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A New Survey of Measures of Noncompactness and Their Applications

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Abstract: We present a survey of the theory of measures of noncompactness and discuss some fixed point theorems of Darbo's type. We apply the technique of measures of noncompactness to the characterization of classes of compact operators between certain sequence spaces, in solving infinite systems of integral equations in some sequence spaces. We also present some recent results related to the existence of best proximity points (pairs) for some classes of cyclic and noncyclic condensing operators in Banach spaces equipped with a suitable measure of noncompactness. Finally, we discuss the existence of an optimal solution for systems of integro–differentials.

Keywords: measures of noncompactness; fixed point theorems; compact operators between BK spaces; best proximity point (pair); cyclic (noncyclic) condensing operator; optimum solution; system of integro–differentials

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1. Introduction, Notations and Preliminaries

Measures of noncompactness play an important role in nonlinear functional analysis. They are important tools in metric fixed point theory, the theory of operator equations in Banach spaces, and the characterizations of classes of compact operators. They are also applied in the studies of various kinds of differential and integral equations.

The first measure of noncompactness, the function α, was defined and studied by Kuratowski [1] in 1930. In 1955, Darbo [2] was the first to use the function α to prove his famous fixed point theorem, Theorem 9.

The second measure of noncompactness was introduced by Goldenštein et al. [3,4], namely the Hausdorff or ball measure of noncompactness denoted by χ.

We refer to [5–10] for comprehensive studies.

Throughout, we use the following standard notations.

Let (X, d) be a metric space, $x \in X$ and $r > 0$. A subset M of X is relatively compact if it has compact closure $\overline{M}$. Further,

$$B(x,r) = B_X(x,r) = \{y \in X : d(y,x) < r\}, \ \overline{B}(x,r) = \overline{B}_X(x,r) = \{y \in X : d(y,x) \leq r\}$$
and
$$S(x,r) = S_X(x,r) = \{y \in X : d(y,x) = r\}$$

denote the open and closed ball, and the sphere of radius r centered at x, respectively. If X is a normed space, $x = 0$ and $r = 1$, then we write $B_X = B_X(0,1)$, $\overline{B}_X = \overline{B}_X(0,1)$ and $S_X = S_X(0,1)$. Let S and $\tilde{S}$ be subsets of a metric space (X,d), then:

$$\operatorname{diam}(S) = \sup\{d(s_1,s_2) : s_1,s_2 \in S\},\ \operatorname{dist}(S,\tilde{S}) = \inf\{d(s,\tilde{s}) : s \in S, \tilde{s} \in \tilde{S}\}$$
and
$$\operatorname{dist}(x,S) = \operatorname{dist}(\{x\},S)$$

are called the diameter of S, the distance of S, and $\tilde{S}$, and the distance of the point x and the set S, respectively.

If $M, S \subset X$ and $\varepsilon > 0$, then S is called an ε–net of M, if, for every $x \in M$, there exists $s \in S$ such that $d(x,s) < \varepsilon$; if S is finite, then S is a finite ε–net of M.

A sequence (b_n) in a linear metric space X is called a Schauder basis for X if for every $x \in X$ there exists a unique sequence $(\lambda_n)_{n=1}^{\infty}$ of scalars such that:

$$x = \sum_{n=1}^{\infty} \lambda_n x_n.$$

Let X and Y be Banach spaces. Then, $\mathcal{B}(X,Y)$ denotes the Banach space of all bounded linear operators from X into Y with the operator norm:

$$\|L\| = \sup\{\|L(x)\| : \|x\| = 1\} \text{ for all } L \in \mathcal{B}(X,Y);$$

we write $\mathcal{B}(X) = \mathcal{B}(X,X)$, for short. In particular, $X^* = \mathcal{B}(X,\mathbb{C})$ denotes the set of all continuous linear functionals on X with the norm:

$$\|f\| = \sup\{|f(x)| : \|x\| = 1\} \text{ for all } f \in X^*;$$

X^* is also referred to as the continuous dual of X.

An operator $L : X \to Y$ is compact if L maps bounded subsets of X to relatively compact subsets of Y, or equivalently, for any bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. The set of all compact operators from X to Y is denoted by $\mathcal{C}(X,Y)$; we write $\mathcal{C}(X) = \mathcal{C}(X,Y)$, for short.

Bk Spaces

The study of operators, in particular of matrix transformations, between sequence spaces is an important field of applications of measures of noncompactness. Here, we mention the standard notations and list the necessary results concerning *BK* spaces. We recommend the monographs [9,11–15] for the study of the theory of *BK* spaces.

We denote by ω the set of all complex sequences $x = (x_k)_{k=1}^{\infty}$, and by ℓ_∞, c, c_0, and ϕ the subsets of ω of all bounded, convergent, null, and finite sequences, and write:

$$\ell_p = \left\{ x = (x_k)_{k=1}^{\infty} \in \omega : \sum_{k=1}^{\infty} |x_k|^p < \infty \right\} \text{ for } 1 \le p < \infty.$$

Moreover, cs and bs denote the sets of all convergent and bounded series of complex numbers, respectively. Finally we write:

$$bv = \{x = (x_k)_{k=1}^{\infty} \in \omega : \Delta x = (x_k - x_{k+1})_{k=1}^{\infty} \in \ell_1\}$$

for the set of all sequences of bounded variation, and $bv_0 = bv \cap c_0$.

We write $e = (e_k)_{k=1}^{\infty}$ and $e^{(n)} = (e_k^{(n)})_{k=1}^{\infty}$ $(n \in \mathbb{N})$ for the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

Example 1. *The following facts are well known.*
(a) The set ω is a Fréchet space, that is, a complete linear metric space, with respect to:

$$d_\omega(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{|x_k - y_k|}{1 + |x_k - y_k|} \ (x, y \in \omega) \tag{1}$$

and convergence in (ω, d_ω) and coordinatewise convergence are equivalent; this means:

$$\lim_{n\to\infty} d\Big((x_k^{(n)})_{k=1}^{\infty}, (x_k)_{k=1}^{\infty}\Big) = 0 \text{ if and only if } \lim_{n\to\infty} x_k^{(n)} = x_k \text{ for each } k.$$

(b) The sets ℓ_∞, c, c_0, ℓ_p for $1 \le p < \infty$, bs, cs, bv and bv_0 are Banach spaces with respect to their natural norms defined by:

$$\begin{aligned} \|x\|_\infty &= \sup_k |x_k| \text{ on } \ell_\infty, c, c_0, \\ \|x\|_p &= \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \text{ on } \ell_p, \\ \|x\|_{bs} &= \sup_n \left|\sum_{k=1}^{n} x_k\right| \text{ on } bs, cs, \\ \|x\|_{bv} &= \sum_{k=1}^{\infty} |x_k - x_{k+1}| + \left|\lim_{k\to\infty} x_k\right| \text{ on } bv, \end{aligned}$$

and

$$\|x\|_{bv_0} = \sum_{k=1}^{\infty} |x_k - x_{k+1}| \text{ on } bv_0.$$

Now, we recall the definition of FK spaces, and their special cases BK spaces. FK and BK were first studied by Zeller [16–18].

Definition 1. *A Fréchet sequence space (X, d) is called an FK* space *if d is stronger than d_ω, that is, if the inclusion map $\iota : (X, d) \to (\omega, d_\omega)$ with $\iota(x) = x$ for all $x \in X$ is continuous. An FK space is called a BK* space *if its metric is given by a norm.*

We note that, by Example 1 (a), a Fréchet sequence space (X, d) is an FK space if convergence in d implies coordinatewise convergence.

Now, we recall the concept of the AK property.

Definition 2. *An FK space X has AK, if every sequence $x = (x_k)_{k=1}^{\infty} \in X$ has a unique representation:*

$$x = \sum_{k=1}^{\infty} x_k e^{(k)}, \text{ that is, } x = \lim_{n\to\infty} x^{[n]},$$

*where $x^{[n]} = \sum_{k=1}^{n} x_k e^{(k)}$ is the n–*section *x.*

Example 2. *The following facts are well known.*
(a) The FK space (X, d_ω) has AK.
(b) The Banach spaces of Example 1 (a) are BK space with respect to their natural norms.
(c) The BK spaces c_0, ℓ_p $(1 \le p < \infty)$, cs and bv_0 have AK; every sequence $x = (x_k)_{k=1}^{\infty} \in c$ has a unique representation:

$$x = \xi e + \sum_{k=1}^{\infty} (x_k - \xi) e^{(k)}, \tag{2}$$

where $\xi = \lim_{k\to\infty} x_k$; ℓ_∞ and bs have no Schauder bases.

We also recall the following concepts.

Let $X \supset \phi$. Then, the set,

$$X^{\beta} = \{a = (a_k)_{k=1}^{\infty} : ax = (a_k x_k)_{k=1}^{\infty} \in cs \text{ for all } x = (x_k)_{k=1}^{\infty} \in X\}$$

is called the *β–dual of X*.

Theorem 1 ([13], Theorem 7.2.9)**.**
Let $X \supset \phi$ be an FK space. Then, $X^{\beta} \subset X'$; this means that there is a linear one–to–one map $T : X^{\beta} \to X'$. If X has AK then T is onto.

Let $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex entries, $A_n = (a_{nk})_{k=1}^{\infty}$ denote the sequence in the n^{th} row of A, $x = (x_k)_{k=1}^{\infty}$ be a sequence and X and Y be subsets of ω. Then

$$A_n x = \sum_{k=1}^{\infty} a_{nk} x_k \text{ for } n \in \mathbb{N}$$

is called the n^{th} *A transform of the sequence x*, and $Ax = (A_n x)_{n=1}^{\infty}$ is called the *A transform of the sequence x* (provided all the series converge). Furthermore,

$$X_A = \{x = (x_k)_{k=1}^{\infty} \in \omega : Ax \in X\}$$

is the *matrix domain of A in X*. Finally (X, Y) denotes the class of all infinite matrices A with $X \subset Y_A$.

Now, we state the probably most important result concerning matrix transformations.

Theorem 2 ([13], Theorem 4.2.7)**.** *Matrix transformations between FK spaces are continuous.*

Finally, we state the relation between the classes $\mathcal{B}(X, Y)$ and (X, Y) for BK spaces X and Y; the first part is a special case of Theorem 2, and the second part is ([9], Theorem 9.3.3).

Theorem 3. *Let X and Y be BK spaces.*
(a) Then, $(X, Y) \subset \mathcal{B}(X, Y)$; this means, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$, where:

$$L_A(x) = Ax \text{ for all } x = (x_k)_{k=1}^{\infty} \in X. \tag{3}$$

(b) If X has AK then $\mathcal{B}(X, Y) \subset (X, Y)$; this means, every operator $L \in \mathcal{B}(X, Y)$ is given by *a matrix $A \in (X, Y)$, where:*

$$Ax = L(x) \text{ for all } x = (x_k)_{k=1}^{\infty} \in X. \tag{4}$$

Example 3 ([13], 8.4.1D)**.** *We have $L \in \mathcal{B}(\ell_1)$ if and only if:*

$$\|L\| = \|A\|_{(1,1)} = \sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty, \tag{5}$$

where $A = (a_{nk})_{n,k=1}^{\infty}$ represents L as in (4).

Proof. By Example 2 (c), ℓ_1 is a BK space with AK, hence $L \in \mathcal{B}(\ell_1)$ if and only if $A \in (\ell_1, \ell_1)$ with $L(x) = Ax$ for all $x \in \ell_1$.

(i) Let $L \in \mathcal{B}(\ell_1)$. Then we obtain for each $k \in \mathbb{N}$,

$$\|L(e^{(k)})\|_1 = \sum_{n=1}^{\infty} |A_n e^{(k)}| = \sum_{n=1}^{\infty} |a_{nk}| \leq \|L\| \cdot \|e^{(k)}\|_1 = \|L\|,$$

whence,

$$\sup_k \sum_{n=1}^{\infty} |a_{nk}| \leq \|L\| < \infty, \tag{6}$$

since $k \in \mathbb{N}$ was arbitrary.

(ii) Conversely, we assume that $\sup_k \sum\limits_{n=1}^{\infty} |a_{nk}| < \infty$. Then it follows for all $x \in \ell_1$ that:

$$\begin{aligned}\|L(x)\|_1 &= \sum_{n=1}^{\infty} |A_n x| \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |a_{nk}x_k| = \sum_{k=1}^{\infty}\left(|x_k| \sum_{n=1}^{\infty} |a_{nk}|\right) \\ &\leq \left(\sup_k \sum_{n=1}^{\infty} |a_{nk}|\right) \cdot \|x\|_1 < \infty,\end{aligned}$$

hence $L \in \mathcal{B}(\ell_1)$ and:

$$\|L\| \leq \sup_k \sum_{n=1}^{\infty} |a_{nk}|. \tag{7}$$

Finally (6) and (7) yield (5). □

2. Measures of Noncompactness and Their Properties

We start with the axioms of a measure of noncompactness on $\mathcal{M}_X$, the bounded subsets of a complete metric space (X, d); they can be found, for instance, in ([7], Definition II, 1.1).

We will also consider the axioms of measures of noncompactness in Banach spaces as in [5,6].

Definition 3. *Let X be a complete metric space. A set function $\phi : \mathcal{M}_X \to [0, \infty)$ is a measure of noncompactness on $\mathcal{M}_X$, if the following conditions are satisfied for all sets $Q, Q_1, Q_2 \in \mathcal{M}_X$,*

(MNC.1)	$\phi(Q) = 0$ *if and only if* $\bar{Q}$ *is compact*	*(regularity)*
(MNC.2)	$\phi(Q) = \phi(\overline{Q})$	*(invariance under closure)*
(MNC.3)	$\phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\}$	*(semi–additivity).*

Example 4. *Let X be a complete metric space and ϕ for all $Q \in \mathcal{M}_X$ be defined by $\phi(Q) = 0$ if Q is relatively compact, and $\phi(Q) = 1$ otherwise. Then ϕ is a measure of noncompactness, the so–called trivial measure of noncompactness.*

The following properties are easily obtained from Definition 3.

Proposition 1. *Let ϕ be a measure of noncompactness on a complete metric space X. Then ϕ has the following properties:*

$$Q \subset \tilde{Q} \text{ implies } \phi(Q) \leq \phi(\tilde{Q}) \quad \text{(monotonicity)}, \tag{8}$$

$$\phi(Q_1 \cap Q_2) \leq \min\{\phi(Q_1), \phi(Q_2)\} \text{ for all } Q_1, Q_2 \in \mathcal{M}_X. \tag{9}$$

$$\text{If } Q \text{ is finite then } \phi(Q) = 0 \quad \text{(non–singularity)}. \tag{10}$$

$$\left\{\begin{array}{l}\text{Generalized Cantor's intersection property} \\ \textit{If } (Q_n) \textit{ is a decreasing sequence of nonempty, bounded and closed sets in } X, \\ \textit{and } \lim_{n\to\infty}\phi(Q_n) = 0, \textit{ then the intersection} \\ \qquad\qquad Q_\infty = \bigcap Q_n \neq \emptyset \\ \textit{is compact.}\end{array}\right\} \tag{11}$$

Definition 4. *Let (X, d) be a complete metric space.*
(a) The function $\alpha : \mathcal{M}_X \to [0, \infty)$ with:

$$\alpha(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} S_k,\ S_k \subset X,\ diam(S_k) < \varepsilon\ (k = 1, 2, \ldots, n; n \in \mathbb{N})\right\}$$

for all $Q \in \mathcal{M}_X$ is called the Kuratowski measure of noncompactness; the real number $\alpha(Q)$ is called the Kuratowski measure of noncompactness of Q.
(b) The function $\chi : \mathcal{M}_X \to [0, \infty)$ with:

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} B(x_k, r_k),\ B(x_k, r_k) \subset X,\ r_k < \varepsilon\ (k = 1, 2, \ldots, n; n \in \mathbb{N})\right\}$$

for all $Q \in \mathcal{M}_X$ is called the Hausdorff or ball measure of noncompactness; the real number $\chi(Q)$ is called the Hausdorff or ball measure of noncompactness of Q.
(c) We recall that a subset S of (X, d) is said to be r–separated or r–discrete, if $d(x, y) \geq r$ for all distinct elements of S; the set S is called an r–separation. The function $\beta : \mathcal{M}_X \to [0, \infty)$ with:

$$\beta(Q) = \sup\{r > 0 : Q \text{ has an infinite } r\text{–separation}\},$$

or equivalently,

$$\beta(Q) = \inf\{r > 0 : Q \text{ does not an infinite } r\text{–separation}\}$$

for all $Q \in \mathcal{M}_X$ is called the separation or Istrăţescu measure of noncompactness; the real number $\beta(Q)$ is called the separation or Istrăţescu measure of noncompactness of Q.

Remark 1. *(a) If it is required that the centers of the balls that cover Q belong to Q then the real number $\chi_i(Q)$ is referred to a the inner Hausdorff measure on noncompactness of Q, and the function $\chi_i : \mathcal{M}_X \to [0, \infty)$ is called the inner Hausdorff measure on noncompactness.*

(b) (([9], Remark 7.7.3) The function χ_i is not a measure of noncompactness in the sense of Definition 3; it satisfies the conditions in (MNC.1) and (MNC.2), but (MNC.3) and (8) do not hold, in general. It can be shown that:

$$\chi(Q) \leq \chi_i(Q) \leq \alpha(Q) \text{ for all } Q \in \mathcal{M}_X.$$

The following results hold (([9], Theorems 7.6.3, 7.7.5 (a)) for α and χ, and ([7], Remark II.3.2) for β).

Theorem 4. *Let X be a complete metric space, and ϕ be any of the functions α, χ or β. Then ϕ is a measure of noncompactness which also satisfies the conditions in (8)–(11).*

If X is a Banach space, then some measures of noncompactness my satisfy some additional conditions. The convex hull of a subset M of a linear space is denoted and defined by:

$$\text{co}(M) = \bigcap\{C \supset M : C \text{ convex}\}.$$

The following results hold for α and χ by ([7], Proposition II.2.3 and Theorem II.2.4) and for β by ([7], Remark II.3.2 and Theorems II.3.4 and II.3.6).

Theorem 5. *Let X be a Banach space, and ϕ be any of the functions α, χ or β. Then we have for all $Q, Q_1, Q_2 \in \mathcal{M}_X$:*

$$\phi(\lambda Q) = |\lambda|\phi(Q) \text{ for all } \lambda \in \mathbb{C} \quad \text{(semi–homogeneity)} \tag{12}$$
$$\phi(Q_1 + Q_2) \leq \phi(Q_1) + \phi(Q_2) \quad \text{(algebraic semi–additivity)} \tag{13}$$
$$\phi(x + Q) = \phi(Q) \text{ for all } x \in X \quad \text{(translation invariance)} \tag{14}$$
$$\phi(co(Q)) = \phi(Q) \quad \text{(invariance under the passage to the convex hull).} \tag{15}$$

Remark 2. *Let X be an infinite dimensional Banach space.*
(a) ([7], Corollary II.2.6) Then,

$$\alpha(B_X) = \alpha(\overline{B}_X) = \alpha(S_X) = 2 \text{ and } \chi(B_X) = \chi(\overline{B}_X) = \chi(S_X) = 1.$$

(b) ([7], Remark II.3.2) The functions α, β and χ are equivalent, that is,

$$\chi(Q) \leq \beta(Q) \leq \alpha(Q) \leq 2\chi(Q) \text{ for all } Q \in \mathcal{M}_X.$$

(c) The Kuratowski and Hausdorff measures of noncompactness are closely related to the geometric properties of the space; the inequality $\chi(Q) \leq \alpha(Q)$ can be improved in some spaces ([19,20]). For instance, in Hilbert spaces H ([5,21]):

$$\sqrt{2}\chi(Q) \leq \alpha(Q) \leq 2\chi(Q) \text{ for all } Q \in \mathcal{M}_H,$$

and in ℓ_p for $1 \leq p < \infty$,

$$\sqrt[p]{\chi}(Q) \leq \alpha(Q) \leq 2\chi(Q) \text{ for all } Q \in \mathcal{M}_{\ell_p}.$$

(d) Studies on inequivalent measures of noncompactness can be found, for instance, in [22,23].

Whereas $\alpha(B_X)$ and $\chi(B_X)$ in infinite dimensional Banach spaces X are independent of the space, this is not true for β. The following result holds by ([7], Remark II.3.11 and Theorem II.3.12) for $1 \leq p \leq 2$ and $2 < p < \infty$, respectively.

Remark 3. *Let $1 \leq p < \infty$. Then $\beta(B_{\ell_p}) = 2^{1/p}$.*

There is a relation between the Hausdorff distance (Definition 5) and χ.

Definition 5. Let (X, d) be a metric space. The function $d_H : \mathcal{M}_X \times \mathcal{M}_X \to \mathbb{R}$ defined by:

$$d_H(S, \tilde{S}) = \max\left\{\sup_{s \in S} \text{dist}(s, \tilde{S}), \sup_{\tilde{s} \in \tilde{S}} \text{dist}(\tilde{s}, S)\right\} \text{ for all } S, \tilde{S} \in \mathcal{M}_X$$

is called the *Hausdorff distance*; the value $d_H(S, \tilde{S})$ is called the *Hausdorff distance of the sets S and $\tilde{S}$.*

Remark 4 ([9], Theorem 7.4.2)**.** *It is well known that if (X, d) is a metric space, then $(\mathcal{M}_X, d_H)$ is a semimetric space and $(\mathcal{M}^c_X, d_H)$ is a metric space, where $\mathcal{M}^c_X$ denotes the class of closed subsets in $\mathcal{M}_X$.*

We also mention the following result.

Theorem 6 ([9], Theorem 7.7.14)**.** *Let (X, d) be a complete metric space, and $\mathcal{N}^c_X$ denote the class of all nonempty compact sets in $\mathcal{M}_X$. Then we have:*

$$|\chi(Q_1) - \chi(Q_2)| \leq d_H(Q_1, Q_2) \text{ for all } Q_1, Q_2 \in \mathcal{M}_X,$$

and,

$$\chi(Q) = d_H(Q, \mathcal{N}_X^c) \text{ for all } Q \in \mathcal{M}_X.$$

Now, we list the axioms for measures of noncompactness in as stated by *Banaś and Goebel* [5].

Definition 6 ([5], Definition 3.1.1)**.** *Let X be a Banach space.*
A function $\psi : \mathcal{M}_X \to [0, \infty)$ is a measure of noncompactness on X if it satisfies the conditions (MNC.2) (invariance under closure), (8) (monotonicity), (14) (invariance under the passage to the convex hull), and,

(i) The family $ker(\psi) = \{Q \in \mathcal{M}_X : \psi(Q) = 0\} \neq \emptyset$ is contained in the family of all relatively compact subsets of X (compare this with (MNC.1));
(ii) i=If (Q_n) is a decreasing sequence of sets in $\mathcal{M}_X^c$ with $\lim_{n\to\infty} \psi(Q_n) = 0$, then

$$\emptyset \neq Q_\infty = \bigcap_{n=1}^{\infty} Q_n \in ker(\psi)$$

(compare with (11) (Cantor's generalized intersection property));
(iii) $\psi(\lambda Q + (1-\lambda)\tilde{Q}) \leq \lambda\psi(Q) + (1-\lambda)\psi(\tilde{Q})$ for all $\lambda \in (0,1)$ and all $Q, \tilde{Q} \in \mathcal{M}_X$ (convexity condition).

Remark 5. *(a) The functions α, χ, and β are measures on noncompactness in the sense of Definition 6. (b) The family $ker(\psi)$ is referred to as the kernel of the measure of noncompactness ψ.*
(c) A measure of noncompactness is said to be sublinear if it satisfies the conditions (12) and (13) (semi–homogeneity and algebraic semi–additivity). If $ker(\psi) = \mathcal{N}$, the family of all nonempty relatively compact sets, then ψ is said to be full.

Remark 6. *The term measure of noncompactness will always be used in the sense of Definition 3 unless explicitly stated otherwise.*

As an important application of the Hausdorff measure of noncompactness χ we are now going to state the famous *Goldenštein, Go'hberg, Markus theorem* [3] which provides an estimate for $\chi(Q)$ in Banach spaces with a Schauder basis.

Theorem 7 (Goldenštein, Go'hberg, Markus ([3] or [9], Theorem 7.9.3))**.**
Let X be a Banach space with a Schauder basis (b_k) and the functions $\mu_n : \mathcal{M}_X \to [0, \infty)$ for $n = 1, 2, \ldots$ be defined by:

$$\mu_n(Q) = \sup_{x \in Q} \|\mathcal{R}_n(x)\|,$$

where $\mathcal{R}_n : X \to X$ for each n is the function with:

$$\mathcal{R}_n(x) = \sum_{k=n+1}^{\infty} \lambda_k x_k \text{ for all } x = \sum_{k=1}^{\infty} \lambda_k x_k \in X.$$

Then, we have for all $Q \in \mathcal{M}_X$:

$$\frac{1}{a} \limsup_{n\to\infty} \mu_n(Q) \leq \chi(Q) \leq \inf_n \mu_n(Q) \leq \limsup_{n\to\infty} \mu_n(Q), \tag{16}$$

where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$ is the basis constant.

The following corollary of Theorem 7 is very useful for *BK* spaces with *AK* with a so–called *monotonous norm* $\|\cdot\|$, that is, a norm for which $\|x\| \leq \|\tilde{x}\|$ whenever $x, \tilde{x} \in X$ with $|x_k| \leq |\tilde{x}_k|$ for all k.

Corollary 1 ([9], Lemma 9.8.1)**.**
(a) Let $(X, \|\cdot\|)$ *be a monotonous BK space with AK and* $\mathcal{R}_n(x) = x - x^{[n]}$ $(x \in X)$ *for each* n*. Then we have:*

$$\chi(Q) = \lim_{n\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_n(x)\|\right) \text{ for all } Q \in \mathcal{M}_X.$$

(b) Let $\mathcal{R}_n : c \to c$ *for* $n = 1, 2, \ldots$ *be defined by* $R_n(x) = \sum_{k=n+1}^{\infty}(x_k - \xi)e^{(k)}$ *for all* $x = \xi e + \sum_{k=1}^{\infty}(x_k - \xi)e^{(k)} \in c$*, where* $\xi = \lim_{k\to\infty} x_k$*. Then,*

$$\lim_{n\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_n(x)\|\right) \text{ exists for all } Q \in \mathcal{M}_c,$$

and $a = \lim_{n\to\infty}\|\mathcal{R}_n\| = 2$*.*

Example 5. *(a) Since* ℓ_p $(1 \le p < \infty)$ *and* c_0 *are monotonous BK spaces with AK, Corollary 1 (a) yields:*

$$\chi(Q) = \begin{cases} \lim\limits_{n\to\infty}\left(\sup\limits_{x\in Q}\left(\sum\limits_{k=n+1}^{\infty}|x_k|^p\right)^{1/p}\right) & (Q \in \mathcal{M}_{\ell_p}) \\ \lim\limits_{n\to\infty}\left(\sup\limits_{x\in Q}\left(\sup\limits_{k\ge n+1}|x_k|\right)\right) & (Q \in \mathcal{M}_{c_0}). \end{cases}$$

(b) We obtain from Corollary 1(b):

$$\frac{1}{2}\lim_{n\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_n(x)\|_\infty\right) \le \chi(Q) \le \inf_n\left(\sup_{x\in Q}\|\mathcal{R}_n(x)\|_\infty\right) \ (Q \in \mathcal{M}_c),$$

where, for each $x \in c$ *with* $\xi_x = \lim_{k\to\infty} x_k$*,*

$$\|\mathcal{R}_n(x)\|_\infty = \sup_{k\ge n}|x_k - \xi|.$$

Measures of Noncompactness of Operators

Contractive and *condensing maps* play an important role in fixed point theory, for instance in *Banach's* and *Darbo's* eminent fixed point theorems. Now, we are going to introduce these concepts, and *measures of noncompactness of operators*.

Definition 7 ([7], Definition II.5.1)**.** *Let* X *and* Y *be complete metric spaces,* ϕ *and* ψ *be measures of noncompactness on* X *and* Y*, respectively, and* $L : D \subset X \to Y$ *be a map. Then:*
(a) L *is a* (ϕ, ψ)*–contractive operator with constant* $k > 0$*, or* $k - (\phi, \psi)$*–contractive, for short, if* L *is continuous and satisfies:*

$$\psi(L(Q)) \le k \cdot \phi(Q) \text{ for every } Q \in \mathcal{M}_D.$$

If $X = Y$ *and* $\psi = \phi$*,* L *is referred to as a* $k - \phi$*–contractive operator.*
(b) L *is a* (ϕ, ψ)*–condensing operator with constant* $k > 0$*, or* $k - (\phi, \psi)$*–condensing, for short, if* L *is continuous and satisfies*

$$\psi(L(Q)) < k \cdot \phi(Q) \text{ for every non–relatively compact set } Q \in \mathcal{M}_D.$$

If $X = Y$ *and* $\psi = \phi$*,* L *is referred to as a* $k - \phi$*–condensing operator. Moreover, if* $k = 1$*, then* L *is said to be a* ϕ*–condensing operator.*

Remark 7 ([7], Proposition II.5.3)**.**
(a) If $\phi = \alpha$*, the Kuratowski measure of noncompactness, then the* $k - \alpha$*–contractive (condensing)*

operators are called k–set contractive (condensing).
(b) If $\phi = \chi$*, the Hausdorff measure of noncompactness, then the* $k - \chi$*–contractive (condensing) operators are called* k–ball contractive (condensing).
(c) Every compact operator is $k - (\phi, \psi)$*–contractive and* $k - (\phi, \psi)$*–condensing for all* $k > 0$.
(d) Every $k - (\phi, \psi)$*–condensing operator is* $k - (\phi, \psi)$*–contractive, but the converse is not true, in general.*
(e) An example of a set–condensing operator which is not k–set–contractive for any $k \in [0, 1)$ *can be found in ([7], Example II.6).*

We recall that a map f from a metric space (X, d) into itself is called a *contraction* if there exists a constant $c \in (0, 1)$ such that:

$$d(f(x), f(y)) \leq c \cdot d(x, y) \text{ for all } x, y \in X.$$

Using the above concepts, we can now state the famous fixed point theorems by Banach et al. Banach's fixed point theorem is also referred to as the *Banach contraction principle*. We recommend the monographs [24–28] and the survey paper [29] for further studies on fixed point theorems.

Theorem 8 (Banach's fixed point theorem)**.** *Every contraction from a complete metric space into itself has a unique fixed point.*

Theorem 9 (Darbo's fixed point theorem [2])**.** *Let* X *be a Banach space and* $C \in \mathcal{M}_X^c$ *be nonempty and convex. If* $L : C \to C$ *is a k–contractive set operator for some* $k \in (0, 1)$*, then* L *has a fixed point in* C*.*

Darbo's fixed point theorem is a generalization of Schauder's fixed point theorem.

Theorem 10 (Schauder's fixed point theorem) ([30], Theorem 1)**.** *Every continuous map from a nonempty, compact and convex subset* C *of a Banach space into* C *has a fixed point.*

The next result is a generalization of Theorem 9.

Theorem 11 (Darbo–Sadovskiĭ ([31,32] or ([7], Theorem II.5.4)))**.**
Let X *be a Banach space,* ϕ *be a measure of noncompactness which is invaraint under the passage to the convex hull,* $C \in \mathcal{M}_X^c$ *be nonempty and convex, and* $L : M \to M$ *be a* ϕ*–condensing operator. Then* L *has a fixed point.*

The following example shows that Theorem 11 need not hold for one–contractive operators f.

Example 6 ([7], Example II.7)**.**
We define the operator $f : \overline{B}_{\ell_2} \to \overline{B}_{\ell_2}$ *by:*

$$f(x) = f((x_k)_{k=1}^{\infty}) = \left(\sqrt{1 - \|x\|_2^2}, x_1, x_2, \dots\right) \text{ for all } x = (x_k)_{k=1}^{\infty} \in \overline{B}_{\ell_2}.$$

Then, we can write $f = g + h$*, where* g *is the mapping with:*

$$g(x) = g((x_k)_{k=1}^{\infty}) = \sqrt{1 - \|x\|_2^2}\, e^{(1)},$$

and $h(x) = (0, x_1, x_2, \dots)$ *is an isometry.*
Then f *is a well–defined, continuous operator, and every bounded subset* Q *in* $\overline{B}_{\ell_2}$ *satisfies:*

$$\alpha(f(Q)) \leq \alpha(g(Q)) + \alpha(h(Q)) = \alpha(h(Q)) = \alpha(Q).$$

Consequently, f is a one–set–contractive operator, but f has no fixed points.

If f had a fixed point $x \in \overline{B}_{\ell_2}$, then we would have $x_k = x_{k+1}$ for all k. Since $x \in \ell_{\ell_2}$, this would imply $x_k = 0$ for all k, and then $f(x) = \sqrt{1 - \|x\|_2^2}\, e^{(0)} = e^{(0)} = (0, 0, \cdots)$. This is a contradiction.

Definition 8 ([9], Definition 7.11.1)**.**
Let ϕ and ψ be measures of noncompactness on the Banach spaces X and Y, respectively.
*(a) An operator $L : X \to Y$ is said to be (ϕ, ψ)–*bounded*, if:*

$$L(Q) \in \mathcal{M}_Y \text{ for all } Q \in \mathcal{M}_X,$$

and if there exist a nonnegative real number c such that:

$$\psi(L(Q)) \leq c \cdot \phi(Q) \text{ for all } Q \in \mathcal{M}_X. \tag{17}$$

(b) If an operator L is (ϕ, ψ)–bounded, then the number,

$$\|L\|_{(\phi,\psi)} = \inf\{c \geq 0 : (17) \text{ holds}\} \tag{18}$$

*is called the (ϕ, ψ)–*operator norm of L *or* (ϕ, ψ)–measure of noncompactness of L.
If $\psi = \phi$, we write $\|L\|_\phi = \|L\|_{(\phi,\psi)}$, for short.

Remark 8. *A (ϕ, ψ)–bounded operator is a c–contractive (ϕ, ψ)– operator between Banach spaces for some c by Definitions 8 (a) and 7 (a).*

Theorem 12 ([9], Theorem 7.11.4)**.** *Let X and Y be infinite dimensional Banach spaces and $L \in \mathcal{B}(X, Y)$. Then we have:*

$$\|L\|_\chi = \chi(L(S_X)) = \chi(L(B_X)) = \chi(L(\overline{B}_X)). \tag{19}$$

Theorem 13 ([9], Theorem 7.11.5)**.** *Let X, Y, and Z be Banach spaces, $L \in \mathcal{B}(X, Y)$ and $\tilde{L} \in \mathcal{B}(Y, Z)$. Then $\|\cdot\|_\chi$ is a seminorm on $\mathcal{B}(X, Y)$, and:*

$$\begin{aligned} \|L\|_\chi &= 0 \text{ if and only if } L \in \mathcal{C}(X, Y); \\ \|L\|_\chi &\leq \|L\| \text{ for all } L \in \mathcal{B}(X, Y); \\ \|\tilde{L} \circ L\|_\chi &\leq \|\tilde{L}\|_\chi \cdot \|L\|_\chi \text{ for all } L \in \mathcal{B}(X, Y) \text{ and } \tilde{L} \in \mathcal{B}(Y, Z). \end{aligned} \tag{20}$$

In Example (3), we characterized the class $\mathcal{B}(\ell_1)$ and established a formula for the norm of operators in $\mathcal{B}(\ell_1)$. Now we characterize the class $\mathcal{C}(\ell_1)$.

Example 7 (Goldenštein, Go'hberg, Markus)**.** *([3] or ([9], Theorem 7.9.3)) Let $L \in \mathcal{B}(\ell_1)$. Then:*

$$\|L\|_\chi = \lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty} |a_{nk}|\right), \tag{21}$$

where $A = (a_{nk})_{n,k=1}^{\infty}$ represents L.

Furthermore, $L \in \mathcal{C}(\ell_1)$ if and only if:

$$\lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty} |a_{nk}|\right) = 0. \tag{22}$$

Proof. Let $A = (a_{nk})_{n,k=1}^{\infty}$ be any infinite matrix. Then, for each $m \in \mathbb{N}$, let $A^{<m>}$ be the matrix with the rows $A_n^{<m>} = 0$ for $n \leq m$ and $A_n^{<m>} = A_n$ for $n \geq m + 1$. It is clear that:

$$(\mathcal{R}_m \circ L)(x) = A^{<m>}(x) \text{ for all } x \in \ell_1.$$

Now (19), Example 5 (a) and (5) in Example 3 imply:

$$\|L\|_\chi = \chi\big(L(\overline{B}_{\ell_1})\big) = \lim_{m\to\infty}\left(\sup_{\|x\|_1=1}\|(\mathcal{R}_m \circ L)(x)\|_1\right) = \lim_{m\to\infty}\|A^{<m>}\|_{(1,1)}$$
$$= \lim_{m\to\infty}\left(\sup_k \sum_{n=m+1}^{\infty}|a_{nk}|\right),$$

whence (21).

Furthermore, it follows from (21) and (20) that $L \in \mathcal{C}(\ell_1)$ if and only if (22) is satisfied. □

3. Bounded and Compact Operators on the Generalized Hahn Space

Here, we apply the results of Sections 1 and 2 to the characterizations of classes of bounded and compact linear operators from the generalized Hahn space h_d into itself and into the spaces of sequences that are strongly summable by the Cesàro method of order one, with index $p \geq 1$, and into the spaces of strongly convergent sequences. We also establish identities or estimates for the Hausdorff measure of noncompactness of those operators.

For further studies on the generalized Hahn space we recommend the research papers [33–35].

The Properties of Our Sequence Spaces

We recall the definition of the operators $\Delta, \Delta^- : \omega \to \omega$ of the forward and backward differences given for all sequences $x = (x_k)_{k=1}^\infty$ by:

$$\Delta x_k = x_k - x_{k+1} \text{ and } \Delta^- x_k = x_k - x_{k-1} \text{ for } k = 1, 2, \ldots, \text{ respectively.}$$

Throughout, we use the convention that every term with an index ≤ 0 is equal to 0.

The *original Hahn space*:

$$h = \left\{x = (x_k)_{k=1}^\infty \in \omega : \sum_{k=1}^\infty k|\Delta x_k| < \infty\right\} \cap c_0$$

was introduced by Hahn in 1922 [36] in connection with the theory of singular integrals. K. C. Rao showed [37] that h is a BK space with AK with the norm:

$$\|x\| = \sum_{k=1}^\infty k|\Delta x_k| \ (x \in h).$$

Goes [38] introduced and studied the generalized Hahn space:

$$h_d = \left\{x = (x_k)_{k=1}^\infty \in \omega : \sum_{k=1}^\infty d_k|\Delta x_k| < \infty\right\} \cap c_0,$$

where $d = (d_k)_{k=1}^\infty$ is a given sequence of positive real numbers d_k $(k = 1, 2, \ldots)$. If $d_k = k$ for all k, then h_d reduces to the original Hahn space h, and if $d = e$ then $h_e = bv_0$.

Let $1 \leq p < \infty$. The sets:

$$w_0^p = \left\{x = (x_k)_{k=1}^\infty \in \omega : \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^n |x_k|^p\right) = 0\right\},$$
$$w^p = \left\{x = (x_k)_{k=1}^\infty \in \omega : x - \xi e \in w_0^p \text{ for some } \xi \in \mathbb{C}\right\}$$

and

$$w_\infty^p = \left\{ x = (x_k)_{k=1}^\infty \in \omega : \sup_n \left(\frac{1}{n} \sum_{k=1}^n |x_k|^p \right) < \infty \right\}$$

of sequences that are strongly summable to zero, strongly summable and strongly bounded by the Cesàro method of order 1, with index p, were first introduced and studied by I. J. Maddox [39]. We write $w_0 = w_0^1$, $w = w^1$ and $w_\infty = w^\infty$, for short.

The sets:

$$[c_0] = \{x = (x_k)_{k=1}^\infty \in \omega : \Delta^-((kx_k)_{k=1}^\infty) \in w_0\},$$
$$[c] = \{x = (x_k)_{k=1}^\infty \in \omega : x - \xi \in [c_0] \text{ for some } \xi \in \mathbb{C}\}$$

and:

$$[c_\infty] = \{x = (x_k)_{k=1}^\infty \in \omega : \Delta^-((kx_k)_{k=1}^\infty) \in w_\infty\}$$

of sequences that are strongly convergent to zero, strongly convergent, and strongly bounded were introduced and studied by Kuttner and Thorpe [40] and later generalized and studied in [41,42].

Throughout, we assume that the sequence d for h_d is always a monotone increasing unbounded sequence of positive real numbers.

The following result holds.

Theorem 14 ([43], Proposition 2.1). *The space $(h_d, \|\cdot\|_{h_d})$ is a BK space with AK, where:*

$$\|x\|_{h_d} = \sum_{k=1}^\infty d_k |\Delta x_k| \textit{ for all } x \in h_d.$$

The following example shows that h_d may not have AK, in general, if the sequence d is not monotone increasing.

Example 8. *Let $d = (d_k)_{k=1}^\infty$ and $x = (x_k)_{k=1}^\infty$ be the sequences with:*

$$d_k = \begin{cases} k & (k = 2^\nu) \\ 1 & (k \neq 2^\nu) \end{cases} \quad (\nu = 0, 1, \dots)$$

and:

$$x_K = 0 \ (k = 1, 2, 3) \text{ and } x_k = \begin{cases} \frac{1}{k} & (k \neq 2^\nu + 1) \\ [1ex] \frac{1}{k-1} & (k = 2^\nu + 1) \end{cases} \quad (\nu = 2, 3, \dots).$$

Then, clearly, $x \in c_0$, and also,

$$\|x\|_{h_d} = \frac{1}{4} + \sum_{k=4, k\neq 2^\nu}^\infty |x_k - x_{k+1}|,$$

where,

$$x_k - x_{k+1} = \begin{cases} \frac{1}{k-1} - \frac{1}{k+1} = \frac{2}{k^2-1} & (k = 2^\nu + 1) \\ [2ex] \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k-1)} & (k \neq 2^\nu + 1) \end{cases} \quad (\nu = 2, 3, \dots),$$

hence, $\|x\|_{h_d} < \infty$. Thus, we have $x \in h_d$.

On the other hand, let $\nu \geq 2$ be given. Then we have for $x^{[2^\nu]}$,

$$\left\| x - x^{[2^\nu]} \right\|_{h_d} \geq d_{2^\nu} \left| x_{2^\nu} - x_{2^\nu+1} - \left(x^{[2^\nu]}_{2^\nu} - x^{[2^\nu]}_{2^\nu+1} \right) \right| = 2^\nu \left| \frac{1}{2^\nu} - \frac{1}{2^\nu} - \left(\frac{1}{2^\nu} - 0 \right) \right| = 1,$$

hence $x^{[m]} \not\to 0$ as $m \to \infty$.

Let:

$$bs_d = \left\{ a \in \omega : \sup_n \frac{1}{d_n} \left| \sum_{k=1}^{n} a_k \right| < \infty \right\}$$

and:

$$\|a\|_{bs_d} = \sup_n \frac{1}{d_n} \left| \sum_{k=1}^{n} a_k \right| \text{ for all } a \in bs_d.$$

Remark 9. *Since bs is a BK space with $\|a\|_{bs} = \sup_n |\sum_{k=1}^{n} a_k|$ for all $a \in bs$ by ([13], Example 4.3.17), and bs_d is the matrix domain in bs of the triangle $T = (t_{nk})_{n,k=1}^{\infty}$ with $t_{nk} = 1/d_n$ for $1 \leq k \leq n$ and $n = 1, 2, \ldots$, bs_d is a BK space with $\| \cdot \|_{bs_d}$ by ([13], Theorem 4.3.12).*

Theorem 15 ([43], Proposition 2.3)**.** *The spaces h_d^* and h_d^β of h_d are norm isomorphic.*

Now, we list the fundamental topological properties of the sets w_0^p, w^p, w_∞^p $(1 \leq p < \infty)$, $[c_0]$, $[c]$ and $[c_\infty]$. The results are analogous to those for c_0, c and ℓ_∞ in Example 2.

Theorem 16. *(a)* ([39]) *Let $1 \leq p < \infty$. Then the sets w_0^p, w^p, and w_∞^p are BK spaces with their natural norms:*

$$\|x\|_{w_\infty^p} = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |x|^p \right)^{1/p};$$

w_0^p is a closed subspace of w^p and w^p is a closed subspace of w_∞; w_0^p has AK, every sequence $x = (x_k)_{k=1}^{\infty} \in w^p$ has a unique representation (2), where ξ is the unique complex number such that $x - \xi \cdot e \in w_0^p$; w_∞^p has no Schauder basis.
(b) ([42], Theorem 2) *The sets $[c_0]$, $[c]$, and $[c_\infty]$ are BK spaces with their natural norms*

$$\|x\|_{[c_\infty]} = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^-(kx_k)| \right);$$

$[c_0]$ is a closed subspace of $[c]$ and $[c]$ is a closed subspace of $[c_\infty]$; $[c_0]$ has AK, every sequence $x = (x_k)_{k=1}^{\infty} \in [c]$ has a unique representation (2), where ξ is the unique complex number such that $x - \xi \cdot e \in [c_0]$; $[c_\infty]$ has no Schauder basis.

3.1. Some Classes of Bounded Linear Operators on the Generalized Hahn Space

In this subsection, we characterize the classes $\mathcal{B}(h_d, Y)$ where Y is any of the spaces h_d, w_0^p, w^p, w_∞^p for $1 \leq p < \infty$, $[c_0]$, $[c]$ and $[c_\infty]$. We also establish formulas for the norm of the corresponding operators.

We recall the following concept and results needed in the proofs of our characterizations.

Definition 9. *([13], Definition 7.4.2) Let X be a BK space. A subset E of the set ϕ called a determining set for X if $D(X) = \overline{B}_X \cap \phi$ is the absolutely convex hull of E.*

Proposition 2 ([43], Proposition 3.2)**.** *Let,*

$$s(d,k) = \frac{1}{d_k} \cdot e^{[k]} \text{ for each } k \in \mathbb{N}, \text{ and } E = \{s(d,k) : k \in \mathbb{N}\}.$$

Then E is a determining set for h_d.

Proposition 3 ([13], Theorem 8.3.4)**.**
Let X be a BK space with AK, E be a determining set for X, and Y be an FK space. Then, $A \in (X,Y)$ if and only if:

(i) The columns of A belong to Y, that is, $A^k = (a_{nk})_{n=1}^{\infty} \in Y$ for all k,

and,

(ii) $L(E)$ is a bounded subset of Y, where $L(x) = Ax$ for all $x \in X$.

Since (h_d) is a BK space with AK by Theorem 14, and the spaces Y for $Y = w_0^p, w^p, w_\infty^p$ $(1 \leq p < \infty)$, $[c_0]$, $[c]$, and $[c_\infty]$ are BK spaces by Theorem 16, it follows from Theorem 3 that $L \in \mathcal{B}(h_d, Y)$ if and only if $A \in (h_d, Y)$, where A is the infinite matrix that represents L as in (4). We are going to use this throughout.

Theorem 17 ([43], Theorem 3.9 and Corollary 3.15 (a))**.**
We have $L \in \mathcal{B}(h_d)$ if and only if:

$$\lim_{n\to\infty} a_{nk} = 0, \text{ for all } k, \tag{23}$$

and:

$$\|A\|_{(h_d,h_d)} = \sup_m \left(\frac{1}{d_m} \sum_{n=1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| \right) < \infty. \tag{24}$$

Moreover, if $L \in \mathcal{B}(h_d)$ then:

$$\|L\| = \|A\|_{(h_d,h_d)}. \tag{25}$$

Proof. Since h_d is a BK spaces with AK by Theorem 14, we apply Proposition 3 and observe that:

$$E = \left\{ y^{(m)} = \frac{1}{d_m} e^{[m]} : m \in \mathbb{N} \right\}$$

is a determining set for h_d by Proposition 2.

First, the condition in (ii) of Proposition 3 is:

$$\sup_m \|Ay^{(m)}\|_{h_d} < \infty \text{ for all } y^{(m)} \in E \tag{26}$$

and:

$$Ay^{(m)} \in c_0 \text{ for all } y^{(m)} \in E. \tag{27}$$

First, we obtain:

$$\|Ay^{(m)}\|_{h_d} = \sum_{n=1}^{\infty} d_n \left| A_n y^{(m)} - A_{n+1} y^{(m)} \right| = \frac{1}{d_m} \sum_{n=1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right|$$

for all $m \in \mathbb{N}$, and so (26) is (24).

It is easy to see that (27) and (23) are equivalent.

Now, we show that condition (i) in Proposition 3 is redundant. Since $A^k \in c_0$ for each k by (23), it follows from (24) that:

$$\begin{aligned} \|A^k\|_{h_d} &= \sum_{n=1}^{\infty} d_n |a_{nk} - a_{n+1,k}| = \sum_{n=1}^{\infty} d_n \left| \sum_{j=1}^{k} (a_{nj} - a_{n+1,j}) - \sum_{j=1}^{k-1} (a_{nj} - a_{n+1,j}) \right| \\ &\leq d_k \sum_{n=1}^{\infty} d_n \left| A_n y^{(k)} - A_n y^{(k)} \right| + d_{k-1} \sum_{n=1}^{\infty} d_n \left| A_n y^{(k-1)} - A_n y^{(k-1)} \right| \end{aligned}$$

$$= d_k \left\| Ay^{(k)} \right\|_{h_d} + d_{k-1} \left\| Ay^{(k-1)} \right\|_{h_d}$$
$$= d_k \| Ay^{(k)} \|_{h_d} + d_{k-1} \| Ay^{(k-1)} \|_{h_d} < \infty \text{ for all } k.$$

Finally, we show that $L \in \mathcal{B}(h_d)$ implies (25).

We write B for the matrix with the rows $B_n = A_n - A_{n+1}$ for all n. Let $m \in \mathbb{N}$ be given. Then, we have by Abel's summation by parts for each n:

$$L_n(x^{[m]}) - L_{n+1}(x^{[m]}) = \sum_{k=1}^{m} b_{nk} x^{[m]} = \sum_{k=1}^{m-1} \Delta x_k \sum_{j=1}^{k} b_{nj} + x_m \sum_{j=1}^{m} b_{nj}$$
$$= \sum_{k=1}^{m-1} d_k \Delta x_k \frac{1}{d_k} \sum_{j=1}^{k} b_{nj} + d_m x_m \frac{1}{d_m} \sum_{j=1}^{m} b_{nj}.$$

Since h_d has AK and $x \in h_d$, it follows that:

$$0 \leq |d_m x_m| = \sum_{k=m}^{\infty} d_k |\Delta x_k^{[m]}| \leq \sum_{k=1}^{\infty} d_k \left| \Delta (x_k^{[m]} - x_k) \right| + \sum_{k=m}^{\infty} d_k |\Delta x_k|$$
$$= \left\| x^{[m]} - x \right\|_{h_d} + \sum_{k=m}^{\infty} d_k |\Delta x_k| \to 0 \text{ as } m \to \infty.$$

Furthermore, each functional L_n is continuous, since h_d is a BK space, and so for each $n \in \mathbb{N}$ and all $x \in h_d$:

$$L_n(x) - L_{n+1}(x) = \sum_{k=1}^{\infty} d_k \Delta x_k \frac{1}{d_k} \sum_{j=1}^{k} b_{nj},$$

hence for all $x \in h_d$

$$\|L(x)\|_{h_d} = \sum_{n=1}^{\infty} d_n |L_n(x) - L_{n+1}(x)| \leq \sum_{n=1}^{\infty} d_n \sum_{k=1}^{\infty} d_k |\Delta x_k| \frac{1}{d_k} \left| \sum_{j=1}^{k} b_{nj} \right|$$
$$= \sum_{k=1}^{\infty} d_k |\Delta x_k| \left(\frac{1}{d_k} \sum_{n=1}^{\infty} d_n \left| \sum_{j=1}^{k} b_{nj} \right| \right) \leq \sup_k \frac{1}{d_k} \sum_{n=1}^{\infty} d_n \left| \sum_{j=1}^{k} b_{nj} \right| \cdot \|x\|_{h_d}$$
$$= \sup_k \frac{1}{d_k} \sum_{n=1}^{\infty} d_n \left| \sum_{j=1}^{k} (a_{nj} - a_{n+1,j} \right| \cdot \|x\|_{h_d},$$

that is,

$$\|L\| \leq \|A\|_{(h_d, h_d)}. \tag{28}$$

To show the converse inequality, let $m \in \mathbb{N}$ be given and $x^{(m)} = e^{[m]}/d_m$. Then it follows that:

$$\|L(x^{(m)})\|_{h_d} = \sum_{n=1}^{\infty} d_n \left| A_n x^{(m)} - A_{n+1} x^{(m)} \right| = \frac{1}{d_m} \sum_{n=1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| \leq \|L\|.$$

Since $m \in \mathbb{N}$ was arbitrary, we obtain $\|A\|_{(h_d, h_d)} \leq \|L\|$ and this and (28) together imply (25). □

Remark 10. *It was shown in ([37], Proposition 10) that $A \in (h, h)$ if and only if:*

$$\text{(i)}: \quad \lim_{n \to \infty} a_{nk} = 0,$$

$$\text{(ii)}: \quad \sum_{n=1}^{\infty} n |a_{nm} - a_{n+1,m}| \textit{ converges for } m = 1, 2, \ldots,$$

$$\text{(iii)}: \quad \sup_m \left(\frac{1}{m} \sum_{n=1}^{\infty} n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| \right) < \infty.$$

It seems that the condition in (ii) *is redundant.*

Proof. We show more generally that (24) implies:

$$\text{(iv):} \quad \sum_{n=1}^{\infty} d_n |a_{nm} - a_{n+1,m}| \text{ converges for all } m.$$

Let (24) be satisfied. Then:

$$M_m = \sum_{n=1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| < \infty \text{ for all } m,$$

hence:

$$\begin{aligned}\sum_{n=1}^{\infty} d_n |a_{nm} - a_{n+1,m}| &= \sum_{n=1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) - \sum_{k=1}^{m-1} (a_{nk} - a_{n+1,k}) \right| \\ &\leq M_m + M_{m-1} < \infty \text{ for all } m,\end{aligned}$$

that is, (iv) is satisfied. □

Theorem 18 (([44], Theorem 3.3) for $p = 1$ and ([45], Theorem 3.4) for $p > 1$)**.** *We have:*
(a) $L \in \mathcal{B}(h_d, w_\infty^p)$ *if and only if:*

$$\|A\|_{(h_d, w_\infty^p)} = \sup_{l,m} \frac{1}{d_m} \left(\frac{1}{l} \sum_{n=1}^{l} \left| \sum_{k=1}^{m} a_{nk} \right|^p \right)^{1/p} < \infty; \tag{29}$$

(b) $L \in \mathcal{B}(h_d, w^p)$ *if and only if* (29) *holds and:*

$$\left\{ \begin{array}{c} \text{for each } k \in \mathbb{N}, \text{ there exists } \alpha_k \in \mathbb{C} \text{ such that} \\ \lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} |a_{nk} - \alpha_k|^p = 0; \end{array} \right\} \tag{30}$$

(c) $L \in \mathcal{B}(h_d, w_0^p)$ *if and only if* (29) *holds and:*

$$\lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} |a_{nk}|^p = 0 \text{ for each } k. \tag{31}$$

(d) If $L \in \mathcal{B}(h_d, Y)$ *for* $Y \in \{w_\infty^p, w^p, w_0^p\}$*, then,*

$$\|L\| = \|A\|_{(h_d, w_\infty^p)}. \tag{32}$$

Theorem 19 ([46], Theorem 2.4)**.** *We have:*
(a) $L \in \mathcal{B}(h_d, [c_\infty])$ *if and only if,*

$$\|A\|_{(h_d, [c_\infty])} = \sup_{l,m} \frac{1}{l d_m} \sum_{n=1}^{l} \left| n \sum_{k=1}^{m} a_{nk} - (n-1) \sum_{k=1}^{m} a_{n-1,k} \right| < \infty; \tag{33}$$

(b) $L \in \mathcal{B}(h_d, [c])$ *if and only if* (33) *holds and,*

$$\left\{ \begin{array}{l} \text{for each } k \in \mathbb{N}, \text{ there exists } \alpha_k \in \mathbb{C} \text{ such that} \\ \lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} |na_{nk} - (n-1)a_{n-1,k} - \alpha_k| = 0; \end{array} \right\} \tag{34}$$

(c) $L \in \mathcal{B}(h_d, [c_0])$ *if and only if* (33) *holds and,*

$$\lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} |na_{nk} - (n-1)a_{n-1,k}| = 0 \text{ for each } k. \tag{35}$$

(d) If $L \in \mathcal{B}(h_d, Y)$ *for* $Y \in \{[c_0], [c], [c_\infty]\}$*, then,*

$$\|L\| = \|A\|_{(h_d, [c_\infty])}. \tag{36}$$

3.2. Some Classes of Compact Operators on the Generalized Hahn Space

Now, we study the Hausdorff measure of the bounded linear operators of Section 3.1 and the related classes of compact operators.

First, we consider the case of $\mathcal{C}(h_d)$.

Lemma 1 ([43], Lemma 4.5). *Let* $(\alpha_n)_{n=1}^\infty$, $(\beta_n)_{n=1}^\infty$ *and* $(\gamma_n)_{n=1}^\infty$ *be given sequences of complex numbers, and* $A = (a_{nk})_{n,k=1}^\infty$ *be the tridiagonal matrix with:*

$$a_{nk} = \begin{cases} \alpha_n & (k = n) \\ \beta_n & (k = n+1) \\ \gamma_{n-1} & (k = n-1) \\ 0 & (k \neq n, n+1, n-1) \end{cases} \qquad (n = 1, 2, \dots).$$

Putting,

$$c_m = \frac{1}{d_m} \sum_{n=1}^\infty d_n \left| \sum_{k=1}^m (a_{nk} - a_{n+1,k}) \right|,$$

we obtain,

$$c_m = \frac{1}{d_m} \left[\sum_{n=1}^{m-2} d_n |\Delta(\alpha_n + \beta_n + \gamma_{n-1})| + d_{m-1} |\Delta(\alpha_{m-1} + \gamma_{m-2}) + \beta_{m-1}| \right.$$
$$\left. + d_m |\alpha_m + \Delta\gamma_{m-1}| + d_{m+1} |\gamma_m| \right]. \tag{37}$$

For $X, Y \subset \omega$, $M(X, Y) = \{z \in \omega : zx = (z_k x_k) \in Y$ for all $x = (x_k) \in X\}$ is the *multiplier of* X *in* Y.

We obtain some useful special cases of Lemma 1.

Corollary 2 ([43], Remark 4.6). *(a) If* $\alpha_n = z_n$, $\beta_n = \gamma_n = 0$ *for all* n*, then* (37) *reduces to:*

$$c_m = \frac{1}{d_m} \left(\sum_{n=1}^{m-1} d_n |\Delta z_n| \right) + |z_m| \text{ for all } m,$$

so $z \in M(h_d, h_d)$ *if and only if:*

$$\sup_m c_m < \infty, \text{ or equivalently, } \left(\frac{1}{d_m} \cdot \left\| z^{[m-1]} \right\|_{h_d} \right)_{m=1}^\infty \in \ell_\infty.$$

(b) Let $l \in \mathbb{N}$ be given and $z = e - e^{[l]}$, then we obtain from Part (a)*:*

$$c_m^{(l)} = \begin{cases} 0 & (1 \le m \le l) \\ 1 + \dfrac{d_l}{d_m} & (m \ge l+1) \end{cases}$$

and so, since h_d has AK,

$$\limsup_{l\to\infty} \|\mathcal{R}_l\| = \limsup_{l\to\infty}\left(\sup_{m\ge l} c_m^{(l)}\right) = \limsup_{l\to\infty}\left(\sup_{m\ge l}\left(1+\frac{d_l}{d_m}\right)\right) = 2. \tag{38}$$

In the next result, we use the notation introduced at the beginning of the proof of Example 7.

Theorem 20. *(a)* ([43], Theorem 4.8 (a)) *Let $L \in \mathcal{B}(h_d)$. We write:*

$$\gamma_m^{(l)} = \frac{1}{d_m}\left(d_l\left|\sum_{k=1}^m a_{l+1,k}\right| + \sum_{n=l+1}^\infty d_n\left|\sum_{k=1}^m (a_{nk} - a_{n+1,k})\right|\right) \text{ for all } m \text{ and } l.$$

Then, we have:

$$\frac{1}{2}\cdot\limsup_{l\to\infty}\left(\sup_m \gamma_m^{(l)}\right) \le \|L\|_\chi \le \limsup_{l\to\infty}\left(\sup_m \gamma_m^{(l)}\right). \tag{39}$$

(b) ([43], Corollary 4.10 (d)) *The operator $L \in \mathcal{B}(h_d)$ is compact if and only if:*

$$\lim_{l\to\infty}\left(\sup_m \gamma_m^{(l)}\right) = 0;$$

Proof. (a) We apply (16) with $a = 2$ by (38). We have by (24) and (25) for all l:

$$\|L^{<l>}\| = \|A^{<l>}\|_{(h_d,h_d)} = \sup_m \frac{1}{d_m}\sum_{n=1}^\infty d_n\left|\sum_{k=1}^m \left(a_{nk}^{<l>} - a_{n+1,k}^{<l+1>}\right)\right| = \sup_m \gamma_m^{(l)}$$

and (39) follows by (19) and (16).

(b) Part (b) follows from (39) by (20). □

Theorem 21. *(a) Let $L \in \mathcal{B}(h_d, w^p)$. Then we have:*

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{m;l\ge r}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=r}^l\left|\sum_{k=1}^m (a_{nk}-\alpha_k)\right|^p\right)^{1/p}\right) \le \|L\|_\chi$$
$$\le \lim_{r\to\infty}\left(\sup_{m;l\ge r}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=r}^l\left|\sum_{k=1}^m (a_{nk}-\alpha_k)\right|^p\right)^{1/p}\right), \tag{40}$$

where the complex numbers α_k are defined in (30).
(b) Let $L \in \mathcal{B}(h_d, w_0^p)$. Then we have:

$$\|L\|_\chi = \lim_{r\to\infty}\left(\sup_{m;l\ge r}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=r}^l\left|\sum_{k=1}^m a_{nk}\right|^p\right)^{1/p}\right). \tag{41}$$

(c) Let $L \in \mathcal{B}(h_d, w^p)$. Then $L \in \mathcal{C}(h_d, w^p)$ if and only if:

$$\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(a_{nk}-\alpha_k)\right|^p\right)^{1/p}\right)=0,$$

where the complex numbers α_k are defined in (30).
(d) Let $L \in \mathcal{B}(h_d, w_0^p)$. Then $L \in \mathcal{C}(h_d, w^p)$ if and only if:

$$\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}a_{nk}\right|^p\right)^{1/p}\right)=0.$$

Remark 11. *Parts (a) and (b) in Theorem 21 are ([44], Theorem 3.3) for $p = 1$ and ([45], Theorem 3.4) for $p > 1$.*

Parts (c) and (d) in Theorem 21 are ([44], Corollary 3.4) for $p = 1$ and ([45], Corollary 3.5) for $p > 1$.

Theorem 22 ([46], Theorem 3.4 and Corollary 3.5)**.**
(a) Let $L \in \mathcal{B}(h_d, [c])$. Then we have:

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(na_{nk}-(n-1)a_{n-1,k}-\alpha_k)\right|\right)\leq\|L\|_\chi$$
$$\leq\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(na_{nk}-(n-1)a_{n-1,k}-\alpha_k)\right|\right),\qquad(42)$$

where the complex numbers α_k are defined in (34).
(b) Let $L \in \mathcal{B}(h_d, [c_0])$. Then we have:

$$\|L\|_\chi=\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(na_{nk}-(n-1)a_{n-1,k}\right|\right).\qquad(43)$$

(c) Let $L \in \mathcal{B}(h_d, [c])$. Then $L \in \mathcal{C}(h_d, [c])$ if and only if:

$$\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(na_{nk}-(n-1)a_{n-1,k}-\alpha_k)\right|\right)=0,$$

where the complex numbers α_k are defined in (34).
(d) Let $L \in \mathcal{B}(h_d, [c_0])$. Then, $L \in \mathcal{C}(h_d, [c_0])$ if and only if:

$$\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}na_{nk}-(n-1)a_{n-1,k}\right|\right)=0.$$

4. Some Applications

We apply Theorem 17, Corollary 2 (a) and Theorem 20 (b) and get results by Sawano and El–Shabrawy ([47], Corollary 5.1 and Lemma 5.1).

Rhaly [48] defined the generalized Cesàro operator C_t on ω for $t \in [0,1)$ by the triangle $C_t = (a_{nk}(t))_{n,k=0}^{\infty}$, where $a_{nk} = t^{n-k}/(n+1)$ for $0 \leq k \leq n$ and $n = 0, 1, \ldots$.

Example 9 ([47], Corollary 5.1)**.** *We have $C_t \in (h, h)$ for $0 \leq t < 1$.*

Proof. Clearly $\lim_{n\to\infty} a_{nk}(t) = 0$ for each k, so (23) in Theorem 17 holds.

We put $a_{nk} = a_{nk}(t)$ for all n and k. We need show that (24) also holds. We put:

$$c_m(n) = n\left|\sum_{k=1}^{m}(a_{nk} - a_{n+1,k})\right| \text{ and } c_m = \frac{1}{m}\sum_{n=1}^{\infty} c_m(n) \text{ for all } m \text{ and } n.$$

If $t = 0$, then C_0 =diag$(1/(n+1))$ is the diagonal matrix with the entries $1/(n+1)$ on its diagonal.

Let $m \in \mathbb{N}$ be arbitrary.

For $n \leq m-1$, we obtain:

$$c_m(n) = n\left(\frac{1}{n+1} - \frac{1}{n+2}\right).$$

For $n \geq m$, we have $c_m(n) = m/(m+1)$ for $n = m$ and $c_m(n) = 0$ for $n \geq m+1$. For all m, it follows that:

$$c_m = \frac{1}{m}\left(\sum_{n=1}^{m-1} c_m(n) + c_m(m)\right) \leq \sum_{n=1}^{m-1}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \frac{1}{m}\cdot\frac{m}{m+1} \leq \frac{1}{2} + \frac{1}{2} \leq 1,$$

and so (24) also holds.

Now, let $t \in (0,1)$, and $m \in \mathbb{N}$ be arbitrary.

If $n \leq m-1$, then $a_{n,k} - a_{n+1,k} \geq 0$ for $0 \leq k \leq n$ and $a_{n,k} = 0$ for $k \geq n+1$. We get:

$$c_m(n) \leq \left(\sum_{k=1}^{n} t^{n-k} + 1\right) = \sum_{k=0}^{n} t^k \leq \frac{1}{1-t} \tag{44}$$

If $n \geq m$, then $a_{nk} - a_{n+1,k} \geq 0$ for all $k \leq m$. We get:

$$c_m(n) \leq \sum_{k=0}^{m} t^{n-k} \leq t^{n-m}\sum_{k=0}^{m} t^{m-k} \leq \frac{t^{n-m}}{1-t}. \tag{45}$$

Finally (44) and (45) imply:

$$c_m = \frac{1}{m}\left(\sum_{n=1}^{m-1} c_n(m) + \sum_{n=m}^{\infty} c_n(m)\right) \leq \frac{1}{m}\left(m\frac{1}{1-t} + \sum_{n=0}^{\infty}\frac{t^n}{1-t}\right) \leq \frac{2-t}{(1-t)^2} \text{ for all } m,$$

hence, $\sup_m c_m < \infty$. Thus, (24) also holds. □

If $d_k = k$ for all k of the following example gives ([47], Lemma 5.1).

Example 10. *Let* $(\lambda_k)_{k=1}^{\infty}$ *be a decreasing sequence of positive real numbers which converges to* 0 *and* $D(\lambda)$ =*diag*$(\lambda_1, \lambda_2, \dots)$ *denote the diagonal matrix with the sequence* λ *on its diagonal. Then* $L_{D(\lambda)} \in \mathcal{C}(h_d)$.

Proof. Since $d_k \leq d_{k+1}$ and $\lambda_k \geq \lambda_{k+1}$ for all k, we have for all m:

$$c_m = \frac{1}{d_m}\sum_{k=1}^{m-1} d_k|\Delta\lambda_k| + |\lambda_m| \leq \sum_{k=1}^{m-1}(\lambda_k - \lambda_{k+1}) + \lambda_m = \lambda_1,$$

hence, $\lambda \in M(h_d, h_d)$ by Corollary 2 (a), that is, $L_{D(\lambda)} \in \mathcal{B}(h_d)$.

If $l \in \mathbb{N}$ is arbitrary, then $\gamma_m^{(l)} = 0$ for all $m \leq l$, and:

$$\gamma_m^{(l)} = \frac{1}{d_m}\sum_{n=l}^{m-1} d_n|\Delta\lambda_n| + |\lambda_m| \leq \lambda_{l+1} + \sum_{n=l+1}^{m-1}(\lambda_n - \lambda_{n+1}) + \lambda_m$$

$$= 2\lambda_{l+1} - \lambda_m + \lambda_m \leq 2\lambda_{l+1}$$

for all $m \geq l+1$. Hence,

$$0 \leq \lim_{l\to\infty} \sup_m \gamma_m^{(l)} \leq 2 \lim_{l\to\infty} \lambda_{l+1} = 0,$$

and so $L_{D(\lambda)}$ is compact by Theeorem 20 (b). □

We obtain the following results for the classical Hahn space h.

Remark 12. *We have:*
(a) (([44], Example 3.5) for $p = 1$ and ([45], Example 3.6) for $1 < p < \infty$) $L_{C_1} \in \mathcal{C}(h, w_0^p)$ for $1 \leq p < \infty$ and $\|L_{C_1}\| = 1$;
(b) ([46], Example 3.6) $L_{C_1} \in \mathcal{C}(h_d, [c_0])$ and $\|L_{C_1}\| = 1$.

If X and Y are Banach spaces, $L \in \mathcal{B}(X,Y)$, then we denote by $N(L)$ and $R(L)$ denote the null space and the range of L, respectively. Now, L is called a *Fredholm operator*, if $R(L)$ is closed, $\dim N(T), \dim X/R(L)|\infty$. In this case, the *index* L is given by $i(L) = \dim N(L) - \dim X/R(L)$. Furthermore, if $L \in \mathcal{B}(X,X)$ and $\|L\|_\chi < 1$, then $I - L$ is a Fredholm operator with $i(I-L) = 0$ ([49] or ([9], Section 7.13)).

Corollary 3 ([43], Corollary 4.13)**.** *Let $\alpha = (\alpha_n)_{n=1}^\infty$, $\beta = (\beta_n)_{n=1}^\infty$ and $\gamma = (\gamma_n)_{n=1}^\infty$ be given complex sequences, and:*

$$A(\gamma,\alpha,\beta) = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & & & & & \dots & 0 & \cdots \\ \gamma_1 & \alpha_2 & \beta_2 & & & & & \dots & 0 & \cdots \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & 0 & & & \dots & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & & & \\ 0 & & & \gamma_{n-1} & \alpha_n & \beta_n & 0 & \cdots & 0 & \cdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \vdots & \end{pmatrix}.$$

Then, the operator $L \in \mathcal{B}(h_d)$ represented by the matrix:

$$A(\gamma,\alpha,\beta) = A(0,\alpha,0) + A(\gamma,0,0) + A(0,0,\beta)$$

is Fredholm with $i(A(\alpha,\beta,\gamma)) = 0$, if $A(0,\alpha,0)$ is Fredholm with $i(A(0,\alpha,0)) = 0$ and $A(\gamma,0,0)$ and $A(0,0,\beta)$ are compact.

Example 11 ([43], Example 4.14)**.** *If $d_k = k$, $\alpha_k = 1 - 1/k$ and $\beta_k = \gamma_k = 1/k$ for all k, then $L \in \mathcal{B}(h_d)$ represented by $A(\gamma,\alpha,\beta)$ is Fredholm.*

Proof. We write $c_m^{(<l>)}(\alpha - e)$, $c_m^{<l>}(\gamma)$ and $c_m^{<l>}(\beta)$ for the expressions in (37) for the matrices $A(0,\alpha-e,0)$, $A(\gamma,0,0,)$ and $A(0,0,\beta)$. Then we get from (37):

$$\begin{aligned} c_m^{<l>}(\alpha - e) &= \frac{1}{d_m}\left(d_l|\alpha_{l+1} - 1| + \sum_{n=l+1}^{m-1} d_n|\Delta\alpha_n| + d_m|\alpha_m - 1|\right) \\ &= \frac{1}{m}\left(\frac{l}{l+1} + \sum_{n=l+1}^{m-1} n\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{m}{m}\right) \\ &\leq \frac{2}{l} + \sum_{n=l+1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right). \end{aligned}$$

Consequently:

$$\|L_{A(0,\alpha,0)} - I\|_\chi = \limsup_{l\to\infty}\left(\sup_m c_m^{<l>}(\alpha - e)\right) < 1,$$

hence, $L_{A(0,\alpha,0)} - I$ is Fredholm.

Furthermore, (37) yields:

$$\begin{aligned} c_m^{<l>}(\gamma) &= \frac{1}{d_m}\left(d_l|\gamma_l| + \sum_{n=l+1}^{m} d_n|\Delta\gamma_{n-1}| + d_{m+1}|\gamma_m|\right) \\ &\leq \frac{1}{m} + \sum_{n=1}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n}\right) + \frac{m+1}{m^2} \leq \frac{3}{l} + 2\sum_{n=l}^{\infty}\frac{1}{n^2}. \end{aligned}$$

Thus,

$$\|L_{A(\gamma,0,0)}\|_\chi = \limsup_{l\to\infty}\left(\sup_m c_m^{<l>}(\gamma)\right) = 0,$$

and $L_{A(\gamma,0,0)}$ is compact.

Analogously, we can show that the $L_{A(0,0,\beta)}$ is compact.

Thus, $L_{A(\gamma,\alpha,\beta)}$ is Fredholm by Corollary 3. □

5. Some Mathematical Background

Now, we apply measures of noncompactness to the solvability of infinite systems of integral equations.

The notation *MNC* will stand for measures of noncompactness in Banach spaces in the sense of Banaś and Goebel given in Definition 6.

Hyperconvex spaces were introduced by Aronszajn and Panitchpakdi [50]. They are very important in metric fixed point theory, see [51] and the references therein.

Definition 10. *A metric space (X,d) is hyperconvex if every class of closed balls $\{\overline{B}(x_i,r_i)\}_{i\in I}$ with $d(x_i,x_j) \leq r_i + r_j$ satisfies:*

$$\bigcap_{i\in I} B(x_i,r_i) \neq \emptyset.$$

The following result holds.

Theorem 23 ([52])**.** *Let X be a hyperconvex metric space, $x_0 \in X$ and let f be a continuous self–map of X. If the following implication:*

$$(V \text{ is isometric to } \varepsilon f(V) \text{ or } V = f(V)\cup\{x_0\}) \implies (\alpha(V) = 0),$$

where $\varepsilon f(V)$ denotes hyperconvex hull of $f(V)$, holds for every subset $V \subset X$, then f has a fixed point.

Theorem 23 can be applied in certain cases of continuous self–maps in hyperconvex metric spaces, where Darbo's fixed point theorem, Theorem 9, or Darbo–Sadovskiĭ type fixed point theorems such as Theorem 11 are not applicable. This is illustrated in the following example.

Example 12 ([52])**.** *Consider $\mathbb{R}^2$ with the radial metric:*

$$d(v_1,v_2) = \begin{cases} \rho(v_1,v_2) & \text{if } 0, v_1, v_2 \text{ are collinear}, \\ \rho(v_1,0) + \rho(v_2,0) & \text{otherwise}, \end{cases}$$

where ρ denotes the usual Euclidean metric and $v_1 = (x_1,y_1), v_2 = (x_2,y_2) \in \mathbb{R}^2$. Define the map, $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,y) = (hx,hy)$ for $(x,y) \in \mathbb{R}^2$ and $h > 1$. Then f does not satisfy

Darbo's condensing condition, but it satisfies the hypotheses of Theorem 23. Hence, f has a fixed point.

Samadi [53] gave the following extension of Darbo's fixed point theorem.

Theorem 24. *Let $C \neq \emptyset$ be a bounded, closed and convex subset of a Banach space E. Assume $T : C \longrightarrow C$ is a continuous operator satisfying:*

$$\theta(\mu(X)) + f(\mu(T(X))) \leq f(\mu(X)) \tag{46}$$

for all nonempty subsets X of C, where μ is an arbitrary MNC on E and $(\theta, f) \in \Delta$, where Δ is the set of all pairs (θ, f) that satisfy the following conditions:

(Δ_1) $\theta(t_n) \nrightarrow 0$ for each strictly increasing sequence $\{t_n\}$;
(Δ_2) f is strictly increasing function;
(Δ_3) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} f(\alpha_n) = -\infty$.
(Δ_4) If $\{t_n\}$ is a decreasing sequence such that $t_n \to 0$ and $\theta(t_n) < f(t_n) - f(t_{n+1})$, then we have $\sum_{n=1}^{\infty} t_n < \infty$.

5.1. Meir–Keeler Generalization

We continue with the famous result by Meir–Keeler [54] of 1969.

Definition 11. Let (X, d) be a metric space. A self– map T on X is a *Meir–Keeler contraction (MKC)* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon,$$

for all $x, y \in X$.

Theorem 25 ([54])**.** *Let (X, d) be a complete metric space. If $T : X \to X$ is a Meir–Keeler contraction, then T has a unique fixed point.*

Definition 12 ([55])**.** Let C be a nonempty subset of a Banach space E and μ be an *MNC* on E. We say that an operator $T : C \to C$ is a *Meir–Keeler condensing operator* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\varepsilon \leq \mu(X) < \varepsilon + \delta \text{ implies } \mu(T(X)) < \varepsilon,$$

for any bounded subset X of C.

We note that any *MKC* is also a Meir–Keeler condensing operator, if we take the *MNC* as $\operatorname{diam}(X)$.

Theorem 26 ([55])**.** *Let $C \neq \emptyset$ be a closed, bounded, and convex subset of a Banach space E and μ be an arbitrary MNC on E. If $T : C \to C$ is continuous and a Meir–Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in C is compact.*

The characterization of Meir–Keeler contractions in metric spaces was studied by Lim [56] and Suzuki [57] by introducing notion of L–functions.

Definition 13 ([56])**.** A slef–map ϕ on $\mathbb{R}_+$ is called an *L–function* if $\phi(0) = 0$, $\phi(s) > 0$ for $s \in (0, \infty)$, and for every $s \in (0, \infty)$ there exists $\delta > 0$ such that $\phi(t) \leq s$, for any $t \in [s, s + \delta]$.

Theorem 27 ([55])**.** *Let, $C \neq \emptyset$ be a bounded subset of a Banach space E, μ be an arbitrary MNC on E and $T : C \to C$ be a continuous operator. Then, T is a Meir–Keeler condensing operator if and only if there exists an L–function ϕ such that:*

$$\mu(T(X)) < \phi(\mu(X)),$$

for all closed and bounded subset X of C with $\mu(X) \neq 0$.

We need the following concept.

Definition 14 ([58])**.** Let (X, d) be a metric space. Then, a mapping $T : X \to X$ is said to be *contractive* if:

$$d(T(x), T(y)) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$.

Theorem 28 (Edelstein [58])**.** *Let (X, d) be a compact metric space. If T is a contractive map on X, then there exists a unique fixed point $z \in X$.*

Definition 15. *Let $C \neq \emptyset$ be a bounded subset of a Banach space E, and μ an MNC on E. Then, a self–map T on C is an* asymptotic Meir–Keeler condensing operator *if there exists a sequence (ϕ_n) of self–maps on $\mathbb{R}_+$ satisfying the following conditions:*

(A1) For each $\varepsilon > 0$, there exists $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\phi_\nu(t) \leq \varepsilon$ for any $t \in [\varepsilon, \varepsilon + \delta]$,
(A2) $\mu(T^n(C)) < \phi_n(\mu(C))$, $n \in \mathbb{N}$.

In the next theorem, the convexity condition of the set C in the previous results is replaced by assumption that the operator T is contractive.

Theorem 29. *Let $C \neq \emptyset$ be a bounded and closed (not necessarily convex) subset of a Banach space E, and μ be an MNC on E. Let $T : C \to C$ be a contractive and asymptotic Meir–Keeler condensing operator. Then, T has a unique fixed point in C.*

Proof. We define a sequence (C_n) by putting $C_0 = C$ and $C_n = \overline{T^n C}$ for $n \geq 1$. Since T is contractive and continuous, it follows that $T(\overline{A}) \subset \overline{T(A)}$. This inclusion yields $T^{n+1}C \subset T^n C$, so $C_{n+1} \subset C_n$ and $T(C_n) \subset C_n$. If $\mu(C_N) = 0$ for some integer $N \geq 0$, then C_N is compact. Hence, T has a fixed point by Theorem 28. Now we suppose that $\mu(C_n) \neq 0$ for $n \geq 0$. We put $\varepsilon_n = \mu(C_n)$ and $r = \inf_{n \in \mathbb{N}} \varepsilon_n$. We prove $r = 0$. If $r \neq 0$, then by the definition of r, and the conditions in (A1) and (A2), there exist $n_0 \in \mathbb{N}$, $\delta_r > 0$, and $\nu \in \mathbb{N}$ such that $\phi_\nu(t) \leq r$ for any $t \in [r, r + \delta_r]$ and $r \leq \varepsilon_{n_0} < r + \delta_r$. Consequently,

$$\varepsilon_{n_0+\nu} = \mu(C_{n_0+\nu}) = \mu(T^{n_0+\nu}(C)) < \phi_\nu(\mu(T^{n_0}(C))) = \phi_\nu(\mu(C_{n_0})) \leq r.$$

This is a contradiction, so $r = 0$. Hence, $\lim_{n\to\infty} \mu(C_n) = 0$. Since $C_{n+1} \subset C_n$ and $TC_n \subset C_n$ for all $n \geq 1$, the generalized Cantor intersection property of the *MNC* μ yields the $C_\infty = \cap_{n=1}^{\infty} C_n$ is nonempty and closed, invariant under T, and belongs to ker_μ. Then, by Theorem 28, T has a unique fixed point in C_∞. Furthermore, since $F_T = \{x \in X : T(x) = x\} \subset C_n$ for all $n \geq 0$, it follows that $F_T \subset C_\infty$ and T has a unique fixed point in C. □

5.2. Darbo-Type Theorem for Commuting Operators

Now we are going to discuss some fixed point theorems obtained in [59,60] for commuting maps in locally convex spaces and Banach spaces, satisfying the following inequalities:

$$\alpha(S(A)) \leq k \sup_{i \in I}(\alpha(T_i(A)))$$

and:

$$\alpha(S(A)) < \sup_{i\in I}(\alpha(T_i(A), \alpha(A)).$$

We briefly describe MNC's on locally convex spaces. Let X be a Hausdorff complete and locally convex space whose topology is defined by family of equicontinuous seminorms $\mathcal{P}$. A local base of closed 0–neighborhood of X is generated by the sets:

$$\{x \in X \,:\, \max_{1\le i\le n} p_i(x) \le \varepsilon\},\ \varepsilon > 0,\ p_i \in \mathcal{P}.$$

Let $\mathcal{B}$ denote the family of all bounded subsets of X and Φ be the space of all functions $\phi : \mathcal{P} \to \mathbb{R}^+$ with the partial order "$\phi_1 \le \phi_2$ if and only if $\phi_1(p) \le \phi_2(p)$ for all $p \in \mathcal{P}$".

Definition 16. *A measure of noncompactness on a locally convex space is the function γ from $\mathcal{B}$ into Φ such that for each $B \in \mathcal{B}$, we have that $\gamma(B)$ is a function from $\mathcal{P}$ into $\mathbb{R}^+$, such that:*

$$\gamma(B)(p) = \inf\{d > 0 \,:\, B \text{ is a finite union of } B_i,\ \sup\{p(x-y) \,:\, x,y \in B_i\} \le d\ \forall\, p \in \mathcal{P}\}.$$

Remark 13 ([60])**.** *On a Hausdorff, complete locally convex space, γ satisfies the generalized Cantor intersection property.*

Definition 17. *A mapping T of a convex set M is said to be* affine *if:*

$$T(kx + (1-k)y) = kTx + (1-k)Ty,$$

whenever $0 < k < 1$ and $x, y \in M$.

The following result holds.

Theorem 30 ([60])**.** *Let X be a Hausdorff complete and locally convex space, Ω be a convex, closed and bounded subset of X, I be an index set, and $\{T_i\}_{i\in I}$, S be a continuous function from Ω into Ω such that the following conditions hold:*

- *(a) For any $i \in I, T_i$ commutes with S.*
- *(b) For any $A \subset \Omega$ and $i \in I$, we have $T_i(\overline{co}(A)) \subset \overline{co}(T_i(A))$.*
- *(c) There exists $0 < k < 1$ such that for any $A \subset \Omega$ $\alpha(S(A))(p) \le k \sup_{i\in I} \alpha(T_i(A))(p), p \in \mathcal{P}$.*

Then we have:

- *(1) The set $\{x \in \Omega \,:\, Sx = x\}$ is nonempty and compact.*
- *(2) For any $i \in I$, set $\{x \in \Omega \,:\, T_i x = x\}$ is nonempty, closed and invariant by S.*
- *(3) If T_i is affine and $\{T_i\}_{i\in I}$ is a commuting family then T_i and S have a common fixed point and the set $\{x \in \Omega \,:\, T_i(x) = S(x) = x\}$ is compact.*
- *(4) If $\{T_i\}_{i\in I}$ is a commuting family and S is affine, then there exists a common fixed point for the mapping $\{T_i\}_{i\in I}$.*

Remark 14. *If T_i is the identity function for any $i \in I$, above theorem becomes generalization of Darbo's fixed point theorem in the structure of locally convex spaces.*

The following theorem due to [59] generalizes the Sadovskiǐ fixed point theorem for commuting operators.

Theorem 31. *Let X be a Hausdorff complete and locally convex space, Ω be a convex, closed and bounded subset of X, I be an index set, and $\{T_i\}_{i\in I}$, S be a continuous function from Ω into Ω such that:*

- *(a) For each $i \in I$, T_i commutes with S.*
- *(b) For each $i \in I$, T_i is linear map.*

(c) *There exists $j \in I$ such that for each $A \subset \Omega$ and $p \in \mathcal{P}$, with $\alpha(A)(p) \neq 0$, we have:*

$$\alpha(S(A))(p) < \sup(\alpha(T_j(A))(p), \alpha(A)(p)).$$

Then we have:

(1) *T_j and S have a fixed point, and $\{x \in \Omega : T_j x = x\}$ is compact.*
(2) *If $\{T_i\}_{i \in I}$ is a commuting family and S is affine, then there exists a common fixed point for the mapping in $\{T_i\}_{i \in I}$.*

Remark 15. *If T_j is the identity function, then above theorem becomes a generalization of Sadovskii's fixed point theorem.*

It is well known for operators S and T that if the composition operator ST has a fixed point, then S and T do not necessarily poss a fixed point or a common fixed point. It becomes interesting to investigate the conditions which force the operator S, T to have a common fixed point. This result is also helpful in obtaining existence results for common solutions of a certain type of equations.

Theorem 32 ([59]). *Let X be a Banach space and $\Omega \neq \emptyset$ be a convex, closed, and bounded subset of X. Let T and S be two continuous functions from Ω into Ω such that:*

(a) *$ST = TS$;*
(b) *T is affine;*
(c) *There exist $k \in (0,1)$ such that for any $A \subset \Omega$ we have $\alpha(ST(A)) \leq k\alpha(A)$.*

Then, the set $\{x \in \Omega : Tx = Sx = x\}$ is nonempty and compact.

Proof. The operator H with $H(x) = kS(T(x)) + (1-k)T(x)$ is a continuous self–map H on Ω and, commutes with T.
The semi–homogeneity and sub–additive property of the MNC α imply:

$$\alpha(H(A)) = \alpha(kS(T(A))) + (1-k)T(A) \leq k^2\alpha(A) + (1-k)\alpha(T(A))$$

for any $A \subset \Omega$. Since $k \in (0,1)$, $k^2 < k$, and we have $k^2 + 1 - k < k + 1 - k$. Hence, it follows from Theorem 30 that $F_0 = \{x \in \Omega : Hx = Tx = x\} \neq \emptyset$ is compact.
Moreover, we have for any $x \in F_0$:

$$H(x) = kST(x) + (1-k)T(x) = Tx = x \text{ implies } Sx = x.$$

So S and T have a common fixed point. We put $F = \{x \in \Omega : Sx = Tx = x\}$. Then,

$$\alpha(F) = \alpha(ST(F)) \leq k\alpha(F)$$

implies $\alpha(F) = 0$. Since S and T are continuous, F is compact. □

Remark 16. *If the operator T is equal to the identity function, then we obtain Darbo's fixed point theorem from Theorem 32.*

Theorem 33 ([59]). *Let X be a Banach space and $\Omega \neq \emptyset$ be a convex, closed and bounded subset of X. Let T_1, T_2, and S be two continuous self–maps on Ω such that:*

(a) *$T_1T_2 = T_2T_1$;*
(b) *T_1, T_2 are affine;*
(c) *There exist $k \in (0,1)$ such that for any $A \subset \Omega$ we have $\alpha(S(A)) \leq k\alpha(A)$.*

Then, the set $\{x \in \Omega : Sx = T_1x = T_2x = x\} \neq \emptyset$ is compact.

6. Applications to Integral Equations

Now we apply measures of noncompactness to solve some differential and integral equations, and systems of linear equations in sequence spaces. Furthermore, we discuss existence results obtained by various authors, for the solution of integral equations in some sequence spaces.

We use the standard notations and results for functions of bounded variation, their total variation and the Riemann–Stieltjes integral (cf. [55]).

6.1. Infinite System of Integral Equations of Volterra–Stieltjes Type In Sequence Spaces ℓ_p and C_0

We study the solutions for an infinite system of integral equations of the Volterra–Stieltjes type of the form (see [61]):

$$\begin{aligned} u_n(t,x) &= F_n\Big(t,s,f_1(t,u(t,x)) \int_0^t \int_0^x g_n\Big(t,s,x,y,u(t,x)\Big) d_y g_2(x,y) d_s g_1(t,s), \\ &\quad (Tu)(t,x) \int_0^\infty V_n(t,s,u(t,x))ds\Big); \qquad (47) \\ u(t,x) &= \Big\{u_i(t,x)\Big\}_{i=1}^\infty, \quad u_i(t,x) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}), \end{aligned}$$

where $BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ is the space of all real functions $u(t,x) = u : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$, which are defined, continuous, and bounded on the set $\mathbb{R}_+ \times \mathbb{R}_+$ with the supremum norm:

$$\|u\| = \sup\Big\{|u(t,x)| : (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+\Big\}.$$

6.1.1. Solution in the Space ℓ_p $(1 \le p < \infty)$

We consider the following hypotheses:

(H$_1$) : *$F_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there are reals $\tau > 0$ with:*

$$|F_n(t,s,x_1,y_1) - F_n(t,s,x_2,y_2)|^p \le e^{-\tau}(|x_1 - x_2|^p + |y_1 - y_2|^p),$$

for all $t,s \in \mathbb{R}_+$ and *$x_1,x_2,y_1,y_2 \in \mathbb{R}$. Moreover, we have:*

$$\lim_{i \longrightarrow \infty} \Sigma_{i=1}^\infty |F_i(t,s,0,0)|^p = 0, \quad N_1 = \Sigma_{i=1}^\infty |F_i(t,s,0,0)|^p.$$

(H$_2$) : *$f_1 : \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is continuous with $f_0 = \sup_{t \in \mathbb{R}_+} |f(t,0)|$ and there are reals $\tau > 0$ with:*

$$\begin{aligned} |f_1(t,u(t,x)) - f_1(t,v(t,x))|^p &\le e^{-\tau}\|u(t,x) - v(t,x)\|_{\ell_p}, \\ |f_1(t,u(t,x))|^p &\le e^{-\tau}\|u(t,x)\|_{\ell_p}. \end{aligned}$$

for all $t,x \in \mathbb{R}_+$ and:

$$u(t,x) = \Big\{u_i(t,x)\Big\}_{i=1}^\infty, v(t,x) = \Big\{v_i(t,x)\Big\}_{i=1}^\infty \in \ell_p.$$

(H$_3$) : *$T : BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_p) \longrightarrow BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ is a continuous operator satisfying:*

$$\begin{aligned} |(Tu)(t,x) - (Tv)(t,x)| &\le \|u(t,x) - v(t,x)\|_{\ell_p}, \\ |(Tu)(t,x)| &\le 1. \end{aligned}$$

for all $u,v \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_p)$ and $t,x \in \mathbb{R}_+$.

(H$_4$) : *For any fixed $t > 0$ the function $s \longrightarrow g_i(t,s)$ is of bounded variation on the interval $[0,t]$ and the function $t \longrightarrow \bigvee_{s=0}^t g_i(t,s)$ is bounded over $\mathbb{R}_+$.*

(H₅) : $g_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ *is continuous and there exist continuous functions* $a_n : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that:*

$$|g_n(t,s,x,y,u(t,x))| \leq a_n(t,s),$$
$$\lim_{t\longrightarrow\infty} \Sigma_{n\geq 1} \int_0^t |g_n(t,s,x,y,u(t,x)) - g_n(t,s,x,y,v(t,x))| d_s \bigvee_{q=0}^{t} g_1(t,q) = 0,$$
$$\varphi_k = \sup\left\{ \Sigma_{n\geq k} \left[| \int_0^t \int_0^x g_n\Big(t,s,x,y,u(t,x)\Big) d_y g_2(x,y) d_s g_1(t,s)| \right]; \right.$$
$$\left. t,s,x,y \in \mathbb{R}_+, u(t,x) \in \mathbb{R}^\infty \right\}.$$

We also put:

$$A = \sup\left\{ \Sigma_{n=1}^\infty \int_0^t a_n(t,s) d_s \bigvee_{p=0}^{s} g_1(t,p), \;\; t \in \mathbb{R}_+ \right\},$$
$$G = \sup\left\{ \bigvee_{y=0}^{x} g_2(x,y); \;\; x \in \mathbb{R}_+ \right\}, \;\; \lim_{k\longrightarrow\infty} \varphi_k = 0.$$

(H₆) : $V_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ *is a continuous function and there exists a continuous function* $k : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that the function* $s \longrightarrow k(t,s)$ *is integrable over* $\mathbb{R}_+$ *satisfying:*

$$|V_n(t,s,u(t,x))| \leq k(t,s)|u_n(t,x)|^p,$$
$$|V_n(t,s,u(t,x)) - V_n(t,s,v(t,x)| \leq |u_n(t,x) - v_n(t,x)|^p k(t,s).$$

for all $t,s,x \in \mathbb{R}_+$ *and* $u,v \in \ell_p$. *We put:*

$$M = \sup_{t\in\mathbb{R}_+} \int_0^\infty k(t,s)ds.$$

(H₇) : *There exists a solution* $r_0 > 0$ *with:*

$$2^{2p} e^{-2\tau} r_0^p (GA)^p + 2^{2p} e^{-\tau} f_0^p (GA)^p + 2^p e^{-\tau} r_0^p M^p + 2^p N_1 \leq r_0^p,$$

Moreover, assume that $2^p M < 1$.

Theorem 34. *Under the assumptions* (H₁)–(H₇), *Equation (47) has at least one solution* $u(t,x) = \Big\{ u_i(t,x) \Big\}_{i=1}^{\infty}$ *in* ℓ_p.

Example 13. *Here, we investigate the system of integral equations:*

$$\begin{aligned} u_n(t,x) &= \frac{(e^{-\tau}-t-n)^{\frac{1}{p}}}{2} \sin\left(\frac{(e^{-t-\tau})^{\frac{1}{p}} \sin\left(\|u(t,x)\|_{\ell_p} \right)}{2} \right. \\ &\times \int_0^t \int_0^x \arctan\left(\frac{\frac{1}{2^n} \times e^{-3t+s}}{8+|x|+|y|+|u_n(t,x)|} \right) \frac{e^x}{1+y^2e^{2x}} \frac{e^t}{1+t^2} dyds \\ &\left. + \cos\left(\frac{1}{1+\|u(t,x)\|_{l_p}} \right) \int_0^\infty \frac{e^{-s}}{1+\frac{t}{8}} \sin\Big(|u_n(t,x)| \Big) ds \right). \end{aligned} \tag{48}$$

We observe that Equation (48) is a special case of (47) putting:

$$\begin{aligned}
F_n\Big(t,s,x,y\Big) \quad &= \frac{(e^{-\tau-t-n})^{\frac{1}{p}}}{2}\sin\Big(x+y\Big),\\
g_n(t,s,x,y,u(t,x)) \quad &= \arctan\left(\frac{\frac{1}{2^n}\times e^{-3t+s}}{8+|x|+|y|+|u_n(t,x)|}\right),\\
f_1(t,u(t,x)) \quad &= \frac{(e^{-t-\tau})^{\frac{1}{p}}\sin\Big(\|u(t,x)\|_{\ell_p}\Big)}{2},\\
a_n(t,s) \quad &= \frac{1}{2^n}e^{-3t+s},\\
g_1(t,s) \quad &= \frac{se^t}{1+t^2},\\
g_2(x,y) \quad &= \arctan\Big(ye^x\Big),\\
V_n(t,s,u(t,x)) \quad &= \frac{e^{-s}}{1+\frac{t}{8}}\sin\Big(|u_n(t,x)|\Big),\\
&k(t,s) = \frac{e^{-s}}{1+\frac{t}{8}},\\
(Tu)(t,x) \quad &= \cos\left(\frac{1}{1+\|u(t,x)\|_{l_p}}\right).
\end{aligned}$$

Obviously, F_n and f_1 satisfy (H_1) *and* (H_2) *with $N_1 = 0$ and $f_0 = 0$, T satisfies* (H_3)*. To check* (H_5)*, we assume $t, sx, y \in \mathbb{R}_+$ and $u, u \in \ell_p$. It follows that:*

$$|g_n(t,s,x,y,u(t,x))| \le \frac{1}{2^n}e^{-3t+s} = a_n(t,s).$$

We obtain from $\frac{\partial g_1}{\partial s} = \frac{e^t}{1+t^2} > 0$ that: $\bigvee_{q=0}^{s} g_1(t,q) = g_1(t,s) - g_1(t,0) = \frac{se^t}{1+t^2}$. Consequently, we have:

$$\begin{aligned}
\lim_{t\longrightarrow\infty}\int_0^t a_n(t,s)d_s\bigvee_{q=0}^{s} g_1(t,q) \quad &= \quad \lim_{t\longrightarrow\infty}\int_0^t \frac{1}{2^n}e^{-3t+s}(\frac{e^t}{1+t^2})ds\\
&= \quad \lim_{t\longrightarrow\infty}\frac{1}{2^n}\frac{e^{-2t+s}}{1+t^2}|_0^t = 0,
\end{aligned}$$

hence,

$$\begin{aligned}
&\lim_{t\longrightarrow\infty}\Sigma_{n\ge 1}\int_0^t |g_n(t,s,x,y,u(t,x)) - g_n(t,s,x,y,v(t,x))|d_s\bigvee_{q=0}^{t} g_1(t,q) = 0,\\
&A = \sup\left\{\Sigma_{i=1}^{\infty}\int_0^t a_n(t,s)d_s\bigvee_{p=0}^{s} g_1(t,s),\ \ t\in\mathbb{R}_+\right\},\\
&\varphi_k = \sup\Bigg\{\Sigma_{n\ge k}\bigg[\int_0^t\int_0^x g_n\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s)\bigg];\\
&\quad t,s,x,y\in\mathbb{R}_+, u(t,x)\in\ell_p\Bigg\} \le G\bigg(\frac{e^{-2t}}{1+t^2} - \frac{e^{-t}}{1+t^2}\bigg)\Sigma_{n\ge k}\frac{1}{2^n}.
\end{aligned}$$

Thus, $\varphi_k \longrightarrow 0$. Furthermore, $V_n(t,s,u(t,x)) = \frac{e^{-s}}{1+\frac{t}{8}}\sin\Big(|u_n(t,x)|\Big)$ verifies (H_6) *with $k(t,s) = \frac{e^{-s}}{1+\frac{t}{8}}$ and $M = 1$. To establish that g_1 and g_2 satisfy assumption* (H_4)*, we observe that g_1*

and g_2 are increasing on every interval $[0,t]$ and g_2 is bounded on the triangle $\triangle_2$. Therefore, the function $y \longrightarrow g_2(x,y)$ is of bounded variation on $[0,x]$ and:

$$\bigvee_{y=0}^{x} g_2(x,y) = g_2(x,y) - g_2(x,0) = g_2(x,y) \le \frac{\pi}{4}.$$

Thus, $G \le \pi/4$ and we may choose $G = \pi/4$.
Therefore, by Theorem 34, the infinite system (48) has at least one solution in ℓ_p.

6.1.2. Solution in the Space C_0

Now we study the system (47) and consider the following assumptions.

(D_1) $F_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist positive reals τ with:

$$|F_n(t,s,x_1,y_1) - F_n(t,s,x_2,y_2)| \le e^{-\tau}(|x_1 - x_2| + |y_1 - y_2|),$$

for all $t,s \in \mathbb{R}_+$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$. Moreover, assume:

$$\lim_{i \longrightarrow \infty} |F_i(t,s,0,0)| = 0, \quad M_1 = \sup\left\{|F_i(t,s,0,0)|; t,s \in \mathbb{R}_+, i \ge 1\right\}.$$

(D_2) $f_1 : \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is continuous with $f_0 = \sup_{t\in\mathbb{R}_+} |f(t,0)|$ and there exist positive reals τ with:

$$\begin{aligned} |f_1(t,u(t,x)) - f_1(t,v(t,x))| &\le e^{-\tau} \sup_{n\ge1}\left\{|u_i(t,x) - v_i(t,x)|;\ i \ge n\right\}, \\ |f_1(t,u(t,x))| &\le e^{-\tau} \sup_{n\ge1}\left\{|u_i(t,x)|;\ i \ge n\right\} \end{aligned}$$

for all $t,x \in \mathbb{R}_+$ and $u(t,x) = \left\{u_i(t,x)\right\}, v(t,x) = \left\{v_i(t,x)\right\} \in c_0$.

(D_3) $T : BC(\mathbb{R}_+ \times \mathbb{R}_+, c_0) \longrightarrow BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ is a continuous operator satisfying:

$$\begin{aligned} &|(Tu)(t,x) - (Tv)(t,x)| \le \sup_{n\ge1}\left\{|u_i(t,x) - v_i(t,x)|;\ i \ge n\right\}, \\ &|(Tu)(t,x)| \le 1. \end{aligned}$$

for all $u,v \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c_0)$ and $t,x \in \mathbb{R}_+$.

(D_4) For any fixed $t > 0$ the functions $s \longrightarrow g_i(t,s)$ are of bounded variation on $[0,t]$ and the functions $t \longrightarrow \bigvee_{s=0}^{t} g_i(t,s)$ are bounded on $\mathbb{R}_+$. Furthermore, for arbitrary, fixed positive T, the function $w \longrightarrow \bigvee_{z=0}^{w} g_i(w,z)$ is continuous on $[0,T]$ for $i = 1,2$.

(D_5) $g_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is continuous and there exist continuous functions $a_n : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with:

$$\begin{aligned} &|g_n(t,s,x,y,u(t,x))| \le a_n(t,s), \\ &\lim_{t \longrightarrow \infty} \int_0^t |g_n(t,s,x,y,u(t,x)) - g_n(t,s,x,y,v(t,x))| d_s \bigvee_{q=0}^{t} g_1(t,q) = 0, \end{aligned}$$

for all $t,s,x,y \in \mathbb{R}_+$ and $u,v \in \mathbb{R}^\infty$. Furthermore, we suppose that:

$$\lim_{n \longrightarrow \infty} \int_0^t a_n(t,s) d_s \bigvee_{p=0}^{s} g_1(t,p) = 0, \quad A = \sup\left\{\int_0^t a_n(t,s) d_s \bigvee_{p=0}^{s} g_1(t,p); n \in \mathbb{N}\right\},$$

$$G = \sup\left\{\bigvee_{y=0}^{x} g_2(x,y);\ x \in \mathbb{R}_+\right\}, \quad G_1 = \sup\left\{\bigvee_{z=0}^{w} g_1(w,z);\ w \in [0,T]\right\}.$$

where T is an arbitrary fixed positive real number.

(D_6) $V_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is a continuous function and there exists continuous function $k : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the function $s \longrightarrow k(t,s)$ is integrable over $\mathbb{R}_+$ and the following conditions hold:

$$|V_n(t,s,u(t,x))| \le k(t,s) \sup_{n\ge1}\Big\{|u_i(t,x)|;\ \ i \ge n\Big\},$$
$$|V_n(t,s,u(t,x)) - V_n(t,s,v(t,x)| \le \sup_{n\ge1}\Big\{|u_i(t,x) - v_i(t,x); i \ge n\Big\}k(t,s).$$

for all $t,s,x \in \mathbb{R}_+$ and $u,v \in c_0$. Furthermore, we suppose:

$$M = \sup_{t\in\mathbb{R}_+}\int_0^\infty k(t,s)ds < 1,\ \ e^{-2\tau}GA + f_0GAe^{-\tau} + Me^{-\tau} + Me^{-\tau} < 1.$$

Theorem 35. *If the infinite system (47) satisfies* $(D_1) - (D_6)$*, then it has at least one solution* $u(t) = (u_i(t,x))_{i=1}^\infty$ *in* c_0.

Example 14. *Now we investigate:*

$$\begin{aligned} & u_n(t,x) \\ & = e^{-t-s-\tau-n}\sqrt[3]{\sqrt[5]{\arctan\left(e^{-\tau}\Sigma_{k\ge n}\frac{|u_k(t,x)|}{1+k^2}\right)}(H_n)(u) + \sqrt[7]{(D_n)(u)}} \end{aligned} \tag{49}$$

in c_0*. Writing:*

$$\begin{aligned} (D_n)(u) &= e^{-100}\Sigma_{k\ge n}\frac{\sin\Big(|u_k(t,x)|\Big)}{(1+k^2)}\int_0^\infty e^{-t-s-n}\Sigma_{k\ge n}\frac{|u_k(t,x)|}{10^n(1+k^2)}ds, \\ (H_n)(u) &= \int_0^t\int_0^x \arctan\left(\frac{e^{s+t}2^{-n}}{8+|u(t,x)|}\right)\frac{e^{-2t}}{1+t^2}\times\frac{e^x}{1+y^2e^{2x}}dyds, \\ F_n(t,s,x,y) &= e^{-\tau-t-s-n}\sqrt[3]{\sqrt[5]{x}+\sqrt[7]{y}}, \\ f_1(t,u(t,x)) &= \arctan\left(e^{-\tau}\Sigma_{k\ge n}\frac{|u_k(t,x)|}{1+k^2}\right), \\ g_n(t,s,x,y,u(t,x)) &= \arctan\left(\frac{e^{s+t}2^{-n}}{8+|u(t,x)|}\right), \\ g_1(t,s) &= \frac{se^{-2t}}{1+t^2}, \\ g_2(x,y) &= \arctan\Big(ye^x\Big), \\ V_n(t,s,u(t,x)) &= e^{-t-s-n}\Sigma_{k\ge n}\frac{|u_k(t,x)|}{10^n(1+k^2)}, \\ k(t,s) &= e^{-t-s}, \\ (Tu)(t,x) &= e^{-100}\Sigma_{k\ge n}\frac{\sin\Big(|u_k(t,x)|\Big)}{(1+k^2)}\ \ n\in\mathbb{N}, \end{aligned}$$

in (47), we obtain (49). We observe that F_n *and* f_1 *satisfy* (D_1) *and* (D_2)*. Indeed, we have:*

$$|F_n(t,x_1,y_1) - F_n(t,x_2,y_2)| \quad = \quad e^{-\tau-n-t}\left[\left|\sqrt[3]{\sqrt[5]{x_1}+\sqrt[7]{y_1}} - \sqrt[3]{\sqrt[5]{x_1}+\sqrt[7]{y_1}}\right|\right]$$

$$
\begin{aligned}
&\leq e^{-\tau}\left[\sqrt[3]{|\sqrt[5]{x_1}+\sqrt[7]{y_1}-\sqrt[5]{x_2}-\sqrt[7]{y_2}|}\right]\\
&\leq e^{-\tau}\left[\sqrt[3]{\sqrt[5]{|x_1-x_2|}+\sqrt[7]{|y_1-y_2|}}\right]\\
&\leq e^{-\tau}\left[|x_1-x_2|+|y_1-y_2|\right],
\end{aligned}
$$

$$
\begin{aligned}
M_1 &= 0, \lim_{n\longrightarrow\infty} F_n\Big(t,s,0,0\Big)=0,\\
|f_1\Big(t,u(t,x)\Big)| &\leq \sup_{n\geq 1}\Big\{|u_i(t,x)|;\ \ i\geq n\Big\},\\
|f_1\Big(t,u(t,x)\Big)-f_1\Big(t,v(t,x)\Big)| &\leq \sup_{n\geq 1}\Big\{|u_i(t,x)|-|v_i(t,x)|;\ \ i\geq n\Big\}.
\end{aligned}
$$

Obviously, T satisfies (D_3) *and:*

$$
\begin{aligned}
|(Tu)(t,x)| &\leq e^{-100}\frac{\pi^2}{6}\sup_{n\geq 1}\Big\{|u_i(t,x)|;\ \ i\geq n\Big\},\\
|(Tu)(t,x)-(Tv)(t,x)| &\leq e^{-\tau}\frac{\pi^2}{6}\sup_{n\geq 1}\Big\{|u_i(t,x)-v_i(t,x)|;\ \ i\geq n\Big\}.
\end{aligned}
$$

Moreover, since:

$$\frac{\partial g_1}{\partial s}=\frac{e^{-2t}}{1+t^2}>0,$$

the function g_1 is increasing and we obtain:

$$\bigvee_{q=0}^{s} g_1(t,q)=g_1(t,s)-g_1(t,0)=g_1(t,s)=\frac{se^{-2t}}{1+t^2}>0.$$

Consequently,

$$
\begin{aligned}
&|g_n(t,s,x,y,u(t,x))|\leq e^{s+t}2^{-n},\\
&\lim_{t\longrightarrow\infty}\int_0^t |g_n(t,s,x,y,u(t,x))-g_n(t,s,x,y,v(t,x))|d_s\bigvee_{q=0}^{t} g_1(t,q)\\
\leq\ & 2\lim_{t\longrightarrow\infty}\int_0^t e^{t+s}\frac{e^{-2t}}{1+t^2}ds=0.
\end{aligned}
$$

Again, we have:

$$
\begin{aligned}
&\bigvee_{q=0}^{y} g_2(x,y)=g_2(x,y)-g_2(x,0)=g_2(x,y)\leq\frac{\pi}{4},\\
&\lim_{n\longrightarrow\infty}\int_0^t a_n(t,s)d_s\bigvee_{q=0}^{s} g_1(t,q)=\lim_{n\longrightarrow\infty}2^{-n}\left(\frac{1}{1+t^2}-\frac{e^{-t}}{1+t^2}\right)=0.
\end{aligned}
$$

Hence, $G=\pi/4$ and $A<\infty$. We also have that:

$$V_n(t,s,u(t,x))=e^{-t-s-n}\Sigma_{k\geq n}\frac{|u_k(t,x)|}{10^n(1+k^2)}$$

satisfies assumption (D_6) *with* $k(t,s) = e^{-t-s}$ *and* $M = 1$*. Since the function* $h \longrightarrow \bigvee_{z=0}^{w} g_i(h,z)$ *is continuous on* $[0,T]$*, we can put* $G_1 = \sup\left\{ \bigvee_{z=0}^{w} g_1(w,z) : \ w \in [0,T] \right\}$*, where* T *is an arbitrary, fixed, psoitive real number. Thus, Theorem 35 implies that the infinite system (49) has at least one solution in* c_0*.*

6.2. Infinite System of Integral Equations in Two Variables of Hammerstein Type in Sequence Spaces C_0 *and* ℓ_1

In this subsection, we study the following infinite system of Hammerstein-type integral equations in two variables:

$$v_n(s,t) = \mathrm{r}_n(s,t) + \int_a^b \int_a^b K_n(s,t,\tau_1,\tau 2) f_n(\tau_1,\tau_2,v(\tau_1,\tau_2)) d\tau_1 d\tau_2, \tag{50}$$

where $(s,t) \in [a,b] \times [a,b]$ in c_0 and ℓ_1. The solvability of (50) is studied in [62] using the idea of measure of noncompactness (MNC).

To find the condition under which (50) has a solution in c_0 we need the following assumptions:

(A_1) The functions $(f_j)_{j=1}^{\infty}$ are real valued and continuous defined on the set $I^2 \times \mathbb{R}^{\infty}$. The operator Q defined on the space $I^2 \times c_0$ as:

$$(s,t,v) \mapsto (\mathrm{Q}v)(s,t) = (f_1(s,t,v), f_2(s,t,v), f_3(s,t,v), \dots)$$

maps $I^2 \times c_0$ into c_0. The set of all such functions $\{(\mathrm{Q}v)(s,t)\}_{(s,t)\in I^2}$ is equicontinuous at every point of c_0, that is, given $\epsilon, \delta > 0$:

$$\|u - v\|_{c_0} \le \delta \text{ implies } \|(\mathrm{Q}u)(s,t) - (\mathrm{Q}v)(s,t)\|_{c_0} \le \epsilon.$$

(A_2) For each fixed $(s,t) \in I^2$, $v(s,t) = \big(v_j(s,t)\big) \in C(I^2, c_0)$:

$$|f_n(s,t,v(s,t))| \le p_n(s,t) + q_n(s,t) \sup_{j\ge n}\{|v_j|\} \qquad n \in \mathbb{N},$$

where $p_j(s,t)$ and $q_j(s,t)$ are real–valued continuous functions on I^2. The function sequence $\big(q_j(s,t)\big)_{j\in\mathbb{N}}$ is equibounded on I^2 and the function sequence $\big(p_j(s,t)\big)_{j\in\mathbb{N}}$ converges uniformly on I^2 to a function vanishing identically on I^2.

(A_3) The functions $K_n : I^4 \to \mathbb{R}$ are continuous on $I^4, (n = 1,2,\dots)$, and $K_n(s,t,x,y)$ are equicontinuous with respect to (s,t) that is, for every $\epsilon > 0$ there exists $\delta > 0$ with:

$$|K_n(s_2,t_2,x,y) - K_n(s_1,t_1,x,y)| \le \epsilon, \text{ whenever } |s_2 - s_1| \le \delta, |t_2 - t_1| \le \delta,$$

for all $(x,y) \in I^2$. Furthermore, the function sequence $(K_n(s,t,x,y))$ is equibounded on the set I^4 and:

$$\mathrm{K} = \sup\Big\{|K_n(s,t,x,y)| : (s,t),(x,y) \in I^2,\ n = 1,2,\dots\Big\} < \infty.$$

(A_4) The functions $\mathrm{r}_n : I^2 \to \mathbb{R}$ are continuous and the function sequence (r_n) is uniformly convergent to zero on I^2. Moreover,

$$\mathrm{R} = \sup\Big\{|\mathrm{r}_n(s,t)| : (s,t) \in I^2 : n = 1,2,\dots\Big\} < \infty.$$

Keeping assumption (A_2) under consideration, we define the following finite constants:

$$\mathcal{Q} = \sup\Big\{q_n(s,t) : (s,t) \in I^2, n \in \mathbb{N}\Big\},$$

$$\mathcal{P} = \sup\Big\{ p_n(s,t) : (s,t) \in I^2, n \in \mathbb{N} \Big\}.$$

Theorem 36. *If the infinite system (50) satisfies (A_1)–(A_4), then it has at least one solution $v(s,t) = \big(v_j(s,t)\big)_{j\in\mathbb{N}}$ in c_0 for fixed $(s,t) \in I^2$, whenever $(b-a)^2 \mathsf{K}\mathcal{Q} < 1$.*

Example 15. *We study the infinite system of Hammerstein-type integral equations in two variables:*

$$v_n(s,t) = \frac{1}{n}\arctan(s+t)^n + \int_1^2\int_1^2 \sin\left(\frac{s+t+\tau_1+\tau_2}{n}\right) \ln\left(\frac{1+4n^2+(\tau_1+\tau_2)^2\Big[4+\sup_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}\Big]}{4[(\tau_1+\tau_2)^2+n^2]}\right) d\tau_1 d\tau_2 \tag{51}$$

for $(s,t) \in [1,2]\times[1,2]$ and $n = 1,2,\cdots$.

Comparing (51) with (50) we have:

$$\mathrm{r}_n(s,t) = \frac{1}{n}\arctan(s+t)^n \ , \ K_n(s,t,x,y) = \sin\left(\frac{s+t+x+y}{n}\right),$$

$$f_n(\tau_1,\tau_2,v(\tau_1,\tau_2)) = \ln\left(\frac{1+4n^2+(\tau_1+\tau_2)^2\Big[4+\sup_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}\Big]}{4[(\tau_1+\tau_2)^2+n^2]}\right)$$
$$= \ln\left(1+\frac{1+(\tau_1+\tau_2)^2\sup_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}}{4[(\tau_1+\tau_2)^2+n^2]}\right).$$

Denoting, by I_2 the interval $[1,2]$, we show that the assumptions of the Theorem 36 are satisfied. It is obvious that the operator F_1 defined by:

$$(\mathsf{F}_1 v)(s,t) = (f_n(s,t,v(s,t))),$$

transforms the space $I_2^2 \times c_0$ into c_0.

Now, we establish that the family of functions $\{(\mathsf{F}_1 v)(s,t)\}_{(s,t)\in I_2^2}$ is equicontinuous at an arbitrary point $v \in c_0$. Fix $\epsilon > 0, n \in \mathbb{N}, v \in c_0$ and $(s,t) \in I_2^2$, let $u \in c_0$ such that $\|u-v\|_{c_0} \leq \epsilon$. Then,

$$\begin{aligned}
&|f_n(s,t,v) - f_n(s,t,u)| \\
&= \left|\ln\left(1+\frac{1+(\tau_1+\tau_2)^2\sup\limits_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}}{4[(\tau_1+\tau_2)^2+n^2]}\right) - \ln\left(1+\frac{1+(\tau_1+\tau_2)^2\sup\limits_{k\geq n}\{|u_k(\tau_1,\tau_2)|\}}{4[(\tau_1+\tau_2)^2+n^2]}\right)\right| \\
&\leq \left|\frac{(\tau_1+\tau_2)^2}{4[(\tau_1+\tau_2)^2+n^2]}\left[\sup\limits_{k\geq n}\{|v_k(\tau_1,\tau_2)|\} - \sup\limits_{k\geq n}\{|u_k(\tau_1,\tau_2)|\}\right]\right| \\
&\leq \frac{1}{16}\sup\limits_{k\geq n}\{|v_k - u_k|\}.
\end{aligned}$$

Hence,

$$\|f_n(s,t,v) - f_n(s,t,u)\| \leq \frac{1}{16}\|v-u\|_{c_0} \leq \frac{\epsilon}{16},$$

so the family $\{(\mathsf{F}_1 v)(s,t)\}_{(s,t)\in I_2^2}$ is equicontinuous.

Now, fix $(s,t) \in I_2^2, v \in c_0$ and $n \in \mathbb{N}$, then:

$$|f_n(s,t,v)| = \left|\ln\left(1+\frac{1+(\tau_1+\tau_2)^2\sup\limits_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}}{4[(\tau_1+\tau_2)^2+n^2]}\right)\right|$$

$$\leq \frac{1 + (\tau_1 + \tau_2)^2 \sup\limits_{k \geq n}\{|v_k(\tau_1, \tau_2)|\}}{4[(\tau_1 + \tau_2)^2 + n^2]}$$

$$= \frac{1}{4[(\tau_1 + \tau_2)^2 + n^2]} + \frac{(\tau_1 + \tau_2)^2}{4[(\tau_1 + \tau_2)^2 + n^2]} \sup_{k \geq n}\{|v_k(\tau_1, \tau_2)|\}$$

We put $p_n(s,t) = \frac{1}{4[(s+t)^2 + n^2]}$ *and* $q_n(s,t) = \frac{(s+t)^2}{4[(s+t)^2 + n^2]}$. *Then, clearly* $p_n(s,t)$ *and* $q_n(s,t)$ *are real–valued functions and* $p_n(s,t)$ *converges uniformly to zero.*

Further, $|q_n(s,t)| \leq 1/4$ *for all* $n = 1, 2, \cdots$.

Hence, $\mathcal{P} = 1/4$ *and* $\mathcal{Q} = \sup\limits_{s,t) \in I^2} \{q_n(s,t)\} = 1/4$.

The functions $K_n(s,t,x,y)$ *are continuous on* $I_2^4 = [1,2] \times [1,2] \times [1,2] \times [1,2]$ *and the function sequence* $(K_n(s,t,x,y))$ *is equibounded on* I_2^4. *Moreover,*

$$\mathsf{K} = \sup\Big\{|K_n(s,t,x,y)| : (s,t), (x,y) \in I_2^2, n \in \mathbb{N}\Big\} = 1.$$

Now, fix $\epsilon > 0, (x,y) \in I_2^2$ *and* $n \in \mathbb{N}$ *then for arbitrary* $(s_1, t_1), (s_2, t_2) \in I^2$ *with:*

$$|s_2 - s_1| \leq \frac{\epsilon}{2}, |t_2 - t_1| \leq \frac{\epsilon}{2}.$$

We have:

$$\begin{aligned} |K_n(s_2, t_2, x, y) - K_n(s_1, t_1, x, y)| &\leq \left| \frac{s_2 + t_1 + x + y}{n} - \frac{s_1 + t_1 + x + y}{n} \right| \\ &= \frac{1}{n}|(s_2 - s_1) + (t_2 - t_1)| \\ &\leq \frac{1}{n}(|s_2 - s_1| + |t_2 - t_1|) \\ &\leq \epsilon. \end{aligned}$$

Therefore, $(K_n(s,t,x,y))$ *is equicontinuous.*

Thus, $\mathsf{r}_n(s,t)$, *is continuous for all* $(s,t) \in I_2^2$ *and for all* n *and* $\mathsf{r}_n(s,t)$ *converges uniformly to zero.*

The value of the factor $(b-a)^2 \mathsf{K}\mathcal{Q} = 1/4 < 1$. *Thus, by Theorem 36, the infinite system in (50) has a solution in* c_0, *which belongs to the ball* $\mathsf{B}_{R_0} \subset c_0$ *where:*

$$R_0 = \frac{\mathsf{R} + (b-a)^2 \mathsf{K}\mathsf{Q}}{1 - (b-a)^2 \mathsf{K}\mathcal{Q}} = \frac{\arctan 4 + \frac{1}{4}}{1 - \frac{1}{4}} = \frac{4}{3}\arctan(4).$$

6.2.1. Solution in the Space ℓ_1

The existence of a solution for the system (50) is found in the space ℓ_1 keeping the following assumptions under consideration:

(C_1) The functions $(f_j)_{j=1}^{\infty}$ are real valued and continuous defined on the set $I^2 \times \mathbb{R}^{\infty}$. The operator Q defined on the space $I^2 \times \ell_1$ as:

$$(s,t,v) \mapsto (\mathsf{Q}v)(s,t) = (f_1(s,t,v), f_2(s,t,v), f_3(s,t,v), \ldots),$$

maps $I^2 \times \ell_1$ into ℓ_1. The set of all such functions $\{(\mathsf{Q}v)(s,t)\}_{(s,t) \in I^2}$ is equicontinuous at every point of the space ℓ_1, that is, given $\epsilon, \delta > 0$,

$$\|u - v\|_{\ell_1} \leq \delta \text{ implies } \|(\mathsf{Q}u)(s,t) - (\mathsf{Q}v)(s,t)\|_{\ell_1} \leq \epsilon.$$

(C_2) For fixed $(s,t) \in I^2$, $v(s,t) = \big(v_j(s,t)\big) \in C(I^2, \ell_1)$, the following inequality holds:

$$|f_n(s,t,v(s,t))| \leq a_n(s,t) + d_n(s,t)|v_n|, \qquad n = 1,2,3,\ldots,$$

where $a_j(s,t)$ and $d_j(s,t)$ are real–valued continuous functions on I^2. The function series $\sum_{n=1}^{\infty} a_n(s,t)$ is uniformly convergent on I^2 and the function sequence $\big(d_j(s,t)\big)_{j\in\mathbb{N}}$ is equibounded on I^2. The function $a(s,t)$ given by $a(s,t) = \sum_{n=1}^{\infty} a_n(s,t)$ is continuous on I^2 and the constants D , A defined as:

$$\mathsf{D} = \sup\Big\{d_n(s,t) : (s,t) \in I^2 \,,\, n \in \mathbb{N}\Big\},$$
$$\mathsf{A} = \max\Big\{a(s,t) : (s,t) \in I^2\Big\},$$

are finite.

(C_3) The functions $K_n : I^4 \to \mathbb{R}$ are continuous on I^4 $(n = 1,2,\ldots)$. Furthermore, these functions $K_n(s,t,x,y)$ are equicontinuous with respect to (s,t), that is, for all $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$|K_n(s_2,t_2,x,y) - K_n(s_1,t_1,x,y)| \leq \epsilon \text{ whenever } |s_2 - s_1| \leq \delta, |t_2 - t_1| \leq \delta,$$

for all $(x,y) \in I^2$. Moreover, the function sequence $(K_n(s,t,x,y))$ is equibounded on the set I^4 and:

$$\mathsf{K} = \sup\Big\{|K_n(s,t,x,y)| : (s,t),(x,y) \in I^2, n = 1,2,\ldots\Big\} < \infty.$$

(C_4) The functions $\mathsf{r}_n : I^2 \to \mathbb{R}$ are continuous and the function sequence $(\mathsf{r}_n) \in C(I^2, \ell_1)$.

Remark 17. *Since $I^2 = [a,b] \times [a,b]$ is a compact subset of $\mathbb{R}^2$, so the assumption of continuity in (C_4) implies that $\mathsf{r}_n : I^2 \to \mathbb{R}$ is uniformly continuous, which implies that the function sequence $(\mathsf{r}_n(s,t))$ is equicontinuous on I^2, as for every $\epsilon > 0$ there is a $\delta > 0$, such that for all $(s_1,t_1),(s_2,t_2) \in I^2$,*

$$\|(\mathsf{r}_n(s_1,t_1)) - (\mathsf{r}_n(s_2,t_2))\|_{\ell_1} \leq \sum_{n=1}^{\infty} |\mathsf{r}_n(s_2,t_2) - \mathsf{r}_n(s_2,t_2)| \leq \epsilon, \tag{52}$$

whenever $|(s_1,t_1) - (s_2,t_2)| < \delta$. Furthermore, by (52), the function series $\sum_{n=1}^{\infty} \mathsf{r}_n(s,t)$ is obviously convergent on I^2 and the function:

$$\mathsf{r}(s,t) = \sum_{n=1}^{\infty} \mathsf{r}_n(s,t),$$

is continuous on I^2. Furthermore,

$$R = \max\{\mathsf{r}(s,t) : (s,t) \in I^2\} < \infty.$$

Theorem 37. *If the system (50) satisfies (C_1)–(C_4), then it has at least one solution $v(s,t) = \big(v_j(s,t)\big)_{j\in\mathbb{N}}$ in ℓ_1 for fixed $(s,t) \in I^2$, whenever $(b-a)^2\mathsf{K}\mathsf{D} < 1$.*

Example 16. *We study the infinite system of Hammerstein-type integral equations in two variables:*

$$v_n(s,t) = \frac{\alpha}{n^2}\ln[(s+t)+n] + \int_1^2\int_1^2 \arctan(s+t+\tau_1+\tau_2+n)\Big((\tau_1+\tau_2)^2 e^{-n(\tau_1+\tau_2)} + \frac{\sin n(\tau_1+\tau_2)}{(\tau_1+\tau_2)^2+n^3}\cdot\frac{v_n^2(\tau_1,\tau_2)}{1+v_1^2(\tau_1,\tau_2)+\cdots+v_n^2(\tau_1,\tau_2)}\Big)d\tau_1 d\tau_2 \quad (53)$$

for $(s,t)\in[1,2]\times[1,2]$, $\alpha>0$ *a constant.*

Comparing the system with (50) *we have:*

$$\begin{aligned} \mathsf{r}_n(s,t) &= \frac{\alpha}{n^2}\ln[(s+t)+n], \\ K_n(s,t,x,y) &= \arctan(s+t+x+y+n), \\ f_n(s,t,v_1,v_2,\ldots) &= (s+t)^2 e^{-n(s+t)} + \frac{\sin n(s+t)}{(s+t)^2+n^3}\cdot\frac{v_n^2(s,t)}{1+v_1^2(s,t)+\cdots+v_n^2(s,t)}. \end{aligned}$$

for $(s,t),(\tau_1,\tau_2)\in[1,2]\times[1,2]$ *and* $n=1,2,\cdots$.

Clearly, $\mathsf{r}_n(s,t)$ *is continuous on* $I_1^2=[1,2]\times[1,2]$.

Moreover, for fixed $(s_1,t_1),(s_2,t_2)\in I_1^2$, *we see that:*

$$\begin{aligned} \Big\|(\mathsf{r}_n)(s_1,t_1)-(\mathsf{r}_n)(s_2,t_2)\Big\| &= \sum_{n=1}^{\infty}|\mathsf{r}_n(s_1,t_1)-\mathsf{r}_n(s_2,t_2)| \\ &= \alpha\sum_{n=1}^{\infty}\frac{1}{n^2}|\ln[(s_1+t_1)+n]-\ln[(s_2+t_2)+n]| \\ &= \alpha\sum_{n=1}^{\infty}\frac{1}{n^2}\left|\ln\left(1+\frac{s_1+t_1-s_2-t_2}{s_2+t_2+n}\right)\right| \\ &\le \alpha\sum_{n=1}^{\infty}\frac{1}{n^3}|s_1+t_1-s_2-t_2| \\ &\le \alpha[|s_1-s_2|+|t_1-t_2|]\zeta(3), \end{aligned}$$

where $\zeta(s)$ *denotes Riemann zeta function.*

Choosing $\delta=\epsilon/(\alpha\zeta(3))$, *so that* $|s_1-s_2|<\frac{\delta}{2}$, $|t_1-t_2|<\delta/2$, *we obtain:*

$$\|(\mathsf{r}_n)(s_1,t_1)-(\mathsf{r}_n)(s_2,t_2)\|<\epsilon.$$

Furthermore, for every $(s,t)\in I_1^2$ *we have:*

$$\mathsf{r}_n(s,t)\le\frac{\alpha}{n^2}\ln(4+n)\le\frac{\alpha}{n^2}\sqrt{4+n}\le\alpha\left(\frac{2}{n^2}+\frac{1}{n^{3/2}}\right).$$

Hence,

$$\mathsf{R}=\max\left\{\sum_{n=1}^{\infty}\mathsf{r}_n(s,t):(s,t)\in I_1^2\right\}=\alpha(2\zeta(2)+\zeta(1.5))<\infty. \quad (54)$$

Thus, assumption (C_4) *and Remark 17 are satisfied.*

Then, the function $K_n(s,t,x,y)$ *is continuous in* I_1^4 *and:*

$$K_n(s,t,x,y)=|\arctan(s+t+x+y+n)|\le\frac{\pi}{2}.$$

Thus, the function sequence (K_n) *is equibounded on* I_1^4. *Moreover, for fixed* $(s_1,t_1),(s_2,t_2)\in I_1^2$ *and* $n\in\mathbb{N}$, *we have for* $(x,y)\in I_1^2$:

$$|K_n(s_1,t_1,x,y)-K_n(s_2,t_2,x,y)|$$

$$= |\arctan(s_1 + t_1 + x + y + n) - \arctan(s_2 + t_2 + x + y + n)|$$
$$\leq |s_1 - s_2| + |t_1 - t_2|.$$

Therefore, the function sequence $K_n(s,t,x,y)$ is equicontinuous with respect to $(s,t) \in I_1^2$ uniformly with respect to $(x,y) \in I_1^2$, the value of the constant K *given as:*

$$\mathsf{K} = \sup\{K_n(s,t,x,y) : (s,t),(x,y) \in I_1^2, n \in \mathbb{N}\} = \frac{\pi}{2}. \tag{55}$$

Hence, all assumptions of (C_3) are satisfied.
Again,

$$\begin{aligned}
|f_n(s,t,v)| &\leq (s+t)^2 e^{-n(s+t)} + \left| \frac{\sin n(s+t)}{(s+t)^2 + n^3} \cdot \frac{v_n^2}{1 + v_1^2 + \cdots + v_n^2} \right| \\
&\leq (s+t)^2 e^{-n(s+t)} + \frac{1}{(s+t)^2 + n^3} \cdot \left| \frac{v_n^2}{1 + v_1^2 + \cdots + v_n^2} \right| \\
&\leq (s+t)^2 e^{-n(s+t)} + \frac{1}{(s+t)^2 + n^3} \cdot \frac{|v_n|}{1 + v_n^2}(|v_n|) \\
&\leq (s+t)^2 e^{-n(s+t)} + \frac{1}{2[(s+t)^2 + n^3]} |v_n|.
\end{aligned}$$

Taking, $a_n(s,t) = (s+t)^2 e^{-n(s+t)}$ and $d_n(s,t) = \dfrac{1}{2[(s+t)^2 + n^3]}$ gives:

$$|f_n(s,t,v)| \leq a_n(s,t) + d_n(s,t)|v_n|.$$

Obviously, the functions $a_n(s,t)$ are continuous on I_1^2, for any $(s,t) \in I_1^2$ we have $|a_n(s,t)| \leq (4/n^3) \cdot e^{-2}$, and the function series $a(s,t) = \sum_{n=1}^{\infty} a_n(s,t) = \dfrac{(s+t)^2}{e^{s+t} - 1}$ is uniformly convergent on the interval I_1^2.
Furthermore,

$$|d_n(s,t)| = \frac{1}{2[(s+t)^2 + n^3]} \leq \frac{1}{2n^3} \leq \frac{1}{2},$$

for all $n \in \mathbb{N}$. Hence, the function sequence $(h_n(s,t))$ is equibounded on I_1^2. The value of the constants A, D *are:*

$$\mathsf{A} = \max\left\{ a(s,t) : (s,t) \in I_1^2 \right\} = \frac{16}{e^2 - 1}; \mathsf{D} = \frac{1}{2}, \tag{56}$$

and $(b-a)^2 \mathsf{KD} = \dfrac{\pi}{8}$. Using (54), (55), (56), and equation (11) of [62], we obtain:

$$\begin{aligned}
R_1 &= \frac{\alpha(2\zeta(2) + \zeta(3)) + (2-1)^2 \times \frac{1}{2} \times \frac{16}{e^2 - 1}}{1 - \frac{\pi}{8}} \\
&\approx 1.84 \qquad\qquad \textit{for } \alpha = 0.10.
\end{aligned} \tag{57}$$

Finally, we check whether the assumption (C_1) is satisfied. Fix $v = (v_n) \in \mathsf{B}_{R_1} \subset \ell_1$ and $\epsilon > 0$, then for any $u = (u_n) \in \mathsf{B}_{R_1}$ with $\|u - v\|_{\ell_1} \leq \epsilon$, then for fixed $(s,t) \in I_1^2$, we have:

$$\begin{aligned}
\left\| (\mathsf{Q}u)(s,t) - (\mathsf{Q}v)(s,t) \right\|_{\ell_1} &= \sum_{n=1}^{\infty} |f_n(s,t,u) - f_n(s,t,v)| \\
\leq \sum_{n=1}^{\infty} \left| \frac{\sin n(s+t)}{(s+t)^2 + n^3} \right| &\left| \frac{u_n^2}{1 + u_1^2 + \cdots + u_n^2} - \frac{v_n^2}{1 + v_1^2 + \cdots + v_n^2} \right|
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^3} |u_n^2(1+v_1^2+\cdots+v_n^2) - v_n^2(1+u_1^2+\cdots+u_n^2)|$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[|u_n^2 - v_n^2| + |u_n^2(v_1^2+\cdots+v_n^2) - u_n^2(u_1^2+\cdots+u_n^2)|$$
$$+ |u_n^2(u_1^2+\cdots+u_n^2) - v_n^2(u_1^2+\cdots+u_n^2)| \Big]$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[|u_n^2 - v_n^2| + u_n^2(|v_1^2 - u_1^2| + \cdots + |v_n^2 - u_n^2|) + |u_n^2 - v_n^2|(u_1^2+\cdots+u_n^2) \Big].$$

Since, $v_n, u_n \in \mathsf{B}_{R_1}, n \in \mathbb{N}$ so $|v_n| \leq R_1, |u_n| < R_1$ so:

$$\Big\| (\mathrm{Q}u)(s,t) - (\mathrm{Q}v)(s,t) \Big\|_{\ell_1} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \Big(|u_n - v_n|(|u_n| + |v_n|)(1 + u_1^2 + \cdots + u_n^2) +$$
$$u_n^2(|v_1 - u_1|(|v_1| + |u_1|) + \cdots + |v_n - u_n|(|v_n| + |u_n|) \Big)$$
$$< 2R_1 \sum_{n=1}^{\infty} \left[\frac{1}{n^3} |u_n - v_n|(1 + nR_1^2) + R_1^2 \left(\sum_{i=1}^{n} |v_i - u_i| \right) \right]$$
$$= 2R_1 \|u - v\|_{\ell_1} \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[(1 + nR_1^2) + R_1^2 \Big]$$
$$= 2R_1 \|u - v\|_{\ell_1} \Big([1 + R_1^2]\zeta(3) + R_1^2\zeta(2) \Big).$$

Thus, choose:

$$\delta = \frac{\epsilon}{2R_1([1 + R_1^2]\zeta(3) + R_1^2\zeta(2))},$$

then for $\|u - v\|_{\ell_1} < \delta$ we have:

$$\Big\| (\mathrm{Q}u)(s,t) - (\mathrm{Q}v)(s,t) \Big\|_{\ell_1} < \epsilon.$$

Hence, the assumption (C_1) is also satisfied, therefore by Theorem 37, we conclude that the system in (53) has a solution in $\mathsf{B}_{R_1} \subset \ell_1$, where R_1 is given by (57).

6.3. Solvability of an Infinite System Of Integral Equations of Volterra–Hammerstein Type on the Real Half–Axis

Here, we consider one more recent application of a measure of noncompactness and Darbo's fixed point theorem to the solvability of an infinite system of integral equations of Volterra–Hammerstein type:

$$x_n(t) = a_n(t) + f_n(t, x_1, x_2 \ldots) \int_0^t k_n(t,s) g_n(s, x_1, x_2, \ldots)\, ds \tag{58}$$

where $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, on the real half–axis ([63], Theorem 3.4). The paper [63] is in continuation of the papers [64,65].

In [63], the authors construct a measure of noncompactness on the space $BC(\mathbb{R}_+, \ell_\infty)$ of all functions $x : \mathbb{R}_+ \to \ell_\infty$ that are continuous and bounded on $\mathbb{R}_+$. If $x \in BC(\mathbb{R}_+, \ell_\infty)$, then $x(t) = (x_n(t)) \in \ell_\infty$ for each $t \in \mathbb{R}_+$; $BC(\mathbb{R}_+, \ell_\infty)$ is a Banach space with:

$$\|x\| = \sup_{t \in \mathbb{R}_+} \|x\|_\infty = \sup_{t \in \mathbb{R}_+} \left(\sup_n |x_n(t)| \right) \text{ for all } x \in BC(\mathbb{R}_+, \ell_\infty).$$

The following assumptions are made for the system (58):

(i) The sequence $(a_n(t)) \in BC(\mathbb{R}_+, \ell_\infty)$ satisfies $\lim_{t\to\infty} a_n(t) = 0$ uniformly in n, that is,

$$\text{for all } \varepsilon > 0 \text{ there exists } T > 0 \text{ such that for all } t \geq T \text{ and all } n \in \mathbb{N}\ |a_n(t)| \leq \varepsilon,$$

and also $(a_n(t)) \in c_0$ for all $t \in \mathbb{R}_+$.

(ii) The functions $k_n(t,s) = k_n : \mathbb{R}_+^2 \to \mathbb{R}$ are continuous on $\mathbb{R}_+^2$ for $n = 1, 2, \ldots$. Moreover the functions $t \to k_n(t,s)$ are equicontinuous on $\mathbb{R}_+$ uniformly with respect to $s \in \mathbb{R}_+$, that is,

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that for all } n \in \mathbb{N}, \text{ all } s \in \mathbb{R}_+ \text{ and all } t_1, t_2 \in \mathbb{R}_+ \\ |t_1 - t_2| \leq \delta \text{ implies } |k_n(t_2, s) - k_n(t_1, s)| \leq \varepsilon.$$

(iii) There exists a positive constant K_1 such that:

$$\int_0^t |k_n(t,s)|\, ds \leq K_1$$

for any $t \in \mathbb{R}_+$ and $n = 1, 2, \ldots$.

(iv) The sequence $(k_n(t,s))$ is equibounded on $\mathbb{R}_+^2$, that is, there exists a positive constant K_2 such that $|k_n(t,s)| \leq K_2$ for all $t, s \in \mathbb{R}^+$ and $n = 1, 2, \ldots$.

(v) The functions f_n are defined on the $\mathbb{R}_+ \times \mathbb{R}$ and take real values for $n = 1, 2, \ldots$. Moreover, the function $t \to f_n\ (t, x_1, x_2, \ldots)$ is uniformly continuous on $\mathbb{R}_+$ with respect to $x = (x_n) \in \ell_\infty$ and uniformly with respect to $n \in \mathbb{N}$, that is, the following condition is satisfied:

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that for all } (x_i) \in \ell_\infty, \text{ all } n \in \mathbb{N} \text{ and all } t, s \in \mathbb{R}_+ \\ |t - s| \leq \delta \text{ implies } |f_n(t, x_1, x_2, \ldots) - f_n(s, x_1, x_2, \ldots)| \leq \varepsilon.$$

(vi) There exists a function $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that l is nondecreasing on $\mathbb{R}_+$, continuous at 0 and there exists a sequence of functions (f_n) in $BC(\mathbb{R}_+, \ell_\infty)$, taking nonnegative values and such that $\lim_{t\to\infty} f_n(t) = 0$ uniformly with respect to $n \in \mathbb{N}$ (cf. assumption (i)) and $\lim_{n\to\infty} f_n(t) = 0$ for any $t \in \mathbb{R}_+$. Moreover, for any $r > 0$ the following inequality is satisfied:

$$|f_n(t, x_1, x_2, \ldots)| \leq \bar{f}_n(t) + l(r) \sup\{|x_i| : i > n\}$$

for each $x = (xi) \in \ell_\infty$ such that $\|x\|_\infty \leq r$, for every $t \in \mathbb{R}_+$ and for $n = 1, 2, \ldots$. Let $\bar{F} = \sup\{\bar{f}_n(t) : n \in \mathbb{N}, t \in \mathbb{R}_+$.

(vii) There exists a nondecreasing function $m : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous at 0 and satisfies:

$$|f_n(t, x_1, x_2, \ldots) - f_n(t, y_1, y_2, \ldots)| \leq m(r)\|x - y\|_\infty$$

for any $r > 0$, for $x = (x_i), y = (y_i) \in \ell_\infty$ such that $\|x\|_\infty, \|y\|_\infty \leq r$ and for all $t \in \mathbb{R}_+$ and $n = 1, 2, \ldots$.

(viii) The functions g_n are defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and take real values for $n = 1, 2, \ldots$. Moreover, the operator g defined on $\mathbb{R}_+ \times \ell_\infty$ by:

$$(gx)(t) = (g_n(t,x)) = (g_1(t,x), g_2(t,x), \ldots)$$

transforms the set $\mathbb{R}_+ \times \ell_\infty$ into ℓ_∞ and is such that the family of functions $\{(gx)(t)\}_{t\in\mathbb{R}_+}$ is equicontinuous on ℓ_∞, that is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\|(gy)(t) - (gx)(t)\|_\infty \leq \varepsilon$$

for all $t \in \mathbb{R}_+$ and all $x, y \in \ell_\infty$ such that $\|x - y\|_\infty \leq \delta$.

(ix) The operator g defined in assumption (viii) is bounded on the set $\mathbb{R}_+ \times \ell_\infty$, that is, there exists a positive constant G such that $\|(gx)(t)\|_\infty \leq G$ for all $x \in \ell_\infty$ and all $t \in \mathbb{R}_+$.

(x) There exists a positive solution r_0 of the inequality:

$$A + \bar{F}\bar{G}K_1 + \bar{G}K_1 r l(r) \leq r$$

such that $\bar{G}K_1 \max\{l(r_0), m(r_0)\} < 1$, where the constants $\bar{F}, \bar{G}, K1$ were defined above and the constant A is defined by:

$$A = \sup\{|a_n(t)| : t \in \mathbb{R}_+, n = 1, 2, \dots\}.$$

Theorem 38. ([63], Theorem 3.4) *Under the assumptions (i)–(x), the infinite system (58) has at least one solution $x(t) = (x_n(t))$ in $BC(\mathbb{R}_+, \ell_\infty)$.*

Remark 18. *An example of the application of Theorem 38 can be found in ([63], Section 4).*

We also recommend the paper [66].

Recently, in 2021 [67], a new sequence space related to the space ℓ_p $(1 \leq p < \infty)$ was defined. The authors showed that it is a BK space with a Schauder basis. They established a formula for the Hausdorff measure of noncompactness for the bounded sets in the new sequence space. Then, Darbo's fixed point theorem is applied to study the existence results for some infinite system of Langevin equations.

6.4. Periodic Mild Solutions for a Class of Functional Evolution Equations

In [68], the authors showed that the Poincaré operator is condensing with respect to the Kuratowski measure of noncompactness in a determined phase space. They also obtained periodic solutions from bounded solutions by applying Sadovskiĭ's fixed point theorem.

Consider the existence of periodic mild solutions to the class of functional differential equations with infinite delay and non-instantaneous impulses:

$$\begin{cases} u'(t) + A(t)u(t) = f(t, u(t), u_t) & \text{if} \quad t \in I_k,\ k = 0, 1, \dots, \\ u(t) = g_k(t, u(t_k^-)) & \text{if} \quad t \in J_k,\ k = 1, 2, \dots, \\ u(t) = f(t) & \text{if} \quad t \in \mathbb{R}_- := (-\infty, 0], \end{cases} \tag{59}$$

where $I_0 = [0, t_1]$, $I_k = (s_k, t_{k+1}]$, $J_k = (t_k, s_k]$, $0 = s_0 < t_1 = s_1 = t_2 < \cdots < s_{m-1} = t_m = s_m = t_{m+1} = T = s_{m+1} = t_{m+2} = \cdots < +\infty$, $(E, \|\cdot\|_E)$ is a real Banach space, $f : I_k \times E \times \mathcal{B} \to E$, $k = 0, \dots, g_k : J_k \times E \to E$, $k = 1, 2, \dots$, are given functions T–periodic in $t, T > 0$, $\mathcal{B}$ is an abstract phase space to be specified later, and $\phi : \mathbb{R}_- \to E$ is a given function. Here, $\{A(t)\}_{t>0}$ is a T–periodic family of unbounded operators from E into E that generate an evolution system of operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+}$ for $(t,s) \in \Lambda = \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \leq s \leq t < +\infty\}$, where $\mathbb{R}_+ := [0, +\infty)$.

For any continuous function u and any $t \in \mathbb{R}_+$, we denote by u_t the element of $\mathcal{B}$ defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in \mathbb{R}_- = (-\infty, 0]$. Here, $u_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories u_t belong to $\mathcal{B}$.

By a periodic mild solution of problem (59), we mean a measurable and T–periodic function u that satisfies:

$$u(t) = \begin{cases} U(t,0)\phi(0) + \int_0^t U(t,s) f(s, u(s), u_s)\, ds & \text{if} \quad t \in I_0 \\ U(t, s_k) g_k(s_k, u(s_k^-)) + \int_{s_k}^t U(t,s) f(s, u(s), u_s)\, ds, & \text{if} \quad t \in I_k,\ k = 1, \dots, m \\ g_k(t, u(t_k^-)) & \text{if} \quad t \in J_k,\ k = 1, \dots, m \\ \phi(t) & \text{if} \quad t \in \mathbb{R}_-. \end{cases}$$

We use the following assumptions.

(H_1) The functions f and g_k are continuous, and map bounded sets into bounded sets.

(H_2) The function $t \to f(t,u,v)$ is measurable on I_k for $k = 0,\dots,m$ and for each $u,v \in E \times \mathcal{B}$. Furthermore, the functions $u \to f(t,u,v)$ and $v \to f(t,u,v)$ are continuous on $E \times \mathcal{B}$ for a.e. $t \in I_k$ for $k = 0,\dots,m$.

(H_3) There is a positive constant T with $f(t+T,u,v) = f(t,u,v)$, $A(t+T) = A(t)$ for $t \in I_k$ and $k = 0,\dots,m, u,v \in E \times \mathcal{B}$, and $g_k(t+T,z) = g_k(t,z)$ for $t \in J_k$, $k = 1,\dots$ and $m, z \in E$.

(H_4) There exist continuous functions $p : I_k \to \mathbb{R}_+$ and $q : J_k \to \mathbb{R}_+$ with:

$$\|f(t,u,v)\| \le p(t) \text{ for a.e. } t \in I_k,\ k = 0,...,m, \text{ and each } u,v \in E \times \mathcal{B},$$

and,

$$\|g_k(t,z)\| \le q(t) \text{ for a.e. } t \in J_k, \text{ and each } z \in E,\ k = 0,...,m.$$

(H_5) For bounded and measurable sets $B(t) \subset E$ and $B_t \subset \mathcal{B}$ for $t \in \mathbb{R}_+$

$$B(t) = \{u(t) : u \in C(I)\} \text{ and } B_t = \{u_t : u_t \in \mathcal{B}\},$$

implies,

$$\alpha(f(t,B(t),B_t)) \le p(t)\alpha(B) \text{for a.e. } t \in I_k,\ (k = 0,\dots,m),$$

and,

$$\alpha(g_k(t,B)) \le q(t)\alpha(B) \text{ for a.e. } t \in J_k,\ (k = 1,\dots,m),$$

where α is Kuratowski's measure of noncompactness on the Banach space E.

Further, set:

$$\Delta = \{(t,s) \in J \times J : 0 \le s \le t \le T\},$$

$$M = sup_{(t,s)\in\Delta}\|U(t,s)\|_{B(E)},\ p^* = \sup_{t\in I_k} p(t) \text{ and } q^* = \sup_{t\in J_k} q(t).$$

We shall state the main result of the paper [69].

Theorem 39. ([69], Theorem 3.2) *If (H_1)–(H_5) are satisfied and $4MTp^* < 1$, then Problem (59) has at least one T–periodic mild solution on $\mathbb{R}$.*

The authors also present an example to illustrate Theorem 39.

We also mention that fixed point theorems in b–metric spaces were recently considered.

Remark 19. *Recently, in 2021 [69], the authors introduced and studied two generalized contractions, the generalized F_{t_s}–contraction and the generalized (ψ,ϕ,F_{t_s})–contraction. Two fixed point theorems were established in ordered b–metric spaces. An example is presented to illustrate the fixed point theorem of the generalized F_{t_s}–contraction.*

It would be interesting to prove related results in the framework of measures of noncompactnes.

7. Some Mathematical Background

Here, we present some recent results connected to the existence of best proximity points (pairs) for some classes of cyclic and noncyclic condensing operators in Banach spaces with respect to a suitable measure of noncompactness. We also discuss the existence of an optimal solution for systems of integro–differentials.

Recently, many studies [70–74] applied generalizations of Darbo–Sadovskii's fixed point theorem, Theorem 11, concerning the existence of solutions for several classes of functional integral equations.

In the following survey, we present some recent existence results of best proximity points (pairs) as a generalization of fixed points and obtain other extensions of Schauder's fixed point problem as well as Darbo–Sadovskii's fixed point theorem. As applications of our conclusions, we study the existence of optimal solutions for various classes of differential equations.

We recall that a Banach space X is said to be *strictly convex* provided that the following implication holds for $x, y, p \in X$ and $R > 0$:

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ x \neq y \end{cases} \quad \text{implies } \|\frac{x+y}{2} - p\| < R.$$

It is well known that Hilbert spaces and ℓ_p spaces $(1 < p < \infty)$ are strictly convex Banach spaces. Furthermore, the Banach space ℓ_1 with the norm:

$$\|x\| = \sqrt{\|x\|_1 + \|x\|_2}, \text{ for all } x \in \ell_1,$$

where, $\|.\|_1$ and $\|.\|_2$ are the norms on ℓ_1 and ℓ_2, respectively, is strictly convex.

Suppose A is a nonempty subset of a normed linear space X and T maps A into X. It is clear that the necessary (but not sufficient) condition for the existence of a fixed point of T is that the intersection of A and $T(A)$ is nonempty. If T does not have any fixed point, then the distance between x and Tx is positive for any x in A. In this case, it is our purpose to find an element x in A so that the distance of x and Tx is minimum. Such a point is called a best approximant point of T in A. The first best approximation theorem due to Ky Fan ([75]) states that if $A \neq \emptyset$ is a compact and convex subset of a normed linear space X and $T : A \to X$ is a continuous map from A, then T has a best approximant point in A. An interesting extension of Ky Fan's theorem can be considered when $T : A \to B$, where subset $B \subset X$. In this case, it is interesting to study the existence of the *best proximity points*; that is, points in A that estimate the distance between A and B. The existence of best proximity points for various classes of non-self mappings is a subject in optimization theory, which recently attracted the attention of many authors (see [76–79], and the references therein).

Let $A, B \neq \emptyset$ be subsets of a normed linear space X. We say that a pair (A, B) of subsets of a Banach space X satisfies a certain property if both A and B satisfy that property. For example, (A, B) is convex if and only if both A and B are convex; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C,\ B \subseteq D$. From now on, $\mathcal{B}(x; r)$ will denote the closed ball in the Banach space X centered at $x \in X$ with radius $r > 0$. The *closed and convex hull* of a set A will be denoted by $\overline{\text{con}}(A)$. Furthermore, $\text{diam}(A)$ stands for the diameter of the set A. Moreover, for the pair (A, B) we define:

$$A_0 = \{x \in A : \exists\, y' \in B \mid \|x - y'\| = \text{dist}(A, B)\},$$

$$B_0 = \{y \in B : \exists\, x' \in A \mid \|x' - y\| = \text{dist}(A, B)\}.$$

It is known that if (A, B) is a nonempty, weakly compact, and convex pair in a Banach space X, then the pair (A_0, B_0) is also nonempty, weakly compact, and convex.

Definition 18. *A nonempty pair (A, B) in a normed linear space X is said to be proximinal if $A = A_0$ and $B = B_0$.*

A map $T\colon A \cup B \to A \cup B$ is *cyclic relatively nonexpansive* if T is cyclic, that is, $T(A) \subseteq B$, $T(B) \subseteq A$ and $\|Tx - Ty\| \leq \|x - y\|$, whenever $x \in A$ and $y \in B$. In particular, if $A = B$, then T is called a nonexpansive self–map. A point $x^\star \in A \cup B$ is a best proximity point for the map T if:

$$\|x^\star - Tx^\star\| = \text{dist}(A, B) := \inf\{\|x - y\| \colon x \in A,\ y \in B\}.$$

In fact, best proximity point theorems have been studied to find necessary conditions such that the minimization problem:

$$\min_{x \in A \cup B} \|x - Tx\|, \tag{60}$$

has at least one solution.

A map $T\colon A\cup B\to A\cup B$ is *noncyclic relatively nonexpansive* if T is noncyclic, that is, $T(A)\subseteq A$, $T(B)\subseteq B$ and $\|Tx-Ty\|\leq\|x-y\|$ for any $(x,y)\in A\times B$. Clearly, the class of noncyclic relatively nonexpansive maps contains the class of nonexpansive maps. Noncyclic relatively nonexpansive maps may not necessarily be continuous. A point $(p,q)\in A\times B$ is a *best proximity pair* if it is a solution of the following minimization problem:

$$\min_{x\in A}\|x-Tx\|,\quad \min_{y\in B}\|y-Ty\|,\quad \text{and}\quad \min_{(x,y)\in A\times B}\|x-y\|. \tag{61}$$

Clearly, $(p,q)\in A\times B$ is a solution of the problem (61) if and only if:

$$p=Tp,\quad q=Tq,\quad \text{and}\quad \|p-q\|=\operatorname{dist}(A,B).$$

In 2017, M. Gabeleh, proved the following existence theorems by using a concept of *proximal diametral sequences* (we also refer to [80] for the same results which were based on a geometric notion of proximal normal structure).

Theorem 40 ([81])**.** *Let (A,B) be a nonempty, compact, and convex pair in a Banach space X. If T is cyclic relatively nonexpansive mapping, then T has a best proximity point.*

Theorem 41 ([81])**.** *Let (A,B) be a nonempty, compact, and convex pair in a strictly convex Banach space X. If T is noncyclic relatively nonexpansive mapping, then T has a best proximity pair.*

Finally, we state Mazur's lemma.

Lemma 2 ([82])**.** *Let A be a nonempty and compact subset of a Banach space X. Then $\overline{\operatorname{con}}(A)$ is compact.*

8. Cyclic (Noncyclic) Condensing Operators

We start with an extension of Theorem 40.

Definition 19. *Let $(A,B)\neq\emptyset$ be a bounded pair in a Banach space X and $T:A\cup B\to A\cup B$ a cyclic (noncyclic) map. Then, T is called compact whenever both $T|_A$ and $T|_B$ are compact, that is, the pair $(\overline{T(A)},\overline{T(B)})$ is compact.*

The next result generalizes Schauder's fixed point theorem, Theorem 10.

Theorem 42. ([83], Theorem 3.2) *Let $(A,B)\neq\emptyset$ be a bounded, closed, and convex pair in a Banach space X such that $A_0\neq\emptyset$. Also, let $T:A\cup B\to A\cup B$ be a cyclic relatively nonexpansive map. If T is compact, then T has a best proximity point.*

Proof. Put $K_1=\overline{\operatorname{con}}(T(B))$ and $K_2=\overline{\operatorname{con}}(T(A))$. Let $x\in A_0$. Then there exists $y\in B$ with $\|x-y\|=\operatorname{dist}(A,B)$. Since T is a cyclic relatively nonexpansive map,

$$\operatorname{dist}(K_1,K_2)\leq\|Ty-Tx\|\leq\|x-y\|=\operatorname{dist}(A,B).$$

Thus, $\operatorname{dist}(K_1,K_2)=\operatorname{dist}(A,B)$. It follows from Mazur's lemma that the pair (K_1,K_2) is compact and clearly is convex. Since $T(A)\subseteq B$, we get $\overline{\operatorname{con}}(T(A))\subseteq B$. Hence,

$$T(K_2)=T(\overline{\operatorname{con}}(T(A)))\subseteq T(B)\subseteq\overline{\operatorname{con}}(T(B))=K_1.$$

Analogously, $T(K_1)\subseteq K_2$, and so T is cyclic on $K_1\cup K_2$. It follows from Theorem 40 that there exists a point $x^*\in K_1\cup K_2$ with $\|x^*-Tx^*\|=\operatorname{dist}(K_1,K_2)(=\operatorname{dist}(A,B))$, and the result follows. □

Theorem 43 ([83], Theorem 4.1)**.** *Let $(A, B) \neq \emptyset$ be a bounded, closed, and convex pair in a strictly convex Banach space X such that $A_0 \neq \emptyset$. Furthermore, let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive map. If T is compact, then T has a best proximity pair.*

Proof. We assume $K_1 = \overline{\text{con}}(T(A))$ and $K_2 = \overline{\text{con}}(T(B))$. Then $\text{dist}(K_1, K_2) = \text{dist}(A, B)$. Moreover, $\overline{\text{con}}(T(A)) \subseteq A$, so:

$$T(K_1) = T(\overline{\text{con}}(T(A))) \subseteq T(A) \subseteq \overline{\text{con}}(T(A)) = K_1.$$

Analogously, $T(K_2) \subseteq K_2$. Therefore, T is noncyclic on $K_1 \cup K_2$. On the other hand, from Lemma 2 (K_1, K_2) is compact and convex in a strictly convex Banach space X. By Theorem 41 that there exists $(p, q) \in K_1 \times K_2$ with:

$$p = Tp, \quad q = Tq, \text{ and } \|p - q\| = \text{dist}(K_1, K_2)(= \text{dist}(A, B)),$$

that is, (p, q) is a best proximity pair for the map T. □

Notation. Let $(A, B) \neq \emptyset$ be a pair in a normed linear space X and $T : A \cup B \to A \cup B$ be a cyclic (noncyclic) map. The set of all nonempty, bounded, closed, convex, proximinal, and T–invariant pairs $(C, D) \subseteq (A, B)$ with $\text{dist}(C, D) = \text{dist}(A, B)$ is denoted by $\mathcal{M}_T(A, B)$. Notice that $\mathcal{M}_T(A, B)$ may be empty, but in particular if $(A, B) \neq \emptyset$ is a weakly compact and convex pair in a Banach space X and T is cyclic (noncyclic) relatively nonexpansive, then $(A_0, B_0) \in \mathcal{M}_T(A, B)$ (see [84,85] for more details).

Definition 20 (Gabeleh-Markin, (2018) [83])**.** *Let $(A, B) \neq \emptyset$ be a convex pair in a Banach space X and μ an* MNC *on X. A map $T : A \cup B \to A \cup B$ is said to be a cyclic (noncylic) condensing operator if there exists $r \in (0, 1)$ such that for any $(K_1, K_2) \in \mathcal{M}_T(A, B)$,*

$$\mu(T(K_1) \cup T(K_2)) \leq r\mu(K_1 \cup K_2).$$

Definition 21 (Gabeleh-Vetro, (2019) [86])**.** *Let $(A, B) \neq \emptyset$ be a convex pair in a Banach space X and μ be an* MNC *on X. A map $T : A \cup B \to A \cup B$ is said to be a cyclic (noncyclic) generalized condensing operator provided that T is cyclic (noncyclic) map and for any $(C, D) \in \mathcal{M}_T(A, B)$ there exist $\psi \in \Psi$ and $l \in \mathbb{N}$ such that:*

$$\mu(C_{2l} \cup D_{2l}) \leq \psi(\mu(C \cup D)).$$

Notation. Let Φ denote the set of all functions $\varphi : [0, \infty) \to [0, 1)$ such that:

$$\varphi(t_n) \to 1 \Rightarrow t_n \to 0.$$

Definition 22 (Gabeleh-Moshokoa-Vetro, (2019) [87])**.** *Let $(A, B) \neq \emptyset$ be a convex pair in a Banach space X and μ be an* MNC *on X. A map $T : A \cup B \to A \cup B$ is said to be a noncyclic (cyclic) φ-condensing operator for some $\varphi \in \Phi$ provided that for any $(K_1, K_2) \in \mathcal{M}_T(A, B)$ we have:*

$$\mu(T(K_1) \cup T(K_2)) \leq \varphi(\mu(K_1 \cup K_2))\mu(K_1 \cup K_2).$$

Example 17. *Let $(A, B) \neq \emptyset$ be a convex pair in a Banach space X such that B is compact and α is the Kuratowski measure of noncompactness on X. Assume that $T : A \cup B \to A \cup B$ is a cyclic maps so that $T|_A$ is contraction with the contraction constant $r \in]0, 1[$. Then T is a cyclic condensing operator.*

Proof. Suppose $(H_1, H_2) \subseteq (A, B)$ is a nonempty, bounded, closed, convex, and proximinal pair, which is T-invariant and $\text{dist}(H_1, H_2) = \text{dist}(A, B)$. Since B is compact, $\alpha(T(H_2)) = 0$ and so,

$$\alpha(T(H_1) \cup T(H_2)) = \max\{\alpha(T(H_1)), \alpha(T(H_2))\}$$

$$= \alpha(T(H_1)) \leq r\alpha(H_1) \leq r\alpha(H_1 \cup H_2),$$

and the result follows. □

We recall that $(A, B) \neq \emptyset$ in a metric space (X, d) is be *proximal compactness* ([88]) provided that every net $\{(x_\alpha, y_\alpha)\}$ of $A \times B$ satisfying the condition that $d(x_\alpha, y_\alpha) \to \text{dist}(A, B)$, has a convergent subnet in $A \times B$.

Example 18. *Let $(A, B) \neq \emptyset$ be a convex and a proximal compactness pair in a Banach space X and μ be a measure of noncompactness on X. Then, every cyclic relatively nonexpansive map $T : A \cup B \to A \cup B$ is a condensing operator.*

Proof. Suppose $(H_1, H_2) \subseteq (A, B) \neq \emptyset$ is a bounded, closed, convex, and proximinal pair, which is T–invariant and $\text{dist}(H_1, H_2) = \text{dist}(A, B)$. We prove that $(T(H_1), T(H_2))$ is a relatively compact pair. Let $\{x_n\}$ be a sequence in H_1. Since the (H_1, H_2) is proximinal, there exists a sequence $\{y_n\}$ in H_2 such that $\|x_n - y_n\| = \text{dist}(A, B)$ for all $n \geq 1$. Then,

$$\|Tx_n - Ty_n\| \leq \|x_n - y_n\| = \text{dist}(A, B), \quad \forall n \geq 1.$$

Since (A, B) is a proximal compactness pair, the sequence $\{(Tx_n, Ty_n)\}$ has a convergent subsequence which implies that $(T(H_1), T(H_2))$ is relatively compact. Therefore, $\mu(T(H_1) \cup T(H_2)) = 0$, which concludes that T is a condensing operator for any $r \in [0, 1[$. □

9. Existence Results

In this section, we present some existence theorems of best proximity points for the aforesaid classes of condensing operators, which are new extensions of Darbo's fixed point problem.

Theorem 44 ([83]). *Let $(A, B) \neq \emptyset$ be a bounded, closed, and convex pair in a Banach space X such that $A_0 \neq \emptyset$ and μ is an* MNC *on X. Suppose $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive map, which is condensing in the sense of Definition 21. Then, T has a best proximity point.*

Proof. Note that $(A_0, B_0) \neq \emptyset$ is a closed, convex, and proximinal pair, which is T-invariant because of the fact that T is a cyclic relatively nonexpansive map. Let $(x_0, y_0) \in A_0 \times B_0$ be such that $\|x_0 - y_0\| = \text{dist}(A, B)$ and suppose $\mathcal{C}$ is a family of all nonempty, closed, convex, proximinal, and T-invariant pairs $(E, F) \subseteq (A, B)$ such that $(x_0, y_0) \in E \times F$. Then, $(A_0, B_0) \in \mathcal{C} \neq \emptyset$. Put:

$$(K_1, K_2) = \bigcap_{(E,F) \in \mathcal{C}} (E, F) \in \mathcal{C},$$

and define $N = \overline{\text{con}}(T(K_1) \cup \{y_0\})$ and $M = \overline{\text{con}}(T(K_2) \cup \{x_0\})$. Thus $(x_0, y_0) \in M \times N$ and $(M, N) \subseteq (K_1, K_2)$. Moreover,

$$T(M) \subseteq T(K_1) \subseteq N, \quad T(N) \subseteq T(K_2) \subseteq M,$$

that is, T is cyclic on $M \cup N$. Besides, if $x \in M$, then $x = \sum_{j=1}^{n-1} c_j T(y_j) + c_n x_0$, where $y_j \in K_2$ for all $j \in \{1, 2, \ldots, n-1\}$ for which $c_j \geq 0$, $\sum_{j=1}^{n} c_j = 1$. Since (K_1, K_2) is proximinal, there exists $x_j \in K_1$ so that $\|x_j - y_j\| = \text{dist}(A, B)$ for all $j \in \{1, 2, \ldots, n-1\}$. Now, if $y = \sum_{j=1}^{n-1} c_j T(x_j) + c_n y_0$, then $y \in N$ and we have:

$$\|x - y\| = \|(\sum_{j=1}^{n-1} c_j T(y_j) + c_n x_0) - (\sum_{j=1}^{n-1} c_j T(x_j) + c_n y_0)\|$$

$$\leq \sum_{j=1}^{n-1} c_j \|T(y_j) - T(x_j)\| + c_n \|x_0 - y_0\| \leq [\sum_{j=1}^{n-1} c_j \text{dist}(A,B)] + c_n \text{dist}(A,B)$$
$$= \text{dist}(A,B).$$

Therefore, $M_0 = M$. Similarly, $N_0 = N$ and so (M,N) is proximinal. Hence, $(M,N) \in \mathcal{C}$. It follows from the definition of (K_1,K_2) that $M = K_1$ and $N = K_2$. On the other hand, since T is a condensing operator, we have:

$$\begin{aligned}\mu(M \cup N) &= \max\{\mu(M), \mu(N)\} \\ &= \max\{\mu(\overline{\text{con}}(T(K_2) \cup \{x_0\})), \mu(\overline{\text{con}}(T(K_1) \cup \{y_0\}))\} \\ &= \max\{\mu(T(K_2)), \mu(T(K_1))\} = \max\{\mu(T(N)), \mu(T(M))\} \\ &= \mu(T(N) \cup T(M)) \leq r\mu(M \cup N).\end{aligned}$$

This implies that $\max\{\mu(M), \mu(N)\} = \mu(M \cup N) = 0$. Thereby, $(M,N) \neq \emptyset$ is a compact and convex pair with $\text{dist}(M,N) = \text{dist}(A,B)$ such that $T : M \cup N \to M \cup N$ is a cyclic relatively nonexpansive map. Now from Theorem 42, we conclude that T has a best proximity point. □

In the case that T is noncyclic in the above theorem, we need the strict convexity of the Banach space X.

Theorem 45 ([83]). *Let $(A,B) \neq \emptyset$ be a bounded, closed, and convex pair in a strictly convex Banach space X such that $A_0 \neq \emptyset$ and μ is an* MNC *on X. If $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive map, which is condensing in the sense of Definition 21, then T has a best proximity pair.*

Proof. We note that $(A_0, B_0) \neq \emptyset$ is closed, convex, and proximinal, which is T-invariant. Let $(u,v) \in A_0 \times B_0$ be such that $\|u - v\| = \text{dist}(A,B)$ and suppose $\mathcal{F}$ is a family of all nonempty, closed, convex, proximinal, and T-invariant pairs $(E,F) \subseteq (A,B)$ such that $(u,v) \in E \times F$. Then, $(A_0, B_0) \in \mathcal{F} \neq \emptyset$. Put,

$$(K_1, K_2) = \bigcap_{(E,F) \in \mathcal{C}} (E,F) \in \mathcal{F},$$

and set $H_1 = \overline{\text{con}}(T(K_1) \cup \{u\})$ and $H_2 = \overline{\text{con}}(T(K_2) \cup \{v\})$. Thus, $(u,v) \in H_1 \times H_2$ and $(H_1, H_2) \subseteq (K_1, K_2)$. Further,

$$T(H_1) \subseteq T(K_1) \subseteq H_1, \quad T(H_2) \subseteq T(K_2) \subseteq H_2.$$

Therefore, T is noncyclic on $H_1 \cup H_2$. Moreover, if $x \in H_1$, then $x = \sum_{j=1}^{n-1} c_j T(u_j) + c_n u$, where $u_j \in K_1$ for all $j \in \{1, 2, \ldots, n-1\}$ for which $c_j \geq 0$, $\sum_{j=1}^{n} c_j = 1$. In view of the fact that (K_1, K_2) is proximinal, there exists $v_j \in K_2$ so that $\|u_j - v_j\| = \text{dist}(A,B)$ for all $j \in \{1, 2, \ldots, n-1\}$. Now, if we define $y = \sum_{j=1}^{n-1} c_j T(v_j) + c_n v$, then $y \in H_2$ and:

$$\begin{aligned}\|x - y\| &= \|(\sum_{j=1}^{n-1} c_j T(u_j) + c_n u) - (\sum_{j=1}^{n-1} c_j T(v_j) + c_n v)\| \\ &\leq \sum_{j=1}^{n-1} c_j \|T(u_j) - T(v_j)\| + c_n \|u - v\| \leq \text{dist}(A,B).\end{aligned}$$

Hence, $(H_1)_0 = H_1$. By a similar argument, we can see that $(H_2)_0 = H_2$, that is, (H_1, H_2) is a proximinal pair. This concludes that $(H_1, H_2) \in \mathcal{F}$ and by the definition of (K_1, K_2) we must have $H_1 = K_1$ and $H_2 = K_2$. Now, since T is a condensing operator,

$$\begin{aligned}\mu(H_1 \cup H_2) &= \max\{\mu(H_1), \mu(H_2)\}\\ &= \max\{\mu(\overline{\text{con}}(T(K_1) \cup \{u\})), \mu(\overline{\text{con}}(T(K_2) \cup \{v\}))\}\\ &= \max\{\mu(T(K_1)), \mu(T(K_2))\} = \max\{\mu(T(H_1)), \mu(T(H_2))\} \leq r\mu(H_1 \cup H_2).\end{aligned}$$

Thereby, $\max\{\mu(H_1), \mu(H_2)\} = 0$, and so, $(H_1, H_2) \neq \emptyset$ is a compact and convex pair with $\text{dist}(H_1, H_2) = \text{dist}(A, B)$ and that $T : H_1 \cup H_2 \to H_1 \cup H_2$ is a noncyclic relatively nonexpansive map. Now the result follows from Theorem 43. □

We now present some extensions of Theorem 44 and Theorem 45.

Theorem 46 ([86]). *Let $(A, B) \neq \emptyset$ be a bounded, closed, and convex pair in a Banach space X such that $A_0 \neq \emptyset$ and μ be an* MNC *on X. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive map which is Meir–Keeler condensing. Then, T has a best proximity point.*

Proof. Put $\mathcal{G}_0 := A_0$ and $\mathcal{H}_0 := B_0$, and for all $n \in \mathbb{N}$ define:

$$\mathcal{G}_n = \overline{\text{con}}(T(\mathcal{G}_{n-1})), \quad \mathcal{H}_n = \overline{\text{con}}(T(\mathcal{H}_{n-1})).$$

We now haveL

$$\mathcal{G}_1 = \overline{\text{con}}(T(\mathcal{G}_0)) = \overline{\text{con}}(T(A_0)) \subseteq B_0 = \mathcal{H}_0.$$

Thus, $T(\mathcal{G}_1) \subseteq T(\mathcal{H}_0)$ and so $\mathcal{G}_2 = \overline{\text{con}}(T(\mathcal{G}_1)) \subseteq \overline{\text{con}}(T(\mathcal{H}_0)) = \mathcal{H}_1$. Continuing this process, and by induction, we conclude that $\mathcal{G}_{n+1} \subseteq \mathcal{H}_n$. Similarly, we can see that $\mathcal{H}_n \subseteq \mathcal{G}_{n-1}$ for all $n \in \mathbb{N}$. This implies that:

$$\mathcal{G}_{n+2} \subseteq \mathcal{H}_{n+1} \subseteq \mathcal{G}_n \subseteq \mathcal{H}_{n-1}, \quad \forall n \in \mathbb{N}.$$

Hence, $\{(\mathcal{G}_{2n}, \mathcal{H}_{2n})\}_{n \geq 0}$ is a decreasing sequence of nonempty, closed, and convex pairs in $A_0 \times B_0$. Moreover,

$$T(\mathcal{H}_{2n}) \subseteq T(\mathcal{G}_{2n-1}) \subseteq \overline{\text{con}}(T(\mathcal{G}_{2n-1})) = \mathcal{G}_{2n},$$

$$T(\mathcal{G}_{2n}) \subseteq T(\mathcal{H}_{2n-1}) \subseteq \overline{\text{con}}(T(\mathcal{H}_{2n-1})) = \mathcal{H}_{2n}.$$

Thereby, for all $n \in \mathbb{N}$ the pair $(\mathcal{G}_{2n}, \mathcal{H}_{2n})$ is T-invariant. On the other hand, if $(p, q) \in A_0 \times B_0$ is a proximal pair, then:

$$\text{dist}(\mathcal{G}_{2n}, \mathcal{H}_{2n}) \leq \|T^{2n}p - T^{2n}q\| \leq \|p - q\| = \text{dist}(A, B), \quad \forall n \in \mathbb{N}.$$

We shall show by induction that the pair $(\mathcal{G}_n, \mathcal{H}_n)$ is proximinal for all $n \in \mathbb{N} \cup \{0\}$. It is obvious if $n = 0$. Suppose that $(\mathcal{G}_n, \mathcal{H}_n)$ is proximinal. Let $x \in \mathcal{G}_{n+1} = \overline{\text{con}}(T(\mathcal{G}_n))$ be an arbitrary element. Then $x = \sum_{j=1}^{k} \lambda_j T(x_j)$ with $x_j \in \mathcal{G}_n$ where $k \in \mathbb{N}$, $\lambda_j \geq 0$ and $\sum_{j=1}^{k} \lambda_j = 1$. The proximinality of the pair $(\mathcal{G}_n, \mathcal{H}_n)$ implies that for all $1 \leq j \leq k$ there exists $y_j \in \mathcal{H}_n$ such that $\|x_j - y_j\| = \text{dist}(\mathcal{G}_n, \mathcal{H}_n)\,(= \text{dist}(A, B))$. Put $y = \sum_{j=1}^{k} \lambda_j T(y_j)$. Then $y \in \overline{\text{con}}(T(\mathcal{H}_n)) = \mathcal{H}_{n+1}$ and:

$$\|x - y\| = \|\sum_{j=1}^{k} \lambda_j T(x_j) - \sum_{j=1}^{k} \lambda_j T(y_j)\| \leq \sum_{j=1}^{k} \lambda_j \|x_j - y_j\| = \text{dist}(A, B),$$

and so the pair $(\mathcal{G}_{n+1}, \mathcal{H}_{n+1})$ is proximinal. We now consider the following possible cases.

Case 1. *If* $\max\{\mu(\mathcal{G}_{2k}), \mu(\mathcal{H}_{2k})\} = 0$ *for some* $k \in \mathbb{N}$*, then:*

$$T : \mathcal{G}_{2k} \cup \mathcal{H}_{2k} \to \mathcal{G}_{2k} \cup \mathcal{H}_{2k}$$

is a compact and cyclic relatively nonexpansive map. Now, from Theorem 42, the result follows.

Case 2. *Assume that* $\max\{\mu(\mathcal{G}_{2n}), \mu(\mathcal{H}_{2n})\} > 0$ *for all* $n \in \mathbb{N}$*. Put* $\varepsilon_n := \mu(\mathcal{G}_{2n} \cup \mathcal{H}_{2n})$*. Since* T *is a cyclic Meir–Keeler condensing operator, there exists* $\delta_n := \delta(\varepsilon_n)$ *such that:*

$$\mu(T(\mathcal{G}_{2n}) \cup T(\mathcal{H}_{2n})) < \varepsilon_n, \quad \forall n \in \mathbb{N}.$$

Further, for all $n \in \mathbb{N}$ we have:

$$\varepsilon_{n+1} = \mu(\mathcal{G}_{2n+2} \cup \mathcal{H}_{2n+2}) = \max\{\mu(\mathcal{G}_{2n+2}), \mu(\mathcal{H}_{2n+2})\} \le \max\{\mu(\mathcal{G}_{2n}), \mu(\mathcal{H}_{2n})\} = \varepsilon_n.$$

Thus, $\{\varepsilon_n\}$ is a decreasing sequence of positive real numbers. Assume that $\lim_{n\to\infty} \varepsilon_n = r$. We claim that $r = 0$. Suppose the contrary. Then there exists $l \in \mathbb{N}$ such that $r \le \varepsilon_l < r + \delta(r)$. Again, using the fact that T is a cyclic Meir–Keeler condensing operator, we conclude that:

$$\begin{aligned}
\varepsilon_{l+1} &= \mu(\mathcal{G}_{2l+2} \cup \mathcal{H}_{2l+2}) = \max\{\mu(\mathcal{G}_{2l+2}), \mu(\mathcal{H}_{2l+2})\} \\
&\le \max\{\mu(\mathcal{H}_{2l+1}), \mu(\mathcal{G}_{2l+1})\} = \max\{\mu(\overline{\text{con}}(T(\mathcal{H}_{2l}))), \mu(\overline{\text{con}}(T(\mathcal{G}_{2l})))\} \\
&= \max\{\mu(T(\mathcal{H}_{2l})), \mu(T(\mathcal{G}_{2l}))\} = \mu(T(\mathcal{G}_{2l}) \cup T(\mathcal{H}_{2l})) < r,
\end{aligned}$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} \mu(\mathcal{G}_{2n} \cup \mathcal{H}_{2n}) = \max\{\lim_{n\to\infty} \mu(\mathcal{G}_{2n}), \lim_{n\to\infty} \mu(\mathcal{H}_{2n})\} = 0.$$

Set:

$$\mathcal{G}_\infty = \bigcap_{n=0}^{\infty} \mathcal{G}_{2n} \quad \text{and} \quad \mathcal{H}_\infty = \bigcap_{n=0}^{\infty} \mathcal{H}_{2n}.$$

Then the pair $(\mathcal{C}_\infty, \mathcal{D}_\infty) \neq \emptyset$ is compact. It is also convex and T–invariant with $\text{dist}(A, B) = \text{dist}(\mathcal{G}_\infty, \mathcal{H}_\infty)$. This ensures that T has a best proximity point. □

Theorem 47 ([86])**.** *Let* $(A, B) \neq \emptyset$ *be a weakly compact and convex pair in a Banach space* X *and* μ *be an* MNC *on* X*. Let* $T : A \cup B \to A \cup B$ *be a cyclic relatively nonexpansive map which is a generalized condensing operator in the sense of Definition 20. Then* T *has a best proximity point.*

Proof. Note that $(A_0, B_0) \in \mathcal{M}_T(A, B) \neq \emptyset$. Put:

$$A_n := \overline{\text{con}}(T(A_{n-1})), \quad B_n := \overline{\text{con}}(T(B_{n-1})).$$

By induction, we show that T is cyclic on $A_n \cup B_n$ for all $n \in \mathbb{N}$. Since $A_1 = \overline{\text{con}}(T(A_0)) \subseteq B_0$,

$$T(A_1) \subseteq T(B_0) \subseteq \overline{\text{con}}(T(B_0)) = B_1.$$

Similarly, we can see that $T(B_1) \subseteq A_1$, that is, T is cyclic on $A_1 \cup B_1$. Now, suppose T is cyclic on $A_k \cup B_k$ for some $k \in \mathbb{N}$. Then $T(A_k) \subseteq B_k$ and so $A_{k+1} = \overline{\text{con}}(T(A_k)) \subseteq B_k$ which implies that:

$$T(A_{k+1}) \subseteq T(B_k) \subseteq \overline{\text{con}}(T(B_k)) = B_{k+1}.$$

Equivalently, we can see that $T(B_{k+1}) \subseteq A_{k+1}$, which ensures that T is cyclic on $A_{k+1} \cup B_{k+1}$. Besides,

$$A_{n+1} = \overline{\text{con}}(T(A_n)) \subseteq B_n = \overline{\text{con}}(T(B_{n-1})) \subseteq A_{n-1}, \quad \forall n \in \mathbb{N},$$

which concludes that the sequence $\{A_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ is decreasing and, similarly, we can see that the sequence $\{B_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ is also decreasing. Now, let $(x_0, y_0) \in A_0 \times B_0$ be such that $\|x_0 - y_0\| = \text{dist}(A, B)$. Since T is cyclic relatively nonexpansive, $(T^n x_0, T^n y_0) \in A_n \times B_n$ and:

$$\text{dist}(A_n, B_n) \leq \|T^n x_0 - T^n y_0\| \leq \|x_0 - y_0\| = \text{dist}(A, B), \quad \forall n \in \mathbb{N}.$$

Thus, $\text{dist}(A_n, B_n) = \text{dist}(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$. Again, by mathematical induction, we assert that any pair (A_n, B_n) is proximinal. We note that the pair (A_0, B_0) is proximinal. Let (A_k, B_k) be a proximinal pair. We consider the following observations:

- *The pair* $(\text{con}(T(A_k)), \text{con}(T(B_k)))$ *is proximinal.*

Proof. Let $x \in \text{con}(T(A_k))$ be an arbitrary element. Then, $x = \sum_{j=1}^{m} \lambda_j T(a_j)$ for some $m \in \mathbb{N}$, where $a_j \in A_k$ for all $1 \leq j \leq m$. Since (A_k, B_k) is proximinal, for all $1 \leq j \leq m$ there exists an element $b_j \in B_k$ for which $\|a_j - b_j\| = \text{dist}(A_k, B_k)(= \text{dist}(A, B))$. Put $y := \sum_{j=1}^{m} \lambda_j T(b_j)$. Clearly, $y \in \text{con}(T(B_k))$ and we have:

$$\begin{aligned}\|x - y\| &= \|\sum_{j=1}^{m} \lambda_j T(a_j) - \sum_{j=1}^{m} \lambda_j T(b_j)\| \leq \sum_{j=1}^{m} \lambda_j \|T(a_j) - T(b_j)\| \\ &\leq \sum_{j=1}^{m} \lambda_j \|a_j - b_j\| = \text{dist}(A, B),\end{aligned}$$

and the result follows. □

- *The pair* $(\overline{\text{con}}(T(A_k)), \overline{\text{con}}(T(B_k)))$ *is proximinal.*

Proof. Let $u \in \overline{\text{con}}(T(A_k))$. Then there is a sequence $\{w_n\}$ in $\text{con}(T(A_k))$ such that $w_n \to u$. Since $(\text{con}(T(A_k)), \text{con}(T(B_k)))$ is proximinal, for any $n \in \mathbb{N}$ there exists a point $z_n \in \text{con}(T(B_k))$ such that:

$$\|w_n - z_n\| = \text{dist}(\text{con}(T(A_k)), \text{con}(T(B_k))) = \text{dist}(A, B).$$

By the fact that $\overline{\text{con}}(T(B_k))$ is weakly compact, there exists a subsequence $\{z_{n_j}\}$ of the sequence $\{z_n\}$, which converges weakly to a point $v \in \overline{\text{con}}(T(B_k))$. It now follows from the weakly lower semi-continuity of the norm that:

$$\|u - v\| \leq \liminf_{j\to\infty} \|w_{n_j} - z_{n_j}\| = \text{dist}(A, B).$$

So, the pair (A_{k+1}, B_{k+1}) is proximinal. □

Therefore, $\{(A_{2n}, B_{2n})\}_{n\in\mathbb{N}\cup\{0\}}$ is a descending sequence in $\mathcal{M}_T(A, B)$. Set:

$$r := \lim_{n\to\infty} \mu(A_{2n} \cup B_{2n}).$$

Since T is a cyclic generalized condensing operator, there exist $\psi \in \Psi$ and $l_1 \in \mathbb{N}$ such that $\mu(A_{2l_1} \cup B_{2l_1}) \leq \psi(\mu(A \cup B))$. Note that $(A_{2l_1}, B_{2l_1}) \neq \emptyset$ is a weakly compact, convex, and proximinal pair and that $T : A_{2l_1} \cup B_{2l_1} \to A_{2l_1} \cup B_{2l_1}$ is cyclic. From the above arguments, we can find a positive integer l_2 such that:

$$\mu(A_{2(l_1+l_2)} \cup B_{2(l_1+l_2)}) \leq \psi(\mu(A_{2l_1} \cup B_{2l_1})) \leq \psi^2(\mu(A \cup B)).$$

Continuing this process, there exists $l_i \in \mathbb{N}$ such that:

$$\mu(A_{2(\sum_{j=1}^{i} l_j)} \cup B_{2(\sum_{j=1}^{i} l_j)}) \leq \psi^i(\mu(A \cup B)), \quad \forall i \in \mathbb{N}.$$

In view of the fact that $\psi^i(\mu(A\cup B))\to 0$, we must have $r=0$. Now, if we set:

$$(A_\infty, B_\infty) = (\bigcap_{j=1}^{\infty} A_{2j}, \bigcap_{j=1}^{\infty} B_{2j}),$$

then (A_∞, B_∞) is a nonempty, convex, compact, and T-invariant pair with $\text{dist}(A,B) = \text{dist}(A_\infty, B_\infty)$. Hence, from Theorem 42, we obtain the existence of a best proximity point for the map T, and this completes the proof. □

The noncyclic version of Theorem 47 can be reformulated as below.

Theorem 48. *Let $(A,B)\neq\emptyset$ be a weakly compact and convex pair in a strictly convex Banach space X and μ be an* MNC *on X. Let $T: A\cup B\to A\cup B$ be a noncyclic relatively nonexpansive map, which is a generalized condensing operator in the sense of Definition 21. Then, T has a best proximity pair.*

Proof. As in the proof of Theorem 47, let $A_n=\overline{\text{con}}(T(A_{n-1}))$ and $B_n=\overline{\text{con}}(T(B_{n-1}))$ for all $n\in\mathbb{N}$. Since T is noncyclic, $A_1=\overline{\text{con}}(T(A_0))\subseteq A_0$, and so:

$$T(A_1)\subseteq T(A_0)\subseteq \overline{\text{con}}(T(A_0))=A_1.$$

Similarly, $T(B_1)\subseteq B_1$, that is, T is noncyclic on $A_1\cup B_1$. Continuing this process, and by induction, we can see that T is noncyclic on $A_n\cup B_n$ for all $n\in\mathbb{N}$. For all $n\in\mathbb{N}$ we have:

$$A_{n+1}=\overline{\text{con}}(T(A_n))\subseteq A_n, \quad B_{n+1}=\overline{\text{con}}(T(B_n))\subseteq B_n.$$

Moreover, by an equivalent discussion of Theorem 42, we conclude that (A_n,B_n) is a proximinal pair with $\text{dist}(A_n,B_n)=\text{dist}(A,B)$ for all $n\in\mathbb{N}$. Hence, $\{(A_n,B_n)\}$ is a descending sequence of nonempty, weakly compact, convex, T-invariant, and proximinal pairs and so its even subsequence, that is, $\{(A_{2n},B_{2n})\}$ is a member of $\mathcal{M}_T(A,B)$. By a similar manner of Theorem 42 if we define:

$$(A_\infty, B_\infty) = (\bigcap_{j=1}^{\infty} A_{2j}, \bigcap_{j=1}^{\infty} B_{2j}),$$

then (A_∞, B_∞) is a nonempty, compact, convex, and T-invariant pair in a strictly convex Banach space X and so Theorem 43 guarantees the existence of a best proximity pair for the map T. □

Theorem 49 ([87])**.** *Let $(A,B)\neq\emptyset$ be a bounded, closed, and convex pair in a Banach space X such that $A_0\neq\emptyset$ is nonempty and μ is an* MNC *on X. Let $T: A\cup B\to A\cup B$ be a cyclic relatively nonexpansive map, which is φ–condensing in the sense of Definition 22 for some $\varphi\in\Phi$. Then, T has a best proximity point.*

Proof. For all $n\in\mathbb{N}$ define:

$$\mathcal{C}^n=\overline{\text{con}}(T(\mathcal{C}^{n-1})), \quad \mathcal{D}^n=\overline{\text{con}}(T(\mathcal{D}^{n-1})),$$

where, $\mathcal{C}^0:=A_0$ and $\mathcal{D}^0:=B_0$. Then we have:

$$\mathcal{C}^1=\overline{\text{con}}(T(\mathcal{C}^0))=\overline{\text{con}}(T(A_0))\subseteq B_0=\mathcal{D}^0,$$

and so, $T(\mathcal{C}^1)\subseteq T(\mathcal{D}^0)$ which implies that:

$$\mathcal{C}^2=\overline{\text{con}}(T(\mathcal{C}^1))\subseteq\overline{\text{con}}(T(\mathcal{D}^0))=\mathcal{D}^1.$$

Continuing this process, we obtain: $\mathcal{C}^{n+1} \subseteq \mathcal{D}^n$. We also have:

$$\mathcal{D}^1 = \overline{\text{con}}(T(\mathcal{D}^0)) = \overline{\text{con}}(T(B_0)) \subseteq A_0 = \mathcal{C}^0,$$

and hence, $T(\mathcal{D}^1) \subseteq T(\mathcal{C}^0)$. Thus,

$$\mathcal{D}^2 = \overline{\text{con}}(T(\mathcal{D}^1)) \subseteq \overline{\text{con}}(T(\mathcal{C}^0)) = \mathcal{C}^1.$$

Then by induction we conclude that: $\mathcal{D}^n \subseteq \mathcal{C}^{n-1}$ for all $n \in \mathbb{N}$. Therefore,

$$\mathcal{C}^{n+2} \subseteq \mathcal{D}^{n+1} \subseteq \mathcal{C}^n \subseteq \mathcal{D}^{n-1}, \quad \text{for all } n \in \mathbb{N}.$$

Thereby, $\{(\mathcal{C}^{2n}, \mathcal{D}^{2n})\}_{n \geq 0}$ is a decreasing sequence consisting of closed and convex pairs in $A_0 \times B_0$. Furthermore, for all $n \in \mathbb{N} \cup \{0\}$ we have:

$$T(\mathcal{D}^{2n}) \subseteq T(\mathcal{C}^{2n-1}) \subseteq \overline{\text{con}}(T(\mathcal{C}^{2n-1})) = \mathcal{C}^{2n},$$

$$T(\mathcal{C}^{2n}) \subseteq T(\mathcal{D}^{2n-1}) \subseteq \overline{\text{con}}(T(\mathcal{D}^{2n-1})) = \mathcal{D}^{2n}.$$

So, we deduce that $(\mathcal{C}^{2n}, \mathcal{D}^{2n})$ is T-invariant. Let $(x, y) \in \mathcal{C}^0 \times \mathcal{D}^0$ be such that $\|x - y\| = \text{dist}(A, B)$. Then, $(T^{2n}x, T^{2n}y) \in \mathcal{C}^{2n} \times \mathcal{D}^{2n}$ and by the fact that T is relatively nonexpansive, we have:

$$\text{dist}(\mathcal{C}^{2n}, \mathcal{D}^{2n}) \leq \|T^{2n}x - T^{2n}y\| \leq \|x - y\| = \text{dist}(A, B).$$

We can see that $(\mathcal{C}^{2n}, \mathcal{D}^{2n})$ is also proximinal for all $n \in \mathbb{N}$. Notice that if:

$$\max\{\{\mu(\mathcal{C}^{2k}), \mu(\mathcal{D}^{2k})\} = 0 \text{ for some } k \in \mathbb{N},$$

then the result follows from Theorem 42. So, we assume that $\max\{\{\mu(\mathcal{C}^{2n}), \mu(\mathcal{D}^{2n})\} > 0$ for all $n \in \mathbb{N}$. Then, we obtain $\min\{\{\mu(\mathcal{C}^{2n}), \mu(\mathcal{D}^{2n})\} > 0$ for all $n \in \mathbb{N}$. Since T is cyclic φ-condensing, for all $n \in \mathbb{N} \cup \{0\}$ we have:

$$\begin{aligned}
\mu(\mathcal{C}^{2n+2} \cup \mathcal{D}^{2n+2}) &= \max\{\mu(\mathcal{C}^{2n+2}), \mu(\mathcal{D}^{2n+2})\} \\
&\leq \max\{\mu(\mathcal{D}^{2n+1}), \mu(\mathcal{C}^{2n+1})\} \\
&= \max\{\mu(\overline{\text{con}}(T(\mathcal{D}^{2n}))), \mu(\overline{\text{con}}(T(\mathcal{C}^{2n})))\} \\
&= \max\{\mu((T(\mathcal{C}^{2n}))), \mu((T(\mathcal{D}^{2n})))\} \\
&= \mu(T(\mathcal{C}^{2n}) \cup T(\mathcal{D}^{2n})) \leq \varphi(\mu(\mathcal{C}^{2n} \cup \mathcal{D}^{2n}))\mu(\mathcal{C}^{2n} \cup \mathcal{D}^{2n}) \\
&\leq \mu(\mathcal{C}^{2n} \cup \mathcal{D}^{2n}).
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \mu(\mathcal{C}^{2n} \cup \mathcal{D}^{2n}) = \max\{\lim_{n\to\infty} \mu(\mathcal{C}^{2n}), \lim_{n\to\infty} \mu(\mathcal{D}^{2n})\} = 0.$$

If we set $\mathcal{C}_\infty = \bigcap_{n=0}^{\infty} \mathcal{C}^{2n}$ and $\mathcal{D}_\infty = \bigcap_{n=0}^{\infty} \mathcal{D}^{2n}$ then $(\mathcal{C}_\infty, \mathcal{D}_\infty)$ is nonempty, closed, convex, and T–invariant with $\text{dist}(A, B) = \text{dist}(\mathcal{C}_\infty, \mathcal{D}_\infty)$ for which we have $\max\{\mu(\mathcal{C}_\infty), \mu((\mathcal{D}_\infty))\}$ $= 0$. Hence, T has a best proximity point. □

Theorem 50 ([87]). *Let $(A, B) \neq \emptyset$ be a bounded, closed, and convex pair in a strictly convex Banach space X such that A_0 is nonempty and μ is an* MNC *on X. Let $T : A \cup B \to A \cup B$ be a noncyclic, relatively nonexpansive map, which is φ-condensing in the sense of Definition 22. Then, T has a best proximity pair.*

Proof. Note that $(A_0, B_0) \neq \emptyset$ is closed, convex, and proximinal. Let $x \in A_0$. Then, there exists $y \in B_0$ such that $\|x - y\| = \text{dist}(A, B)$. Since T is relatively nonexpansive,

$\|Tx - Ty\| = \text{dist}(A, B)$, and so $Tx \in A_0$. Thus, $T(A_0) \subseteq A_0$. Similarly, $T(B_0) \subseteq B_0$, which implies that (A_0, B_0) is T-invariant. Set $\mathcal{C}^0 = A_0$ and $\mathcal{D}^0 = B_0$ and for all $n \in \mathbb{N}$ define:

$$\mathcal{C}^n = \overline{\text{con}}(T(\mathcal{C}^{n-1})), \quad \mathcal{D}^n = \overline{\text{con}}(T(\mathcal{D}^{n-1})).$$

Then, we have:

$$\mathcal{C}^1 = \overline{\text{con}}(T(\mathcal{C}^0)) = \overline{\text{con}}(T(A_0)) \subseteq A_0 = \mathcal{C}^0.$$

Continuing this process and by induction we obtain $\mathcal{C}^{n-1} \supseteq \mathcal{C}^n$ for all $n \in \mathbb{N}$. Equivalently, $\mathcal{D}^{n-1} \supseteq \mathcal{D}^n$ for all $n \in \mathbb{N}$. Suppose that there exists $k \in \mathbb{N}$ for which $\max\{\mu(\mathcal{C}^k), \mu(\mathcal{D}^k)\} = 0$. Then, $(\mathcal{C}^k, \mathcal{D}^k)$ is a compact pair. Moreover, we have:

$$T(\mathcal{C}^k) \subseteq \overline{\text{con}}(T(\mathcal{C}^k)) = \mathcal{C}^{k+1} \subseteq \mathcal{C}^k.$$

A similar argument implies that $T(\mathcal{D}^k) \subseteq \mathcal{D}^k$ and so, T is noncyclic relatively nonexpansive on $\mathcal{C}^k \cup \mathcal{D}^k$, where $(\mathcal{C}^k, \mathcal{D}^k)$ is a compact and convex pair in a strictly convex Banach space X. Thus, from Theorem 43, T has a best proximity pair and we are finished. So, we assume that: $\max\{\mu(\mathcal{C}^n), \mu(\mathcal{D}^n)\} > 0$ for any $n \in \mathbb{N}$. If there exist $l_1, l_2 \in \mathbb{N}$ with $l_1 < l_2$ such that $\mu(\mathcal{C}^{l_1}) = \mu(\mathcal{D}^{l_2}) = 0$ then, by the fact that the sequence $\{\mathcal{C}^n\}_{n \in \mathbb{N} \cup \{0\}}$ is a decreasing sequence, we have $\mathcal{C}^{l_2} \subseteq \mathcal{C}^{l_1}$ and so, $\mu(\mathcal{C}^{l_2}) \leq \mu(\mathcal{C}^{l_1})$ which leads to $\mu(\mathcal{C}^{l_2}) = 0$. Hence $\max\{\mu(\mathcal{C}^{l_2}), \mu(\mathcal{D}^{l_2})\} = 0$ which is a contradiction, and so

$$\min\{\mu(\mathcal{C}^n), \mu(\mathcal{D}^n)\} > 0, \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Also, for the pair $(x, y) \in \mathcal{C}^0 \times \mathcal{D}^0$ with $\|x - y\| = \text{dist}(A, B)$ we have $\|T^n x - T^n y\| = \text{dist}(A, B)$ for all $n \in \mathbb{N}$, because of the fact that T is noncyclic relatively nonexpansive. >From the definition of the pair $(\mathcal{C}^n, \mathcal{D}^n)$ we obtain $(T^n x, T^n y) \in \mathcal{C}^n \times \mathcal{D}^n$ which implies that

$$\text{dist}(\mathcal{C}^n, \mathcal{D}^n) = \text{dist}(A, B), \quad \text{for all } n \in \mathbb{N}.$$

Now suppose that $u \in \mathcal{C}^1 = \overline{\text{con}}(T(\mathcal{C}^0))$. Then $u = \sum_{j=1}^m c_j T(u_j)$ where $u_j \in \mathcal{C}^0$ for all $1 \leq j \leq m$ such that $c_j \geq 0$ and $\sum_{j=1}^m c_j = 1$. Since $(\mathcal{C}^0, \mathcal{D}^0)$ is proximinal, for all $1 \leq j \leq m$ there exists $v_j \in \mathcal{D}^0$ such that $\|u_j - v_j\| = \text{dist}(\mathcal{C}^0, \mathcal{D}^0)$ $(= \text{dist}(A, B))$ and so $\|Tu_j - Tv_j\| = \text{dist}(A, B)$. Put $v := \sum_{j=1}^m c_j T(v_j)$. Then $v \in \mathcal{D}^1$ and:

$$\|u - v\| = \|\sum_{j=1}^m c_j T(u_j) - \sum_{j=1}^m c_j T(v_j)\| \leq \sum_{j=1}^m \|T(u_j) - T(v_j)\| = \text{dist}(A, B).$$

Therefore, the pair $(\mathcal{C}^1, \mathcal{D}^1)$ is proximinal. Using a similar discussion, we can see that the pair $(\mathcal{C}^n, \mathcal{D}^n)$ is proximinal for all $n \in \mathbb{N} \cup \{0\}$. Thus, $(\mathcal{C}^n, \mathcal{D}^n)$ is a nonempty, bounded, closed, convex, and proximinal pair, which is T-invariant. Since T is noncyclic φ-condensing, for all $n \in \mathbb{N} \cup \{0\}$ we have:

$$\begin{aligned}
\mu(\mathcal{C}^{n+1} \cup \mathcal{D}^{n+1}) &= \max\{\mu(\mathcal{C}^{n+1}), \mu(\mathcal{D}^{n+1})\} \\
&= \max\{\mu(\overline{\text{con}}(T(\mathcal{C}^n))), \mu(\overline{\text{con}}(T(\mathcal{D}^n)))\} \\
&= \max\{\mu((T(\mathcal{C}^n))), \mu((T(\mathcal{D}^n)))\} \\
&= \mu(T(\mathcal{C}^n) \cup T(\mathcal{D}^n)) \\
&\leq \varphi(\mu(\mathcal{C}^n \cup \mathcal{D}^n))\mu(\mathcal{C}^n \cup \mathcal{D}^n) \\
&\leq \mu(\mathcal{C}^n \cup \mathcal{D}^n).
\end{aligned}$$

Then, $\{\mu(\mathcal{C}^n \cup \mathcal{D}^n)\}$ is a decreasing sequence and bounded below, so there exists a real number $r \geq 0$ such that $\lim_{n\to\infty} \mu(\mathcal{C}^n \cup \mathcal{D}^n) = r$. We claim that $r = 0$. Suppose the contrary. Thus for all $n \in \mathbb{N}$ we have:

$$\frac{\mu(\mathcal{C}^{n+1} \cup \mathcal{D}^{n+1})}{\mu(\mathcal{C}^n \cup \mathcal{D}^n)} \leq \varphi(\mu(\mathcal{C}^n \cup \mathcal{D}^n)).$$

The above inequality yields $\lim_{n\to\infty} \varphi(\mu(\mathcal{C}^n \cup \mathcal{D}^n)) = 1$. In view of the fact that $\varphi \in \Phi$, we conclude that $r = 0$ which is impossible. Hence,

$$\lim_{n\to\infty} \mu(\mathcal{C}^n \cup \mathcal{D}^n) = \max\{\lim_{n\to\infty} \mu(\mathcal{C}^n), \lim_{n\to\infty} \mu(\mathcal{D}^n)\} = 0.$$

So the pair $(\mathcal{C}_\infty, \mathcal{D}_\infty)$ is nonempty, closed, and convex, which is T-invariant, where $\mathcal{C}_\infty = \bigcap_{n=0}^{\infty} \mathcal{C}^n$ and $\mathcal{D}_\infty = \bigcap_{n=0}^{\infty} \mathcal{D}^n$. Furthermore, $\operatorname{dist}(\mathcal{C}_\infty, \mathcal{D}_\infty) = \operatorname{dist}(A, B)$ and it is easy to check that $(\mathcal{C}_\infty, \mathcal{D}_\infty)$ is proximinal. On the other hand, $\max\{\mu(\mathcal{C}_\infty), \mu((\mathcal{D}_\infty))\} = 0$, which ensures that the pair $(\mathcal{C}_\infty, \mathcal{D}_\infty)$ is compact. Finally, the result follows from Theorem 43. □

At the end of this section, we give the following existence theorems which were recently presented in [89] as generalizations of Sadovskii's fixed point problem.

Theorem 51 ([89])**.** *Let $(A, B) \neq \emptyset$ be a bounded, closed, and convex pair in a Banach space X such that $A_0 \neq \emptyset$ and μ be an MNC on X. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive map such that for any $(H_1, H_2) \in \mathcal{M}_T(A, B)$ we have:*

$$\mu(T(H_1) \cup T(H_2)) \neq \mu(H_1 \cup H_2).$$

Then, T has a best proximity point.

Proof. Let $\mathcal{F}$ denote a family of all nonempty, closed, convex proximinal and T-invariant pairs $(C, D) \subseteq (A, B)$. Then $(A_0, B_0) \in \mathcal{F}$. Set:

$$\delta := \inf\{\mu(C \cup D) : (C, D) \in \mathcal{F}\},$$

and assume that $(K_1, K_2) := \bigcap_{(C,D)\in\mathcal{F}} (C, D)$. Then, clearly, $(K_1, K_2) \in \mathcal{F}$ is a nonempty pair for which $\mu(K_1 \cup K_2) = \delta$.

Note that if $\delta = 0$, then $\mu(K_1 \cup K_2) = 0$ and so by Theorem 42, T has a best proximity point in $K_1 \cup K_2$. Suppose that $\mu(K_1 \cup K_2) = \delta > 0$. This follows that $\mu(T(K_1) \cup T(K_2)) \neq \mu(K_1 \cup K_2)$. Since $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$, we have:

$$\mu(T(K_1) \cup T(K_2)) < \mu(K_1 \cup K_2).$$

Let us now define the sets $N := \overline{\text{con}}(T(K_1) \cup \{y_0\})$ and $M := \overline{\text{con}}(T(K_2) \cup \{x_0\})$. Thus, $(x_0, y_0) \in M \times N$ and $(M, N) \subseteq (K_1, K_2)$. Moreover, $T(M) \subseteq T(K_1) \subseteq N$ and $T(N) \subseteq T(K_2) \subseteq M$, that means T is cyclic on $M \cup N$. Furthermore, if $x \in M$, then, $x = \sum_{j=1}^{n-1} c_j T(y_j) + c_n x_0$, where $y_j \in K_2$ for all $j \in \{1, 2, \ldots n-1\}$ for which $c_i \geq 0$, $\sum_{j=1}^{n} c_j = 1$. Since (K_1, K_2) is proximinal, there exists $x_j \in K_1$ so that $\|x_j - y_j\| = \operatorname{dist}(A, B)$ for all $j \in \{1, 2, \ldots, n-1\}$. Now, if $y = \sum_{j=1}^{n-1} c_j T(x_j) + c_n y_0$, then, $y \in N$ and we have $\|x - y\| = \operatorname{dist}(A, B)$. Therefore, $M_0 = M$. Similarly, $N_0 = N$ and so, (M, N) is a proximinal pair. Hence, $(M, N) \in \mathcal{F}$. Considering the definition of (K_1, K_2), it follows that $M = K_1$ and $N = K_2$. Therefore,

$$\begin{aligned}\mu(M \cup N) &= \max\{\mu(M), \mu(N)\} = \max\{\mu(\overline{\text{con}}(T(K_2) \cup \{x_0\})), \mu(\overline{\text{con}}(T(K_1) \cup \{y_0\}))\} \\ &= \max\{\mu(T(K_1)), \mu(T(K_2))\} = \mu(T(K_1) \cup T(K_2)) < \mu(K_1 \cup K_2) = \delta,\end{aligned}$$

which is a contradiction. □

Theorem 52 ([89]). *Let $(A, B) \neq \emptyset$ be a bounded, closed, and convex pair in a strictly convex Banach space X, such that A_0 is nonempty and μ be an* MNC *on X. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive map such that for any $(H_1, H_2) \in \mathcal{M}_T(A, B)$ we have:*

$$\mu(T(H_1) \cup T(H_2)) \neq \mu(H_1 \cup H_2).$$

Then, T has a best proximity pair.

Proof. Let $(u, v) \in A_0 \times B_0$ such that $\|u - v\| = dist(A, B)$ and $\mathcal{G}$ denote the family of all nonempty, closed, convex, proximinal and T–invariant pairs $(E, F) \subseteq (A, B)$ such that $(u, v) \in E \times F$ and $T(E) \subset E$ and $T(F) \subset F$. Then $(A_0, B_0) \in \mathcal{G}$. Let:

$$\delta := \inf\{\mu(E \cup F) : (E, F) \in \mathcal{G}\},$$

and define $(K_1, K_2) = \bigcap_{(E,F)\in\mathcal{G}} (E, F)$. Then, clearly, $(K_1, K_2) \in \mathcal{G}$ is a nonempty pair such that $\mu(K_1 \cup K_2) = \delta$. If $\delta = 0$ then $\mu(K_1 \cup K_2) = 0$ and the result follows from Theorem 43.

Suppose that $\mu(K_1 \cup K_2) = \delta > 0$. It follows that $\mu(T(K_1) \cup T(K_2)) \neq \mu(K_1 \cup K_2)$. Since $T(K_1) \subset K_1$ and $T(K_2) \subset K_2$, we have:

$$\mu(T(K_1) \cup T(K_2)) < \mu(K_1 \cup K_2).$$

Set $H_1 := \overline{\text{con}}(T(K_1) \cup \{u\})$ and $H_2 := \overline{\text{con}}(T(K_2) \cup \{v\})$. Thus, $(u, v) \in H_1 \times H_2$ and $(H_1, H_2) \subseteq (K_1, K_2)$. Moreover, $T(H_1) \subseteq T(K_1) \subseteq H_1$, $T(H_2) \subseteq T(K_2) \subseteq H_2$. Therefore, T is noncyclic on $H_1 \cup H_2$. Thus, if $x \in H_1$, then $x = \sum_{j=1}^{n-1} c_j T(u_j) + c_n u$, where $u_j \in K_1$ for all $j \in \{1, 2, \dots, n-1\}$ for which $c_j \geq 0$ and $\sum_{j=1}^{n} c_j = 1$. From the fact that (K_1, K_2) is proximinal, there exists $v_j \in K_2$ such that $\|u_j - v_j\| = \text{dist}(A, B)$ for all $j \in \{1, 2, \cdots, n-1\}$. Now, if we define $y = \sum_{j=1}^{n-1} c_j T(v_j) + c_n v$, then $y \in H_2$ and $\|x - y\| = \text{dist}(A, B)$. Hence, $(H_1)_0 = H_1$. By similar argument, $(H_2)_0 = H_2$ and hence, (H_1, H_2) is a proximinal pair. Further, from the definition of (K_1, K_2), we have $H_1 = K_1$ and $H_2 = K_2$. Therefore, we have $(H_1, H_2) \in \mathcal{G}$. Thus:

$$\mu(H_1 \cup H_2) = \mu(T(K_1) \cup T(K_2)) < \mu(K_1 \cup K_2) = \delta,$$

That is, $\mu(H_1 \cup H_2) < \delta$ which is contradiction. □

10. Application to a System of Differential Equations

In this section, we present some applications of the existence results of best proximity points in order to establish the optimal solutions for various systems of differential equations.

Application A.

We begin with the following extension of the *Mean-Value Theorem*.

Theorem 53 ([8]). *Let J be a real interval, X be a Banach space, and $f : J \to X$ be a differentiable map. Let $a, b \in J$ with $a < b$. Then:*

$$f(b) - f(a) \in (b - a)\overline{\text{con}}(\{f'(t) : t \in [a, b]\}).$$

Now, we apply the existence theorems of best proximity points to solve the systems of initial-value problems in Banach spaces. To this end, we introduce the following notion.

Definition 23. *Let a and b be real positive numbers, I be the real interval $[t_0 - a, t_0 + a]$ and $V_1 = \mathcal{B}(x_0; b)$, $V_2 = \mathcal{B}(x_1; b)$ be closed balls in a Banach space X, where t_0 is a real number and*

$x_0, x_1 \in X$. Assume that $f : I \times V_1 \to X$ and $g : I \times V_2 \to X$ are continuous maps. Consider the following system of differential equations:

$$x'(t) = f(t, x(t)); \quad x(t_0) = x_0, \tag{62}$$

$$y'(t) = g(t, y(t)); \quad y(t_0) = x_1, \tag{63}$$

defined on a closed real interval $J = [t_0 - h, t_0 + h]$ for some real positive number h. Let us consider the Banach space $\mathcal{C}(J, X)$ of continuous maps from J into X with the supremum norm and define $\mathcal{C}(J, V_1) = \{x \in \mathcal{C}(J, X) : x(t_0) = x_0\}$ and $\mathcal{C}(J, V_2) = \{y \in \mathcal{C}(J, X) : y(t_0) = x_1\}$. In this case, for any $(x, y) \in \mathcal{C}(J, V_1) \times \mathcal{C}(J, V_2)$ we have:

$$\|x - y\|_\infty = \sup_{t \in J} \|x(t) - y(t)\| \geq \|x_0 - x_1\|,$$

and so, $\operatorname{dist}(\mathcal{C}(J, V_1), \mathcal{C}(J, V_2)) = \|x_0 - x_1\|$. *Let:*

$$T : \mathcal{C}(J, V_1) \cup \mathcal{C}(J, V_1) \to \mathcal{C}(J, X),$$

be an operator defined as:

$$Tx(t) = x_1 + \int_{t_0}^{t} g(s, x(s))ds\,; \quad x \in \mathcal{C}(J, V_1),$$

$$Ty(t) = x_0 + \int_{t_0}^{t} f(s, y(s))ds\,; \quad y \in \mathcal{C}(J, V_2).$$

We say that $z \in \mathcal{C}(J, V_1) \cup \mathcal{C}(J, V_2)$ is an optimal solution for the system of differential equations given in (62) and (63) provided that:

$$\|z - Tz\|_\infty = \operatorname{dist}(A, B).$$

Here, we state the following existence theorem.

Theorem 54 ([83]). *Under the assumptions of Definition 23 if,*

$$\alpha(f(I \times W_2) \cup g(I \times W_1)) \leq r\alpha(W_1 \cup W_2),$$

$$\|f(t, x) - g(t, y)\| \leq \frac{1}{h}(\|x(t) - y(t)\| - \|x_1 - x_0\|),$$

for some $r \in]0, 1[$ and for any $(W_1, W_2) \subseteq (V_1, V_2)$ and $h < \min\{a, \frac{b}{M_1}, \frac{b}{M_2}, \frac{1}{r}\}$, where $M_1 = \sup\{\|f(t, x)\| : (t, x) \in I \times V_1\}$ and $M_2 = \sup\{\|g(t, y)\| : (t, y) \in I \times V_2\}$, then the systems (62) and (63) have an optimum solution.

Proof. Clearly, $(\mathcal{C}(J, V_1), \mathcal{C}(J, V_2))$ is a bounded, closed, and convex pair in $\mathcal{C}(J, X)$ and T is cyclic on $\mathcal{C}(J, V_1) \cup \mathcal{C}(J, V_2)$. We now prove that $T(\mathcal{C}(J, V_1))$ is a bounded and equicontinuous subset of $\mathcal{C}(J, V_2)$. Suppose $t, t' \in J$ and $x \in \mathcal{C}(J, V_1)$. Then we have:

$$\begin{aligned}\|Tx(t) - Tx(t')\| &= \|\int_{t_0}^{t} g(s, x(s))ds - \int_{t_0}^{t'} g(s, x(s))ds\| \\ &\leq \int_{t}^{t'} \|g(s, x(s))\|ds \leq M_2|t - t'|,\end{aligned}$$

that is, $T(\mathcal{C}(J, V_1))$ is equicontinuous. Equivalently, we can see that $T(\mathcal{C}(J, V_2))$ is also bounded and equicontinuous. Now, from the Arzela–Ascoli theorem, we conclude that the pair $(\mathcal{C}(J, V_1), \mathcal{C}(J, V_2))$ is relatively compact. In the following, we verify that T is a condensing operator. Let $(K_1, K_2) \subseteq (\mathcal{C}(J, V_1), \mathcal{C}(J, V_2))$ be nonempty, closed, convex, and proximinal pair, which is T-invariant such that $\mathrm{dist}(K_1, K_2) = \mathrm{dist}(\mathcal{C}(J, V_1), \mathcal{C}(J, V_2))$ $(= \|x_0 - x_1\|)$. From ([7], Theorem 2.11) we deduce that:

$$\begin{aligned}\alpha(T(K_1), T(K_2)) &= \max\{\alpha(T(K_1)), \alpha(T(K_2))\}\\ &= \max\left\{\sup_{t\in J}\{\alpha(\{Tx(t) : x \in K_1\})\}, \sup_{t\in J}\{\alpha(\{Ty(t) : y \in K_2\})\}\right\}\\ &= \max\left\{\begin{array}{l}\sup_{t\in J}\{\alpha(\{x_1 + \int_{t_0}^t g(s, x(s))ds : x \in K_1\})\},\\ \sup_{t\in J}\{\alpha(\{x_0 + \int_{t_0}^t f(s, y(s))ds : y \in K_2\})\}\end{array}\right\}.\end{aligned}$$

On the other hand, using Theorem 53 we obtain:

$$x_1 + \int_{t_0}^t g(s, x(s))ds \in x_1 + (t - t_0)\overline{\mathrm{con}}(\{g(s, x(s)) : s \in [t_0, t]\}),$$

$$x_0 + \int_{t_0}^t f(s, y(s))ds \in x_0 + (t - t_0)\overline{\mathrm{con}}(\{f(s, y(s)) : s \in [t_0, t]\}),$$

and thus,

$$\begin{aligned}\alpha(T(K_1), T(K_2)) &\le \max\{\sup_{t\in J}\{\alpha(\{x_1 + (t - t_0)\overline{\mathrm{con}}(\{g(s, x(s)) : s \in [t_0, t]\})\})\},\\ &\quad \sup_{t\in J}\{\alpha(\{x_0 + (t - t_0)\overline{\mathrm{con}}(\{f(s, y(s)) : s \in [t_0, t]\})\})\}\}.\\ &\le \max\{\sup_{0\le\lambda\le h}\{\alpha(\{x_1 + \lambda\overline{\mathrm{con}}(\{g(J \times K_1)\})\})\}, \sup_{0\le\lambda\le h}\{\alpha(\{x_0 + \lambda\overline{\mathrm{con}}(\{f(J \times K_2)\})\})\}\}.\\ &= h\alpha(\{g(J \times K_1) \cup f(J \times K_2)\}) \le hr\alpha(K_1 \cup K_2).\end{aligned}$$

Since $hr < 1$, we conclude that T is a condensing operator. Finally, we show that T is cyclic relatively nonexpansive. From the assumptions of theorem, for any $(x, y) \in \mathcal{C}(J, V_1) \times \mathcal{C}(J, V_1)$ we have:

$$\begin{aligned}\|Tx(t) - Ty(t)\| &= \|(x_1 + \int_{t_0}^t g(s, x(s))ds) - (x_0 + \int_{t_0}^t f(s, y(s))ds)\|\\ &\le \|x_1 - x_0\| + \int_{t_0}^t \|g(s, x(s)) - f(s, y(s)))\|ds\\ &\le \|x_1 - x_0\| + \frac{1}{h}\int_{t_0}^t (\|x(s) - y(s)\| - \|x_1 - x_0\|)ds\\ &\le \|x_1 - x_0\| + (\|x - y\|_\infty - \|x_1 - x_0\|) = \|x - y\|_\infty,\end{aligned}$$

and thereby, $\|Tx - Ty\|_\infty \le \|x - y\|_\infty$. Now the result follows from Theorem 44. □

Application B.

In what follows, let a, b, h be positive real numbers with $h < a$. For a given real number t_0 and a Banach space X, we consider the Banach space $C(I, X)$ of continuous maps from $I = [t_0 - a, t_0 + a]$ into X, endowed with the supremum norm. Furthermore, let $V_1 = B(x_1; b)$ and $V_2 = B(x_2; b)$ be closed balls in X, where $x_1, x_2 \in X$. Assume that $k_i : I \times I \times V_i \to X$ and $f_i : I \times V_i \times V_i \to X$, with $i = 1, 2$, continuous maps, and k_i is k_i-invariant. Here, we consider the problem:

$$\begin{cases} x'(t) = f_1(t, x(t), \int_{t_0}^t k_1(t, s, x(s))ds), & x(t_0) = x_1,\\ y'(t) = f_2(t, y(t), \int_{t_0}^t k_2(t, s, y(s))ds), & y(t_0) = x_2,\end{cases} \tag{64}$$

where the integral is the Bochner integral. Let $J = [t_0 - h, t_0 + h]$ and define $C(J, V_1) = \{x \in C(J, X) : x(t_0) = x_1\}$ and $C(J, V_2) = \{y \in C(J, X) : y(t_0) = x_2\}$. Clearly, $(C(J, V_1),$

$C(J, V_2))$ is a bounded, closed, and convex pair in $C(J, X)$. Thus, for any $(x, y) \in C(J, V_1) \times C(J, V_2)$, we have $\|x_1 - x_2\| \leq \sup_{t \in J} \|x(t) - y(t)\| = \|x - y\|$, and so,

$$\mathrm{dist}(C(J, V_1), C(J, V_2)) = \|x_1 - x_2\|.$$

Now, let $T : C(J, V_1) \cup C(J, V_2) \to C(J, X)$ be the operator defined as:

$$Tx(t) = \begin{cases} x_2 + \int_{t_0}^{t} f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))ds)d\sigma, & x \in C(J, V_1), \\ x_1 + \int_{t_0}^{t} f_2(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_2(\sigma, s, x(s))ds)d\sigma, & x \in C(J, V_2). \end{cases} \tag{65}$$

We show that T is a cyclic operator. Indeed, for $x \in C(J, V_1)$ we have:

$$\begin{aligned} &\|Tx(t) - x_2\| \\ &= \| \int_{t_0}^{t} f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))ds)d\sigma\| \\ &\leq | \int_{t_0}^{t} \|f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))ds)\|d\sigma| \\ &\leq M_1 h, \end{aligned}$$

where $M_i = \sup\{\|f_i(t, x(t), \int_{t_0}^{t} k_i(t, s, x(s))ds)\| : (t, x) \in I \times V_i\}$, $i = 1, 2$. Now, if we assume $h < \frac{b}{\max_{i \in \{1,2\}} M_i}$, we get $\|Tx(t) - x_2\| \leq b$ for all $t \in J$ and so $Tx \in C(J, V_2)$. The same argument shows that $x \in C(J, V_2)$ implies $Tx \in C(J, V_1)$.

Taking into account the above notions and notation, for $0 < h < \frac{b}{\max_{i \in \{1,2\}} M_i}$, the hypotheses are as follows:

(H_1) Let μ be an MNC on $C(J, X)$ such that for any $r > 0$ there exists $\delta(r) > 0$ such that $r \leq \mu(W_1 \cup W_2) < r + \delta(r)$ for any bounded $(W_1, W_2) \subseteq (V_1, V_2)$ implies:

$$\mu(f_1(I \times W_1 \times W_1) \cup f_2(I \times W_2 \times W_2)) < \frac{r}{h};$$

(H_2) also,

$$\begin{aligned} &\|f_1(t, x(t), \int_{t_0}^{t} k_1(t, s, x(s))ds) - f_2(t, y(t), \int_{t_0}^{t} k_2(t, s, y(s))ds)\| \\ &\qquad \leq \tfrac{1}{h}(\|x(t) - y(t)\| - \|x_1 - x_0\|), \text{for all } (x, y) \in C(J, V_1) \times C(J, V_2). \end{aligned}$$

We recall another extension of the Mean–Value Theorem, which we arrange according to our notation and further use.

Theorem 55. *Let I, J, X, V_i, $k_i : I \times I \times V_i \to X$ and $f_i : I \times V_i \times V_i \to X$ with $i = 1, 2$ be given as above. Let $t_0, t \in J$ with $t_0 < t$. Then:*

$$\begin{aligned} &x_j + \int_{t_0}^{t} f_i(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_i(\sigma, s, x(s))ds)d\sigma \\ &\in x_j + (t - t_0)\overline{con}(\{f_i(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_i(\sigma, s, x(s))ds) : \sigma \in [t_0, t]\}), \end{aligned} \tag{66}$$

with $(i, j) \in \{(1, 2), (2, 1)\}$.

We say that $z \in C(J, V_1) \cup C(J, V_2)$ is an optimal solution for the system (64) provided that $\|z - Tz\| = \mathrm{dist}(C(J, V_1), C(J, V_2))$, that is, z is a best proximity point of the operator T in (65). Then we give the following result.

Theorem 56 ([86]). *If the hypotheses (H_1), (H_2) and $h < \frac{b}{\max_{i \in \{1,2\}} M_i}$ are satisfied, then the problem (64) has an optimal solution.*

Proof. Since T is a cyclic operator, it follows trivially that $T(C(J,V_1))$ is a bounded subset of $C(J,V_2)$. So, we prove that $T(C(J,V_1))$ is also an equicontinuous subset of $C(J,V_2)$. Suppose $t,t' \in J$ and $x \in C(J,V_1)$. We observe that:

$$\begin{aligned}
&\|Tx(t) - Tx(t')\| \\
&= \| \int_{t_0}^{t} f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma,s,x(s))ds)d\sigma - \int_{t_0}^{t'} f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma,s,x(s))ds)d\sigma\| \\
&\le | \int_{t}^{t'} \|f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma,s,x(s))ds)\| d\sigma | \\
&\le M_1 |t - t'|,
\end{aligned}$$

that is, $T(C(J,V_1))$ is equicontinuous. The same argument is valid for $T(C(J,V_2))$ and hence, to avoid repetition, we omit the details. Moreover, by use of the Arzelà–Ascoli theorem, it follows that the pair $(C(J,V_1), C(J,V_2))$ is relatively compact. Here, we show that T is a Meir–Keeler condensing operator. Let $(K_1,K_2) \subseteq (C(J,V_1), C(J,V_2)) \neq \emptyset$ be a closed, convex, and proximinal pair, which is T-invariant and such that $\text{dist}(K_1,K_2) = \text{dist}(C(J,V_1), C(J,V_2)) \,(= \|x_1 - x_2\|)$. Using a generalized version of the Arzelà–Ascoli theorem (see Ambrosetti [90]) and hypothesis (H_1), we get:

$$\begin{aligned}
&\mu(T(K_1) \cup T(K_2)) = \max\{\mu(T(K_1)), \mu(T(K_2))\} \\
&= \max\{\sup_{t\in J}\{\mu(\{Tx(t) : x \in K_1\})\}, \sup_{t\in J}\{\mu(\{Ty(t) : y \in K_2\})\}\} \\
&= \max\{\sup_{t\in J}\{\mu(\{x_2 + \int_{t_0}^{t} f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma,s,x(s))ds)d\sigma : x \in K_1\})\}, \\
&\qquad \sup_{t\in J}\{\mu(\{x_1 + \int_{t_0}^{t} f_2(\sigma, y(\sigma), \int_{t_0}^{\sigma} k_2(\sigma,s,y(s))ds)d\sigma : y \in K_2\})\}\}.
\end{aligned}$$

So, in view of (66), it follows that:

$$\begin{aligned}
&\mu(T(K_1) \cup T(K_2)) \\
&\le \max\{\sup_{t\in J}\{\mu(\{x_2 + (t-t_0)\overline{con}(\{f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma,s,x(s))ds) : \sigma \in [t_0,t]\})\})\}, \\
&\qquad \sup_{t\in J}\{\mu(\{x_1 + (t-t_0)\overline{con}(\{f_2(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_2(\sigma,s,x(s))ds) : \sigma \in [t_0,t]\})\})\}\} \\
&\le \max\{\sup_{0\le\lambda\le h}\{\mu(\{x_2 + \lambda\overline{con}(\{f_1(J\times K_1\times K_1)\})\})\}, \\
&\qquad \sup_{0\le\lambda\le h}\{\mu(\{x_1 + \lambda\overline{con}(\{f_2(J\times K_2\times K_2)\})\})\}\} \\
&= \max\{h\mu(f_1(J\times K_1\times K_1)), h\mu(f_2(J\times K_2\times K_2))\} \\
&= h\mu(\{f_1(J\times K_1\times K_1) \cup f_2(J\times K_2\times K_2)\}) < h\frac{r}{h} = r.
\end{aligned}$$

We conclude that T is a Meir–Keeler condensing operator. The last step of the proof is to show that T is cyclic relatively nonexpansive. Indeed, for any $(x,y) \in C(J,V_1) \times C(J,V_2)$ we have:

$$\begin{aligned}
&\|Tx(t) - Ty(t)\| \\
&= \|(x_2 + \int_{t_0}^{t} f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma,s,x(s))ds)d\sigma) \\
&\qquad - (x_1 + \int_{t_0}^{t} f_2(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_2(\sigma,s,x(s))ds)d\sigma\| \\
&\le \|x_1 - x_2\| + | \int_{t_0}^{t} \|f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma,s,x(s))ds)
\end{aligned}$$

$$- f_1(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_2(\sigma, s, y(s))ds)\|d\sigma|$$

$$\leq \|x_1 - x_2\| + \frac{1}{h}|\int_{t_0}^{t}(\|x(s) - y(s)\| - \|x_1 - x_2\|)ds| \quad \text{(by hypothesis } (H_2))$$

$$\leq \|x_1 - x_2\| + (\|x - y\| - \|x_1 - x_2\|) = \|x - y\|,$$

and thereby, $\|Tx - Ty\| \leq \|x - y\|$. All the hypotheses of Theorem 46 hold and so the operator T has a best proximity point $z \in C(J, V_1) \cup C(J, V_2)$, which is an optimal solution for the system (64). □

Application C.

Let a, b, h be positive real numbers with $h < a$. For a given real number t_0 and a Banach space X, we consider the Banach space $C(I, X)$ of continuous maps from $I = [t_0 - a, t_0 + a]$ into X, endowed with the supremum norm. Furthermore, let $V_1 = B(x^*; b)$ and $V_2 = B(x^{**}; b)$ be closed balls in X, where $x^*, x^{**} \in X$. Assume that $f : I \times V_1 \to X$ and $g : I \times V_2 \to X$ are continuous maps. So, we recall the problem:

$$\begin{cases} x'(t) = f(tx(t)), & x(t_0) = x^*, \\ y'(t) = g(t, y(t)), & y(t_0) = x^{**}. \end{cases} \tag{67}$$

Let $J = [t_0 - h, t_0 + h]$ and define $C(J, V_1) = \{x \in C(J, X) : x(t_0) = x^*\}$, $C(J, V_2) = \{y \in C(J, X) : y(t_0) = x^{**}\}$. Clearly, $(C(J, V_1), C(J, V_2))$ is a bounded, closed, and convex pair in $C(J, X)$. Moreover, for any $(x, y) \in C(J, V_1) \times C(J, V_2)$ we have $\|x_1 - x_2\| \leq \sup_{t \in J} \|x(t) - y(t)\| = \|x - y\|$, and so, $\text{dist}(C(J, V_1), C(J, V_2)) = \|x_1 - x_2\|$.

Now, let $T : C(J, V_1) \cup C(J, V_2) \to C(J, X)$ be the operator defined as:

$$Tx(t) = \begin{cases} x^{**} + \int_{t_0}^{t} g(\sigma, x(\sigma))d\sigma, & x \in C(J, V_1), \\ x^* + \int_{t_0}^{t} f(\sigma, x(\sigma))d\sigma, & x \in C(J, V_2). \end{cases} \tag{68}$$

We show that T is a cyclic operator. Indeed, for $x \in C(J, V_1)$ we have:

$$\begin{aligned} \|Tx(t) - x^{**}\| &= \|\int_{t_0}^{t} g(\sigma, x(\sigma))d\sigma\| \\ &\leq \left|\int_{t_0}^{t} \|g(\sigma, x(\sigma))\|d\sigma\right| \\ &\leq M_1 h, \end{aligned}$$

where $M_1 = \sup\{\|g(t, x(t))\| : (t, x) \in I \times V_2\}$ (analogously, $M_2 = \sup\{\|f(t, x(t))\| : (t, x) \in I \times V_1\}$). Now, if we assume $h < \min\{\frac{b}{\max_{i \in \{1,2\}} M_i}, \frac{1}{2b}\}$, we get $\|Tx(t) - x^{**}\| \leq b$ for all $t \in J$, and so, $Tx \in C(J, V_2)$. The same argument shows that $x \in C(J, V_2)$ implies $Tx \in C(J, V_1)$.

Taking into account the above notions and notation, for $0 < h < \min\{\frac{b}{\max_{i \in \{1,2\}} M_i}, \frac{1}{2b}\}$, the hypotheses are as follows:

(H_1) There exists $\psi \in \Psi$ such that $\alpha(f(I \times W_2) \cup g(I \times W_1)) \leq 2b\psi(\alpha(W_1 \cup W_2))$ for any $(H_1, H_2) \subseteq (V_1, V_2)$;

(H_2) $\|f(t, x(t)) - g(t, y(t))\| \leq \frac{1}{h}(\|x(t) - y(t)\| - \|x^{**} - x^*\|)$, for all $(x, y) \in C(J, V_1) \times C(J, V_2)$.

We recall the following extension of the Mean–Value Theorem, which we arrange according to our notation and further use.

Theorem 57. *Let I, J, X, $f : I \times V_1 \to X$, $g : I \times V_2 \to X$ be given as above. Let $t_0, t \in J$ with $t_0 < t$. Then:*

$$x^* + \int_{t_0}^{t} f(\sigma, x(\sigma))d\sigma \in x^* + (t - t_0)\overline{con}(\{f(\sigma, x(\sigma)) : \sigma \in [t_0, t]\}), \tag{69}$$

$$x^{**} + \int_{t_0}^{t} g(\sigma, x(\sigma))d\sigma \in x^{**} + (t - t_0)\overline{con}(\{g(\sigma, x(\sigma)) : \sigma \in [t_0, t]\}). \tag{70}$$

Furthermore, we need the next generalization of the Arzela–Ascoli theorem.

Theorem 58 ([90])**.** *Let X be a Banach space, $D \subseteq \mathbb{R}^n$ compact and $B \subseteq C(D, X)$ a bounded and equicontinuous set. Then $\alpha(B) = \sup_{t \in D} \alpha(\{x(t) : x \in B\})$.*

We say that $z \in C(J, V_1) \cup C(J, V_2)$ is an optimal solution for the system (67) provided that $\|z - Tz\| = \text{dist}(C(J, V_1), C(J, V_2))$, that is, z is a best proximity point of the operator T in (68). Then, we give the following result.

Theorem 59 ([86])**.** *If the hypotheses (H_1),(H_2) and $h < \min\{\frac{b}{\max_{i \in \{1,2\}} M_i}, \frac{1}{2b}\}$ are satisfied, then the problem (67) has an optimal solution.*

Proof. Since T is a cyclic operator, it follows trivially that $T(C(J, V_1))$ is a bounded subset of $C(J, V_2)$. So, we prove that $T(C(J, V_1))$ is also an equicontinuous subset of $C(J, V_2)$. Suppose $t, t' \in J$ and $x \in C(J, V_1)$. We observe that:

$$\begin{aligned}
&\|Tx(t) - Tx(t')\| \\
&= \| \int_{t_0}^{t} g(\sigma, x(\sigma))d\sigma - \int_{t_0}^{t'} g(\sigma, x(\sigma))d\sigma\| \\
&\leq \left| \int_{t}^{t'} \|g(\sigma, x(\sigma))\|d\sigma \right| \\
&\leq M_1|t - t'|,
\end{aligned}$$

that is, $T(C(J, V_1))$ is equicontinuous. The same argument is valid for $T(C(J, V_2))$ and hence, to avoid repetition, we omit the details. Here, we show that T is a generalized condensing operator. Let $(K_1, K_2) \subseteq (C(J, V_1), C(J, V_2)) \neq \emptyset$ be a closed, convex, and proximinal pair, which is T-invariant and such that $\text{dist}(K_1, K_2) = \text{dist}(C(J, V_1), C(J, V_2))$ $(= \|x^{**} - x^*\|)$. By Theorem 58 and hypothesis (H_1), we obtain:

$$\begin{aligned}
&\alpha(T(K_1) \cup T(K_2)) = \max\{\alpha(T(K_1)), \alpha(T(K_2))\} \\
&= \max\{\sup_{t \in J}\{\alpha(\{Tx(t) : x \in K_1\})\}, \sup_{t \in J}\{\alpha(\{Ty(t) : y \in K_2\})\}\} \\
&= \max\{\sup_{t \in J}\{\alpha(\{x^{**} + \int_{t_0}^{t} g(\sigma, x(\sigma))d\sigma : x \in K_1\})\}, \\
&\qquad \sup_{t \in J}\{\alpha(\{x^* + \int_{t_0}^{t} f(\sigma, y(\sigma))d\sigma : y \in K_2\})\}\}.
\end{aligned}$$

So, in view of (69) and (70), it follows that:

$$\begin{aligned}
&\alpha(T(K_1) \cup T(K_2)) \leq \max\{\sup_{t \in J}\{\alpha(\{x^{**} + (t - t_0)\overline{con}(\{g(\sigma, x(t)) : \sigma \in [t_0, t]\})\})\}, \\
&\sup_{t \in J}\{\alpha(\{x^* + (t - t_0)\overline{con}(\{f(t, x(t)) : \sigma \in [t_0, t]\})\})\}\} \\
&\leq \max\{\sup_{0 \leq \lambda \leq h}\{\alpha(\{x^{**} + \lambda\overline{con}(\{g(J \times K_1)\})\})\}, \sup_{0 \leq \lambda \leq h}\{\alpha(\{x^* + \lambda\overline{con}(\{f(J \times K_2)\})\})\}\} \\
&= \max\{h\alpha(g(J \times K_1)), h\alpha(f(J \times K_2))\}
\end{aligned}$$

$$= h\alpha(\{g(J\times K_1)\cup f(J\times K_2)\}) \leq \frac{1}{2b}2b\psi(\alpha(K_1\cup K_2)) = \psi(\alpha(K_1\cup K_2)).$$

We conclude that T is a generalized condensing operator. The last step of the proof is to show that T is cyclic relatively nonexpansive. Indeed, for any $(x,y)\in C(J,V_1)\times C(J,V_2)$ we have:

$$\begin{aligned}
&\|Tx(t)-Ty(t)\| \\
&= \|(x^{**}+\int_{t_0}^{t} g(\sigma,x(\sigma))d\sigma) - (x^{*}+\int_{t_0}^{t} f(\sigma,x(\sigma))d\sigma\| \\
&\leq \|x^{**}-x^{*}\| + \left|\int_{t_0}^{t} \|(g(\sigma,x(\sigma)) - f(\sigma,x(\sigma)))d\sigma\|\right| \\
&\leq \|x^{**}-x^{*}\| + \frac{1}{h}\left|\int_{t_0}^{t} (\|x(s)-y(s)\| - \|x^{**}-x^{*}\|)ds\right| \quad \text{(by hypothesis } (H_2)) \\
&\leq \|x^{**}-x^{*}\| + (\|x-y\| - \|x^{**}-x^{*}\|) = \|x-y\|,
\end{aligned}$$

and thereby, $\|Tx-Ty\| \leq \|x-y\|$. All the hypotheses of Theorem 47 hold and so the operator T has a best proximity point $z\in C(J,V_1)\cup C(J,V_2)$, which is an optimal solution for the system (67). □

An application of a coupled measure of noncompactness can be found in the recent paper [91].

11. Concluding Remarks

We gave a survey of measures of noncompactness and their most important properties. Furthermore, we discussed some fixed point theorems of Darbo type.

First, we applied measures of noncompactness in characterizing classes of compact operators between certain sequence spaces, and in solving infinite systems of integral equations in some sequence and function spaces.

Second, we included some recent results related to the existence of best proximity points (pairs) for some classes of cyclic and noncyclic condensing operators in Banach spaces equipped with a suitable measure of noncompactness.

Finally, we discussed the existence of an optimal solution for systems of integro-differentials.

It is worth mentioning that measures of noncompactness play an important role in nonlinear functional analysis. They are important tools in metric fixed point theory, the theory of operator equations in Banach spaces, and the characterizations of classes of compact operators. They are also applied in the studies of various kinds of differential and integral equations.

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