta) \sum_{n=1}^{\infty} \int_0^1 \frac{4}{5(1+n^2)} \theta \frac{(\sqrt{2} - \sqrt{s+1})^{-\frac{1}{5}}}{2\sqrt{s+1}} e^{\frac{-n^2}{1+n^2}(\sqrt{2} - \sqrt{s+1})^{\frac{4}{5}} \theta} ds \langle x, e_n \rangle e_n d\theta \\
&= \omega \int_0^{\infty} \xi_{\frac{4}{5}}(\theta) \sum_{n=1}^{\infty} \int_0^1 \frac{1}{n^2} \frac{d}{ds} \left[e^{\frac{-n^2}{1+n^2}(\sqrt{2} - \sqrt{s+1})^{\frac{4}{5}} \theta} \right] ds \langle x, e_n \rangle e_n d\theta
\end{aligned}$$

$$= \omega \int_0^\infty \xi_{\frac{4}{5}}(\theta) \sum_{n=1}^{\infty} \frac{1}{n^2} \left[1 - e^{\frac{-n^2}{1+n^2}(\sqrt{2}-1)^{\frac{4}{5}}\theta} \right] \langle x, e_n \rangle e_n d\theta$$

$$= \omega \sum_{n=1}^{\infty} \frac{1}{n^2} \left[1 - E_{\frac{4}{5}} \left(-\frac{n^2}{1+n^2}(\sqrt{2}-1)^{\frac{4}{5}} \right) \right] \langle x, e_n \rangle e_n, \tag{30}$$

where

$$E_{\frac{4}{5}} \left(-\frac{n^2}{1+n^2}(\sqrt{2}-1)^{\frac{4}{5}} \right) := \int_0^\infty e^{\frac{-n^2}{1+n^2}(\sqrt{2}-1)^{\frac{4}{5}}\theta} \xi_{\frac{4}{5}}(\theta) d\theta,$$

is a Mittag–Leffler function (see [43]).

Note that $0 < 1 - e^{\frac{-n^2}{1+n^2}(\sqrt{2}-1)^{\frac{4}{5}}\theta} < 1 - e^{-\theta}$ for $\theta > 0$. Thus one has

$$1 - E_{\frac{4}{5}} \left(-\frac{1}{2}(\sqrt{2}-1)^{\frac{4}{5}} \right) \leq 1 - E_{\frac{4}{5}} \left(-\frac{n^2}{1+n^2}(\sqrt{2}-1)^{\frac{4}{5}} \right) \leq 1 - E_{\frac{4}{5}} \left(-(\sqrt{2}-1)^{\frac{4}{5}} \right). \tag{31}$$

By (30) and (31), we have W is surjective. Denoted $W^{-1} : D(E) \to \mathcal{U}$ by

$$(W^{-1}x)(t, z) := \frac{1}{\omega} \sum_{n=1}^{\infty} \frac{n^2 \langle x, e_n \rangle e_n}{\left[1 - E_{\frac{4}{5}} \left(-\frac{n^2}{1+n^2}(\sqrt{2}-1)^{\frac{4}{5}} \right) \right]},$$

for $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$. Moreover, we can obtain that

$$\begin{aligned}
\|(W^{-1}x)(t, \cdot)\| &= \frac{1}{\omega} \sqrt{\sum_{n=1}^{\infty} \frac{n^4 \langle x, e_n \rangle^2}{\left[1 - E_{\frac{4}{5}} \left(-\frac{n^2}{1+n^2}(\sqrt{2}-1)^{\frac{4}{5}} \right) \right]^2}} \\
&\leq \frac{1}{\omega \left[1 - E_{\frac{4}{5}} \left(-\frac{1}{2}(\sqrt{2}-1)^{\frac{4}{5}} \right) \right]} \sqrt{\sum_{n=1}^{\infty} (1+n^2)^2 \langle x, e_n \rangle^2} \\
&= \frac{1}{\omega \left[1 - E_{\frac{4}{5}} \left(-\frac{1}{2}(\sqrt{2}-1)^{\frac{4}{5}} \right) \right]} \|Ex\| \\
&= \frac{1}{\omega \left[1 - E_{\frac{4}{5}} \left(-\frac{1}{2}(\sqrt{2}-1)^{\frac{4}{5}} \right) \right]} \|x\|_{D(E)}.
\end{aligned}$$

Since $W^{-1}x$ is independent of $t \in J^*$, we have

$$\|W^{-1}\| \leq \frac{1}{\omega \left[1 - E_{\frac{4}{5}} \left(-\frac{1}{2}(\sqrt{2}-1)^{\frac{4}{5}} \right) \right]}.$$

By utilizing the integral representation formula (34) in [44], we define

$$E_\alpha(-z) := \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{s^{\alpha-1}}{1 + 2s^\alpha \cos(\alpha\pi) + s^{2\alpha}} e^{-z^{\frac{1}{\alpha}} s} ds$$

For $\alpha = \frac{4}{5}$ and $z = \frac{1}{2}(\sqrt{2}-1)^{\frac{4}{5}}$, a numerical computation in Matlab shows

$$E_{\frac{4}{5}} \left(-\frac{1}{2}(\sqrt{2}-1)^{\frac{4}{5}} \right) = \frac{\sin\left(\frac{4\pi}{5}\right)}{\pi} \int_0^\infty \frac{s^{-\frac{1}{5}}}{1 + 2s^{\frac{4}{5}} \cos\left(\frac{4\pi}{5}\right) + s^{\frac{8}{5}}} e^{-\frac{\sqrt{2}-1}{2^{\frac{5}{4}}} s} ds \doteq 0.7366.$$

Thus, $\|W^{-1}\| \leq \frac{3.7965}{\omega}$. From (28) and (29), it is easy to know that $\sum\limits_{i=1}^{m} k_i = \frac{1}{9}$, $L = 1$, and $\Lambda = \frac{1}{2}$. Then

$$\begin{aligned}
\rho &= M\sum_{i=1}^{m} k_i + \frac{M\|E^{-1}\|L\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^{\alpha} \\
&\quad + \frac{M^2\|W^{-1}\|\|B\|}{\Gamma(\alpha)} K_{\alpha,\psi}\sqrt{b}\left(\sum_{i=1}^{n} k_i + \frac{L\|E^{-1}\|\Lambda}{\Gamma(\alpha+1)}(\psi(b)-\psi(0))^{\alpha}\right) \\
&\leq \frac{1}{9} + \frac{\frac{1}{2}}{\Gamma(\frac{9}{5})}(\sqrt{2}-1)^{\frac{4}{5}} + \frac{\omega}{\Gamma(\frac{4}{5})}\cdot 0.4911 \cdot \frac{3.7965}{\omega}\left(\frac{1}{9} + \frac{\frac{1}{2}}{\Gamma(\frac{9}{5})}(\sqrt{2}-1)^{\frac{4}{5}}\right) \\
&= 0.9789 < 1,
\end{aligned}$$

that is, (24) holds. Thus, all the assumptions in Theorem 1 are satisfied. Therefore the system (27) is controllable on J.

5. Conclusions

In this study, we constructed a mild solution for a class of impulsive Caputo fractional evolution equations of Sobolev type based on generalized Laplace transform with respect to the ψ-function. By using the boundedness and compactness of two new introduced characteristic solution operators and the fixed point technique, we derive some new controllability results for ψ-fractional impulsive functional evolution equations of Sobolev type. The obtained results generalized the non-impulse and classical Caputo fractional derivative cases. Finally, an example is given to illustrate the effectiveness and feasibility of our criterion.

In the future, we will consider the nonlinear impulsive ψ-Hilfer fractional evolution equations of Sobolev type, and study the controllability of the mild solution for such equations.

Author Contributions: All the authors contributed equally and significantly in the writing of this article. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by Natural Science Foundation of China (11571136).

Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest: The authors declare that there are no conflicts of interest.

References

1. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives Theory and Applications*; Gordon and Breach Science Publishers: Philadelphia, PA, USA, 1993.
2. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
3. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science: Amsterdam, The Netherlands, 2006.
4. Diethelm, K. *The Analysis of Fractional Differential Equations*; Lecture Notes in Mathematics; Springer: New York, NY, USA, 2010.
5. Feckan, M.; Wang, J.; Pospsil, M. *Fractional-Order Equations and Inclusions*; de Gruyter: Berlin, Germany; Boston, MA, USA, 2017.
6. Bouchaud, J.P.; Georges, A. Comment on Stochastic pathway to anomalous diffusion. *Phys. Rev. A* **1990**, *41*, 1156–1157. [CrossRef] [PubMed]
7. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion:a fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77. [CrossRef]
8. Metzler, R. Weak ergodicity breaking, irreproducibility, and ageing in anomalous diffusion processes. *AIP Conf. Proc.* **2014**, *1579*, 89–101.
9. Tomovski, Z.; Sandev, T.; Metzler, R.; Dubbeldam, J. Generalized space-time fractional diffusion equation with composite fractional time derivative. *Phys. A Stat. Mech. Appl.* **2012**, *391*, 2527–2542. [CrossRef]
10. El-Borai, M.M. Some probability densities and fundamental solutions of fractional evolution equations. *Chaos Solitons Fractals* **2002**, *14*, 433–440. [CrossRef]

11. El-Borai, M.M. The fundamental solutions for fractional evolution equations of parabolic type. *J. Appl. Math. Stoch. Anal.* **2004**, *3*, 197–211. [CrossRef]
12. Balachandran, K.; Park, J.Y. Controllability of fractional integrodifferential systems in Banach spaces. *Nonlinear Anal. Hybrid Syst.* **2009**, *3*, 363–367. [CrossRef]
13. Wang, J.; Fan, Z.; Zhou, Y. Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces. *J. Optim. Theory Appl.* **2012**, *154*, 292–302. [CrossRef]
14. Wang, J., Zhou, Y. Complete controllability of fractional evolution systems. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 4346–4355. [CrossRef]
15. Wang, J.; Zhou, Y.; Wei, W. Fractional Schrodinger equations with potential and optimal controls. *Nonlinear Anal. Real World Appl.* **2012**, *3*, 2755–2766. [CrossRef]
16. Zhou, Y.; Jiao, F. Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.* **2010**, *59*, 1063–1077. [CrossRef]
17. Zhou, Y.; Jiao, F. Nonlocal Cauchy problem for fractional evolution equations. *Nonlinear Anal. Real World Appl.* **2010**, *11*, 4465–4475. [CrossRef]
18. Sakthivel, R.; Ren, Y.; Mahmudov, N.I. On the approximate controllability of semilinear fractional differential systems. *Comput. Math. Appl.* **2011**, *62*, 1451–1459. [CrossRef]
19. Debbouchea, A.; Baleanu, D. Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems. *Comput. Math. Appl.* **2011**, *62*, 1442–1450. [CrossRef]
20. Li, K.; Peng, J.; Jia, J. Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives. *J. Funct. Anal.* **2012**, *263*, 476–510. [CrossRef]
21. Kumar, S.; Sukavanam, N. Approximate controllability of fractional order semilinear systems with bounded delay. *J. Differ. Equs.* **2012**, *252*, 6163–6174. [CrossRef]
22. Lord, G.J.; Catherine, E.P.; Tony, S. *An Introduction to Computational Stochastic PDEs*; Cambridge University Press: Cambridge, UK, 2014; Volume 50.
23. Michal Feckan, M.; Wang, J.; Zhou, Y. Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators. *J. Optim. Theory Appl.* **2013**, *156*, 79–95. [CrossRef]
24. Liu, Y. Bifurcation techniques for a class of boundary value problems of fractional impulsive differential equations. *J. Nonlinear Sci. Appl.* **2015**, *8*, 340–353. [CrossRef]
25. Zhao, D.; Liu, Y. Eigenvalues of a class of singular boundary value problems of impulsive differential equations in Banach spaces. *J. Funct. Space* **2014**, *2014*, 720494. [CrossRef]
26. Ji, S.; Li, G.; Wang, M. Controllability of impulsive differential systems with nonlocal conditions. *Appl. Math. Comput.* **2011**, *217*, 6981–6989. [CrossRef]
27. Afrouzi, G.A.; Moradi, S.; Caristi, G. Infinitely many Solutions for impulsive nonlocal elastic beam equations. *Differ. Equ. Dyn. Syst.* **2022**, *30*, 287–300. [CrossRef]
28. Shu, X.B.; Shi, Y. A study on the mild solution of impulsive fractional evolution equations. *Appl. Math. Comput.* **2016**, *273*, 465–476. [CrossRef]
29. Zhao, D. A Study on controllability of a class of impulsive fractional nonlinear evolution equations with delay in Banach spaces. *Fractal Fract.* **2021**, *2021*, 279. [CrossRef]
30. Michal Feckan, M.; Zhou, Y.; Wang, J. On the concept and existence of solution for impulsive fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 3050–3060.
31. Kiryakova, V. *Generalized Fractional Calculus and Applications*; J. Wiley & Sons, Inc.: New York, NY, USA, 1994.
32. Kiryakova, V. A brief story about the operators of generalized fractional calculus. *Fract. Calc. Appl. Anal.* **2008**, *11*, 203–220.
33. Sousa, J.V.da C.; de Oliveira, E.C. On the ψ-Hilfer fractional derivative. *Commun. Nonlinear Sci.* **2018**, *60*, 72–91. [CrossRef]
34. Almeida, R.; Malinowska, A.B.; Monteiro, M.T.T. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Math. Methods Appl. Sci.* **2018**, *41*, 336–352. [CrossRef]
35. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci.* **2017**, *44*, 460–481. [CrossRef]
36. Awadalla, M.; Abuasbeh, K.; Subramanian, M.; Murugesan Manigandan, M. On a system of ψ-Caputo hybrid fractional differential equations with dirichlet boundary conditions. *Mathematics* **2022**, *10*, 1681. [CrossRef]
37. Youssri, Y.H. Orthonormal ultraspherical operational matrix algorithm for fractal-fractional Riccati equation with generalized Caputo derivative. *Fractal Fract.* **2021**, *5*, 100. [CrossRef]
38. Baitiche, Z.; Derbazi, C.; Benchohra, M.; Nieto, J.J. Monotone iterative technique for a new class of nonlinear sequential fractional differential equations with nonlinear boundary conditions under the ψ-Caputo operator. *Mathematics* **2022**, *10*, 1173. [CrossRef]
39. Suechori, A.; Ngiamsunthorn, P.S. Existence uniqueness and stability of mild solutions for semilinear ψ-Caputo fractional evolution equations. *Adv. Differ. Equ.* **2020**, *2020*, 114. [CrossRef]
40. Jarad, F.; Abdeljawad, T. Generalized fractional derivatives and Laplace transform. *Discrete Contin. Dyn. Syst. Ser. S.* **2019**, *13*, 709–722. [CrossRef]
41. Zhao, D.; Liu, Y.; Li, X. Controllability for a class of semilinear fractional evolution systems via resolvent operators. *Commun. Pur. Appl. Anal.* **2019**, *18*, 455–478. [CrossRef]

42. Lightbourne, J.H.; Rankin, S.M. A partial functional differential equation of Sobolev type. *J. Math. Anal. Appl.* **1983**, *93*, 328–337. [CrossRef]
43. Berberan-Santos, M.N. Relation between the inverse Laplace transforms of $I(t^{\beta})$ and $I(t)$: application to the Mittag-Leffler and asymptotic inverse power law relaxation functions. *J. Math. Chem.* **2005**, *38*, 265–270. [CrossRef]
44. Berberan-Santos, M.N. Properties of the Mittag-Leffler relaxation function. *J. Math. Chem.* **2005**, *38*, 629–635. [CrossRef]