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Article
A Selection Principle and Products in Topological Groups

Marion Scheepers

Department of Mathematics, Boise State University, Boise, ID 83725, USA; mscheepe@boisestate.edu

Abstract: We consider the preservation under products, finite powers, and forcing of a selection-principle-based covering property of T_0 topological groups. Though the paper is partly a survey, it contributes some new information: (1) The product of a strictly o-bounded group with an o-bounded group is an o-bounded group—Corollary 1. (2) In the generic extension by a finite support iteration of $\aleph_1$ Hechler reals the product of any o-bounded group with a ground model $\aleph_0$ bounded group is an o-bounded group—Theorem 11. (3) In the generic extension by a countable support iteration of Mathias reals the product of any o-bounded group with a ground model $\aleph_0$ bounded group is an o-bounded group—Theorem 12.

Keywords: selection principle; topological group; consistency

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1. Introduction

In this paper, we consider selection principles for open covers on topological space under the imposition of three major constraints: the topological spaces are assumed to be T_0, are assumed to be topological groups, and these topological groups are $\aleph_0$-bounded (a notion due to Guran [1] and defined below).

Even under these three constraints, there is a broad range of considerations regarding the relevant selection principles, and we also confine attention to a specific class of selection principles and specific concerns regarding these. To give an initial indication of the scope of work considered here: The two following selection principles, among several, are historically well-studied in several mathematical contexts: Let families $\mathcal{A}$ and $\mathcal{B}$ of sets be given. Symbol $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the statement that there is for each sequence $(A_n : n \in \mathbb{N})$ of members of the family $\mathcal{A}$, a corresponding sequence $(B_n : n \in \mathbb{N})$ such that for each n, B_n is a finite subset of A_n, and $\bigcup\{B_n : n \in \mathbb{N}\}$ is a set in family $\mathcal{B}$. Symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the statement that there is for each sequence $(A_n : n \in \mathbb{N})$ of members of the family $\mathcal{A}$, a corresponding sequence $(B_n : n \in \mathbb{N})$ such that for each n, B_n is a member of A_n, and $\{B_n : n \in \mathbb{N}\}$ is a set in the family $\mathcal{B}$. It is well-known, that if $\mathcal{A} \subseteq \mathcal{B}$ and if $\mathcal{C} \subseteq \mathcal{D}$, then the following implications (more broadly illustrated in Figure 1) hold: $\mathsf{S}_{fin}(\mathcal{B},\mathcal{C}) \Rightarrow \mathsf{S}_{fin}(\mathcal{A},\mathcal{D})$, $\mathsf{S}_1(\mathcal{B},\mathcal{C}) \Rightarrow \mathsf{S}_1(\mathcal{A},\mathcal{D})$, and $\mathsf{S}_1(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$.

If instead of giving an entire antecedent sequence $(A_n : n \in \mathbb{N})$ of items from family $\mathcal{A}$ all at once for a selection principle and then producing a consequent sequence $(B_n : n \in \mathbb{N})$ to confirm that for example $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ (or $\mathsf{S}_1(\mathcal{A},\mathcal{B})$) holds, one can define a competition between two players, named ONE and TWO, where in inning n ONE chooses an element A_n from $\mathcal{A}$, and TWO responds with a B_n from TWO's eligible choices. The players play an inning per positive integer n, producing a play

$$A_1\ B_1\ A_2\ B_2\ \cdots\ A_m\ B_m\ \cdots \tag{1}$$

In the game named $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ the play in (1) is won by TWO if for each n, B_n is a finite subset of A_n and $\bigcup\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$—otherwise, ONE wins. In the game named $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ the play in (1) is won by TWO if for each n $B_n \in A_n$, and $\{B_n : n \in \mathbb{N}\}$

is an element of $\mathcal{B}$. When ONE does not have a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, then $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ is true. Similarly, when ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, then $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ is true. The relationship between the existence of winning strategies of a player and the corresponding properties of the associated selection principle is a fundamental question, and answers often reveal significant mathematical information.

$$\begin{tikzcd}
 & \mathsf{S}_{fin}(\mathcal{A},\mathcal{C}) \arrow[rr] & & \mathsf{S}_{fin}(\mathcal{A},\mathcal{D}) \\
\mathsf{S}_{fin}(\mathcal{B},\mathcal{C}) \arrow[ur] \arrow[rr] & & \mathsf{S}_{fin}(\mathcal{B},\mathcal{D}) \arrow[ur] & \\
 & \mathsf{S}_1(\mathcal{A},\mathcal{C}) \arrow[uu] \arrow[rr] & & \mathsf{S}_1(\mathcal{A},\mathcal{D}) \arrow[uu] \\
\mathsf{S}_1(\mathcal{B},\mathcal{C}) \arrow[uu] \arrow[ur] \arrow[rr] & & \mathsf{S}_1(\mathcal{B},\mathcal{D}) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Figure 1. Monotonicity properties: $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{C} \subset \mathcal{D}$.

In this paper, we consider the selection principle $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ in the context where families $\mathcal{A}$ and $\mathcal{B}$ are types of open covers arising in the study of topological groups. In the context of topological groups and the classes of open covers of these considered, there are some equivalences between the $\mathsf{S}_{fin}(\cdot,\cdot)$ and $\mathsf{S}_1(\cdot,\cdot)$ selection principles, as is indicated.

We assume throughout that the topological groups being considered have the T_0 separation property and thus, by the following classical theorem, the $\mathsf{T}_{3\frac{1}{2}}$ separation property:

Theorem 1 (Kakutani, Pontryagin)**.** *Any T_0 topological group is $T_{3\frac{1}{2}}$.*

In Section 2 we briefly describe the resilience of $\aleph_0$-bounded groups under certain mathematical constructions and contrast these with the more constrained classical Lindelöf property. In Section 3, we consider, for groups satisfying the targeted instance of the selection principle $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$, the preservation of the selection property under the product construction. Though there is a significant extant body of work on this topic, only some of these works and motivating mathematical questions relevant to the topic of Section 3 are mentioned. In Section 4, we briefly explore the cardinality of a class of groups emerging from product considerations in Section 3. In Section 5, we focus attention on groups for which finite powers satisfy the instance of $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ being considered in this paper. In Section 6, we briefly return to a specific class of $\aleph_0$ bounded topological groups featured earlier in the paper.

For the background on topological groups, we refer the reader to [2,3]. For relevant background on forcing, we refer the reader to [4–7]. Lastly, this paper is partly a survey of known results and partly an investigation of refining or providing additional context for known results. The author would like to thank the editor of the volume for the flexibility in time to construct this paper.

2. Open Covers and Fundamental Theorems

Besides the typical types of open covers considered for general topological spaces, there are also specific types of open covers considered in the context of topological groups. We introduce notation here for efficient reference to the various types of open covers relevant to this paper. Thus, let $(G,\otimes)$ denote a generic T_0 topological group, where G is the set of elements of the group, and $\otimes$ is the group operation (symbol $\otimes$ used here should not be confused with the tensor product operation in modules. In this paper, $\otimes$ is used exclusively to denote a group operation). Symbol *id* denotes the identity element of the group. It is also common practice to talk about group G without mentioning an explicit symbol for the operation.

For an element x of the group G and for a nonempty subset S of G, define

$$x \otimes S = \{x \otimes g : g \in S\}. \tag{2}$$

If $(G, \otimes)$ is a topological group, then when S is an open subset of G, so is $x \otimes S$ for each element x of G. Moreover, if id is an element of S, then x is an element of $x \otimes S$. Moreover, when S and T are nonempty subsets of G,

$$S \otimes T = \{x \otimes y : x \in S \text{ and } y \in T\}. \tag{3}$$

Now, we introduce notation for types of open covers of $(G, \otimes)$ to be considered here.

- $\mathcal{O}$: the set of all open covers of G.
- $\mathcal{O}_{nbd}$: the set of all open covers of G of the following form: for a neighborhood U of id, $\mathcal{O}(U)$ denotes the open cover $\{x \otimes U : x \in G\}$ of G, and $\mathcal{O}_{nbd}$ denotes the collection $\{\mathcal{O}(U) : U \text{ a neighborhood of } id\}$.
- Ω: an open cover $\mathcal{U}$ of G is an ω-cover (originally defined in [8]) if G itself is not a member of $\mathcal{U}$, and for each finite subset F of G, there is a $U \in \mathcal{U}$ such that $F \subseteq U$. Symbol Ω denotes the set $\{\mathcal{U} : \mathcal{U} \text{ an } \omega \text{ cover of } G\}$
- Ω_{nbd}: the set of all open covers of G of the following form: for a neighborhood U of id, $\Omega(U)$ denotes the open cover $\{F \otimes U : F \subset G \text{ a finite set}\}$. Symbol Ω_{nbd} denotes the set of all open covers of the form $\Omega(U)$ of G.
- Γ: an open cover $\mathcal{U}$ is a γ-cover (also introduced in [8]) if it is infinite and for each $x \in G$, x is a member for all but finitely sets in $\mathcal{U}$. Symbol Γ denotes set $\{\mathcal{U} : \mathcal{U} \text{ a } \gamma \text{ cover of } G\}$
- Λ: an open cover $\mathcal{U}$ is a large cover if for each $x \in G$, x is a member of infinitely sets in $\mathcal{U}$. Λ denotes the collection of large covers of G.

Targeted properties related to topological objects, such as the preservation of a property of factor spaces in product spaces, have led to the identification of several additional types of open covers for topological spaces. Some of these used in this paper are as follows:

- $\mathcal{O}^{gp}$: an open cover $\mathcal{U}$ is an element of $\mathcal{O}^{gp}$ if it is infinite, and there is a partition $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ where for each n the set $\mathcal{U}_n$ is finite, for all $m \neq n$, we have $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$, and each element of the underlying space is in each but finitely many of the sets $\bigcup \mathcal{U}_n$. $\mathcal{U}$ is a *groupable* cover.
- $\mathcal{O}^{wgp}$: An open cover $\mathcal{U}$ is an element of $\mathcal{O}^{wgp}$ if it is infinite, and there is a partition $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ where for each n the set $\mathcal{U}_n$ is finite, for all $m \neq n$, we have $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$, and for each finite subset F of the underlying space there is an n such that $F \subseteq \bigcup \mathcal{U}_n$. We say that $\mathcal{U}$ is a *weakly groupable* cover.

From the definitions, it is evident that the following inclusions hold among these types of open covers: $\Gamma \subset \mathcal{O}^{gp} \subset \mathcal{O}^{wgp} \subset \Lambda \subset \mathcal{O}$, $\Gamma \subset \Omega \subset \mathcal{O}^{wgp} \subset \mathcal{O}$, $\Omega_{nbd} \subset \Omega$, $\Omega_{nbd} \subset \mathcal{O}_{nbd}$ and $\mathcal{O}_{nbd} \subset \mathcal{O}$.

For several traditional covering properties of topological spaces, natural counterparts are defined in the domain of topological groups by restricting the types of open covers considered in defining the covering properties. For example,

Definition 1. *A topological group is*

1. *$\aleph_0$-bounded if it has the Lindelöf property with respect to the family $\mathcal{O}_{nbd}$ of open covers, that is, each member of $\mathcal{O}_{nbd}$ has a countable subset that covers the group.*
2. *totally bounded (or precompact) if it is compact with respect to the family $\mathcal{O}_{nbd}$ of open covers, that is, each element of $\mathcal{O}_{nbd}$ has a finite subset covering the group.*

3. *σ-bounded if it is a union of countably many totally bounded subsets.*

Many of the properties of $\aleph_0$-bounded groups can be obtained from the following fundamental result:

Theorem 2 (Guran)**.** *A topological group is $\aleph_0$-bounded if and only if it embeds as a topological group into the Tychonoff product of second countable groups.*

The $\aleph_0$-boundedness property is resilient under several mathematical constructions. For example, any subgroup of an $\aleph_0$-bounded group is $\aleph_0$-bounded. The Tychonoff product of any number of $\aleph_0$-bounded groups is also an $\aleph_0$-bounded group. These two facts in particular imply:

Lemma 1. *There is for each infinite cardinal number κ an $\aleph_0$-bounded group of cardinality κ.*

The $\aleph_0$-boundedness property and the total boundedness property are also resilient under forcing extensions of the set theoretic universe:

Theorem 3. *If $(G, \otimes)$ is an $\aleph_0$-bounded (totally bounded) topological group and $(\mathbb{P}, <)$ is a forcing notion, then*

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}(\check{G}, \otimes) \textit{ is } \aleph_0\textit{-bounded (respectively totally bounded)”}.$$

Proof. Let $\dot{U}$ be a $\mathbb{P}$-name such that $\mathbf{1}_{\mathbb{P}} \Vdash$ “$\dot{U}$ is a neighborhood of the identity”. Choose a maximal antichain A for $\mathbb{P}$ and, for each $q \in A$, choose a neighborhood U_q of the identity such that $q \Vdash$ “$\dot{U} = \check{U}_q$”. We give an argument for $\aleph_0$-boundedness. The argument for the totally bounded case is similar.

Since $(G, \otimes)$ is $\aleph_0$-bounded, choose for each $q \in A$ a countable set $X_q := \{x_n^q : n < \omega\}$ of elements of G such that $X_q \otimes U_q = G$. Define $\dot{X} = \{(\check{x}_n^q, q) : n < \omega \text{ and } q \in A\}$. Then $\dot{X}$ is a $\mathbb{P}$-name and

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\dot{X} \subseteq \check{G} \text{ is countable and } \dot{X} \otimes \dot{U} = \check{G}\text{”}$$

Thus, $\mathbf{1}_{\mathbb{P}} \Vdash$ “$(\check{G}, \otimes)$ is $\aleph_0$-bounded”. □

Proper forcing posets also preserves the property of not being $\aleph_0$-bounded:

Theorem 4. *Let $(G, \otimes)$ be a topological group which is not $\aleph_0$ bounded. Let $(\mathbb{P}, <)$ be a proper partially ordered set. Then*

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}(\check{G}, \otimes) \textit{ is not } \aleph_0\textit{-bounded”}.$$

Proof. Let U be a neighborhood of the identity witnessing that $(G, \otimes)$ is not $\aleph_0$-bounded. Suppose that $p \in \mathbb{P}$ and $\mathbb{P}$-name $\dot{X}$ are such that $p \Vdash$ “$\dot{X} \otimes \check{U} = \check{G}$ and $\dot{X} \subseteq \check{G}$ is countable”. Since $\mathbb{P}$ is a proper poset there is a countable set $C \subseteq G$ such that $p \Vdash \dot{X} \subseteq \check{C}$”—[6], Proposition 4.1. However, then $p \Vdash$ “$\check{C} \otimes \check{U} = \check{G}$”. Since all the parameters in the sentence forced by p are in the ground model, we find the contradiction that $G = C \otimes U$. □

Thus, when forcing with a proper forcing notion, a ground model topological group is $\aleph_0$-bounded in the generic extension if and only if it is $\aleph_0$-bounded in the ground model. The same argument shows

Theorem 5. *Let (X, τ) be a topological space that is not Lindelöf. Let $(\mathbb{P}, <)$ be a proper partially ordered set. Then,*

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}(\check{X}, \tau) \textit{ is not Lindelöf”}.$$

When considering a strengthening of the $\aleph_0$-boundedness property, the resilience of the stronger property under a corresponding mathematical constructions is more subtle.

For example, the Lindelöf property requires that for any open cover (not only ones from $\mathcal{O}_{nbd}$) there is a countable subset that still is a cover. Every Lindelöf group is an $\aleph_0$-bounded group, but not conversely. The Lindelöf property is not generally preserved by subspaces, products, or forcing extensions. Similarly, for subclasses (determined by selection principles) of the family of $\aleph_0$-bounded groups, the preservation of membership to the subclass under Tychonoff products and behavior under forcing is more subtle. Questions regarding the cardinality of members of the more restricted family are also more delicate.

3. Products and Groups with the Property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$

In the notation established here, a topological group is *o-bounded* if it has the property $\mathsf{S}_1(\Omega_{nbd}, \mathcal{O})$. In the literature, the notion of an o-bounded group is attributed to Okunev. In Theorem 3 of [9], it is proven that, for a topological group, the three properties $\mathsf{S}_1(\Omega_{nbd}, \mathcal{O})$, $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$ and $\mathsf{S}_{fin}(\Omega_{nbd}, \mathcal{O})$ are equivalent. Property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$ is also known as Menger boundedness.

In Problem 5.2 of [10], Hernandez asked:

Problem 1. *Is the product of two topological groups, each satisfying the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, a topological group satisfying the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$?*

Subsequently it was discovered (see Example 2.12 of [11]) that there are groups G and H, each satisfying the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, for which the group $G \times H$ does not satisfy the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$. Since subgroups of a group satisfying $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$ inherit the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, for $G \times H$ to have the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, each of the groups G and H must have at least the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$. Thus, Example 2.12 of [11] demonstrates that G or H should satisfy additional hypotheses to guarantee that the product has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$. Under which conditions on G and H would product group $G \times H$ satisfy property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$? A number of additional ad hoc conditions were discovered on a topological group G that guarantee that its product with a group H also has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$. Here are two examples of such conditions:

Theorem 6 ([10] Theorem 5.3)**.** *If G is a subgroup of a σ-compact topological group and H is an $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$ group, then $G \times H$ satisfies $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$.*

For the next example, recall that a topological group is a P group if and only if the intersection of countably many open neighborhoods of the identity element still is an open neighborhood of the identity element. More generally, a topological space is a P space if each countable intersection of open sets is an open set.

Theorem 7 ([11], Theorem 2.4)**.** *If G is an $\aleph_0$-bounded P group, and group H satisfies the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, then $G \times H$ satisfies $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$.*

Though the conditions in Theorems 6 and 7 at first glance seem very different, a single unifying property in the literature implies both results, namely,

Theorem 8 ([12], Theorem 6)**.** *Let $(G, \otimes)$ be a T_0 topological group satisfying the selection principle $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$. Let $\mathcal{A}$ be any of $\mathcal{O}$, Ω or Γ. If $(H, \triangle)$ is a topological group satisfying $\mathsf{S}_1(\Omega_{nbd}, \mathcal{A})$, then the product group $G \times H$ satisfies $\mathsf{S}_1(\Omega_{nbd}, \mathcal{A})$.*

To obtain Theorem 6 from Theorem 8, observe

Lemma 2. *An infinite σ-compact group, and any of its infinite subgroups, has the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$.*

Proof. We give the argument for infinite σ-compact groups, leaving the proof for subgroups of such groups to the reader. Assume that G is σ-compact, and write G as the union $\bigcup\{G_n : n \in \mathbb{N}\}$, where for each n G_n is compact and $G_n \subset G_{n+1}$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of Ω_{nbd} covers of G. For each n fix a neighborhood U_n of the identity element such that $\mathcal{U}_n = \{F \otimes U_n : F \subset G \text{ finite}\}$.

For each n, as G_n is compact, choose a finite set $F_n \subset G$ such that $G_n \subseteq S_n = F_n \otimes U_n \in \mathcal{U}_n$. Then the sequence $(S_n : n \in \mathbb{N})$ witnesses $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$ for the given sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of Ω_{nbd} covers of G. □

Next, we show how to derive Theorem 7 from Theorem 8. First, using the argument in Theorem 2.4 of [10],

Lemma 3. *If $(G, \otimes)$ is an $\aleph_0$-bounded P group, then it has the property* $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O})$

Proof. Let a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $\mathcal{O}_{nbd}$-covers of G be given. For each n choose a neighborhood M_n of the identity element such that $\mathcal{U}_n = \mathcal{O}(M_n)$. Since G is a P group, $M = \bigcap\{M_n : n \in \mathbb{N}\}$ is an open set, and neighborhood of the identity element. Then $\mathcal{U} = \{x \otimes M : x \in G\}$ is a member of $\mathcal{O}_{nbd}$. Since G is $\aleph_0$-bounded, fix a countable set $\{x_n : n \in \mathbb{N}\}$ of elements of G such that $\{x_n \otimes M : n \in \mathbb{N}\}$ is a cover of G. Then for each n also $x_n \otimes M \subseteq x_n \otimes M_n$. Thus $\{x_n \otimes M_n : n \in \mathbb{N}\}$ witnesses for the sequence $(\mathcal{U}_n : n \in \mathbb{N}\}$ that $(G, \otimes)$ has the property $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O})$. □

Since $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O})$ implies $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, the following lemma extends the conclusion of Lemma 3:

Lemma 4. *If $(G, \otimes)$ is an $\aleph_0$-bounded P group, then it has the property* $\mathsf{S}_1(\Omega_{nbd}, \Omega)$

Proof. Finite products of P spaces are P spaces. However, any (Tychonoff) product of $\aleph_0$-bounded groups is $\aleph_0$-bounded (see for example Proposition 3.2 in the survey [3]). Thus, any finite product of $\aleph_0$-bounded P groups is an $\aleph_0$ bounded P-group. By [11] Theorem 2.4, finite products of $\aleph_0$-bounded P groups are $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$. By Theorems 2 and 4 of [9], G satisfies $\mathsf{S}_1(\Omega_{nbd}, \Omega)$. □

Lastly, we strengthen the conclusion of Lemma 4.

Theorem 9. *(An alternative proof of Theorem 9 is given below by Lemmas 6 and 7). Any $\aleph_0$-bounded P group has the property* $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$

Proof. Let $(G, \otimes)$ be an $\aleph_0$-bounded P group. Let $(V_n : n \in \mathbb{N})$ be a sequence of neighborhoods of the identity element of G. For each n choose a finite set F_n such that $\{F_n \otimes V_n : n \in \mathbb{N}\}$ is an ω-cover of G. Put $V = \bigcap\{U_n : n \in \mathbb{N}\}$. Since G is a P-space, V is an open neighborhood of the identity. For each n $\mathcal{U} = \{F \otimes V : F \subset G \text{ finite}\}$ is an Ω_{nbd} cover refining $\mathcal{U}_n = \{F \otimes V_n : F \subset G \text{ finite}\}$. Applying $\mathsf{S}_1(\Omega_{nbd}, \Omega)$ to the sequence $(\mathcal{U}, \mathcal{U}, \mathcal{U}, \cdots)$ fix for each n a finite set $S_n \subset G$ such that $\{S_n \otimes V : n \in \mathbb{N}\}$ is an ω-cover. For each n, set $G_n = \bigcup\{S_j : j \leq n\}$. Then for each n, $G_n \otimes V_n \in \mathcal{U}_n$, and $\{G_n \otimes V_n : n \in \mathbb{N}\}$ is a γ-cover of G. □

Lastly, Theorems 8 and 9 imply Theorem 7.

Continuing with the theme of providing a single unifying property for questions and claims regarding preserving the property $\mathsf{S}_1(\Omega_{nbd}, \mathcal{O})$ in products, we also give a result on a question from the literature. Tkachenko defined a topological group to be *strictly o-bounded* if player TWO has a winning strategy in the game $\mathsf{G}_1(\Omega_{nbd}, \mathcal{O})$ (Equivalently, TWO has a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$)—[10]. In Problem 2.4 of [11] the authors ask.

Problem 2. *Is it true that whenever $(G, \otimes)$ is a strictly o-bounded group and $(H, \triangle)$ satisfies the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, then $G \times H$ also satisfies the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$?*

Problem 2 was partially answered in Corollary 8 of [12] for the case when the strictly o-bounded group $(G, \otimes)$ is metrizable. Towards answering Problem 2, we generalize a part of Theorem 5 of [12].

Theorem 10. *If player TWO has a winning strategy in the game $\mathsf{G}_1(\Omega_{nbd}, \mathcal{O})$ played on a T_0 topological group, then that group has the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$.*

Proof. Let $(G, \otimes)$ be a strictly o-bounded group. By Lemma 2 we may assume it is not totally bounded. Assume that TWO has a winning strategy in the game, say it is σ. Let $(U_n : n \in \mathbb{N})$ be a sequence of neighborhoods of the identity, each witnessing that the group is not totally bounded. For each n let $\Omega(U_n)$ denote $\{F \otimes U_n : F \subset G \text{ finite}\}$, an element of Ω_{nbd} for G.

Then, $(\Omega(U_n) : n \in \mathbb{N})$ is a sequence of elements of Ω_{nbd}. In game $\mathsf{G}_1(\Omega_{nbd}, \mathcal{O})$ ONE chooses elements of Ω_{nbd}, and TWO selects members of ONE's moves. Following the construction in the proof of 1$\Rightarrow$ 2 of Theorem 5 of [12], define the following subsets of G:

$$G_{\emptyset} = \bigcap_{n \in \mathbb{N}} \sigma(\Omega(U_n)). \tag{4}$$

For $\tau = (n_1, \cdots, n_k)$ a finite sequence of positive integers, define

$$G_{\tau} = \bigcap_{n \in \mathbb{N}} \sigma(\Omega(U_{n_1}), \cdots, \Omega(U_{n_k}), \Omega(U_n)). \tag{5}$$

Claim 1: $G = \bigcup_{\tau \in {}^{<\omega}\mathbb{N}} G_{\tau}$

Suppose that, on the other hand, $x \in G$ is not an element of the union $\bigcup_{\tau \in {}^{<\omega}\mathbb{N}} G_{\tau}$. As x is not in $G_{\emptyset}$, choose n_1 with $x \notin \sigma(\Omega(U_{n_1}))$. Then, as x is not in G_{n_1} choose an n_2 with $x \notin \sigma(\Omega(U_{n_1}), \Omega(U_{n_2}))$, and so on. In this way, we find a σ-play of the game during which TWO never covered x, contradicting the hypothesis that σ is a winning strategy for TWO.
Claim 2: For each finite sequence τ of positive integers and for each n, there is a finite set $F \subseteq G$ such that $G_{\tau} \subseteq F * U_n$.

Let $\tau = (n_1, \cdots, n_k)$ and n be given. Then,

$$G_{\tau} \subseteq \sigma(\Omega(U_{n_1}), \cdots, \Omega(U_{n_k}), \Omega(U_n)) \in \Omega(U_n).$$

Lastly, enumerate the set of finite sequences of positive integers as $\tau_1, \tau_2, \cdots, \tau_n, \cdots$. Choose finite subsets $F_1, F_2, \cdots, F_n, \cdots$ of G so that for each k we have

$$G_{\tau_1} \cup \cdots \cup G_{\tau_k} \subseteq F_k \otimes U_k \in \Omega(U_k).$$

Sequence $(F_k * U_k : k \in \mathbb{N})$ witnesses $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$ for the given sequence of neighborhoods of the identity. $\square$

The following corollary answers Problem 2:

Corollary 1. *If $(G, \otimes)$ is a strictly o-bounded T_0 group, and $(H, \triangle)$ is a T_0 group with the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, then $G \times H$ has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$.*

Proof. Let $(G, \otimes)$ and $(H, \triangle)$ be as in the hypotheses. By Theorem 10 the group $(G, \otimes)$ has the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$. Then, by Theorem 8, $G \times H$ has the property $\mathsf{S}_1(\Omega_{nbd}, \mathcal{O})$. $\square$

4. Cardinality of T_0 Groups with the Property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$

Next, we briefly consider the cardinality of topological groups satisfying the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$. It is useful to first catalogue a few basic behaviors of the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$

under some standard forcing notions. Although one can prove that, in general, any forcing iteration of the length of uncountable cofinality to which cofinality often adds a dominating real converts any ground model $\aleph_0$-bounded group into a group satisfying $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$, we prove it here for a specific partially ordered set:

Theorem 11. *Let κ be a cardinal number of uncountable cofinality. Let $(\mathbb{P}, <)$ be the finite support iteration by κ Hechler reals. If $(G, \otimes)$ is $\aleph_0$-bounded in the ground model, then*

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\check{G} \textit{ has the property } \mathsf{S}_1(\Omega_{nbd}, \Gamma)\text{”}.$$

Proof. By Theorem 3 $\mathbf{1}_{\mathbb{P}} \Vdash$ “$(G, \otimes)\check{}$ is $\aleph_0$ bounded”. Thus, as $\mathbb{P}$ has the countable chain condition, if we take a $\mathbb{P}$-name $(\dot{\mathcal{U}}_n : n < \omega)$ for a sequence of $\mathcal{O}_{nbd}$ members we may assume that this sequence is present in the ground model, since it is a name in an initial segment of the iteration, and we can factor the iteration at this initial segment. Since in this initial segment $(G, \otimes)$ is $\aleph_0$-bounded we may choose for each n a countable subset X_n of G such that $G = \bigcup_{n<\omega} X_n \otimes U_n$, where $\mathcal{U}_n = \mathcal{O}(U_n)$. Define for each $x \in G$ a function f_x from ω to ω as follows: Enumerate X_n as $(x^n_m : m < \omega)$. Then

$$f_x(n) = \min\{m : x \in \{x^n_1, \cdots, x^n_m\} \otimes U_n\}.$$

Family $\{f_x : x \in G\}$ is in an initial segment of the iteration, and so the next Hechler real added eventually dominates each f_x. Let g be the next Hechler real. Then, $\{x^n_j : j \leq g(n)\} \otimes U_n \subseteq \mathcal{U}_n$ is a finite subset of $\mathcal{U}_n$, and for each x, for all but finitely many n, $x \in \{x^n_j : j \leq g(n)\} \otimes U_n$. It follows that the group $(G, \otimes)$ has the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$. $\square$

Incidentally, the Hechler reals partially ordered set does not preserve the Lindelöf property. In Remark 5 of [13], Gorelic indicates that the points G_δ Lindelöf subspace in this model fail to be Lindelöf in the generic extension that forces MA plus not-CH. Indeed, this can be accomplished by a finite support iteration of ω_2 or more Hechler reals over a model of CH. Readers could consult the original paper by Hechler [14] or, for example, [15] on Hechler real generic extensions.

Theorem 12. *Let $(\mathbb{P}, <)$ be the countable support iteration by $\aleph_2$ Mathias reals over a model of CH. If $(G, \otimes)$ is $\aleph_0$-bounded in the ground model, then*

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}(G, \otimes)\check{}\ \textit{has the property } \mathsf{S}_1(\Omega_{nbd}, \Gamma)\text{”}.$$

Proof. By Theorem 3 $\mathbf{1}_{\mathbb{P}} \Vdash$ “$\check{G}$ is $\aleph_0$ bounded”. Thus, as CH holds, and antichains of the poset $(\mathbb{P}, <)$ have cardinality at most $\aleph_1$, for any $\mathbb{P}$-name $(\dot{U}_n : n < \omega)$ for a sequence of neighborhoods of the identity, we may assume that this sequence of neighborhoods of the identity is present in the ground model (the name of the sequence is a name in an initial segment of the iteration), and factor the iteration over this initial segment. Since by Theorem 3 $(G, \otimes)$ is $\aleph_0$-bounded in this initial segment choose (in the generic extension by this initial segment) for each n a countable subset X_n of G such that $G = \bigcup_{n<\omega} X_n \otimes U_n$, where $\mathcal{U}_n = \mathcal{O}(U_n)$. Define for each $x \in G$ a function f_x from ω to ω as follows: Enumerate X_n as $(x^n_m : m < \omega)$. Then

$$f_x(n) = \min\{m : x \in \{x^n_1, \cdots, x^n_m\} \otimes U_n\}.$$

Family $\{f_x : x \in G\}$ is in the generic extension by the initial segment (the “ground model" for the remaining generic extension), and so the next Mathias real added by the generic extension eventually dominates each f_x. Let g be such a dominating real. Then, $\{x^n_j : j \leq g(n)\} \otimes U_n \subseteq \mathcal{U}_n$ is a finite subset of $\mathcal{U}_n$, and for each x, for all but finitely many n, $x \in \{x^n_j : j \leq g(n)\} \otimes U_n$. It follows that, in the generic extension, $(G, \otimes)$ has the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$. $\square$

As a consequence, we obtain

Theorem 13. *It is consistent, relative to the consistency of ZFC, that there is for each cardinal number κ a group with property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$.*

Proof. By Lemma 1 there exists, for each infinite cardinal number κ, an $\aleph_0$-bounded group of cardinality κ. By either Theorem 11 or Theorem 12, in the corresponding generic extension, each ground model $\aleph_0$-bounded group has property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$. Since the forcing partially ordered set in either case preserves cardinal numbers, the result follows. □

To round off the consideration of the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$ under forcing:

Theorem 14. *If the group $(G, \otimes)$ has the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$ and if $(\mathbb{P}, <)$ is a partially ordered set with the countable chain condition, then*

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}(\check{G}, \check{\otimes}) \textit{ has the property } \mathsf{S}_1(\Omega_{nbd}, \Gamma)\text{”}.$$

Proof. Let $(\dot{U}_n : n < \omega)$ be a $\mathbb{P}$-name for a sequence of neighborhoods of the identity element of the group $(G, \otimes)$. For each n, choose (in the ground model) a sequence $(U^n_m : m < \omega)$ of neighborhoods of the identity element of $(G, \otimes)$, and a maximal antichain $(q^n_m : m < \omega)$ of $\mathbb{P}$, such that, for each n and m $q^n_m \Vdash$ “$\check{U}^n_m \subseteq \dot{U}_n$”. Then $\dot{V}_n = \{(\check{U}^n_m, q^n_m) : m < \omega\}$ is a $\mathbb{P}$-name and

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\dot{V}_n \subseteq \dot{U}_n \text{ is a neighborhood of the identity element of } \check{G}\text{”}$$

For each n, define $N_n = \bigcap_{k,\ell \leq n} U^k_\ell$, a (ground model) neighborhood of the identity element of $(G, \otimes)$. Applying the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$, choose finite sets $F_1 \subseteq F_2 \subseteq \cdots$ such that for each $x \in G$, for all but finitely many k, x is a member of $F_k \otimes N_k$. Then, for each n, define the $\mathbb{P}$-name $\dot{F}_n$ for a finite subset of $\check{G}$ by $\{(\check{F}_{n+m}, q^n_m) : m < \omega\}$.

Claim: $\mathbf{1}_{\mathbb{P}} \Vdash$ “$(\forall x \in \check{G})(\forall^\infty_n)(x \in \dot{F}_n \otimes \dot{V}_n)$”.

Let H be a $\mathbb{P}$-generic filter. For each n, choose m_n with $q^n_{m_n} \in H$. Then, we have that, for each n, $(\dot{F}_n)_H = F_{n+m_n}$. Consider any $x \in G$. Choose k to be so large that, for $n \geq k$, we have $n + m_n > k$ and $x \in F_{n+m_n} \otimes N_{n+m_n}$. Since $N_{n+m_n} \subseteq U^n_{m_n} = (\dot{V})_H$ it follows that $x \in (\dot{F}_n \otimes \dot{V}_n)_H$. □

5. Finite Powers of Groups with the Property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$

Consider a topological group $(G, \otimes)$ that has the property that, whenever $(H, \triangle)$ is a topological group with the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, then the product group $G \times H$ also has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$. Then, group $G \times G$ necessarily has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$: Indeed, every finite power of the group $(G, \otimes)$ has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$.

Recall Example 2.12 of [11], which illustrates that the product of two groups, each with the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, does not necessarily have the property of $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$. This example in fact gives a group $(G, \otimes)$ that has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, but $(G, \otimes) \times (G, \otimes)$ does not have the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$ (and the group is even metrizable). One might ask whether the phenomenon exhibited by this example ($(G, \otimes)$ has property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, but $(G, \otimes) \times (G, \otimes)$ does not) is the only obstruction to a topological group $(G, \otimes)$ having a property such as

(A) the product of $(G, \otimes)$ with any group with property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$ has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$
(B) each finite power of $(G, \otimes)$ has property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$.

The two following prior results shed significant light on version (B) of this question:

Theorem 15 (Banakh and Zdomskyy [16], Mildenberger and Shelah)**.** *The following statement is consistent, relative to the consistency of ZFC:*

For each T_0 group $(G,\otimes)$, if $(G,\otimes)\times(G,\otimes)$ has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O})$, then the group $(G,\otimes)$ in fact has the property $\mathsf{S}_{fin}(\Omega_{nbd},\mathcal{O}^{wgp})$.

Regarding Theorem 15: prior results (Theorems 3, 6, and 7 of [9]) that show that every finite power of a topological group has property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O})$ if and only if the group has the property $\mathsf{S}_1(\Omega_{nbd},\mathcal{O}^{wgp})$. Moreover,

Lemma 5. *For a topological group $(G,\otimes)$ the following are equivalent:*

1. *$(G,\otimes)$ has the property $\mathsf{S}_{fin}(\Omega_{nbd},\mathcal{O}^{wgp})$*
2. *$(G,\otimes)$ has the property $\mathsf{S}_1(\Omega_{nbd},\mathcal{O}^{wgp})$*

Proof. We must show that (1) implies (2). Thus, let $(\mathcal{U}_n : n\in\mathbb{N})$ be a sequence of elements of Ω_{nbd}, say for each n the set V_n is a neighborhood of the identity element of G and $\mathcal{V}_n = \{F\otimes V_n : F\subset G \text{ finite}\}$.

Then, for each n define $U_n = \bigcap\{V_j : j\le n\}$, a neighborhood of the identity of G, and define $\mathcal{U}_n = \{F\otimes U_n : n\in\mathbb{N}\}$. As each $\mathcal{U}_n$ is an element of Ω_{nbd}, apply $\mathsf{S}_{fin}(\Omega_{nbd},\mathcal{O}^{wgp})$ to the sequence $(\mathcal{U}_n : n\in\mathbb{N})$. For each n choose a finite subset $\mathcal{F}_n$ of $\mathcal{U}_n$ such that $\mathcal{G} = \bigcup\{\mathcal{F}_n : n\in\mathbb{N}\}$ is a weakly groupable cover of G.

Fix a partition $(\mathcal{G}_n : n\in\mathbb{N})$ of $\mathcal{G}$ into finite sets $\mathcal{G}_n$, such that there is for each finite subset S of G an n with $S\subset\bigcup\mathcal{G}_n$. □

Thus, Theorem 15 establishes the consistency of the statement that if a T_0 group $(G,\otimes)$ is such that $(G,\otimes)\times(G,\otimes)$ has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O})$, then every finite power of $(G,\otimes)$ has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O})$. This statement is in fact independent of ZFC, since, on the other hand,

Theorem 16 ([17], Theorem 11)**.** *It is consistent, relative to the consistency of ZFC, that there is, for each positive integer k, a separable metrizable topological group $(G,\otimes)$, such that G^k has the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O})$, while G^{k+1} does not have the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O})$.*

Less is known about version (A) of the question above. Interestingly, for the subclass of metrizable groups that have the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O})$ in all finite powers, it is consistent that a product of finitely many groups in this subclass is still in this subclass. In fact, an equiconsistency criterion was identified.

Theorem 17 (He, Tsaban and Zang [18], Theorem 2.1)**.** *The following statements are equivalent:*

1. *NCF.*
2. *The product of two metrizable groups, each with the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O}^{wgp})$, is a topological group with the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O}^{wgp})$.*

This result raises the following, potentially more modest, analog of the version (A) question:

Problem 3. *Is it consistent that product of any two T_0 groups, each with the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O}^{wgp})$, is a topological group with the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O}^{wgp})$?*

6. Further Remarks on $\aleph_0$ Bounded P Groups

Towards further strengthening the results about $\aleph_0$-bounded P groups, we next consider products of topological groups with the property $\mathsf{S}_1(\mathcal{O}_{nbd},\mathcal{O})$, a stronger property than $\mathsf{S}_{fin}(\mathcal{O}_{nbd},\mathcal{O})$. For finite powers there is the following prior result

Theorem 18 ([9], Theorem 15)**.** *For a topological group $(G,\otimes)$ the following are equivalent:*

1. *Each finite power of $(G,\otimes)$ has the property $\mathsf{S}_1(\mathcal{O}_{nbd},\mathcal{O})$*
2. *$(G,\otimes)$ has the property $\mathsf{S}_1(\mathcal{O}_{nbd},\mathcal{O}^{wgp})$*

Lemma 3 can be strengthened as follows:

Lemma 6. *Any $\aleph_0$-bounded P group has the property* $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O}^{gp})$

Proof. Let $(G, \otimes)$ be an $\aleph_0$-bounded P group. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\mathcal{O}_{nbd}$-covers of G. For each n choose a neighborhood U_n of the identity such that $\mathcal{U}_n = \mathcal{O}(U_n) = \{g \otimes U_n : g \in G\}$. Since G is a P space the G_δ set $U = \bigcap\{U_n : n \in \mathbb{N}\}$ is an open neighborhood of the identity, and the $\mathcal{O}_{nbd}$-cover $\{g \otimes U : g \in G\}$ of G is a refinement of each of the $\mathcal{O}_{nbd}$ covers $\mathcal{U}_n$. For each n set $\mathcal{V}_n$ be the Ω_{nbd} cover $\{F \otimes U : F \subset G \text{ finite}\}$.

As was shown in Theorem 9, this group has the selection property of $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$. Applying this selection property of G to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ we find for each n a set $V_n \in \mathcal{V}_n$ such that $\{V_n : n \in \mathbb{N}\}$ is a γ-cover of G, that is, for each $g \in G$, we have for all but finitely many n that $g \in V_n$. For each n, fix the finite set $F_n \subset G$ such that $V_n = F_n \otimes U$. Now let $n_1, n_2, \cdots, n_k, \cdots$ be natural numbers such that for each i we have $n_i = |F_i|$. Next, choose elements $g_1, g_2, \cdots, g_n, \cdots$ from G as follows: $g_1, \cdots, g_{n_1}$ lists the distinct elements of F_1, $g_{n_1+1}, \cdots, g_{n_1+n_2}$ lists the distinct elements of F_2, and in general $\{g_{n_1+\cdots+n_{k-1}+1}, \cdots, g_{n_1+n_2+\cdots+n_k}\}$ lists the distinct elements of F_k, and so on. Thus for each k we have $V_k = \cup\{g_i \otimes U : n_1 + \cdots + n_{k-1} < i \leq n_1 + \cdots + n_k\}$.

Claim: $\{g_i \otimes U_i : i \in \mathbb{N}\}$ is a groupable open cover of G. For let an $h \in G$ be given. Since $(V_n : n \in \mathbb{N})$ is a γ cover of G, fix a k such that for all $m \geq k$ it is true that $h \in V_k$. Then for all $m \geq k$, the element g of G is in $\bigcup\{g_i \otimes U_i : n_1 + \cdots + n_{k-1} < i \leq n_1 + \cdots + n_k\}$, confirming that the selector $(g_i \otimes U_i : i \in \mathbb{N})$ of the original sequence of $\mathcal{O}_{nbd}$ covers is a groupable open cover of G. $\square$

Lemma 6 provides the following alternative derivation that $\aleph_0$-bounded P groups have the property $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$:

Lemma 7. *If a topological group has the property* $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O}^{gp})$*, then it has the property* $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$.

Proof. Let $(G, \otimes)$ be a topological group which has the property $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O}^{gp})$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of Ω_{nbd} covers of G. For each n fix U_n, the neighborhood of the identity element for which $\mathcal{U}_n = \{F \otimes U_n : F \subset G \text{ finite}\}$.

For each n set $V_n = \bigcap\{U_j : j \leq n\}$, a neighborhood if the identity element of the group $(G, \otimes)$. Set $\mathcal{V}_n = \{F \otimes V_n : F \subset G \text{ finite}\}$, a member of Ω_{nbd} that refines $\mathcal{U}_n$.

Now apply to selection principle $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O}^{gp})$ to each of the $\mathcal{O}_{nbd}$ covers $\mathcal{A}_n = \{g \otimes V_n : n \in \mathbb{N}\}$: For each n choose an $A_n \in \mathcal{A}_n$ such that $\{A_n : n \in \mathbb{N}\} \in \mathcal{O}^{gp}$. Fix a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of natural numbers such that for each $g \in G$, for all but finitely many k, g is an element of $\bigcup\{A_m : n_{k-1} < m \leq n_k\}$. For each m fix $g_m \in G$ such that $A_m = g_m \otimes V_m$. Then define finite sets $F_1 = \{g_n : n \leq n_1\}$ and for each k, $F_k = \{g_j : n_{k-1} < j \leq n_k\}$.

For each k set $U_k = F_k \otimes V_k$, an element of $\mathcal{U}_k$. Then $\{U_k : k \in \mathbb{N}\}$ is a γ-cover of G, for let an $x \in G$ be given. Choose k so large that for all $m \geq k$, x is a member of $\bigcup_{n_{m-1} < j \leq n_m} g_j \otimes V_j$. Since $\bigcup_{n_{m-1} < j \leq n_m} g_j \otimes V_j \subseteq U_m$, it follows that for all $m \geq k$, x is a member of U_m. $\square$

The classical examples of $\aleph_0$-bounded P groups were given by Comfort and collaborators in, for example, [19,20]. These examples are indeed Lindelöf P-groups and have the property that TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$—[21]. The answers to the two following problems appear to be unknown (for T_0 groups):

Problem 4. *Is every $\aleph_0$-bounded P group Lindelöf?*

Problem 5. *Does Player TWO have a winning strategy in the game* $\mathsf{G}_1(\mathcal{O}_{nbd}, \mathcal{O})$ *in any $\aleph_0$-bounded P group?*

7. Conclusions

In this paper, we merely touched on four extensively explored topics in the area of $\aleph_0$-bounded groups. Among the numerous exploration possibilities, we pose here only the following one about cardinalities:

There are no a priori theoretical restrictions on the cardinality that a T_0 group with the property $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$ or even $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$ can have. Each infinite cardinality is possible. As was indicated in Theorem 8 and Corollary 17 of [21], the same holds for T_0 groups with the property $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O})$, or even the much stronger property that the group is an $\aleph_0$-bounded P group, or a group in which TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{nbd}, \mathcal{O})$. However, the following issue regarding the achievable cardinality for a given type of $\aleph_0$-bounded group is much more subtle (in the class of Lindelöf spaces, for example, there are no constraints on the cardinalities achievable in the class of T_0 Lindelöf spaces, yet there are constraints on the cardinalities of subspaces that are Lindelöf, as can, for example, be gleaned from [22]): Let an $\aleph_0$-bounded T_0 group be given. It necessarily has subgroups with properties $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$, $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$, $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O})$, and any of the other nonempty-selection-based classes obtained by varying the types of open covers appearing. The question of what cardinality restrictions there may be on subgroups of an $\aleph_0$ bounded T_0 group was extensively studied in the case of the property $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O})$. For example, in [23,24], the following hypothesis (this hypothesis is a generalization of the classical Borel Conjecture) was investigated:

> Each subgroup with the property $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O})$ of an $\aleph_0$-bounded group of weight κ has cardinality at most κ.

It would be interesting to know if there are similar feasible hypotheses of cardinality bounds for subgroups with the property of $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$ or $\mathsf{S}_{fin}(\mathcal{O}_{nbd}, \mathcal{O})$ of $\aleph_0$ bounded groups that are not σ totally bounded.

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