Article

New Theorems in Solving Families of Improper Integrals

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Abstract: Many improper integrals appear in the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik. It is a challenge for some researchers to determine the method in which these integrations are formed or solved. In this article, we present some new theorems to solve different families of improper integrals. In addition, we establish new formulas of integrations that cannot be solved by mathematical software such as Mathematica or Maple. In this article, we present three main theorems that are essential in generating new formulas for solving improper integrals. To show the efficiency and the simplicity of the presented techniques, we present some applications and examples on integrations that cannot be solved by regular methods. Furthermore, we acquire new results for integrations and compare them to that obtained in the classical table of integrations. Some previous results, become special cases of our outcomes or generalizations to acquire new integrals.

Keywords: improper integrals; special functions; Cauchy residue theorem; power series; analytic function

1. Introduction

In the history of mathematics, there are many improper integrals that appeared in the fields of science, physics, engineering and etc., [1–5]. Some of these integrals can be solved easily and others need some techniques to be solved. some of these integrals can't be solved up to now, some of them are solved numerically, and there still some integrations that cannot be determined exactly or need much effort to be solved. The importance of computing improper integrals has come from the wide usage in applied math, physics, engineering and etc., [6–15].

Evaluation of Improper integrals is not typically subjected to general rules or assignable methods at first glance. Various ways, techniques and methods have been studied and presented by scientists and researchers for providing a closed-form expression for definite integrals. Series methods, double integrals technique, Cauchy residue theorem, differentiating under the integral sign, and plenty of other techniques are used in solving complicated improper integrals exactly or approximately, [16–20]. It is nearly always that one blindly looks for a way that leads to the answer, without certainty on the success.

Within these theorems, we present the closed form of the integral that can be found only by identifying the function that formed the improper integral. These theorems are considered a solid tool for unraveling some families of improper integrals and creating an infinite number of challenging integrals.

In 1826. A.L. Cauchy came up with the Residue Theorem, which is one of the most important achievements in complex analysis. Nevertheless, applications of the residue theorem to solve integrals over real line require rigorous conditions that must be met to solve the integrals, such as determining the appropriate closed contour, finding the poles, and also that the process of finding residues may be difficult and exhausting in some integrals. In his published memoir, Cauchy had come up with mathematical formulas using the residue theorem [21–27], the formulas that he achieved are considered very simple

comparing to the results that will be presented; considering that the theorems in this article don't depend on the residue theorem.

Another important achievement in the world of improper integrals, is Ramanujan's master theorem, which provides an explicit expression for the Mellin transform of a function in terms of the analytic continuation of its Taylor coefficients [16]. It was widely used by Ramanujan as a tool in computing definite integrals and infinite series. The theorems presented in this article are just as powerful and efficient as Ramanujan's master theorem in solving some families of improper integrals, for more about integral transforms, see [5–8].

This article is organized as follows, in the second section, we present some basic definitions and theorems that are needed in our research, Main results and theorems are presented in section three. Some applications and examples are presented in section four. Finally, some mathematical remarks are presented in section five.

2. Basic Definitions and Theorems

Before going into our results, we need to introduce some basic definitions and theorems, to be used in the construction of the new results.

Definition 1. ([18]). *Let f be analytic function in an open set Ω which is a subset of the complex plane, and D is a disc centered at z_0 subset of D; whose closure is contained in the region Ω, then f has a power series expansion at z_0 as,*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \ n \in N^*, \ a_n \text{ is a real coefficient of the series.} \tag{1}$$

Definition 2. ([19]). *Let f be a real analytic function, that is, infinitely differentiable such that Taylor series at any point x_0 in it is domain is,*

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \tag{2}$$

converges to $f(x)$ in a neighborhood of x_0 pointwisely.

Definition 3. ([20]). *Cauchy principal value of a finite integral of a function f about a point c, with $a \le c \le b$ is given by,*

$$PV \int_a^b f(x)dx \equiv \lim_{\varepsilon \to 0^+} \left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right]. \tag{3}$$

Definition 4. ((Gamma function) [21]). *Euler integral of the second kind, is defined by the integral,*

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx, \text{ where } Re(z) > 0,$$

and Euler's reflection formula for the gamma function is,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z} \tag{4}$$

Definition 5. ((Beta function) [21]). *It is also called Euler integral of the first kind that is defined by the integral,*

$$\beta(p, y) = \int_0^1 t^{p-1}(1 - t)^{y-1} dt, \tag{5}$$

for complex numbers inputs p and y such that Re p > 0 and Re y > 0.

Integral (5) can be rewritten as,

$$\beta(p,y) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+y}} dt \qquad ; Re(p),\ Re(y) > 0. \tag{6}$$

There is a relation between the gamma function and the beta function as follows,

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{7}$$

In the following arguments, we present some formulas of integrations, that are essential in our work. Herein, we mention that these integrals could be evaluated by many methods, beta function, residue theorem and other techniques.

Lemma 1. *For any real number x and a constant p, we have the following integrals,*

(i)

$$I_1 = \int_0^\infty \frac{x^p}{x^2 + 1} dx = \frac{\pi}{2\cos\left(\frac{\pi p}{2}\right)}, \text{ where } -1 < \operatorname{Re}(p) < 1. \tag{8}$$

(ii)

$$I_2 = PV \int_0^\infty \frac{x^{p-1}}{x^2 - 1} dx = \frac{-\pi\cot\left(\frac{\pi p}{2}\right)}{2}, \text{ where } 0 < Re(p) < 2. \tag{9}$$

Proof of (i). This is a direct proof that can be obtained from the relations between the gamma function and the beta function. To prove the first integral, let $x^2 = t$, then I_1 using the fact in Equation (6) becomes,

$$I_1 = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{1-p}{2}\right).$$

Using the relation between the beta and the gamma function in Equation (7), we acquire,

$$I_1 = \frac{1}{2}\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right).$$

Now, using the reflection formula (4) of the the gamma function we have,

$$I_1 = \frac{\pi}{2\cos\left(\frac{\pi p}{2}\right)}.$$

□

Proof of (ii). By factorizing the denominator of I_2, and decompose the integral into partial fractions, we have,

$$I_2 = PV \int_0^\infty \frac{x^{p-1}}{2(x-1)} - \frac{x^{p-1}}{2(x+1)} dx = PV\left(\int_0^\infty \frac{x^{p-1}}{2(x-1)} dx - \int_0^\infty \frac{x^{p-1}}{2(x+1)} dx\right).$$

Let $M = \int_0^\infty \frac{x^{p-1}}{x+1} dx$, and $= PV \int_0^\infty \frac{x^{p-1}}{(x-1)}$.

M can be solved easily, by using Equation (6), followed by the relation between beta and gamma function in Equation (7), and then using the reflection formula (4) we acquire,

$$M = \int_0^\infty \frac{x^{p-1}}{x+1} dx = \frac{\pi}{\sin(\pi p)}. \tag{10}$$

The value of N can be obtained as follows, using the value in Equation (10), we acquire,

$$M^2 = \int_0^\infty \int_0^\infty \frac{x^{p-1} y^{p-1}}{(x+1)(y+1)} dx \, dy.$$

Let $y = \frac{z}{x} \to dy = \frac{dz}{x}$, thus after simple computations and using the idea of interchanging the order of integration, we have,

$$M^2 = PV \int_0^\infty \frac{x^{p-1}}{x+1} \int_0^\infty \frac{x^{1-p} z^{p-1}}{\left(1 + \frac{z}{x}\right)x} dz \, dx = PV \int_0^\infty \int_0^\infty \frac{z^{p-1}}{(x+1)(z+x)} dx \, dz.$$

After simple computations, one can obtain,

$$M^2 = PV \int_0^\infty \frac{z^{p-1}}{z-1} ln(z) dz,$$

which can be re written as,

$$M^2 = \frac{d}{dp} \int_0^\infty \frac{z^{p-1}}{z-1} ln(z) dz.$$

Thus,

$$\int M^2 dp = PV \int_0^\infty \frac{z^{p-1}}{z-1} dz$$

Now, using the value of M in Equation (10), we have,

$$N = PV \int_0^\infty \frac{z^{p-1}}{z-1} dz = \int_0^\infty \frac{\pi^2}{\sin^2(\pi p)} dp = -\pi \cot(p\,\pi). \tag{11}$$

Thus, we the proof is complete and we have,

$$I_2 = -\frac{\pi}{2} \cot(p\,\pi) - \frac{\pi}{2\sin(\pi p)} = -\frac{\pi \cot\left(\frac{p\pi}{2}\right)}{2}.$$

□

3. Main Theorems

In this section, we introduce new attractive results for integrations, that can be used to generate improper integrals and solve difficult applications. To achieve our goal, we need to present some relations about analytic functions that can be found in [20–22].

Let $f(z) = f\left(\beta e^{\theta x}\right)$ be analytic function around 0, then according to Taylor's series where θ and β denote positive or negative real quantities, we have,

$$f(z) = \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} (z)^k, \text{ and } f\left(\beta e^{\theta x}\right) = \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} \beta^k e^{\theta k x} \tag{12}$$

Using the formula $e^{i\theta x} + e^{-i\theta x} = 2\cos(\theta x)$ and $e^{i\theta x} - e^{-i\theta x} = 2i\sin(\theta x)$, one can acquire for any p,

$$
\begin{aligned}
e^{\frac{-i\pi p}{2}} f\left(\beta e^{i\theta x}\right) + e^{\frac{i\pi p}{2}} f\left(\beta e^{-i\theta x}\right) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \beta^k \left(e^{\frac{-i\pi p}{2}} e^{i\theta k x} + e^{\frac{i\pi p}{2}} e^{-i\theta k x}\right) \\
&= 2\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \beta^k \cos\left(k\theta x - \frac{\pi p}{2}\right).
\end{aligned}
\tag{13}
$$

Similarly,

$$
\begin{aligned}
\frac{1}{i}\left(e^{\frac{-i\pi p}{2}} f\left(\beta e^{i\theta x}\right) - e^{\frac{i\pi p}{2}} f\left(\beta e^{-i\theta x}\right)\right) \\
= \frac{1}{i} \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} \beta^k \left(e^{\frac{-i\pi p}{2}} e^{i\theta k x} - e^{\frac{i\pi p}{2}} e^{-i\theta k x}\right) \\
= 2\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \beta^k \sin\left(k\theta x - \frac{\pi p}{2}\right).
\end{aligned}
\tag{14}
$$

The results obtained in Equations (13) and (14) can be extended in a modified form as follows.

Lemma 2. *Let $g(z)$ be analytic function that can be expanded in a series form,*

$$
g(z) = \sum_{k=0}^{\infty} M_k e^{-kz},
\tag{15}
$$

whetherz be real or imaginary , and $\sum_{k=0}^{\infty} M_k$ be absolutely convergent, then Equations (13) and (14) can be re written in a modified form as follows,

$$
\begin{aligned}
\frac{1}{2}\left(e^{\frac{-i\pi p}{2}} g(-i\theta x) + e^{\frac{i\pi p}{2}} g(i\theta x)\right) &= \frac{1}{2} \sum_{k=0}^{\infty} M_k \left(e^{-\frac{i\pi p}{2}} e^{ik\theta x} + e^{\frac{i\pi p}{2}} e^{-ik\theta x}\right) \\
&= \sum_{k=0}^{\infty} M_k \cos\left(k\theta x - \frac{\pi p}{2}\right)
\end{aligned}
\tag{16}
$$

and,

$$
\frac{1}{2i}\left(e^{\frac{-i\pi p}{2}} g(-i\theta x) - e^{\frac{i\pi p}{2}} g(i\theta x)\right) = \frac{1}{2i} \sum_{k=0}^{\infty} M_k \left(e^{\frac{-i\pi p}{2}} e^{ik\theta x} - e^{\frac{i\pi p}{2}} e^{-ik\theta x}\right) = \sum_{k=0}^{\infty} M_k \sin\left(k\theta x - \frac{\pi p}{2}\right),
\tag{17}
$$

where $\theta > 0$ andxis any real number.

The following theorems are new results that can be used for solving some difficult improper integrals. Herein, it is worth mentioning that some of Cauchy's results in [26], can be obtained as corollaries from the following theorems as will be stated.

Theorem 1. *Let $f(z)$ be an analytic function around 0, then we have the following result,*

$$
I = \int_0^{\infty} \frac{x^p \left(e^{\frac{-i\pi p}{2}} f\left(\beta e^{i\theta x}\right) + e^{\frac{i\pi p}{2}} f\left(\beta e^{-i\theta x}\right)\right)}{1 + x^2} dx = \pi f\left(\beta e^{-\theta}\right),
\tag{18}
$$

where $-1 < Re(p) < 1$, and $\theta > 0$.

Proof. Since f is analytic around 0, thus, using the fact in Equation (13), we have,

$$
e^{\frac{-i\pi p}{2}} f\left(\beta e^{i\theta x}\right) + e^{\frac{i\pi p}{2}} f\left(\beta e^{-i\theta x}\right) = 2\left(f(0)\cos\left(\frac{\pi p}{2}\right) + \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} \beta^k \cos\left(k\theta x - \frac{\pi p}{2}\right)\right).
\tag{19}
$$

Substituting Equation (19) in Equation (18) to acquire,

$$I = \int_0^\infty \frac{x^p \left(2 \left(f(0) \cos\left(\frac{\pi p}{2}\right) + \sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \cos\left(k\theta x - \frac{\pi p}{2}\right) \right) \right)}{1+x^2} dx.$$

$$= 2f(0) \cos\left(\frac{\pi p}{2}\right) \int_0^\infty \frac{x^p}{1+x^2} dx + 2 \sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \int_0^\infty \frac{x^p \cos\left(k\theta x - \frac{\pi p}{2}\right)}{1+x^2} dx.$$

Use the result in Equation (8) to evaluate $\int_0^\infty \frac{x^p}{1+x^2} dx$, thus we obtain,

$$I = \pi f(0) + 2 \sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \int_0^\infty \frac{x^p \cos\left(k\theta x - \frac{\pi p}{2}\right)}{1+x^2} dx. \tag{20}$$

The integral $\int_0^\infty \frac{x^p \cos\left(k\theta x - \frac{\pi p}{2}\right)}{1+x^2} dx$, can be solved using the Laplace transform as follows. Define,

$$j(t) = \int_0^\infty \frac{x^p \left(\cos\left(\frac{\pi p}{2}\right) \cos(\theta t k x) + \sin\left(\frac{\pi p}{2}\right) \sin(\theta t k x) \right)}{1+x^2} dx, \tag{21}$$

where $\theta, t > 0$, $-1 < \text{Re}\,(p) < 1$, $k \in \mathbb{N}$, $k \neq 0$, and our desired integral is $j(1)$.

Thus, by taking Laplace transform for both sides of Equation (21), we have,

$$J(s) = \int_0^\infty \int_0^\infty \frac{x^p e^{-st} \left[\cos\left(\frac{\pi p}{2}\right) \cos(\theta t k x) + \sin\left(\frac{\pi p}{2}\right) \sin(\theta t k x) \right]}{1+x^2} dx\, dt.$$

Using Fubini's theorem, the double integral yields a finite answer when the integral is replaced by its absolute value, i.e., converges in the Riemann sense. Thus, we can interchange the order of the integrations, to acquire

$$J(s) = \int_0^\infty \frac{x^p}{1+x^2} \left(\int_0^\infty e^{-st} \left[\cos\left(\frac{\pi p}{2}\right) \cos(\theta t k x) + \sin\left(\frac{\pi p}{2}\right) \sin(\theta t k x) \right] dt \right) dx.$$

$$= \int_0^\infty \frac{x^p}{1+x^2} \mathcal{L}\left\{ \cos\left(\frac{\pi p}{2}\right) \cos(\theta t k x) + \sin\left(\frac{\pi p}{2}\right) \sin(\theta t k x) \right\} dx. \tag{22}$$

Running the Laplace transform on both sides of Equation (22), we acquire,

$$J(s) = \int_0^\infty \frac{x^p s \cos\left(\frac{p\pi}{2}\right)}{(s^2 + k^2 x^2 \theta^2)(1+x^2)} + \frac{x^{p+1} k\theta \sin\left(\frac{p\pi}{2}\right)}{(1+x^2)(s^2 + k^2 x^2 \theta^2)} dx. \tag{23}$$

Now, use the partial fraction decomposition as,

$$\frac{1}{(1+x^2)(s^2 + k^2 x^2 \theta^2)} = \frac{1}{(s^2 - k^2 \theta^2)} \left(\frac{1}{x^2 + 1} - \frac{(k\theta)^2}{s^2 + k^2 x^2 \theta^2} \right). \tag{24}$$

Substituting Equation (24) in Equation (23) to acquire,

$$J(s) = \frac{s \cos\left(\frac{\pi p}{2}\right)}{s^2 - (\theta k)^2} \int_0^\infty \left(\frac{x^p}{x^2 + 1} - \frac{\theta^2 k^2 x^p}{\theta^2 k^2 x^2 + s^2} \right) dx + \frac{\theta k \sin\left(\frac{\pi p}{2}\right)}{s^2 - (\theta k)^2} \int_0^\infty \left(\frac{x^{p+1}}{x^2 + 1} - \frac{\theta^2 k^2 x^{p+1}}{\theta^2 k^2 x^2 + s^2} \right) dx.$$

Using the results in Equation (8), one can obtain,

$$J(s) = \frac{s\cos\left(\frac{\pi p}{2}\right)}{s^2 - (\theta k)^2}\left(\frac{\pi}{2\cos\left(\frac{\pi p}{2}\right)} - \frac{\left(\frac{s}{\theta k}\right)^{p-1}\pi}{2\cos\left(\frac{\pi p}{2}\right)}\right) + \frac{\theta k\sin\left(\frac{\pi p}{2}\right)}{s^2 - (\theta k)^2}\left(\frac{-\pi}{2\sin\left(\frac{\pi p}{2}\right)} + \frac{\pi\left(\frac{s}{\theta k}\right)^{p}}{2\sin\left(\frac{\pi p}{2}\right)}\right). \tag{25}$$

After simple computations, Equation (25) becomes,

$$J(s) = \frac{\pi}{2(s + \theta k)}. \tag{26}$$

Taking the inverse Laplace transform for both sides of Equation (24), we acquire,

$$j(t) = \frac{\pi}{2}e^{-\theta k t}.$$

Thus, $j(1) = \frac{\pi}{2}e^{-\theta k}$, which is the desired integral.
Substituting the value of $j(1)$ in Equation (20), we have,

$$I = \pi f(0) + \pi\sum_{k=1}^{\infty}\frac{f^{(k)}(0)}{k!}\beta^k\,e^{-\theta k}.I = \pi\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\beta^k\,e^{-\theta k}. \tag{27}$$

Using the fact in Equation (12), we acquire,

$$I = \pi f\left(\beta e^{-\theta}\right).$$

This completes the proof of Theorem 1. □

In the following arguments, we present some results and corollaries from Theorem (1) that are useful in solving integrals and acquire new applications.

Remark 1. *Taking the simple case $p = 0$, and $\beta = 1$ in Theorem 1, we obtain,*

$$\int_0^{\infty}\frac{\left(f\left(e^{i\theta x}\right) + f\left(e^{-i\theta x}\right)\right)}{1 + x^2}dx = \pi\,f\left(e^{-\theta}\right), \tag{28}$$

which is identical to Cauchy's theorem obtained in [26] *(P.62 Theorem 8), and in the classical table of integrals* [27] *(3.037 Theorem 1).*

Corollary 1. *Let $g(z)$ be analytic function that has the series representation in Equation (15), then,*

$$\int_0^{\infty}\frac{x^p\left(e^{\frac{-i\pi p}{2}}g(-i\theta x) + e^{\frac{i\pi p}{2}}g(i\theta x)\right)}{1 + x^2}dx = \pi\,g(\theta), \tag{29}$$

where $-1 <Re\,(p) < 1$, and $\theta > 0$.

Corollary 2. *Let $f(z)$ be analytic function around 0, then we have the following result,*

$$\int_0^{\infty}\frac{x^{ip}\left(e^{\frac{\pi p}{2}}f\left(\beta e^{i\theta x}\right) + e^{\frac{-\pi p}{2}}f\left(\beta e^{-i\theta x}\right)\right)}{1 + x^2}dx = \pi\,f\left(\beta e^{-\theta}\right), \tag{30}$$

where $p \in \mathbb{R}$, $\theta > 0$

Proof. The proof comes directly form Theorem 1. Since, we have $0 <Re\,(p) < 1$, thus if we set $p = a + ib$ where a and b $\in \mathbb{R}$, $-1 < a < 1$. Then $ip = ia - b$, and Re $(ip) = -\text{Im}(p)$. Thus, Equation (31) fulfills for $p \in \mathbb{R}$. □

Corollary 3. *Let $g(z)$ be analytic function that has the series representation in Equation (15), then, Corollary 2 can be rewritten as,*

$$\int_0^\infty \frac{x^{ip}\left(e^{\frac{\pi p}{2}}g(-i\theta x)+e^{\frac{-\pi p}{2}}g(i\theta x)\right)}{1+x^2}dx = \pi\, g(\theta),\tag{31}$$

where $p\in\mathbb{R}$, $\theta > 0$.

Corollary 4. *Let $f(z)$ be analytic function around 0, then we have the following result,*

$$\int_0^\infty \frac{x^{p-1}\left(e^{\frac{-i\pi p}{2}}f\left(\beta e^{i\theta x}\right)-e^{\frac{i\pi p}{2}}f\left(\beta e^{-i\theta x}\right)\right)}{i\left(1+x^2\right)}dx = -\pi f\left(\beta e^{-\theta}\right),\tag{32}$$

where $0 < Re\,(p) < 2$, and $\theta > 0$.

Proof. This is a direct result of Theorem 1, that can be deduced by setting $p = p - 1$ in Equation (18). $\square$

Corollary 5. *Let $g(z)$ be analytic function that has the series representation in Equation (15) and using the fact in Equation (17) then, we have the following result:*

$$\int_0^\infty \frac{x^{p-1}\left(e^{\frac{-i\pi p}{2}}g(-i\theta x)-e^{\frac{i\pi p}{2}}g(i\theta x)\right)}{i\left(1+x^2\right)}dx = -\pi g(\theta),\tag{33}$$

where $0 < Re\,(p) < 2$, and $\theta > 0$.

Proof. The proof comes directly from Theorem 1 and Corollary 4. $\square$

Corollary 6. *Let $f(z)$ be analytic function around 0, then we have the following result:*

$$\int_0^\infty \frac{x^{ip-1}\left(e^{\frac{\pi p}{2}}f\left(\beta e^{i\theta x}\right)-e^{\frac{-\pi p}{2}}f\left(\beta e^{-i\theta x}\right)\right)}{i(1+x^2)}dx = -\pi f\left(\beta e^{-\theta}\right),\tag{34}$$

where $-2 < Im(p) < 0$, and $\theta > 0$.

Proof. The proof comes directly from Corollary 4. Since, we have $0 < Re\,(p) < 2$, thus if we set $p = a + ib$ where a and b $\in\mathbb{R}$, $0 < a < 2$. Then $ip = ia - 1$, and Re $(ip) = -$Im(p). Thus, Equation (31) fulfills for $-2 <$ Im$(p) < 0$. $\square$

Corollary 7. *Let $g(z)$ be an analytic function that has the series representation in Equation (15) and using the fact in Equation (17), then Corollary 6. can be rewritten as,*

$$\int_0^\infty \frac{x^{ip-1}\left(e^{\frac{\pi p}{2}}g(-i\theta x)-e^{\frac{-\pi p}{2}}g(i\theta x)\right)}{1+x^2}dx = -\pi\, g(\theta),\tag{35}$$

where $-2 < Im(p)\langle 0$ and $\theta\rangle 0$.

Theorem 2. *Let $f(z)$ be analytic function around 0, then we have the following result,*

$$PV \int_0^\infty \frac{x^{p-1}\left(e^{\frac{-i\pi p}{2}} f\left(\beta e^{i\theta x}\right) - e^{\frac{i\pi p}{2}} f\left(\beta e^{-i\theta x}\right)\right)}{i(x^2-1)} dx = \frac{\pi}{2}\left(e^{\frac{i\pi p}{2}} f\left(\beta\, e^{-i\theta}\right) + e^{\frac{-i\pi p}{2}} f\left(\beta e^{i\,\theta}\right)\right), \tag{36}$$

where $\theta > 0$, and $0 < Re(p) < 2$.

Proof. Since f is analytic around 0, thus, using the facts in Equations (12) and (14), we acquire,

$$\frac{1}{i}\left(e^{\frac{-i\pi p}{2}} f\left(\beta e^{i\theta x}\right) - e^{\frac{i\pi p}{2}} f\left(\beta e^{-i\theta x}\right)\right) = 2\left(\sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \sin\left(k\theta x - \frac{\pi p}{2}\right) - f(0)\sin\left(\frac{\pi p}{2}\right)\right) \tag{37}$$

Substituting Equation (37) in Equation (36) to acquire,

$$I = PV \int_0^\infty \frac{x^{p-1}\left(2\left(\sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \sin\left(k\theta x - \frac{\pi p}{2}\right) - f(0)\sin\left(\frac{\pi p}{2}\right)\right)\right)}{x^2-1} dx.$$

$$= -2f(0)\sin\left(\frac{\pi p}{2}\right) \int_0^\infty \frac{x^{p-1}}{x^2-1} dx + 2\sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \left(PV \int_0^\infty \frac{x^{p-1}\,\sin\left(k\theta x - \frac{\pi p}{2}\right)}{x^2-1} dx\right).$$

Using the result in Equation (9) to evaluate $PV \int_0^\infty \frac{x^{p-1}}{x^2-1} dx$, thus we obtain,

$$I = \pi f(0)\cos\left(\frac{\pi p}{2}\right) + 2\sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \left(PV \int_0^\infty \frac{x^{p-1}\,\sin\left(k\theta x - \frac{\pi p}{2}\right)}{x^2 - 1} dx\right). \tag{38}$$

The integral $PV \int_0^\infty \frac{x^{p-1}\,\sin\left(k\theta x - \frac{\pi p}{2}\right)}{x^2-1} dx$, can be solved using the Laplace transform as follows.

Define,

$$j(t) = PV \int_0^\infty \frac{x^{p-1}\left(\cos\left(\frac{\pi p}{2}\right)\sin(\theta t k x) - \sin\left(\frac{\pi p}{2}\right)\cos(\theta t k x)\right)}{x^2-1} dx, \tag{39}$$

where $t, \theta > 0$, $0 < Re(p) < 2$, *and* $k \in \mathbb{N}, k \neq 0$, thus our desired integral is $j(1)$.

Therefore, by taking Laplace transform for both sides of Equation (39), we acquire,

$$J(s) = PV \int_0^\infty \int_0^\infty \frac{x^{p-1}\,e^{-st}\left[\cos\left(\frac{\pi p}{2}\right)\sin(\theta t k x) - \sin\left(\frac{\pi p}{2}\right)\cos(\theta t k x)\right]}{x^2-1} dx\, dt.$$

Again, using Fubini's theorem, we interchange the order of the integration to acquire,

$$J(s) = PV \int_0^\infty \frac{x^{p-1}}{x^2-1}\left(\int_0^\infty e^{-st}\left[\cos\left(\frac{\pi p}{2}\right)\sin(\theta t k x) - \sin\left(\frac{\pi p}{2}\right)\cos(\theta t k x)\right] dt\right) dx.$$

$$= PV \int_0^\infty \frac{x^{p-1}}{x^2-1}\mathcal{L}\left\{\cos\left(\frac{\pi p}{2}\right)\sin(\theta t k x) - \sin\left(\frac{\pi p}{2}\right)\cos(\theta t k x)\right\} dx. \tag{40}$$

Running the laplace transform on Equation (40), to acquire,

$$J(s) = PV \left(\int_0^\infty \frac{x^p\,\theta k \cos\left(\frac{p\pi}{2}\right)}{(s^2 + k^2 x^2 \theta^2)(x^2-1)} dx - \int_0^\infty \frac{s\,x^{p-1}\sin\left(\frac{p\pi}{2}\right)}{(x^2-1)(s^2 + k^2 x^2 \theta^2)} dx\right). \tag{41}$$

Now, using the partial fraction decomposition,

$$\frac{1}{(x^2-1)(s^2+k^2x^2\theta^2)} = \frac{1}{(s^2+k^2\theta^2)(x^2-1)} - \frac{k^2\theta^2}{(s^2+k^2\theta^2)(s^2+k^2x^2\theta^2)}. \tag{42}$$

Substituting Equation (42) in Equation (41), we have,

$$J(s) = \frac{\theta k \cos\left(\frac{\pi p}{2}\right)}{(s^2+k^2\theta^2)} PV \int_0^\infty \left(\frac{x^p}{(x^2-1)} - \frac{k^2\theta^2 x^p}{(s^2+k^2x^2\theta^2)} \right) dx - \frac{s \sin\left(\frac{\pi p}{2}\right)}{(s^2+k^2\theta^2)} PV \int_0^\infty \left(\frac{x^{p-1}}{(x^2-1)} - \frac{x^{p-1}}{(s^2+k^2x^2\theta^2)} \right) dx.$$

Using the facts in Equations (8) and (9), one can obtain:

$$J(s) = \frac{\theta k \cos\left(\frac{\pi p}{2}\right)}{(s^2+k^2\theta^2)} \left(\frac{\pi}{2} \tan\left(\frac{\pi p}{2}\right) - \frac{\pi \left(\frac{s}{\theta k}\right)^{p-1}}{2\cos\left(\frac{\pi p}{2}\right)} \right) - \frac{s \sin\left(\frac{\pi p}{2}\right)}{(s^2+k^2\theta^2)} \left(-\frac{\pi}{2} \cot\left(\frac{\pi p}{2}\right) - \frac{\pi \left(\frac{s}{\theta k}\right)^{p-2}}{2\sin\left(\frac{\pi p}{2}\right)} \right). \tag{43}$$

After simple computations, Equation (43) becomes,

$$J(s) = \frac{\pi \left(s \cos\left(\frac{p\pi}{2}\right) + k\theta \sin\left(\frac{p\pi}{2}\right) \right)}{2(s^2+k^2\theta^2)}. \tag{44}$$

Taking the inverse Laplace transform for both sides of Equation (44), we acquire,

$$j(t) = \frac{\pi}{2} \cos\left(kt\theta - \frac{p\pi}{2} \right).$$

Thus, $j(1) = \frac{\pi}{2} \cos\left(k\theta - \frac{\pi p}{2} \right)$, which is the desired integral.
Substituting the value of j(1) in Equation (38), we have,

$$I = \pi f(0) \cos\left(\frac{\pi p}{2}\right) + \pi \sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \cos\left(k\theta - \frac{\pi p}{2} \right)$$
$$I = \pi \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \cos\left(k\theta - \frac{\pi p}{2} \right) \tag{45}$$

By rewriting $\cos\left(k\theta - \frac{\pi p}{2} \right)$ in the exponential form, Equation (45) becomes,

$$I = \frac{\pi}{2} \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} \beta^k \left(e^{\frac{i\pi p}{2}} e^{-i\theta k} + \frac{1}{2} e^{-\frac{i\pi p}{2}} e^{i\theta k} \right)$$

and using the fact in Equation (12), we acquire,

$$I = \frac{\pi}{2} \left(e^{\frac{i\pi p}{2}} f\left(\beta e^{-i\theta} \right) + e^{\frac{-i\pi p}{2}} f\left(\beta e^{i\theta} \right) \right)$$

This completes the proof of Theorem 2. □

Now, we introduce some corollaries that comes directly after simple assumptions and computations in the same manner that used in the previous results.

Corollary 8. *Let $g(z)$ be analytic function that has the series representation in Equation (15) and using the fact in Equation (17) then, Theorem 2 can be rewritten as,*

$$PV \int_0^\infty \frac{x^{p-1} \left(e^{\frac{-i\pi p}{2}} g(-i\theta x) - e^{\frac{i\pi p}{2}} g(i\theta x) \right)}{i(x^2-1)} dx = \frac{\pi}{2} \left(e^{\frac{i\pi p}{2}} g(i\theta) + e^{\frac{-i\pi p}{2}} g(-i\theta) \right). \tag{46}$$

Corollary 9. *Let $f(z)$ be an analytic function around 0, then we have the following result,*

$$PV \int_0^\infty \frac{x^p \left(e^{\frac{-i\pi p}{2}} f(\beta e^{i\theta x}) + e^{\frac{i\pi p}{2}} f(\beta e^{-i\theta x}) \right)}{(x^2 - 1)} dx = \frac{-\pi}{2i} \left(e^{\frac{i\pi p}{2}} f\left(\beta e^{-i\theta} \right) - e^{\frac{-i\pi p}{2}} f\left(\beta e^{i\theta} \right) \right), \tag{47}$$

where $\theta > 0, -1 < Re(p) < 1$.

Remark 2. *Taking the simple case $p = 0$ and $\beta = 1$ in Corollary 9, we have,*

$$PV \int_0^\infty \frac{f(e^{i\theta x}) + f(e^{-i\theta x})}{(x^2 - 1)} dx = \frac{\pi}{2i} \left(f\left(e^{i\theta} \right) - f\left(e^{-i\theta} \right) \right), \tag{48}$$

where $\theta > 0$.
Which is identical to Cauchy's theorem obtained in [26] (P.62 Theorem 9).

Corollary 10. *Let $g(z)$ be an analytic function that has the series representation in Equation (15), and using the fact in Equation (16), then Corollary 9 can be rewritten as,*

$$PV \int_0^\infty \frac{x^p \left(e^{\frac{-i\pi p}{2}} g(-i\theta x) + e^{\frac{i\pi p}{2}} g(i\theta x) \right)}{(x^2 - 1)} dx = \frac{-\pi}{2i} \left(e^{\frac{i\pi p}{2}} g(i\theta) - e^{\frac{-i\pi p}{2}} g(-i\theta) \right). \tag{49}$$

Corollary 11. *Let $f(z)$ be an analytic function around 0, then we have the following result,*

$$PV \int_0^\infty \frac{x^{ip} \left(e^{\frac{\pi p}{2}} f(\beta e^{i\theta x}) + e^{\frac{-\pi p}{2}} f(\beta e^{-i\theta x}) \right)}{(x^2 - 1)} dx = \frac{-\pi}{2i} \left(e^{\frac{-\pi p}{2}} f\left(\beta e^{-i\theta} \right) - e^{\frac{\pi p}{2}} f\left(\beta e^{i\theta} \right) \right), \tag{50}$$

where $\theta > 0$ and $p \in \mathbb{R}$.

Corollary 12. *Let $g(z)$ be an analytic function that has the series representation in Equation (15) and using the fact in Equation (16), then Corollary 11 can be rewritten as,*

$$PV \int_0^\infty \frac{x^p \left(e^{\frac{\pi p}{2}} g(-i\theta x) + e^{\frac{-\pi p}{2}} g(i\theta x) \right)}{(x^2 - 1)} dx = \frac{-\pi}{2i} \left(e^{\frac{-\pi p}{2}} g(i\theta) - e^{\frac{\pi p}{2}} g(-i\theta) \right). \tag{51}$$

Corollary 13. *Let $f(z)$ be an analytic function around 0, then we have the following result,*

$$PV \int_0^\infty \frac{x^{ip-1} \left(e^{\frac{\pi p}{2}} f(\beta e^{i\theta x}) - e^{\frac{-\pi p}{2}} f(\beta e^{-i\theta x}) \right)}{x^2 - 1} dx = \frac{\pi}{2} \left(e^{\frac{-\pi p}{2}} f\left(\beta e^{-i\theta} \right) + e^{\frac{\pi p}{2}} f\left(\beta e^{i\theta} \right) \right), \tag{52}$$

where $-2 < Im(p) < 0$, and $\theta > 0$.

Corollary 14. *Let $g(z)$ be an analytic function that has the series representation in Equation (15) and using the fact in Equation (17), then Corollary 13 can be written as,*

$$PV \int_0^\infty \frac{x^{ip-1} \left(e^{\frac{\pi p}{2}} g(-i\theta x) - e^{\frac{-\pi p}{2}} g(i\theta x) \right)}{x^2 - 1} dx = \frac{\pi}{2} \left(e^{\frac{-\pi p}{2}} g(i\theta) + e^{\frac{\pi p}{2}} g(-i\theta) \right). \tag{53}$$

Theorem 3. *Let $f(z)$ be an analytic function around 0, then we have the following result,*

$$I = \int\limits_{-\infty}^{\infty} \frac{f\left(\beta e^{i\theta x}\right) + f\left(\beta e^{-i\theta x}\right)}{x^2 - 2x\cos(\alpha) + 1} dx = \frac{\pi}{\sin(\alpha)}\left(f\left(\beta e^{-\theta \sin(\alpha) - i\theta \cos(\alpha)}\right) + f\left(\beta e^{-\theta \sin(\alpha) + i\cos(\alpha)\theta}\right)\right), \tag{54}$$

where $0 < |\alpha| < \pi$ and $\theta > 0$.

Proof. Since f is analytic around 0, thus Substituting Equation (13) in Equation (54), we acquire,

$$I = 2\int\limits_{-\infty}^{\infty} \frac{\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \beta^k \cos(k\theta x)}{x^2 - 2x\cos(\alpha) + 1} dx. = 2\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\beta^k \int\limits_{-\infty}^{\infty} \frac{\cos(k\theta x)}{x^2 - 2x\cos(\alpha) + 1} dx. \tag{55}$$

The integral $\int_{-\infty}^{\infty} \frac{\cos(k\theta x)}{x^2 - 2x\cos(\alpha) + 1} dx$ can be solved as follows, let,

$$J = \int\limits_{-\infty}^{\infty} \frac{\cos(k\theta x)}{x^2 - 2x\cos(\alpha) + 1} dx, \tag{56}$$

and $x = \cos(\alpha) + y\sin(\alpha) \rightarrow dx = \sin(\alpha)\, dy$.

$$(x - \cos(\alpha))^2 = x^2 - 2x\cos(\alpha) + \cos^2(\alpha) = y^2 \sin^2(\alpha).$$

Thus,

$$x^2 - 2\cos(\alpha)x + 1 = \sin^2(\alpha)\left(1 + y^2\right)$$

Therefore, Equation (56) becomes,

$$J = \int\limits_{-\infty}^{\infty} \frac{\cos(k\theta(\cos(\alpha) + y\sin(\alpha)))}{\sin(\alpha)(1 + y^2)} dy$$

$$= \frac{1}{\sin(\alpha)} \int\limits_{-\infty}^{\infty} \frac{\cos(k\theta\cos(\alpha))\cos(k\theta\, y\sin(\alpha)) - \sin(k\theta\cos(\alpha))\sin(k\theta\, y\sin(\alpha))}{(1 + y^2)} dy \tag{57}$$

$$= \frac{1}{\sin(\alpha)}\left(\cos(k\theta\cos(\alpha)) \int\limits_{-\infty}^{\infty} \frac{\cos(k\theta\, y\sin(\alpha))}{(1 + y^2)} dy - \sin(k\theta\cos(\alpha)) \int\limits_{-\infty}^{\infty} \frac{\sin(k\theta\, y\sin(\alpha))}{(1 + y^2)} dy\right).$$

Since the function $\frac{\sin(k\theta\, y\sin(\alpha))}{(1 + y^2)}$ is odd, then,

$$\int\limits_{-\infty}^{\infty} \frac{\sin(k\theta\, y\sin(\alpha))}{(1 + y^2)} dy = 0. \tag{58}$$

Using the fact that can be obtained in [2],

$$\int\limits_{-\infty}^{\infty} \frac{\cos(\theta x)}{1 + x^2} dx = \pi e^{-\theta}, \tag{59}$$

and substituting Equations (58) and (59) in Equation (57), to acquire,

$$J = \frac{\pi e^{-\theta k \sin(\alpha)}}{\sin(\alpha)} \cos(k\theta\cos(\alpha)). \tag{60}$$

Substituting Equation (60) in Equation (55), we acquire,

$$I = 2 \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \beta^k \frac{\pi e^{-\theta k \sin(\alpha)}}{sin(\alpha)} \cos(k\theta \cos(\alpha)).$$

Rewriting $\cos(k\theta \cos(\alpha))$ in the exponential form and after simple computations, we have,

$$I = \frac{\pi}{sin(\alpha)} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \beta^k \left(e^{-\theta k \sin(\alpha) - i\theta \cos(\alpha)k} + e^{-\theta k \sin(\alpha) + i \cos(\alpha)\theta k} \right).$$

Using the fact in Equation (12), we acquire,

$$I = \frac{\pi}{sin(\alpha)} \left(f\left(\beta e^{-\theta \sin(\alpha) - i\theta \cos(\alpha)} \right) + f\left(\beta e^{-\theta \sin(\alpha) + i \cos(\alpha)\theta} \right) \right).$$

This completes the proof of Theorem 3. □

Remark 3. *Taking the simple case $\beta = 1$, $\alpha = \frac{\pi}{2}$ in Theorem 3, we obtain Equation (28) which is similar to Cauchy's theorem obtained in [26] (P.62 Theorem 8) and in the classical table of integrals [27] (3.037 Theorem 1).*

Corollary 15. *Let $g(z)$ be an analytic function that has the series representation in Equation (15), then we have,*

$$\int_0^{\infty} \frac{(g(-i\theta x) + g(i\theta x))}{x^2 - 2x \cos(\alpha) + 1} dx = = \frac{\pi}{sin(\alpha)}(g(\theta \sin(\alpha) + i\theta \cos(\alpha)) + g(\theta \sin(\alpha) - i \cos(\alpha)\theta)), \tag{61}$$

where $0 < |\alpha| < \pi$ and $\theta > 0$.

Corollary 16. *Let $h(z)$ be analytic function around 0, then we have the following result,*

$$\int_0^{\infty} \frac{x^p \left(e^{\frac{-i\pi p}{2}} h(\beta e^{i\theta x}) + e^{\frac{i\pi p}{2}} h(\beta e^{-i\theta x}) \right)}{x^2 - 2x \cos(\alpha) + 1} dx = \pi g(\theta), \tag{62}$$

where $-1 < Re(p) < 1, 0 < |\alpha| < \pi$, and $\theta > 0$, where $h(z)$ is analytic function around zero.

Proof. The proof comes directly form Theorem 3 by letting,

$$f\left(\beta e^{\theta x} \right) = (-x)^p h\left(\beta e^{\theta x} \right),$$

Thus, we have,

$$f\left(\beta e^{i\theta x} \right) + f\left(\beta e^{-i\gamma x} \right) = (-ix)^p h(\beta e^{i\theta x}) + (ix)^p h(\beta e^{-i\theta x}) = x^p \left(e^{\frac{-i\pi p}{2}} h\left(\beta e^{i\theta x} \right) + e^{\frac{i\pi p}{2}} h\left(\beta e^{-i\theta x} \right) \right). \tag{63}$$

Now, substituting Equation (62) in Theorem 3, we acquire,

$$\int_0^{\infty} \frac{x^p \left(e^{\frac{-i\pi p}{2}} h(\beta e^{i\theta x}) + e^{\frac{i\pi p}{2}} h(\beta e^{-i\theta x}) \right)}{x^2 - 2x \cos(\alpha) + 1} dx$$
$$= \frac{\pi}{sin(\alpha)} \left((\sin(\alpha) + i \cos(\alpha))^p h\left(\beta e^{\theta(-\sin(\alpha) - i \cos(\alpha))} \right) + (\sin(\alpha) - i \cos(\alpha))^p h\left(\beta e^{\theta(-\sin(\alpha) + i \cos(\alpha))} \right) \right).$$

□

4. Applications and Examples

In this section, we introduce some examples on some complicated integrals, that can't be solved easily by familiar methods or may take huge efforts to be solved. Herein, we show that using the obtained results in this article, the solution can be determined directly in a simple way.

Example 1. *Solve the following integral,*

$$PV \int\limits_{0}^{\infty} \frac{\left(e^{b \tan^{-1}(\theta x)} - e^{-b \tan^{-1}(\theta x)}\right) \cos\left(\frac{b}{2}\ln\left(1 + \theta^2 x^2\right)\right)}{x^2 - 1} dx,$$

Where $\theta > 0$, and $b \in \mathbb{R}$.

Solution. *Let $g(z) = \cos(b\ln(1 + z))$, then we have,*

$$g(-i\theta x) + g(i\theta x) = \cos(b\ln(1 - i\theta x)) + \cos(b\ln(1 + i\theta x))$$
$$= \cos\left(\frac{b}{2}\ln\left(1 + \theta^2 x^2\right) + ib \tan^{-1}(\theta x)\right) + \cos\left(\frac{b}{2}\ln\left(1 + \theta^2 x^2\right) - ib \tan^{-1}(\theta x)\right)$$
$$= 2 \cosh\left(b \tan^{-1}(\theta x)\right) \cos\left(\frac{b}{2}\ln\left(1 + \theta^2 x^2\right)\right)$$

Using Corollary (10) and letting $p = 0$, we acquire,

$$PV \int\limits_{0}^{\infty} \frac{(g(-i\theta x) + g(i\theta x))}{(x^2 - 1)} dx = \frac{-\pi}{2i}(g(i\theta) - g(-i\theta)).$$
$$PV \int\limits_{0}^{\infty} \frac{\left(2 \cosh\left(b \tan^{-1}(\theta x)\right) \cos\left(\frac{b}{2}\ln\left(1 + \theta^2 x^2\right)\right)\right)}{1 - x^2} dx$$
$$= \pi\left(\sin\left(\frac{1}{2}b\ln\left(\theta^2 + 1\right)\right) \sinh\left(b\tan^{-1}(\theta x)\right)\right).$$

Example 2. *Solve the following integral,*

$$\int\limits_{0}^{\infty} \frac{x^p e^{\cos(\theta x)} \cos\left(\frac{\pi p}{2} - \sin\theta x\right)}{x^2 + 1} dx,$$

where $-1 < p < 1$.

Solution. *Let $f(z) = e^z$ then, we have,*

$$e^{\frac{-i\pi p}{2}} f\left(e^{i\theta x}\right) + e^{\frac{i\pi p}{2}} f\left(e^{-i\theta x}\right) = 2e^{\cos(\theta x)} \cos\left(\frac{\pi p}{2} - \sin\theta x\right).$$

Using Theorem 1 and setting $\beta = 1$, we have,

$$\int\limits_{0}^{\infty} \frac{x^p e^{\cos(\theta x)} \cos\left(\frac{\pi p}{2} - \sin\theta x\right)}{x^2 + 1} dx = \frac{1}{2}\pi f\left(\beta e^{-\theta}\right) = \frac{\pi}{2} e^{e^{-\theta}}$$

Example 3. *Solve the following integral,*

$$\int\limits_{0}^{\infty} \frac{x^p \left(\alpha \cos\left(\frac{\pi p}{2}\right) - \cos\left(\frac{\pi p}{2} + x\right)\right)}{(1 + x^2)(1 - 2\alpha \cos(x) + \alpha^2)} dx,$$

where $-1 < Re(p) < 1$ and $\alpha < 1$.

Solution. *Using Theorem 1 and letting $\beta = 1$, $\theta = 1$ and $f(z) = \frac{1}{\alpha - z}$, we have,*

$$e^{\frac{-i\pi p}{2}} f\left(e^{i\theta x}\right) + e^{\frac{i\pi p}{2}} f\left(e^{-i\theta x}\right) = \frac{2\left(\alpha \cos\left(\frac{p\pi}{2}\right) - \cos\left(\frac{p\pi}{2} + x\theta\right)\right)}{(1 - 2\alpha \, \cos(x) + \alpha^2)}.$$

$$\therefore \int_0^\infty \frac{x^p\left(\alpha \cos\left(\frac{\pi p}{2}\right) - \cos\left(\frac{\pi p}{2} + x\right)\right)}{(1 + x^2)(1 - 2\alpha \, \cos(x) + \alpha^2)} dx = \frac{\pi}{2}\left(\frac{1}{\alpha - e^{-1}}\right).$$

5. Conclusions

In this research, we introduced three interesting theorems in complex analysis. The main purpose of these results is to acquire the solutions of some families of improper integrals that take much effort and time to solve, and sometimes they can't be solved directly even with computer software. The proof of these theorems was obtained in the article using idea of Laplace transform and some techniques depends on some theorems such as Fubini's theorem. Some corollaries and remarks were discussed to prove the applicability of these theorems and comparisons with previous results are also illustrated. Three applications are presented, to show the reliability and the simplicity of the introduced theorems. These results can be generalized to solve wide families of improper integrals on different domains. Finally, we mention that the restrictions on the domain and the parameters of these results can mitigated, so that we can acquire better results in the future. In the following, we introduce some mathematical remarks about the proposed theorems.

The theorems can be extended for analytic functions around α, where $\alpha \in \mathbb{R}$, and we are working on such results.

If $\theta < 0$, then Theorems 1, Theorems 3 and the corollaries after, can be generalized by switching θ to $|\theta|$.

If $\theta < 0$, then Theorems 2 and the corollaries after, can be generalized by switching θ to $-\theta$.

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