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Existence Results for a Multipoint Fractional Boundary Value Problem in the Fractional Derivative Banach Space

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Abstract: We study a class of nonlinear implicit fractional differential equations subject to nonlocal boundary conditions expressed in terms of nonlinear integro-differential equations. Using the Krasnosel'skii fixed-point theorem we prove, via the Kolmogorov–Riesz criteria, the existence of solutions. The existence results are established in a specific fractional derivative Banach space and they are illustrated by two numerical examples.

Keywords: fractional differential equations; boundary value problems; Kolmogorov–Riesz theorem; fixed point theorems; Nemytskii operator

MSC: 34B10; 34K37; 45J05

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1. Introduction

It is noticeable, in recent years, that the field of fractional calculus has been swept for research by many mathematicians, due to its effectiveness in describing many physical phenomena, see, e.g., [1–7].

A fractional derivative is a generalization of the ordinary one. Despite the emergence of several definitions of fractional derivative, the content is one that depends entirely on Volterra integral equations and their kernel, which facilitates the description of each phenomenon as a temporal lag, such as rheological phenomena [8–10].

The study of differential equations is considered of primary importance in mathematics. In applications, differential equations serve as mathematical models for all natural phenomena. Regardless of their type (ordinary, partial, or fractional), the study of differential equations is developed in three directions: existence, uniqueness, and stability of solutions. Therefore, to investigate boundary value problems is always a central question in mathematics [11–16].

Often, it is of central importance to know the behavior of any solution, of the equation under study, at the boundary of the domain, because that makes it easier to find the solution. In 2009, Ahmad and Nieto considered the following boundary value problem [17]:

$$
\begin{aligned}
{}^{C}D^{\alpha}y(t) &= f\left(t, y(t), \int_0^t \varphi(t,s)y(s)ds\right), \quad 1<\alpha<2,\\
ay(0)+by'(0) &= \int_0^1 q_1(y(s))ds,\\
ay(1)+by'(1) &= \int_0^1 q_2(y(s))ds.
\end{aligned}
\tag{1}
$$

Their results of existence are obtained via Krasnosel'skii fixed-point theorem in the space of continuous functions. For that, they apply Ascoli's theorem in order to provide the compactness of the first part of the Krasnosel'skii operator.

The pioneering work of Ahmad and Nieto of 2009 [17] gave rise to several different investigations. These include: inverse source problems for fractional integrodifferential equations [18]; the study of positive solutions for singular fractional boundary value problems with coupled integral boundary conditions [19]; the expression and properties of Green's function for nonlinear boundary value problems of fractional order with impulsive differential equations [20]; existence of solutions to several kinds of differential equations using the coincidence theory [21]; existence and uniqueness of solution for fractional differential equations with Riemann–Liouville fractional integral boundary conditions [22]; sufficient conditions for the existence and uniqueness of solutions for a class of terminal value problems of fractional order on an infinite interval [23]; existence of solutions, and stability, for fractional integro-differential equations involving a general form of Hilfer fractional derivative with respect to another function [24]; existence of solutions for a boundary value problem involving mixed generalized fractional derivatives of Riemann–Liouville and Caputo, supplemented with nonlocal multipoint boundary conditions [25]; existence conditions to fractional order hybrid differential equations [26]; and an existence analysis for a nonlinear implicit fractional differential equation with integral boundary conditions [27]. Motivated by all these existence results, we consider here a more general multipoint fractional boundary value problem in the fractional derivative Banach space.

Let $1 < p < \infty$ and $1 \geq \gamma > \frac{1}{p}$ and consider the following fractional boundary value problem:

$$\begin{aligned} {}^{C}D^{\alpha}y(t) &= f\left(t, y(t), {}^{C}D^{\gamma}y(t)\right) + {}^{C}D^{\alpha-2}g(t, y(t)), \quad 2 < \alpha < 3, \\ y(0) + y'(0) &= \int_0^1 q_1(y(s))ds, \\ y(1) + y'(1) &= \int_0^1 q_2(y(s))ds, \\ y''(0) &= 0, \end{aligned} \tag{2}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo derivative, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ are given functions such that $g(t,0) = g(0,y) = q_1(0) = q_2(0) = 0$ for any $(t,y) \in [0,1] \times \mathbb{R}$. Our problem (2) generalizes (1) and finds applications in viscoelasticity, where the fractional operators are associated with delay kernels that make the fractional differential equations the best models for several rheological Maxwell phenomena. In particular, for $\alpha \in (1,2)$, we can model oscillatory processes with fractional damping [28].

We prove existence of a solution to problem (2) in the special Banach space $E^{\gamma,p}$ that is known in the literature as the fractional derivative space [29]. This Banach space is equipped with the norm

$$\|u\|_{\gamma,p} = \left(\int_0^T |u(t)|^p + \int_0^T \left|{}^{C}D_0^{\gamma}u(t)\right|^p\right)^{\frac{1}{p}}. \tag{3}$$

The paper is organized as follows. In Section 2, we recall some useful definitions and lemmas to prove our main results. The original contributions are then given in Section 3. The main result is Theorem 1, which establishes the existence of solutions to the fractional boundary value problem (2) using Krasnosel'skii fixed point theorem. Two illustrative examples are given. We end with Section 4, discussing the obtained existence result.

2. Preliminaries

For the convenience of the reader, and to facilitate the analysis of problem (2), we begin by recalling the necessary background from the theory of fractional calculus [30,31].

Definition 1. *The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2. *The Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds = I_0^{n-\alpha} f^{(n)}(t),$$

where $n = [\alpha] + 1$, with $[\alpha]$ denoting the integer part of α.

Lemma 1 (See [17])**.** *Let $\alpha > 0$. Then, the fractional differential equation ${}^C D_{0^+}^\alpha u(t) = 0$ has*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, \quad i = 1, 2, \ldots, n-1,$$

as solution.

Definition 3. *A map $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if*

(a) $t \to f(t, u; v)$ *is measurable for each $u, v \in \mathbb{R}$;*
(b) $(u, v) \to f(t, u; v)$ *is continuous for almost all $t \in [0,1]$.*

Definition 4. *Let J be a measurable subset of $\mathbb{R}$ and $f : J \times \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ satisfies the Carathéodory condition. By a generalized Nemytskii operator we mean the mapping N_f taking a (measurable) vector function $u = (u_1, \ldots, u_{d_1})$ to the function $N_f u(t) = f(t, u(t))$, $t \in J$.*

The following lemma is concerned with the continuity of the operator N_f with $d_1 = 2$ and $d_2 = 1$.

Lemma 2 (See [32])**.** *Consider the same data of Definition 4. Assume there exists $w \in L^1([0,1])$ with $1 \le p < \infty$ and a constant $c > 0$ such that $|f(t, u, v)| \le w(t) + c(|u|^p + |v|^p)$ for almost all $t \in [0,1]$ and $u, v \in \mathbb{R}$. Then, the Nemytskii operator*

$$N_f u(t) = f(t, u(t)), \quad u = (u_1, u_2) \in L^p(0,1) \times L^p(0,1), \quad t \in [0,1] \text{ a.e.},$$

is continuous from $L^p([0,1]) \times L^p([0,1])$ to $L^1(0,1)$.

Lemma 3 (See [33])**.** *Let $\mathcal{F}$ be a bounded set in $L^p([0,1])$ with $1 \le p < \infty$. Assume that*

$$\lim_{|h| \to 0} \|\tau_h f - f\|_p = 0 \text{ uniformly on } \mathcal{F}.$$

Then, $\mathcal{F}$ is relatively compact in $L^p([0,1])$.

For any $1 \le p < \infty$, we denote

$$\|u\|_{L^p[0,T]} := \left(\int_0^T |u(t)|^p \right)^{\frac{1}{p}}, \quad \|u\|_\infty := \max_{t \in [0,T]} |u(t)|.$$

Now, we give the definition and some properties of $E^{\gamma,p}$. For more details about the following lemmas, see [29,34] and references therein.

Definition 5. *Let $0 < \gamma \leq 1$ and $1 < p < \infty$. The fractional derivative space $E^{\gamma,p}$ is defined by the closure of $C^\infty([0,T])$ with respect to the norm*

$$\|u\|_{\gamma,p} = \left(\int_0^T |u(t)|^p + \int_0^T \left|{}^C D_0^\gamma u(t)\right|^p\right)^{\frac{1}{p}}. \tag{4}$$

Lemma 4 (See [29,34]). *Let $0 < \gamma \leq 1$ and $1 < p < \infty$. The fractional derivative space $E^{\gamma,p}$ is a reflexive and separable Banach space.*

Lemma 5 (See [29,34]). *Let $0 < \gamma \leq 1$ and $1 < p < \infty$. For all $u \in E^{\gamma,p}$, we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\gamma+1)}\left\|{}^C D_0^\gamma u\right\|_{L^p}. \tag{5}$$

Moreover, if $\gamma > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\gamma)((\gamma-1)q+1)^{\frac{1}{q}}}\left\|{}^C D_0^\gamma u\right\|_{L^p}. \tag{6}$$

According to the inequality (5), we can also consider the space $E^{\gamma,p}$ with respect to the equivalent norm

$$\|u\|_{\gamma,p} = \left\|{}^C D_0^\gamma u\right\|_{L^p} = \left(\int_0^T \left|{}^C D_0^\gamma u(t)\right|^p\right)^{\frac{1}{p}}, \quad u \in E^{\gamma,p}.$$

3. Main Results

We begin by considering a linear problem and obtain its solution in terms of a Green function.

Lemma 6. *Assume $h, k \in C([0,1])$, $k(0) = 0$ and $\alpha \in (2,3)$. Then, the solution to the boundary value problem*

$$\begin{aligned} {}^C D^\alpha y(t) &= h(t) + {}^C D^{\alpha-2} k(t), \quad t \in (0,1), \\ y''(0) &= 0, \\ y(0) + y'(0) &= \int_0^1 \eta_1(s)ds, \\ y(1) + y'(1) &= \int_0^1 \eta_2(s)ds, \end{aligned} \tag{7}$$

is given by

$$y(t) = \int_0^1 G(t,s)h(s)ds + \int_0^1 H(t,s)k(s)ds + (2-t)\int_0^1 \eta_1(s)ds + (t-1)\int_0^1 \eta_2(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1} + (1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{8}$$

and

$$H(t,s) = \begin{cases} (t-s) + (1-t)(2-s), & 0 \leq s \leq t \leq 1, \\ (1-t)(2-s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{9}$$

Proof. Let y be a solution of problem (7). By Lemma 1, we have

$$y(t) = c_0 + c_1 t + c_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + I_0^2 k(t).$$

Taking the conditions (7) into account, it follows that

$$c_2 = 0,$$

$$y(0) + y'(0) = c_0 + c_1 = \int_0^1 \eta_1(s) ds,$$

and

$$\begin{aligned} y(1) + y'(1) &= c_0 + 2c_1 + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds + \int_0^1 (1-s) k(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s) ds + \int_0^1 k(s) ds \\ &= \int_0^1 \eta_2(s) ds, \end{aligned}$$

which implies

$$\begin{aligned} c_0 = {} & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s) ds \\ & + \int_0^1 (2-s) k(s) ds + 2 \int_0^1 \eta_1(s) ds - \int_0^1 \eta_2(s) ds, \end{aligned}$$

and

$$\begin{aligned} c_1 = {} & -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s) ds \\ & - \int_0^1 (2-s) k(s) ds + \int_0^1 \eta_2(s) ds - \int_0^1 \eta_1(s) ds. \end{aligned}$$

Hence, the solution of problem (7) is

$$\begin{aligned} y(t) &= \int_0^t \left(\frac{(t-s)^{\alpha-1} + (1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) h(s) ds \\ &\quad + \int_t^1 \left(\frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) h(s) ds \\ &\quad + \int_0^t ((t-s) + (1-t)(2-s)) k(s) ds + \int_t^1 (1-t)(2-s) k(s) ds \\ &\quad + (2-t) \int_0^1 \eta_1(s) ds + (t-1) \int_0^1 \eta_2(s) ds \\ &= \int_0^1 G(t,s) h(s) ds + \int_0^1 H(t,s) k(s) ds \\ &\quad + (2-t) \int_0^1 \eta_1(s) ds + (t-1) \int_0^1 \eta_2(s) ds. \end{aligned}$$

The proof is complete. □

Lemma 7. *Functions $G, H, \frac{\partial^\gamma}{\partial t} G$ and $\frac{\partial^\gamma}{\partial t} H$ are continuous on $[0,1] \times [0,1]$ and satisfy for all $t, s \in [0,1]$:*

1. $|G(t,s)| \le \frac{3}{\Gamma(\alpha-1)}, \quad |H(t,s)| \le 3.$

2. $\left|\frac{\partial^\gamma}{\partial t}G(t,s)\right| \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} + \frac{2}{\Gamma(2-\gamma)\Gamma(\alpha-1)}$, $\left|\frac{\partial^\gamma}{\partial t}H(t,s)\right| \leq \frac{3}{\Gamma(2-\gamma)}$.

Proof. We have

$$ {}^C D_0^\gamma(1-t) = I_0^{1-\gamma}(1-t)' = -\frac{1}{\Gamma(2-\gamma)}t^{1-\gamma}, \tag{10} $$

and

$$ \frac{\partial^\gamma}{\partial t}(t-s)^{\alpha-1} = I_0^{1-\gamma}\frac{\partial}{\partial t}(t-s)^{\alpha-1} = (\alpha-1)I_0^{1-\gamma}(t-s)^{\alpha-2}. $$

Thus, for $0 \leq s \leq t \leq 1$, we get $\frac{\partial^\gamma}{\partial t}(t-s)^{\alpha-1} \geq 0$ and

$$ \frac{\partial^\gamma}{\partial t}(t-s)^{\alpha-1} \leq^C D_0^\gamma t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}. \tag{11} $$

On the other hand, we have $\Gamma(\alpha-1) \leq \Gamma(\alpha)$ for $2 \leq \alpha \leq 3$. Now, we give the bound of functions $|G(t,s)|$ and $\left|\frac{\partial^\gamma}{\partial t}G(t,s)\right|$. From the definition of function G and (10) and (11), we obtain:

- For $0 \leq s \leq t \leq 1$,

$$ \begin{aligned} |G(t,s)| &= \frac{(t-s)^{\alpha-1}+(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\leq \frac{(1-s)^{\alpha-1}(1+(1-t))}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\leq \frac{1+(1-t)}{\Gamma(\alpha)} + \frac{(1-t)}{\Gamma(\alpha-1)} \\ &\leq \frac{3}{\Gamma(\alpha-1)}, \end{aligned} $$

  and

$$ \begin{aligned} \left|\frac{\partial^\gamma}{\partial t}G(t,s)\right| &\leq \left|\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}\right| + \left|\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\right| \\ &\leq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} + \frac{1}{\Gamma(2-\gamma)}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right) \\ &\leq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} + \frac{2}{\Gamma(2-\gamma)\Gamma(\alpha-1)}. \end{aligned} $$

- For $0 \leq t \leq s \leq 1$,

$$ \begin{aligned} |G(t,s)| &= \frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\leq \frac{(1-t)}{\Gamma(\alpha)} + \frac{(1-t)}{\Gamma(\alpha-1)} \\ &\leq \frac{2}{\Gamma(\alpha-1)}, \end{aligned} $$

  and

$$ \begin{aligned} \left|\frac{\partial^\gamma}{\partial t}G(t,s)\right| &= \left|-\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\right| \\ &\leq \frac{1}{\Gamma(2-\gamma)}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right) \\ &\leq \frac{2}{\Gamma(2-\gamma)\Gamma(\alpha-1)}. \end{aligned} $$

By using the same above calculation, we obtain the estimation of $|H(t,s)|$ and $\left|\frac{\partial^\gamma}{\partial t}H(t,s)\right|$. The proof is complete. □

In the sequel, we denote

$$ G_\gamma(t,s) := \frac{\partial^\gamma}{\partial t}G(t,s), \quad t,s \in [0,1]\times[0,1]. $$

Moreover, we also use the following notations: $G^* := \max_{t,s\in[0,1]\times[0,1]}|G(t,s)|$ and

$$ G_\gamma^* := \max_{t,s\in[0,1]\times[0,1]}|G_\gamma(t,s)|. $$

Theorem 1. *Assume that the following four hypotheses hold:*

(H1) $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory condition.*

(H2) There exist $w \in L^1(0,1)$ *and* $c > 0$ *such that*

$$|f(t,u,v)| \leq w(t) + c\big(|u|^p + |v|^p\big) \text{ for } t \in (0,1) \text{ and } u, v \in \mathbb{R}. \tag{12}$$

(H3) There exist two strictly positive constants k_1 *and* k_2 *and a function* $\varphi_1 \in L^q((0,1), \mathbb{R}_+)$, $\frac{1}{p} + \frac{1}{q} = 1$, *such that for all* $t \in [0,1]$ *and* $x, y \in \mathbb{R}$, *we have*

$$\begin{aligned} |g(t,x) - g(t,y)| &\leq \varphi_1(t)|x-y|, \\ |q_1(x) - q_1(y)| &\leq k_1|x-y|, \\ |q_2(x) - q_2(y)| &\leq k_2|x-y|. \end{aligned}$$

(H4) There exists a real number $R > 0$ *such that*

$$\frac{R\Big[3\|\varphi_1\|_q + k_1 + k_2\Big]}{\Gamma(2-\gamma)\Gamma(1+\gamma)} + G^*_\gamma\left(\|w\|_1 + c\left(1 + \left(\frac{1}{\Gamma(\gamma+1)}\right)^p\right)R^p\right) \leq R. \tag{13}$$

Then, if

$$\frac{3\|\varphi_1\|_q + k_1 + k_2}{\Gamma(2-\gamma)\Gamma(1+\gamma)} < 1, \tag{14}$$

the boundary value problem (2) *has a solution in* $E^{\gamma,p}$.

Proof. We transform problem (2) into a fixed-point problem. Define two operators $F, L : E^{\gamma,p} \to E^{\gamma,p}$ by

$$Fy(t) = \int_0^1 G(t,s) f(s, y(s), D^\gamma y(s)) ds,$$

and

$$Ly(t) = \int_0^1 H(t,s) g(s, y(s)) ds + (2-t) \int_0^1 q_1(y(s)) ds + (t-1) \int_0^1 q_2(y(s)) ds.$$

Then, y is a solution of problem (2) if, and only if, y is a fixed point of $F + L$. We define the set B_R as follows:

$$B_R := \{u \in E^{\gamma,p}, \|u\|_{E^{\gamma,p}} \leq R\},$$

where R is the same constant defined in (H_3). It is clear that B_R is convex, closed, and a bounded subset of $E^{\gamma,p}$. We shall show that F, G satisfy the assumptions of Krasnosel'skii fixed-point theorem. The proof is given in several steps.

(i) We prove that F is continuous. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence such that $y_n \to y$ in $E^{\gamma,p}$. From (12) and Lemma 2, and for each $t \in [0,1]$, we obtain

$$\begin{aligned} &\left|\left({}^C D^\gamma_0 F y_n\right)(t) - \left({}^C D^\gamma_0 F y\right)(t)\right| \\ &\leq \int_0^1 |G_\gamma(t,s)|\, |f(s, y_n(s), D^\gamma y_n(s)) - f(s, y(s), D^\gamma y(s))| ds \\ &\leq G^*_\gamma \left\|N_f y_n - N_f y\right\|_1. \end{aligned}$$

Applying the L^p norm, we obtain that $\|Fy_n - Fy\|_{E^{\gamma,p}} \to 0$ when $y_n \to y$ in$E^{\gamma,p}$. Thus, the operator F is continuous.

(ii) Now, we prove that $Fx + Ly \in B_R$ for $x, y \in B_R$. Let $x, y \in B_R$, $t \in (0,1)$. In view of hypothesis $(H3)$, we obtain

$$\begin{aligned}\left|{}^C D_0^\gamma Fy(t)\right| &\le \int_0^1 |G_\gamma(t,s)||f(s,y(s),D^\gamma y(s))|ds \\ &\le G_\gamma^*\left(\|w\|_1 + c\left(\|y\|_p^p + \left\|{}^C D_0^\gamma y\right\|_p^p\right)\right) \\ &\le G_\gamma^*\left(\|w\|_1 + c\left(1 + \left(\frac{1}{\Gamma(\gamma+1)}\right)^p\right)R^p\right).\end{aligned}$$

Applying the L^p norm, we obtain that

$$\|Fy\|_{E^{\gamma,p}} \le G_\gamma^*\left(\|w\|_1 + c\left(1 + \left(\frac{1}{\Gamma(\gamma+1)}\right)^p\right)R^p\right). \tag{15}$$

Also,

$$\begin{aligned}\left|{}^C D_0^\gamma L(x)(t)\right| \le{}& \frac{3}{\Gamma(2-\gamma)}\int_0^1 |g(s,x(s))|ds \\ &+ \frac{1}{\Gamma(2-\gamma)}\int_0^1 |q_1(x(s))|ds + \frac{1}{\Gamma(2-\gamma)}\int_0^1 |q_2(x(s))|ds \\ \le{}& \frac{3}{\Gamma(2-\gamma)}\int_0^1 \varphi_1(s)|x(s)|ds + \frac{1}{\Gamma(2-\gamma)}\int_0^1 k_1|x(s)|ds \\ &+ \frac{1}{\Gamma(2-\gamma)}\int_0^1 k_2|x(s)|ds.\end{aligned}$$

Applying again the L^p norm, we obtain from Holder's inequality that

$$\|L(x)\|_{E^{\gamma,p}} \le \frac{3}{\Gamma(2-\gamma)}\left(\|\varphi_1\|_q\|x\|_p\right) + \frac{k_1}{\Gamma(2-\gamma)}\|x\|_p + \frac{k_2}{\Gamma(2-\gamma)}\|x\|_p.$$

In view of (5), we obtain

$$\|L(x)\|_{E^{\gamma,p}} \le \left[\frac{3\|\varphi_1\|_q}{\Gamma(2-\gamma)\Gamma(1+\gamma)} + \frac{k_1}{\Gamma(2-\gamma)\Gamma(1+\gamma)} + \frac{k_2}{\Gamma(2-\gamma)\Gamma(1+\gamma)}\right]\|x\|_{E^{\gamma,p}}.$$

Then,

$$\|L(x)\|_{E^{\gamma,p}} \le \frac{R\left[3\|\varphi_1\|_q + k_1 + k_2\right]}{\Gamma(2-\gamma)\Gamma(1+\gamma)}. \tag{16}$$

From (13), (15) and (16), we conclude that $Fx + Ly \in B_R$ whenever $x, y \in B_R$.

(iii) Let us prove that $F(B_R) = \{F(u) : u \in B_R\}$ is relatively compact in $E^{\gamma,p}$. Let $t \in (0,1)$ and $h > 0$, where $t + h \le 1$, and let $u \in D_R$. From (12), we obtain that

$$\begin{aligned}&\left|{}^C D_0^\gamma Fy(t+h) - {}^C D_0^\gamma Fy(t)\right| \\ &\le \int_0^1 |G_\gamma(t+h,s) - G_\gamma(t,s)||f(s,y(s),D^\gamma y(s))|ds \\ &\le \int_0^1 |G_\gamma(t+h,s) - G_\gamma(t,s)|\left[w(s) + c\left(|y(s)|^p + |D^\gamma y(s)|^p\right)\right]ds \\ &\le \sup_{t\in[0,1]}\left[\sup_{s\in[0,1]}|G_\gamma(t+h,s) - G_\gamma(t,s)|\right]\left(\|w\|_1 + c\left(1 + \left(\frac{1}{\Gamma(\gamma+1)}\right)^p\right)R^p\right).\end{aligned}$$

Therefore,

$$\frac{\|Fu(\cdot+h)-Fu(\cdot)\|_{E^{\gamma,p}}}{\left(\|w\|_1+c\left(1+\left(\frac{1}{\Gamma(\gamma+1)}\right)^p\right)R^p\right)} \leq \sup_{t\in[0,1]}\left[\sup_{s\in[0,1]}|G_\gamma(t+h,s)-G_\gamma(t,s)|\right]. \tag{17}$$

Then, $\|Fu(\cdot+h)-Fu(\cdot)\|_{E^{\gamma,p}} \to 0$ as $h\to 0$ for any $u\in B_R$, since G_γ is a continuous function on $[0,1]\times[0,1]$. From Lemma 3, we conclude that $F: B_R\to B_R$ is compact.

(iv) Finally, we prove that L is a contraction. Let $x,y\in D_R$ and $t\in(0,1)$. Then,

$$\begin{aligned}\left|{}^C D_0^\gamma L(x)(t) - {}^C D_0^\gamma L(y)(t)\right| &\leq \frac{3}{\Gamma(2-\gamma)}\int_0^1 |g(s,x(s))-g(s,y(s))|ds \\ &\quad + \frac{1}{\Gamma(2-\gamma)}\int_0^1 |q_1(x(s))-q_1(x(s))|ds \\ &\quad + \frac{1}{\Gamma(2-\gamma)}\int_0^1 |q_2(x(s))-q_2(x(s))|ds \\ &\leq \frac{3}{\Gamma(2-\gamma)}\int_0^1 \varphi_1(s)|x(s)-y(s)|ds \\ &\quad + \frac{k_1}{\Gamma(2-\gamma)}\int_0^1 |x(s)-x(s)|ds \\ &\quad + \frac{k_2}{\Gamma(2-\gamma)}\int_0^1 |x(s)-x(s)|ds.\end{aligned}$$

Applying the L^p norm and Holder's inequality, we obtain that

$$\begin{aligned}\|L(x)-L(y)\|_{E^{\gamma,p}} \leq{}& \frac{3}{\Gamma(2-\gamma)}\left(\|\varphi_1\|_q\|x-y\|_p\right)+\frac{k_1}{\Gamma(2-\gamma)}\left(\|x-y\|_p\right) \\ &+\frac{k_2}{\Gamma(2-\gamma)}\left(\|x-y\|_p\right).\end{aligned}$$

Then, from (5), we obtain

$$\|L(x)-L(y)\|_{E^{\gamma,p}} \leq \frac{3\|\varphi_1\|_q+k_1+k_2}{\Gamma(2-\gamma)\Gamma(1+\gamma)}\|x-y\|_{E^{\gamma,p}}.$$

From (14), the operator L is a contraction.

As a consequence of (i)–(iv), we conclude that $F: B_R\to B_R$ is continuous and compact. As a consequence of Krasnosel'skii fixed point theorem, we deduce that $F+G$ has a fixed point $y\in B_R\subset E^{\gamma,p}$, which is a solution to problem (2). □

We now illustrate our Theorem 1 with two examples.

Example 1. *Consider the fractional boundary value problem* (2) *with*

$$\begin{aligned}\alpha = 2.5,\quad \gamma &= 0.5,\quad p=3,\quad q=\frac{3}{2}, \\ f(t,x,y) &= \frac{\exp(-t)}{5}-\frac{1}{164\pi}\arctan\left(x^3+y^3\right), \\ g(t,x) &= \frac{1}{10}t^{\frac{2}{3}}x, \\ q_1(x)=q_2(x) &= \frac{1}{20}x,\end{aligned}$$

which we denote by (P_1). *Hypotheses* $(H1)$ *and* $(H2)$ *are satisfied for*

$$w(t)=\frac{\exp(-t)}{5}\in L^1(0,1),\quad c=\frac{1}{164\pi},\quad \varphi_1(t)=\frac{t^{\frac{2}{3}}}{10},\quad \textit{and}\quad k_1=k_2=\frac{1}{20}.$$

Moreover, we have

$$\frac{\left[3\|\varphi_1\|_q+k_1+k_2\right]}{\Gamma(2-\gamma)\Gamma(1+\gamma)}=\frac{\left[\frac{3}{10}\left(\frac{1}{2}\right)^{\frac{2}{3}}+\frac{1}{10}\right]}{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}\simeq 0.368<1.$$

If we choose $R=2$, *then we obtain*

$$\begin{aligned}&\frac{R\left[3\|\varphi_1\|_q+k_1+k_2\right]}{\Gamma(2-\gamma)\Gamma(1+\gamma)}+G^*_\gamma\left(\|w\|_1+c\left(1+\left(\frac{1}{\Gamma(\gamma+1)}\right)^p\right)R^p\right)-R\\ &\leq\frac{2\left[\frac{3}{10}\left(\frac{1}{2}\right)^{\frac{2}{3}}+\frac{1}{10}\right]}{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}+4.047\left(\frac{1}{5}+\frac{1}{164\pi}\left(1+\left(\frac{1}{\Gamma\left(\frac{3}{2}\right)}\right)^3\right)2^3\right)-2\\ &\simeq -0.301.\end{aligned}$$

Since all conditions of our Theorem 1 are satisfied, we conclude that the fractional boundary value problem (P_1) *has a solution in* $E^{\gamma,p}$.

Example 2. *Consider the fractional boundary value problem (2) with*

$$\begin{aligned}\alpha=2.7,\quad \gamma=0.7,\quad p=4,\quad q=\frac{4}{3},\\ f(t,x,y)=\frac{1}{10}\sin(t)+\frac{1}{200}\cos\left(x^4+y^4\right),\\ g(t,x)=\frac{1}{9\pi}t^{\frac{3}{4}}\arctan(x),\\ q_1(x)=q_2(x)=\frac{1}{10}\sin(x),\end{aligned}$$

which we denote by (P_2). *Hypotheses* $(H1)$ *and* $(H2)$ *are satisfied for*

$$w(t)=\frac{1}{10}\sin(t)\in L^1(0,1),\quad c=\frac{1}{200},\quad \varphi_1(t)=\frac{t^{\frac{3}{4}}}{9\pi}\quad \textit{and}\quad k_1=k_2=\frac{1}{10}.$$

Moreover, we have

$$\frac{\left[3\|\varphi_1\|_q+k_1+k_2\right]}{\Gamma(2-\gamma)\Gamma(1+\gamma)}=\frac{\left[\frac{1}{3\pi}\left(\frac{1}{2}\right)^{\frac{3}{4}}+\frac{1}{5}\right]}{\Gamma(1.3)\Gamma(1.7)}\simeq 0.323<1.$$

If we choose $R=2$, *then we obtain*

$$\begin{aligned}&\frac{R\left[3\|\varphi_1\|_q+k_1+k_2\right]}{\Gamma(2-\gamma)\Gamma(1+\gamma)}+G^*_\gamma\left(\|w\|_1+c\left(1+\left(\frac{1}{\Gamma(\gamma+1)}\right)^p\right)R^p\right)-R\\ &\leq\frac{2\left[\frac{1}{3\pi}\left(\frac{1}{2}\right)^{\frac{3}{4}}+\frac{1}{5}\right]}{\Gamma(1.3)\Gamma(1.7)}+3.9995\left(\frac{1}{10}+\frac{1}{200}\left(1+\left(\frac{1}{\Gamma(1.7)}\right)^4\right)2^4\right)-2\\ &\simeq -0.163.\end{aligned}$$

Since all conditions of our Theorem 1 are satisfied, we conclude that the fractional boundary value problem (P_2) *has a solution in* $E^{\gamma,p}$.

4. Discussion

The celebrated existence result of Ahmad and Nieto [17] for problem (1) is obtained via Krasnosel'skii fixed-point theorem in the space of continuous functions. For that, they needed to apply Ascoli's theorem in order to provide the compactness of the first part of the Krasnosel'skii operator. Here, we proved existence for the more general problem (2) in the fractional derivative Banach space $E^{\gamma,p}$, equipped with the norm (3). From norm (3), it is natural to deal with a subspace of $L^p \times L^p$. Since Ascoli's theorem is limited to the space of continuous functions for the compactness, we had to make use of a different approach to ensure existence of solution in the fractional derivative space $E^{\gamma,p}$. Our tool was the Kolmogorov–Riesz compactness theorem, which turned out to be a powerful tool to address the problem. To the best of our knowledge, the use of the Kolmogorov–Riesz compactness theorem to prove existence results for boundary value problems involving nonlinear integrodifferential equations of fractional order with integral boundary conditions is a completely new approach. In this direction, we are only aware of the work [35], where a necessary and sufficient condition of pre-compactness in variable exponent Lebesgue spaces is established and, as an application, the existence of solutions to a fractional Cauchy problem is obtained in the Lebesgue space of variable exponent. As future work, we intend to generalize our existence result to the variable-order case [36].

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