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Some Generalized Euclidean Operator Radius Inequalities

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Abstract: In this work, some generalized Euclidean operator radius inequalities are established. Refinements of some well-known results are provided. Among others, some bounds in terms of the Cartesian decomposition of a given Hilbert space operator are proven.

Keywords: Euclidean operator radius; numerical radius; self-adjoint operator

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1. Introduction

In recent decades, the field of values has become increasingly important in numerical analysis, particularly in numerical linear algebra issues requiring matrices and iterative approaches for solving large systems of linear equations. One must deal with increasing-dimensional matrices in such cases. For example, matrices may result from the discretization of differential or integral operations, and their dimension approaches infinity as the discretization is refined; in other circumstances, the discretization is fixed but the computing domain grows without bounds. In numerical linear algebra, analyzing the behavior of techniques for approximating functions of such matrices as their size grows is critical. Indeed, the spectral theorem for normal matrices (or bounded operators) allows one to convert the approximation problem for matrices into a problem for functions of a real (or complex) variable and apply classical approximation theory results.

On the other hand, the quadratic forms and their applications are used in many branches of mathematics and physical sciences. Most researchers in this area of mathematics have studied many types of quadratic forms, such as the numerical range and its radius. In recent years, the concept of the generalized Euclidean operator radius has attracted the serious attention of many researchers. In fact, this type of radius generalizes the classical numerical radius but for multivariable Hilbert space operators and their extensions to infinite dimensions; which is indeed considered one of the most recent concepts in the field of values studied in literature.

This work provides some new theoretical developments in this direction. To highlight the significance of these developments, some mathematical background and current state of the art on the Euclidean operator radius and related inequalities must be presented. Below are the essentials.

Let $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{H};\langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{B}(\mathscr{H})$.

For a bounded linear operator S on a Hilbert space $\mathscr{H}$, the numerical range $W(S)$ is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $z \to \langle Sz, z\rangle$ associated with the operator. More precisely,

$$W(S) = \{\langle Sz, z\rangle : z \in \mathscr{H}, \|z\| = 1\}.$$

Moreover, the numerical radius is defined by

$$\omega(S) = \sup\{|\lambda| : \lambda \in W(S)\} = \sup_{\|z\|=1} |\langle Sz, z\rangle|.$$

We recall that the usual operator norm of an operator S is

$$\|S\| = \sup\{\|Sz\| : z \in \mathscr{H}, \|z\| = 1\}.$$

It is well known that $\omega(\cdot)$ defines an operator norm on $\mathscr{B}(\mathscr{H})$ which is equivalent to the operator norm $\|\cdot\|$. Moreover, we have

$$\frac{1}{2}\|S\| \leq \omega(S) \leq \|S\| \tag{1}$$

for any $S \in \mathscr{B}(\mathscr{H})$ and this inequality is sharp.

Denote $|S| = (S^*S)^{\frac{1}{2}}$ the absolute value of the operator S. Then, we have

$$\omega(|S|) = \|S\|.$$

It is well known that $\omega(\cdot)$ defines an operator norm on $\mathscr{B}(\mathscr{H})$ which is equivalent to the operator norm $\|\cdot\|$. Moreover, we have

$$\frac{1}{2}\|S\| \leq \omega(S) \leq \|S\| \tag{2}$$

for any $S \in \mathscr{B}(\mathscr{H})$ and this inequality is sharp.

In 2003, Kittaneh [1] refined the right-hand side of (2); he proved that

$$\omega(S) \leq \frac{1}{2}\left(\|S\| + \|S^2\|^{\frac{1}{2}}\right) \tag{3}$$

for any $S \in \mathscr{B}(\mathscr{H})$.

After that, in 2005, the same author in [2] proved that

$$\frac{1}{4}\|S^*S + SS^*\| \leq \omega^2(S) \leq \frac{1}{2}\|S^*S + SS^*\|. \tag{4}$$

The inequality is sharp. For recent further inequalities regarding (4) and other related results, the reader may refer to [3–12].

In 2009, Popsecu [13] introduced the concept of Euclidean operator radius of an n-tuple $\mathbf{S} = (S_1, \cdots, S_n) \in \mathscr{B}(\mathscr{H})^n := \mathscr{B}(\mathscr{H}) \times \cdots \times \mathscr{B}(\mathscr{H})$. Namely, for $S_1, \cdots, S_n \in \mathscr{B}(\mathscr{H})$, the Euclidean operator radius of $S_1, \cdots, S_n$ is defined by

$$\omega_{\mathrm{e}}(S_1, \cdots, S_n) := \sup_{\|z\|=1}\left(\sum_{i=1}^{n} |\langle S_i z, z\rangle|^2\right)^{\frac{1}{2}}.$$

The Euclidean operator radius was generalized in [9] as follows:

$$\omega_p(S_1, \cdots, S_n) := \sup_{\|z\|=1}\left(\sum_{i=1}^{n} |\langle S_i z, z\rangle|^p\right)^{\frac{1}{p}}, \qquad p \geq 1.$$

In [14] Moslehian, Sattari and Shebrawi proved several inequalities regarding n-tuples operators. Among others they proved the following two results

$$w_p(S_1,\cdots,S_n)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}\left(|S_i|^{2\alpha}+|S_i^*|^{2(1-\alpha)}\right)^p\right\|^{\frac{1}{p}} \tag{5}$$

and

$$w_p(S_1,\cdots,S_n)\leq\left\|\sum_{i=1}^{n}(\alpha|S_i|^p+(1-\alpha)|S_i^*|^p)\right\|^{\frac{1}{p}} \tag{6}$$

for $\alpha\in[0,1]$ and $p\geq 1$.

In [15], Sheikhhosseini, Moslehian and Shebrawi refined the above two inequalities by proving that

$$w_p(S_1,\cdots,S_n)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}\left(|S_i|^{2\alpha}+|S_i^*|^{2(1-\alpha)}\right)^p\right\|^{\frac{1}{p}}-\inf_{\|x\|=1}\xi(x), \tag{7}$$

where

$$\xi(x)=\frac{1}{2}\sum_{i=1}^{n}\left(\left\langle|S_i|^{2\alpha p}x,x\right\rangle^{\frac{1}{2}}-\left\langle|S_i^*|^{2(1-\alpha)p}x,x\right\rangle^{\frac{1}{2}}\right)^2$$

and

$$w_p^p(S_1,\cdots,S_n)\leq\left\|\sum_{i=1}^{n}\left(\alpha|S_i|^{\frac{p}{m}}+(1-\alpha)|S_i^*|^{\frac{p}{m}}\right)^m\right\|-\inf_{\|x\|=1}\xi(x), \tag{8}$$

where

$$\xi(x)=\min\{\alpha,1-\alpha\}\sum_{i=1}^{n}\left(\left\langle|S_i|^{\frac{p}{m}}x,x\right\rangle^{\frac{m}{2}}-\left\langle|S_i^*|^{\frac{p}{m}}x,x\right\rangle^{\frac{m}{2}}\right)^2.$$

For further properties of the Euclidean operator radius combined with several basic properties, the reader may refer to [13–16].

In this work, we prove several new inequalities for the generalized Euclidean operator radius. Among others, some bounds in terms of Cartesian decomposition of a given Hilbert space operator are proven. More precisely, Section 2 is devoted to inequalities for the generalized Euclidean operator radius which gives an equivalent version of the inequalities (5)–(8), and Section 3 is focused on diverse upper and lower bounds for quantities involving this radius; and this gives an extension of [6] (Theorem 5) and [15] (Theorem 4.1). The paper is concluded in Section 4.

2. Inequalities for the Generalized Euclidean Operator Radius

In order to prove our main results, we need the following sequence of lemmas.

Lemma 1 ([17])**.** *Let $C\in\mathscr{B}(\mathscr{H})$. If k and ℓ are nonnegative continuous functions on $[0,\infty)$ satisfying $k(t)\ell(t)=t$ $(t\geq 0)$, then we have*

$$|\langle Cz,y\rangle|\leq\|k(|C|)z\|\|\ell(|C^*|)y\| \tag{9}$$

for any vectors $z,y\in\mathscr{H}$.

Lemma 2 ([3])**.** *Let $C \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $C = G + iF$. If k and ℓ are nonnegative continuous functions on $[0,\infty)$ satisfying $k(t)\ell(t) = t$ $(t \geq 0)$, then we have*

$$|\langle Cz, y\rangle| \leq \{\|k(|G|)z\|\|\ell(|G|)y\| + \|k(|F|)z\|\|\ell(|F|)y\|\} \tag{10}$$

for all $z, y \in \mathscr{H}$.

Lemma 3. *Let $S \in \mathscr{B}(\mathscr{H})$, $S \geq 0$ and $z \in \mathscr{H}$ be a unit vector. Then, the operator Jensen inequalities are given by*

$$\langle Sz, z\rangle^r \leq \langle S^r z, z\rangle, \qquad r \geq 1 \tag{11}$$

and

$$\langle Sz, z\rangle^r \geq \langle S^r z, z\rangle, \qquad 0 \leq r \leq 1. \tag{12}$$

Lemma 4 ([7])**.** *Let $c, d \geq 0$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$cd + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}(c^{\frac{p}{2}} - d^{\frac{q}{2}})^2 \leq \frac{c^p}{p} + \frac{d^q}{q}. \tag{13}$$

Lemma 5 ([18])**.** *If $c, d > 0$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then, for $m = 1, 2, 3, \ldots,$*

$$\left(c^{\frac{1}{p}} d^{\frac{1}{q}}\right)^m + r_0^m \left(c^{\frac{m}{2}} - d^{\frac{m}{2}}\right)^2 \leq \left(\frac{c^r}{p} + \frac{d^r}{q}\right)^{\frac{m}{r}}, \ r \geq 1, \tag{14}$$

where $r_0 = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$. In particular, if $p = q = 2$, we obtain

$$\left(c^{\frac{1}{2}} d^{\frac{1}{2}}\right)^m + \frac{1}{2^m}\left(c^{\frac{m}{2}} - d^{\frac{m}{2}}\right)^2 \leq 2^{-\frac{m}{r}}(c^r + d^r)^{\frac{m}{r}}. \tag{15}$$

Lemma 6. *For $c, d > 0$, $0 \leq \alpha \leq 1$. Let*

$$M_r(c, d, \alpha) := \begin{cases} (\alpha c^r + (1-\alpha) d^r)^{\frac{1}{r}}, & r \geq 1 \\ c^\alpha d^{1-\alpha}, & r = 0 \end{cases}.$$

Then, for all $r \leq s$, we have

$$M_r(c, d, \alpha) \leq M_s(c, d, \alpha). \tag{16}$$

We are in a position to state our first main result which combines (5) and (6).

Theorem 1. *Let $C_j \in \mathscr{B}(\mathscr{H})$ $(1 \leq j \leq n)$. Then, we have*

$$\begin{aligned} &\omega_p^p(C_1, \cdots, C_n) \\ &\leq \frac{1}{2}\left\|\sum_{j=1}^n \left(\alpha|C_j|^{2r\beta} + (1-\alpha)\left|C_j^*\right|^{2r(1-\beta)}\right)^{\frac{p}{r}} + \left(\alpha|C_j|^{2r(1-\beta)} + (1-\alpha)\left|C_j^*\right|^{2r\beta}\right)^{\frac{p}{r}}\right\| \end{aligned} \tag{17}$$

for all $\alpha, \beta \in [0,1]$ and $p \geq r \geq 1$ such that $r\beta \geq 1$.

Proof. Let $y = z$ in (9), we get

$$
\begin{aligned}
&\sum_{j=1}^{n}|\langle C_j z,z\rangle|^p \\
&\leq \sum_{j=1}^{n}\left\langle |C_j|^{2\alpha}z,z\right\rangle^{\frac{p}{2}}\left\langle \left|C_j^*\right|^{2(1-\alpha)}z,z\right\rangle^{\frac{p}{2}} \\
&\leq \frac{1}{2^p}\sum_{j=1}^{n}\Bigg[\left\langle |C_j|^{2\alpha}z,z\right\rangle^{\beta}\left\langle \left|C_j^*\right|^{2(1-\alpha)}z,z\right\rangle^{1-\beta} && \left(\text{since } \sqrt{cd} \leq \frac{c^\gamma d^{1-\gamma}+c^{1-\gamma}d^\gamma}{2}\right) \\
&\qquad +\left\langle |C_j|^{2\alpha}z,z\right\rangle^{1-\beta}\left\langle \left|C_j^*\right|^{2(1-\alpha)}z,z\right\rangle^{\beta}\Bigg]^p \\
&\leq \frac{1}{2^p}\sum_{j=1}^{n}\Bigg[\left\langle |C_j|^{2}z,z\right\rangle^{\beta\alpha}\left\langle \left|C_j^*\right|^{2}z,z\right\rangle^{(1-\beta)(1-\alpha)} && (\text{by } (12)) \\
&\qquad +\left\langle |C_j|^{2}z,z\right\rangle^{\alpha(1-\beta)}\left\langle \left|C_j^*\right|^{2(1-\alpha)}z,z\right\rangle^{(1-\alpha)\beta}\Bigg]^p \\
&\leq \frac{1}{2^p}\sum_{j=1}^{n}\Bigg[\alpha\left\langle |C_j|^{2}z,z\right\rangle^{\beta}+(1-\alpha)\left\langle \left|C_j^*\right|^{2}z,z\right\rangle^{1-\beta} && (\text{by the AM–GM inequality}) \\
&\qquad +\alpha\left\langle |C_j|^{2}z,z\right\rangle^{1-\beta}+(1-\alpha)\left\langle \left|C_j^*\right|^{2}z,z\right\rangle^{\beta}\Bigg]^p \\
&\leq \frac{1}{2^p}\sum_{j=1}^{n}\Bigg[\left(\alpha\left\langle |C_j|^{2}z,z\right\rangle^{r\beta}+(1-\alpha)\left\langle \left|C_j^*\right|^{2}z,z\right\rangle^{r(1-\beta)}\right)^{\frac{1}{r}} && (\text{by } (16)) \\
&\qquad +\left(\alpha\left\langle |C_j|^{2}z,z\right\rangle^{r(1-\beta)}+(1-\alpha)\left\langle \left|C_j^*\right|^{2}z,z\right\rangle^{r\beta}\right)^{\frac{1}{r}}\Bigg]^p \\
&\leq \frac{1}{2^p}\sum_{j=1}^{n}\Bigg[\left(\alpha\left\langle |C_j|^{2r\beta}z,z\right\rangle+(1-\alpha)\left\langle \left|C_j^*\right|^{2r(1-\beta)}z,z\right\rangle\right)^{\frac{1}{r}} && (\text{by } (11)) \\
&\qquad +\left(\alpha\left\langle |C_j|^{2r(1-\beta)}z,z\right\rangle+(1-\alpha)\left\langle \left|C_j^*\right|^{2r\beta}z,z\right\rangle\right)^{\frac{1}{r}}\Bigg]^p \\
&= \frac{1}{2^p}\sum_{j=1}^{n}\Bigg[\left(\left\langle \alpha|C_j|^{2r\beta}+(1-\alpha)\left|C_j^*\right|^{2r(1-\beta)}z,z\right\rangle\right)^{\frac{1}{r}} \\
&\qquad +\left(\left\langle \alpha|C_j|^{2r(1-\beta)}+(1-\alpha)\left|C_j^*\right|^{2r\beta}z,z\right\rangle\right)^{\frac{1}{r}}\Bigg]^p \\
&\leq \frac{1}{2}\sum_{j=1}^{n}\Bigg[\left(\left\langle \alpha|C_j|^{2r\beta}+(1-\alpha)\left|C_j^*\right|^{2r(1-\beta)}z,z\right\rangle\right)^{\frac{p}{r}} && \left(\text{since } \left(\frac{c+d}{2}\right)^p \leq \frac{c^p+d^p}{2}\right) \\
&\qquad +\left(\left\langle \alpha|C_j|^{2r(1-\beta)}+(1-\alpha)\left|C_j^*\right|^{2r\beta}z,z\right\rangle\right)^{\frac{p}{r}}\Bigg]
\end{aligned}
$$

$$\leq \frac{1}{2}\sum_{j=1}^{n}\left[\left\langle\left(\alpha\left|C_j\right|^{2r\beta}+(1-\alpha)\left|C_j^*\right|^{2r(1-\beta)}\right)^{\frac{p}{r}}z,z\right\rangle\right. \qquad \text{(by (11))}$$

$$\left.+\left\langle\left(\alpha\left|C_j\right|^{2r(1-\beta)}+(1-\alpha)\left|C_j^*\right|^{2r\beta}\right)^{\frac{p}{r}}z,z\right\rangle\right]$$

$$=\frac{1}{2}\left[\left\langle\sum_{j=1}^{n}\left(\alpha\left|C_j\right|^{2r\beta}+(1-\alpha)\left|C_j^*\right|^{2r(1-\beta)}\right)^{\frac{p}{r}}z,z\right\rangle\right.$$

$$\left.+\left\langle\sum_{j=1}^{n}\left(\alpha\left|C_j\right|^{2r(1-\beta)}+(1-\alpha)\left|C_j^*\right|^{2r\beta}\right)^{\frac{p}{r}}z,z\right\rangle\right].$$

Taking the supremum over all unit vectors $z \in \mathscr{H}$, we obtain the desired result. □

Corollary 1. *Let $C_j \in \mathscr{B}(\mathscr{H})$ $(1 \leq j \leq n)$. Then, we have*

$$\omega_p^p(C_1,\cdots,C_n) \leq \left\|\sum_{j=1}^{n}\left(\alpha\left|C_j\right|^r+(1-\alpha)\left|C_j^*\right|^r\right)^{\frac{p}{r}}\right\| \tag{18}$$

for all $\alpha,\beta \in [0,1]$ and $p \geq r \geq 2$.

In particular, we have

$$\omega_p^p(C_1,\cdots,C_n) \leq \frac{1}{2^{\frac{p}{r}}}\left\|\sum_{j=1}^{n}\left(\left|C_j\right|^r+\left|C_j^*\right|^r\right)^{\frac{p}{r}}\right\|. \tag{19}$$

Proof. The proof follows by setting $\beta = \frac{1}{2}$ in (17). □

Remark 1. *Setting $r = 2$, then $|C_j|^2 + |C_j^*|^2 = C_j^*C_j + C_jC_j^*$, so that the inequality (19) becomes*

$$\omega_p(C_1,\cdots,C_n) \leq \frac{1}{\sqrt{2}}\left\|\sum_{j=1}^{n}\left(C_j^*C_j+C_jC_j^*\right)^{\frac{p}{2}}\right\|^{\frac{1}{p}}$$

for all $p \geq 2$. In particular, in case we choose $p = 2$, we obtain

$$\omega_{\mathrm{e}}(C_1,\cdots,C_n) \leq \frac{1}{\sqrt{2}}\left\|\sum_{j=1}^{n}\left(C_j^*C_j+C_jC_j^*\right)\right\|^{\frac{1}{2}}, \tag{20}$$

which is the multivariable version of the right-hand side of Kittaneh inequality (4).

Example 1. *Let $C_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ be 2×2-matrices. Employing (20) with $n = 2$, $\alpha = \frac{1}{2}$ and $p = 2$, we obtain*

$$\omega_{\mathrm{e}}^2(C_1,C_2) = \sup_{\|z\|=1}\left(|\langle C_1z,z\rangle|^2+|\langle C_2z,z\rangle|^2\right) = 4,$$

i.e., $\omega_{\mathrm{e}}(C_1,C_2) = 2$. However,

$$2 = \omega_{\mathrm{e}}(C_1,C_2) \leq \frac{1}{\sqrt{2}}\left\|\sum_{j=1}^{n}\left(C_j^*C_j+C_jC_j^*\right)\right\|^{\frac{1}{2}} = 2.1213,$$

which verifies (20).

Our next goal is to generalize the inequality (4).

Theorem 2. *Let $C_j \in \mathscr{B}(\mathscr{H})$ $(j = 1, \cdots, n)$. Assume $C_j = G_j + iF_j$ be the Cartesian decomposition of C_j for all $j = 1, \cdots, n$. Then, we have*

$$\begin{aligned} \frac{1}{2^p n^{p-1}} \left\| \sum_{j=1}^{n} (G_j + F_j)^2 \right\|^p &\leq \frac{1}{2^p} \left\| \sum_{j=1}^{n} (G_j + F_j)^{2p} \right\| \\ &\leq \omega_{2p}^{2p}(C_1, \cdots, C_n) \\ &\leq 2^{p-1} \left\| \sum_{j=1}^{n} \left(|G_j|^{2p} + |F_j|^{2p} \right) \right\| \end{aligned} \tag{21}$$

for all $p \geq 1$.

Proof. We start by proving the left-side inequality. We have

$$\begin{aligned} \sum_{j=1}^{n} |\langle C_j z, z\rangle|^{2p} &= \sum_{j=1}^{n} \left(|\langle G_j z, z\rangle|^2 + |\langle F_j z, z\rangle|^2 \right)^p \\ &\geq \sum_{j=1}^{n} \left(\frac{1}{2} (|\langle G_j z, z\rangle| + |\langle F_j z, z\rangle|)^2 \right)^p \\ &\geq \frac{1}{2^p} \sum_{j=1}^{n} (|\langle G_j z, z\rangle + \langle F_j z, z\rangle|)^{2p} \\ &= \frac{1}{2^p} \sum_{j=1}^{n} |\langle (G_j + F_j) z, z\rangle|^{2p} \\ &\geq \frac{1}{2^p n^{p-1}} \left(\sum_{j=1}^{n} |\langle (G_j + F_j) z, z\rangle|^2 \right)^p. \text{ (Jensen's inequality)} \end{aligned}$$

Taking the supremum over all unit vectors $z \in \mathscr{H}$, we obtain the left hand side of (21). To prove the right-hand side of (21), we have

$$\begin{aligned} \left(\sum_{j=1}^{n} \left(\frac{|\langle C_j z, z\rangle|^2}{2} \right)^p \right)^{\frac{1}{p}} &= \left(\sum_{j=1}^{n} \left(\frac{|\langle G_j z, z\rangle|^2 + |\langle F_j z, z\rangle|^2}{2} \right)^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{j=1}^{n} \left(\frac{|\langle G_j z, z\rangle|^{2p} + |\langle F_j z, z\rangle|^{2p}}{2} \right) \right)^{\frac{1}{p}} \\ &\leq 2^{-\frac{1}{p}} \left(\sum_{j=1}^{n} \left(\langle |G_j| z, z\rangle^{2p} + \langle |F_j| z, z\rangle^{2p} \right) \right)^{\frac{1}{p}} \\ &\leq 2^{-\frac{1}{p}} \left(\sum_{j=1}^{n} \left(\left\langle |G_j|^{2p} z, z \right\rangle + \left\langle |F_j|^{2p} z, z \right\rangle \right) \right)^{\frac{1}{p}} \\ &= 2^{-\frac{1}{p}} \left(\sum_{j=1}^{n} \left(\left\langle \left(|G_j|^{2p} + |F_j|^{2p} \right) z, z \right\rangle \right) \right)^{\frac{1}{p}} \\ &= 2^{-\frac{1}{p}} \left\langle \sum_{j=1}^{n} \left(|G_j|^{2p} + |F_j|^{2p} \right) z, z \right\rangle^{\frac{1}{p}}. \end{aligned}$$

Taking the supremum over all unit vectors $z \in \mathscr{H}$ we obtain the right-hand side of (21), and thus the proof of Theorem 2 is completely finished. □

Example 2. *Let* $C_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ *and* $C_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ *be* 2×2*-matrices. Then it is easy to observe that*

$$C_1 = G_1 + iF_1 = \begin{bmatrix} 0 & \frac{3}{2} \\ \frac{3}{2} & 0 \end{bmatrix} + i\begin{bmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix},$$

and

$$C_2 = G_2 + iF_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Employing (21) *with* $n = 2$ *and* $p = 1$*, we obtain*

$$\begin{aligned} \frac{1}{2}\left\|\sum_{j=1}^{2}(G_j + F_j)^2\right\| &= \frac{1}{2}\left\|(G_1 + F_1)^2 + (G_2 + F_2)^2\right\| \\ &= \frac{1}{2}\left\|\begin{bmatrix} \frac{5}{2} & 0 \\ 0 & \frac{5}{2} \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right\| \\ &= 2.25, \end{aligned}$$

and

$$\omega_e^2(C_1, C_2) = \sup_{\|z\|=1}\left(|\langle C_1 z, z\rangle|^2 + |\langle C_2 z, z\rangle|^2\right) = 4$$

while

$$\begin{aligned} \left\|\sum_{j=1}^{2}\left(|G_j|^2 + |F_j|^2\right)\right\| &= \left\|\left(|G_1|^2 + |F_1|^2\right) + \left(|G_2|^2 + |F_2|^2\right)\right\| \\ &= \left\|\left(\begin{bmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\right) + \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right\| \\ &= \left\|\begin{bmatrix} \frac{18}{4} & 0 \\ 0 & \frac{18}{4} \end{bmatrix}\right\| \\ &= 4.5, \end{aligned}$$

which verifies that

$$2.25 := \frac{1}{2}\left\|\sum_{j=1}^{2}(G_j + F_j)^2\right\| \le \omega_e^2(C_1, C_2) = 4 \le \left\|\sum_{j=1}^{2}\left(|G_j|^2 + |F_j|^2\right)\right\| := 4.5.$$

Corollary 2. *Let* $C \in \mathscr{B}(\mathscr{H})$*. Assume* $C = G + iF$ *be the Cartesian decomposition of* C*. Then, we have*

$$\frac{1}{2^p}\|G + F\|^{2p} \le \omega^{2p}(C) \le 2^{p-1}\left\||G|^{2p} + |F|^{2p}\right\|$$

for all $p \ge 1$*. In particular, we have*

$$\frac{1}{2}\|G+F\|^2 \le \omega^2(C) \le \left\| |G|^2 + |F|^2 \right\|. \tag{22}$$

Proof. Choosing $n = 1$ in (21) and set $C_1 = C$, $G_1 = G$ and $F_1 = F$, this yields that $\omega_{2p}^{2p}(C_1, \cdots, C_n) = \omega^2(C)$. The particular case holds with $n = 1$ and $p = 1$. □

Example 3. *As in Example 2, let $C = G + iF$. Then, by employing (22) we obtain*

$$1.25 = \frac{1}{2}\|G+F\|^2 \le \omega^2(C) = 2.25 \le \left\| |G|^2 + |F|^2 \right\| = 2.5.$$

Our next result can be stated as follows:

Theorem 3. *Let $C_j \in \mathscr{B}(\mathscr{H})$ $(j = 1, \cdots, n)$. Assume $C_j = G_j + iF_j$ be the Cartesian decomposition of C_j for all $j = 1, \cdots, n$. Then, we have*

$$\omega_p^{rp}(C_1, \cdots, C_n) \le \frac{1}{2}\left\| \sum_{j=1}^n \left\{ \left[k^2(|G_j|) + k^2(|F_j|)\right]^{pr} + \left[\ell^2(|G_j|) + \ell^2(|F_j|)\right]^{pr} \right\} \right\| \tag{23}$$

for all $r \ge 1$ and $p \ge 2$.

Proof. Setting $y = z$ in (10). Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\begin{aligned}
|\langle Cz, z\rangle| &\le \{\|k(|G|)z\|\|\ell(|G|)z\| + \|k(|F|)z\|\|\ell(|F|)z\|\} \\
&\le \left(\|k(|G|)z\|^p + \|k(|F|)z\|^p\right)^{\frac{1}{p}} \\
&\qquad \times \left(\|\ell(|G|)z\|^q + \|\ell(|F|)z\|^q\right)^{\frac{1}{q}} \qquad \text{(by the Hölder inequality)} \\
&\le \left(\langle k^2(|G|)z, z\rangle^{\frac{p}{2}} + \langle k^2(|F|)z, z\rangle^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\qquad \times \left(\langle \ell^2(|G|)z, z\rangle^{q/2} + \langle \ell^2(|F|)z, z\rangle^{q/2}\right)^{\frac{1}{q}} \\
&\le \left(\langle k^p(|G|)z, z\rangle + \langle k^p(|F|)z, z\rangle\right)^{\frac{1}{p}} \\
&\qquad \times \left(\langle \ell^q(|G|)z, z\rangle + \langle \ell^q(|F|)z, z\rangle\right)^{\frac{1}{q}} \qquad \text{(by (11))} \\
&\le \langle [k^p(|G|) + k^p(|F|)]z, z\rangle^{\frac{1}{p}} \langle [\ell^q(|G|) + \ell^q(|F|)]z, z\rangle^{\frac{1}{q}}
\end{aligned} \tag{24}$$

In particular, for $p = q = 2$, we have

$$|\langle Cz, z\rangle| \le \left\langle \left[k^2(|G|) + k^2(|F|)\right]z, z \right\rangle^{\frac{1}{2}} \left\langle \left[\ell^2(|G|) + \ell^2(|F|)\right]z, z \right\rangle^{\frac{1}{2}}. \tag{25}$$

Applying (25) for $p \ge 2$, we obtain

$$\begin{aligned}
&\sum_{j=1}^n |\langle C_j z, z\rangle|^p \\
&\le \sum_{j=1}^n \left\langle \left[k^2(|G_j|) + k^2(|F_j|)\right]z, z \right\rangle^{\frac{p}{2}} \left\langle \left[\ell^2(|G_j|) + \ell^2(|F_j|)\right]z, z \right\rangle^{\frac{p}{2}} \\
&\le \sum_{j=1}^n \left\langle \left[k^2(|G_j|) + k^2(|F_j|)\right]^p z, z \right\rangle^{\frac{1}{2}} \left\langle \left[\ell^2(|G_j|) + \ell^2(|F_j|)\right]^p z, z \right\rangle^{\frac{1}{2}} \qquad \text{(by (11))}
\end{aligned}$$

$$\leq \frac{1}{2^{\frac{1}{r}}} \sum_{j=1}^{n} \left[\left\langle \left[k^2(|G_j|) + k^2(|F_j|) \right]^p z, z \right\rangle^r + \left\langle \left[\ell^2(|G_j|) + \ell^2(|F_j|) \right]^p z, z \right\rangle^r \right]^{\frac{1}{r}} \qquad \text{(by Lemma 6)}$$

$$\leq \frac{1}{2^{\frac{1}{r}}} \sum_{j=1}^{n} \left[\left\langle \left[k^2(|G_j|) + k^2(|F_j|) \right]^{pr} z, z \right\rangle + \left\langle \left[\ell^2(|G_j|) + \ell^2(|F_j|) \right]^{pr} z, z \right\rangle \right]^{\frac{1}{r}} \qquad \text{(by (11))}$$

$$= \left[\frac{1}{2} \left\langle \sum_{j=1}^{n} \left\{ \left[k^2(|G_j|) + k^2(|F_j|) \right]^{pr} + \left[\ell^2(|G_j|) + \ell^2(|F_j|) \right]^{pr} \right\} z, z \right\rangle \right]^{\frac{1}{r}}.$$

Taking the supremum over all unit vectors $z \in \mathscr{H}$, we obtain the desired result. □

Corollary 3. *Let $C_j \in \mathscr{B}(\mathscr{H})$ $(j = 1, \cdots, n)$. Assume $C_j = G_j + iF_j$ be the Cartesian decomposition of C_j for all $j = 1, \cdots, n$. Then, we have*

$$\omega_p^{rp}(C_1, \cdots, C_n) \leq \frac{1}{2} \left\| \sum_{j=1}^{n} \left\{ \left[|G_j|^{2\alpha} + |F_j|^{2\alpha} \right]^{pr} + \left[|G_j|^{2(1-\alpha)} + |F_j|^{2(1-\alpha)} \right]^{pr} \right\} \right\| \qquad (26)$$

for all $r \geq 1$, $p \geq 2$ and $\alpha \in [0, 1]$.

Proof. The desired result follows by setting $k(t) = t^{\alpha}$ and $\ell(t) = t^{1-\alpha}$ $(0 \leq \alpha \leq 1)$ in Theorem 3. □

Corollary 4. *Let $C_j \in \mathscr{B}(\mathscr{H})$ $(j = 1, \cdots, n)$. Assume $C_j = G_j + iF_j$ be the Cartesian decomposition of C_j for all $j = 1, \cdots, n$. Then, we have*

$$\omega_p^p(C_1, \cdots, C_n) \leq \left\| \sum_{j=1}^{n} \left[|G_j| + |F_j| \right]^p \right\| \qquad (27)$$

for all $p \geq 1$.

Proof. Setting $r = 1$ an $\alpha = \frac{1}{2}$ in (26), we obtain the desired result. □

Example 4. *Consider $C_1 = G_1 + iF_1$ and $C_2 = G_2 + iF_2$ as given in Example 2. Then, by employing (27) with $p = 2$, we obtain*

$$4 = \omega_e^2(C_1, C_2) \leq \left\| (|G_1| + |F_1|)^2 + (|G_2| + |F_2|)^2 \right\| = 8,$$

or it is more appropriate to write

$$2 = \omega_e(C_1, C_2) \leq \sqrt{\left\| (|G_1| + |F_1|)^2 + (|G_2| + |F_2|)^2 \right\|} = 2.8284.$$

3. Upper and Lower Bounds for the Generalized Euclidean Operator Radius

In this section, we provide some upper and lower bounds for quantities involving the generalized Euclidean operator radius. Let us start, with the following result.

Theorem 4. *Let $C_j \in \mathscr{B}(\mathscr{H})$ $(j = 1, \cdots, n)$. Assume $C_j = G_j + iF_j$ be the Cartesian decomposition of C_j for all $j = 1, \cdots, n$. If k and ℓ are nonnegative continuous functions on $[0, \infty)$ satisfying $k(t)\ell(t) = t$ $(t \geq 0)$, then*

$$\begin{aligned}
&\frac{1}{n^{2r-1}}\left\|\sum_{i=1}^{n} C_i\right\|^{2r} \\
&\leq \omega_p\Big(\Big[k^2(|G_1|) + k^2(|F_1|)\Big]^r, \cdots, \Big[k^2(|G_n|) + k^2(|F_n|)\Big]^r\Big) \\
&\qquad \times \omega_q\Big(\Big[\ell^2(|G_1|) + \ell^2(|F_1|)\Big]^r, \cdots, \Big[\ell^2(|G_n|) + \ell^2(|F_n|)\Big]^r\Big) \\
&\leq \frac{1}{p}\left\|\sum_{i=1}^{n}\Big[k^2(|G_i|) + k^2(|F_i|)\Big]^{rp}\right\| + \frac{1}{q}\left\|\sum_{i=1}^{n}\Big[\ell^2(|G_i|) + \ell^2(|F_i|)\Big]^{rq}\right\| - \inf_{\|z\|=\|y\|=1}\Phi(z,y),
\end{aligned} \tag{28}$$

for all $r \geq 1$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where

$$\Phi(z,y) := \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\sqrt{\sum_{i=1}^{n}\langle[k^2(|G_i|) + k^2(|F_i|)]z, z\rangle} - \sqrt{\sum_{i=1}^{n}\langle[\ell^2(|G_i|) + \ell^2(|F_i|)]y, y\rangle}\right)^2.$$

Proof. Let $z, y \in \mathscr{H}$. Applying inequality (10) and the convexity of t^{2r}, we have

$$\begin{aligned}
&\frac{1}{n^{2r-1}}\left|\sum_{i=1}^{n}\langle C_i z, y\rangle\right|^{2r} \\
&\leq \sum_{i=1}^{n}|\langle C_i z, y\rangle|^{2r} \\
&\leq \sum_{i=1}^{n}\Big\langle\Big[k^2(|G_i|) + k^2(|F_i|)\Big]z, z\Big\rangle^r\Big\langle\Big[\ell^2(|G_i|) + \ell^2(|F_i|)\Big]y, y\Big\rangle^r \\
&\leq \sum_{i=1}^{n}\Big\langle\Big[k^2(|G_i|) + k^2(|F_i|)\Big]^r z, z\Big\rangle\Big\langle\Big[\ell^2(|G_i|) + \ell^2(|F_i|)\Big]^r y, y\Big\rangle \qquad \text{(by (11))} \\
&\leq \left(\sum_{i=1}^{n}\Big\langle\Big[k^2(|G_i|) + k^2(|F_i|)\Big]^r z, z\Big\rangle^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\Big\langle\Big[\ell^2(|G_i|) + \ell^2(|F_i|)\Big]^r y, y\Big\rangle^q\right)^{\frac{1}{q}} \ \text{(by the Hölder inequality)} \\
&\leq \frac{1}{p}\left(\sum_{i=1}^{n}\Big\langle\Big[k^2(|G_i|) + k^2(|F_i|)\Big]^r z, z\Big\rangle^p\right) + \frac{1}{q}\left(\sum_{i=1}^{n}\Big\langle\Big[\ell^2(|G_i|) + \ell^2(|F_i|)\Big]^r y, y\Big\rangle^q\right) \qquad \text{(by (14))} \\
&\quad - \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\sqrt{\sum_{i=1}^{n}\langle[k^2(|G_i|) + k^2(|F_i|)]z, z\rangle} - \sqrt{\sum_{i=1}^{n}\langle[\ell^2(|G_i|) + \ell^2(|F_i|)]y, y\rangle}\right)^2 \\
&\leq \frac{1}{p}\left(\sum_{i=1}^{n}\Big\langle\Big[k^2(|G_i|) + k^2(|F_i|)\Big]^{rp} z, z\Big\rangle\right) + \frac{1}{q}\left(\sum_{i=1}^{n}\Big\langle\Big[\ell^2(|G_i|) + \ell^2(|F_i|)\Big]^{rq} y, y\Big\rangle\right) \qquad \text{(by (11))} \\
&\quad - \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\sqrt{\sum_{i=1}^{n}\langle[k^2(|G_i|) + k^2(|F_i|)]z, z\rangle} - \sqrt{\sum_{i=1}^{n}\langle[\ell^2(|G_i|) + \ell^2(|F_i|)]y, y\rangle}\right)^2
\end{aligned}$$

Taking the supremum over all unit vectors $z, y \in \mathscr{H}$, we obtain the desired result. which proves the required result. □

Corollary 5. *Let $C_j \in \mathscr{B}(\mathscr{H})$ $(j = 1, \cdots, n)$. Assume $C_j = G_j + iF_j$ be the Cartesian decomposition of C_j for all $j = 1, \cdots, n$. Then, we have*

$$\begin{aligned}&\frac{1}{n^{2r-1}}\left\|\sum_{i=1}^{n}C_i\right\|^{2r}\\&\leq \omega_p\left(\left[|G_1|^{2\alpha}+|F_1|^{2\alpha}\right]^r,\cdots,\left[|G_n|^{2\alpha}+|F_n|^{2\alpha}\right]^r\right)\\&\qquad\times\omega_q\left(\left[|G_1|^{2(1-\alpha)}+|F_1|^{2(1-\alpha)}\right]^r,\cdots,\left[|G_n|^{2(1-\alpha)}+|F_n|^{2(1-\alpha)}\right]^r\right)\\&\leq \max\left\{\frac{1}{p},\frac{1}{q}\right\}\left[\left\|\sum_{i=1}^{n}\left[|G_i|^{2\alpha}+|F_i|^{2\alpha}\right]^{rp}\right\|+\left\|\sum_{i=1}^{n}\left[|G_i|^{2(1-\alpha)}+|F_i|^{2(1-\alpha)}\right]^{rq}\right\|\right]-\inf_{\|z\|=\|y\|=1}\Psi_{p,q,\alpha}(z,y),\end{aligned}\tag{29}$$

for all $r \geq 1$, $p,q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, where

$$\Psi_{p,q,\alpha}(z,y) := \min\left\{\frac{1}{p},\frac{1}{q}\right\}\left(\sqrt{\sum_{i=1}^{n}\left\langle\left[|G_i|^{2\alpha}+|F_i|^{2\alpha}\right]z,z\right\rangle}-\sqrt{\sum_{i=1}^{n}\left\langle\left[|G_i|^{2(1-\alpha)}+|F_i|^{2(1-\alpha)}\right]y,y\right\rangle}\right)^2.$$

Proof. Setting $k(t)=t^{\alpha}$ and $\ell(t)=t^{1-\alpha}$ $(0\leq\alpha\leq 1)$ in (28) yields the desired result. □

Corollary 6. *Let $C_j \in \mathscr{B}(\mathscr{H})$ $(j=1,\cdots,n)$. Assume $C_j = G_j + iF_j$ be the Cartesian decomposition of C_j for all $j=1,\cdots,n$. Then, we have*

$$\begin{aligned}&\frac{1}{n^{2r-1}}\left\|\sum_{i=1}^{n}C_i\right\|^{2r}\\&\leq \omega_p\left([|G_1|+|F_1|]^r,\cdots,[|G_n|+|F_n|]^r\right)\\&\qquad\times\omega_q\left([|G_1|+|F_1|]^r,\cdots,[|G_n|+|F_n|]^r\right)\\&\leq \max\left\{\frac{1}{p},\frac{1}{q}\right\}\left[\left\|\sum_{i=1}^{n}[|G_i|+|F_i|]^{rp}\right\|+\left\|\sum_{i=1}^{n}[|G_i|+|F_i|]^{rq}\right\|\right]-\inf_{\|z\|=\|y\|=1}\Psi_{p,q,\frac{1}{2}}(z,y),\end{aligned}\tag{30}$$

for all $r\geq 1$, $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, where

$$\Psi_{p,q,\frac{1}{2}}(z,y) := \min\left\{\frac{1}{p},\frac{1}{q}\right\}\left(\sqrt{\sum_{i=1}^{n}\langle[|G_i|+|F_i|]z,z\rangle}-\sqrt{\sum_{i=1}^{n}\langle[|G_i|+|F_i|]y,y\rangle}\right)^2.$$

Proof. Setting $\alpha=\frac{1}{2}$ in (29) yields the stated result. □

Remark 2. *Setting $r=1$ and $p=q=2$ in Corollary 6, we obtain*

$$\begin{aligned}\frac{1}{n}\left\|\sum_{i=1}^{n}C_i\right\|^2 &\leq \omega_2^2([|G_1|+|F_1|],\cdots,[|G_n|+|F_n|])\\&\leq \left\|\sum_{i=1}^{n}[|G_i|+|F_i|]^2\right\|-\inf_{\|z\|=\|y\|=1}\Psi_{1,2,2,\frac{1}{2}}(z,y),\end{aligned}\tag{31}$$

where

$$\Psi_{1,2,2,\frac{1}{2}}(z,y) := \frac{1}{2}\left(\sqrt{\sum_{i=1}^{n}\langle[|G_i|+|F_i|]z,z\rangle}-\sqrt{\sum_{i=1}^{n}\langle[|G_i|+|F_i|]y,y\rangle}\right)^2.$$

Example 5. *Consider $C_1 = G_1 + iF_1$ and $C_2 = G_2 + iF_2$ as given in Example 2. Therefore, by employing (31) with $r=1$ and $p=q=2$, then we have*

$$\frac{1}{2}\|C_1+C_2\|^2 = 4.5,$$

$$\omega_2^2([|G_1|+|F_1|],[|G_2|+|F_2|]) = \sup_{\|z\|=1}\left(|\langle[|G_1|+|F_1|]z,z\rangle|^2+|\langle[|G_2|+|F_2|]z,z\rangle|^2\right) = 8,$$

and

$$\left\|(|G_1|+|F_1|)^2+(|G_2|+|F_2|)^2\right\| = 8,$$

with

$$\inf_{\|z\|=\|y\|=1}\Psi_{1,2,2,\frac{1}{2}}(z,y) = 0.$$

This gives that

$$\begin{aligned}4.5 = \frac{1}{2}\|C_1+C_2\|^2 &\le \omega_2^2([|G_1|+|F_1|],[|G_2|+|F_2|]) = 8\\ &\le \left\|(|G_1|+|F_1|)^2+(|G_2|+|F_2|)^2\right\| - \inf_{\|z\|=\|y\|=1}\Psi_{1,2,2,\frac{1}{2}}(z,y) = 8,\end{aligned}$$

or we may write

$$\begin{aligned}2.1213 = \frac{1}{\sqrt{2}}\|C_1+C_2\| &\le \omega_e(C_1,C_2) = 2.8284\\ &\le \sqrt{\left\|(|G_1|+|F_1|)^2+(|G_2|+|F_2|)^2\right\| - \inf_{\|z\|=\|y\|=1}\Psi_{1,2,2,\frac{1}{2}}(z,y)} = 2.8284.\end{aligned}$$

In 2007, El-Hadad and Kittaneh in [6] proved the corresponding version of the Kittaned inequality (4) in terms of the Cartesian decomposition. Indeed, they proved

$$2^{-\frac{r}{2}-1}\left\||G+F|^r+|G-F|^r\right\| \le \omega^r(C) \le \frac{1}{2}\left\||G+F|^r+|G-F|^r\right\| \tag{32}$$

for all $r \ge 2$, where F, G are the Cartesian decomposition of C.

In the next result, we generalize (32) in terms of the generalized Euclidean operator radius.

Theorem 5. *Let $S_j = G_j + iF_j \in \mathbb{B}(\mathscr{H})$ be the Cartesian decomposition of S_j $(1 \le j \le n)$. Then*

$$\frac{2^{-\frac{p}{2}}}{n^{p-1}}\left\|\sum_{j=1}^{n}|G_j+F_j|\right\|^p \le \omega_p^p(S_1,\ldots,S_n) \le \frac{1}{2}\sum_{j=1}^{n}\left\||G_j+F_j|^p+|G_j-F_j|^p\right\| \tag{33}$$

for all $p \ge 2$.

Proof. Let z be a unit vector in $\mathscr{H}$. Then, the right-hand side inequality could be obtained as follows:

$$\begin{aligned}&\omega_p^p(S_1,\ldots,S_n)\\ &= \sup_{\|z\|=1}\sum_{j=1}^{n}|\langle S_jz,z\rangle|^p\\ &= \sup_{\|z\|=1}\sum_{j=1}^{n}\left(\langle G_jz,z\rangle^2+\langle F_jz,z\rangle^2\right)^{\frac{p}{2}}\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{n} \sup_{\|z\|=1} \left(\langle G_j z, z\rangle^2 + \langle F_j z, z\rangle^2 \right)^{\frac{p}{2}} && \text{(by properties of sup)} \\
&= 2^{-p/2} \sum_{j=1}^{n} \sup_{\|z\|=1} \left(|\langle (G_j + F_j) z, z\rangle|^2 + |\langle (G_j - F_j) z, z\rangle|^2 \right)^{\frac{p}{2}} \\
&\leq 2^{-p/2+p/2-1} \sum_{j=1}^{n} \sup_{\|z\|=1} \left(|\langle (G_j + F_j) z, z\rangle|^p + |\langle (G_j - F_j) z, z\rangle|^p \right) && \left(\text{by convexity of } t^{\frac{p}{2}}\right) \\
&\leq \frac{1}{2} \sum_{j=1}^{n} \sup_{\|z\|=1} \left(\langle |G_j + F_j| z, z\rangle^p + \langle |G_j - F_j| z, z\rangle^p \right) && (\text{since } G_j, F_j \text{ are selfadjoint}) \\
&\leq \frac{1}{2} \sum_{j=1}^{n} \sup_{\|z\|=1} \left(\left\langle |G_j + F_j|^p z, z\right\rangle + \left\langle |G_j - F_j|^p z, z\right\rangle \right) && \text{(by McCarthy inequality)} \\
&= \frac{1}{2} \sum_{j=1}^{n} \sup_{\|z\|=1} \left(\left\langle \left(|G_j + F_j|^p + |G_j - F_j|^p \right) z, z\right\rangle \right) \\
&= \frac{1}{2} \sum_{j=1}^{n} \left\| |G_j + F_j|^p + |G_j - F_j|^p \right\|,
\end{aligned}
$$

which proves the right-hand side of (33). To prove the left-hand side, since we have

$$
\begin{aligned}
&\omega_p^p(S_1, \ldots, S_n) \\
&= \sup_{\|z\|=1} \sum_{j=1}^{n} \left(|\langle S_j z, z\rangle|^2 \right)^{\frac{p}{2}} \\
&= \sup_{\|z\|=1} \sum_{j=1}^{n} \left(|\langle G_j z, z\rangle|^2 + |\langle F_j z, z\rangle|^2 \right)^{\frac{p}{2}} \\
&\geq 2^{-\frac{p}{2}} \sup_{\|z\|=1} \sum_{j=1}^{n} |\langle G_j z, z\rangle + \langle F_j z, z\rangle|^p && \left(\text{since } \frac{c^2+d^2}{2} \geq \left(\frac{c+d}{2}\right)^2\right) \\
&\geq \frac{2^{-\frac{p}{2}}}{n^{p-1}} \sup_{\|z\|=1} \left(\sum_{j=1}^{n} |\langle (G_j + F_j) z, z\rangle| \right)^p, && \text{(by Jensen's inequality)}
\end{aligned}
$$

which proves the left-hand side inequality of (33). Hence, the proof is established. □

Example 6. *Let $S_1 = C_1$ and $S_2 = C_2$ as given in Example 2. Employing (33) with $n = 2$ and $p = 2$, we obtain*

$$
\begin{aligned}
\frac{1}{4} \left\| \sum_{j=1}^{2} |G_j + F_j| \right\|^2 &= \frac{1}{4} \| |G_1 + F_1| + |G_2 + F_2| \|^2 \\
&= 2.24303,
\end{aligned}
$$

and

$$
\omega_e^2(S_1, S_2) = \sup_{\|z\|=1} \left(|\langle S_1 z, z\rangle|^2 + |\langle S_2 z, z\rangle|^2 \right) = 4
$$

while

$$
\begin{aligned}
\frac{1}{2} \sum_{j=1}^{2} \left\| |G_j + F_j|^2 + |G_j - F_j|^2 \right\| &= \frac{1}{2} \left[\left\| |G_1 + F_1|^2 + |G_1 - F_1|^2 \right\| + \left\| |G_2 + F_2|^2 + |G_2 - F_2|^2 \right\| \right] \\
&= 4.5,
\end{aligned}
$$

which verifies that

$$2.24303 := \frac{1}{4}\left\|\sum_{j=1}^{2}\left|G_j + F_j\right|\right\|^2 \le \omega_e^2(S_1, S_2) = 4 \le \frac{1}{2}\sum_{j=1}^{2}\left\|\left|G_j + F_j\right|^2 + \left|G_j - F_j\right|^2\right\| := 4.5.$$

4. Conclusions

In this work, we proved several new inequalities for the generalized Euclidean operator radius. Among others, some bounds in terms of Cartesian decomposition of a given Hilbert space operator were established. More precisely, Section 2 was devoted to inequalities for the generalized Euclidean operator radius which gives an equivalent version of the inequalities (5)–(8), and Section 3 was focused on diverse upper and lower bounds for quantities involving this radius; and this gives an extension of [6] (Theorem 5) and [15] (Theorem 4.1).

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