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Several Double Inequalities for Integer Powers of the Sinc and Sinhc Functions with Applications to the Neuman–Sándor Mean and the First Seiffert Mean

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Abstract: In the paper, the authors establish a general inequality for the hyperbolic functions, extend the newly-established inequality to trigonometric functions, obtain some new inequalities involving the inverse sine and inverse hyperbolic sine functions, and apply these inequalities to the Neuman–Sándor mean and the first Seiffert mean.

Keywords: Neuman–Sándor mean; Seiffert mean; inequality; sinc function; sinhc function; inverse hyperbolic function; trigonometric function; necessary and sufficient condition

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1. Introduction

For $s,t>0$ with $s\ne t$, the Neuman–Sándor mean $M(s,t)$, the first Seiffert mean $P(s,t)$, and the second Seiffert mean $T(s,t)$ are, respectively, defined in [1–3] by

$$M(s,t)=\frac{s-t}{2\operatorname{arcsinh}\frac{s-t}{s+t}},\quad P(s,t)=\frac{s-t}{4\arctan\sqrt{\frac{s}{t}}-\pi},\quad T(s,t)=\frac{s-t}{2\arctan\frac{s-t}{s+t}},$$

where $\operatorname{arcsinh}x=\ln\left(x+\sqrt{x^2+1}\,\right)$ denotes the inverse hyperbolic sine function. The first Seiffert mean $P(s,t)$ can be rewritten ([1], Equation (2.4)) as

$$P(s,t)=\frac{s-t}{2\arcsin\frac{s-t}{s+t}}.$$

Recently, these bivariate mean values have been the subject of intensive research. In particular, many remarkable inequalities and properties for the means $M(s,t)$, $P(s,t)$, and $T(s,t)$ can be found in the literature [4–20].

Let $A(s,t)=\frac{s+t}{2}$, $H(s,t)=\frac{2st}{s+t}$, and $C(s,t)=\frac{s^2+t^2}{s+t}$ be the arithmetic, harmonic, and contra-harmonic mean of two positive numbers s and t. The inequalities

$$H(s,t)<P(s,t)<A(s,t)<T(s,t)<C(s,t) \tag{1}$$

hold for all $s,t>0$ with $s\ne t$.

In [1,21], it was established that

$$P(s,t)<M(s,t)<T^2(s,t),\quad A(s,t)<M(s,t)<T(s,t), \tag{2}$$

$$A(s,t)T(s,t)<M^2(s,t)<\frac{A^2(s,t)+T^2(s,t)}{2}$$

for $s, t > 0$ with $s \neq t$.

For $z \in \mathbb{C}$, the functions

$$\operatorname{sinc} z = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases} \quad \text{and} \quad \operatorname{sinhc} z = \begin{cases} \frac{\sinh z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

are called the sinc function and hyperbolic sinc function, respectively. The function $\operatorname{sinc} z$ is also called the sine cardinal or sampling function, and the function $\operatorname{sinhc} z$ is also called the hyperbolic sine cardinal; see [22]. The sinc function $\operatorname{sinc} z$ arises frequently in signal processing, the theory of Fourier transforms, and other areas in mathematics, physics, and engineering. It is easy to see that these two functions $\operatorname{sinc} z$ and $\operatorname{sinhc} z$ are analytic on $\mathbb{C}$, that is, they are entire functions.

In [23], the authors obtained double inequalities of the Neuman–Sándor meansin terms of the arithmetic and contra-harmonic means, and they deduced that the inequalities

$$\begin{gathered} 1 - \beta_1 \left(1 - \frac{1}{\cosh^2 \theta}\right) < \frac{1}{\operatorname{sinhc} \theta} < 1 - \alpha_1 \left(1 - \frac{1}{\cosh^2 \theta}\right), \\ 1 - \beta_2 \left(1 - \frac{1}{\cosh^4 \theta}\right) < \frac{1}{\operatorname{sinhc}^2 \theta} < 1 - \alpha_2 \left(1 - \frac{1}{\cosh^4 \theta}\right), \\ 1 + \alpha_3 \left(\cosh^4 \theta - 1\right) < \operatorname{sinhc}^2 \theta < 1 + \beta_3 \left(\cosh^4 \theta - 1\right) \end{gathered} \tag{3}$$

hold for $\theta \in (0, \ln(1 + \sqrt{2}))$ if and only if

$$\begin{gathered} \alpha_1 \leq \frac{1}{6} \quad \text{and} \quad \beta_1 \geq 2\left[1 - \ln\left(1 + \sqrt{2}\right)\right] = 0.237253\ldots, \\ \alpha_2 \leq \frac{1}{6} \quad \text{and} \quad \beta_2 \geq \frac{4}{3}\left[1 - \ln^2\left(1 + \sqrt{2}\right)\right] = 0.297574\ldots, \\ \alpha_3 \leq \frac{1 - \ln^2\left(1 + \sqrt{2}\right)}{3 \ln^2\left(1 + \sqrt{2}\right)} = 0.095767\ldots \quad \text{and} \quad \beta_3 \geq \frac{1}{6} \end{gathered}$$

respectively.

In this paper, motivated by those double inequalities in (3), we will obtain necessary and sufficient conditions on α and β such that double inequalities

$$1 - \alpha + \alpha \cosh^{2r} x < \operatorname{sinhc}^r x < 1 - \beta + \beta \cosh^{2r} x \tag{4}$$

and

$$1 - \alpha + \alpha \cos^{2r} x < \operatorname{sinc}^r x < 1 - \beta + \beta \cos^{2r} x \tag{5}$$

are valid on $(-\infty, \infty)$ for some ranges of $r \in \mathbb{R}$. Hereafter, substituting the double inequalities (4) and (5) into the Neuman–Sándor mean $M(s, t)$ and the first Seiffert means $P(s, t)$, we will derive generalizations of some inequalities for the Neuman–Sándor mean $M(s, t)$ and the first Seiffert means $P(s, t)$.

2. Lemmas

To achieve our main purposes, we need the following lemmas.

Lemma 1 ([24], Theorem 1.25). *For $-\infty < s < t < \infty$, let f, g be continuous on $[s, t]$, differentiable on (s, t), and $g'(x) \neq 0$ on (s, t). If the ratio $\frac{f'(x)}{g'(x)}$ is increasing on (s, t), so are the functions $\frac{f(x) - f(s)}{g(x) - g(s)}$ and $\frac{f(x) - f(t)}{g(x) - g(t)}$.*

Lemma 2 ([25], Lemma 1.1)**.** *Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *have the radius* $r > 0$ *of convergence and* $b_n > 0$ *for all* $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$*. Let* $h(x) = \frac{f(x)}{g(x)}$*. Then the following statements are true.*

1. *If the sequence* $\{\frac{a_n}{b_n}\}_{n=0}^{\infty}$ *is increasing, so is the function* $h(x)$ *on* $(0, r)$*.*
2. *If the sequence* $\{\frac{a_n}{b_n}\}$ *is increasing for* $0 < n \le n_0$ *and decreasing for* $n > n_0$*, then there exists* $x_0 \in (0, r)$ *such that* $h(x)$ *is increasing on* $(0, x_0)$ *and decreasing on* (x_0, r)*.*

The classical Bernoulli numbers B_n for $n \ge 0$ are generated in ([26], p. 3) by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi.$$

In the recent papers [27–29], some novel results for the even-indexed Bernoulli numbers B_{2n} were discovered.

Lemma 3 ([30])**.** *Let* B_{2n} *be the even-indexed Bernoulli numbers. Then*

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n}, \quad 0 < |x| < \pi. \tag{6}$$

Lemma 4 ([30–32])**.** *Let* B_{2n} *be the even-indexed Bernoulli numbers. Then*

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}$$

and

$$\frac{1}{\sin^2 x} = \csc^2 x = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n-2} \tag{7}$$

for $0 < |x| < \pi$*.*

Lemma 5. *The function*

$$h_1(x) = \frac{2\sinh^2 x \cosh x - x \sinh x - x^2 \cosh^3 x}{(x - \sinh x \cosh x - x \sinh^2 x)(x \cosh x - \sinh x)}$$

is increasing on $(0, \infty)$ *and has the limits*

$$\lim_{x \to 0^+} h_1(x) = \frac{17}{25} \quad \text{and} \quad \lim_{x \to \infty} h_1(x) = 1. \tag{8}$$

Proof. Let

$$A(x) = 2\sinh^2 x \cosh x - x \sinh x - x^2 \cosh^3 x$$

and

$$B(x) = (x - \sinh x \cosh x - x \sinh^2 x)(x \cosh x - \sinh x).$$

Straightforward computation gives

$$\begin{aligned} A(x) &= 2\cosh^3 x - 2\cosh x - x \sinh x - x^2 \cosh^3 x \\ &= \frac{\cosh 3x}{2} - \frac{\cosh x}{2} - \frac{x^2 \cosh 3x}{4} - \frac{3x^2 \cosh x}{4} - x \sinh x \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(3x)^{2n}}{(2n)!} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \frac{x^2}{4} \sum_{n=0}^{\infty} \frac{(3x)^{2n}}{(2n)!} - \frac{3x^2}{4} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - x \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(3x)^{2n+2}}{(2n+2)!} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+2)!} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n}x^{2n+2}}{(2n)!} \\
&\quad - \frac{3}{4}\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n)!} - \sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+1)!} \\
&= \frac{1}{2}\sum_{n=2}^{\infty}\frac{3^{2n}(-2n^2-3n+8)-6n^2-13n-8}{(2n+2)!}x^{2n+2}
\end{aligned}
$$

and

$$
\begin{aligned}
B(x) &= x^2\cosh x - 2x\sinh x - x^2\sinh^2 x\cosh x + \sinh^2 x\cosh x \\
&= x^2\cosh x - 2x\sinh x - \frac{x^2\cosh 3x}{4} + \frac{x^2\cosh x}{4} + \frac{\cosh 3x}{4} - \frac{\cosh x}{4} \\
&= \frac{5}{4}\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n)!} - 2\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+1)!} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n}x^{2n+2}}{(2n)!} + \frac{1}{4}\sum_{n=0}^{\infty}\frac{(3x)^{2n}}{(2n)!} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \\
&= \frac{1}{4}\sum_{n=2}^{\infty}\frac{3^{2n}(-4n^2-6n+7)+20n^2+14n-7}{(2n+2)!}x^{2n+2}.
\end{aligned}
$$

Let

$$a_n = \frac{3^{2n}(-2n^2-3n+8)-6n^2-13n-8}{2(2n+2)!}$$

and

$$b_n = \frac{3^{2n}(-4n^2-6n+7)+20n^2+14n-7}{4(2n+2)!}.$$

Simple computation leads to

$$
\begin{aligned}
a_n = \frac{3^{2n}(-2n^2-3n+8)-6n^2-13n-8}{2(2n+2)!} &\le \frac{3^4(-2n^2-3n+8)-6n^2-13n-8}{2(2n+2)!} \\
&= \frac{-168n^2-256n+640}{2(2n+2)!} \le -\frac{272}{(2n+2)!} < 0
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $n \ge 2$, whereas, for all $n \in \mathbb{N}$ and $n \ge 2$,

$$
\begin{aligned}
b_n = \frac{3^{2n}(-4n^2-6n+7)+20n^2+14n-7}{4(2n+2)!} &\le \frac{3^4(-4n^2-6n+7)+20n^2+14n-7}{4(2n+2)!} \\
&= \frac{-304n^2-472n+560}{4(2n+2)!} \le -\frac{400}{(2n+2)!} < 0.
\end{aligned} \tag{9}
$$

Consequently, we obtain

$$
\begin{aligned}
c_n = \frac{-a_n}{-b_n} &= 2\times\frac{3^{2n}(2n^2+3n-8)+6n^2+13n+8}{3^{2n}(4n^2+6n-7)-20n^2-14n+7} \\
&= \frac{9^n(4n^2+6n-16)+12n^2+26n+16}{9^n(4n^2+6n-7)-20n^2-14n+7} \\
&= 1 + \frac{-9^{n+1}+32n^2+40n+9}{9^n(4n^2+6n-7)-20n^2-14n+7} \\
&\triangleq 1 + k(n)
\end{aligned} \tag{10}
$$

for $n \in \mathbb{N}$ and $n \ge 2$. Let

$$k(x) = \frac{-9^{x+1}+32x^2+40x+9}{9^x(4x^2+6x-7)-20x^2-14x+7}$$

for $x \in [2, \infty)$. Then

$$k'(x) = \frac{\ell(x)}{[9^x(4x^2 + 6x - 7) - 20x^2 - 14x + 7]^2},$$

where

$$\begin{aligned}
\ell(x) = & \left(-9^{x+1}\ln 9 + 64x + 40\right)\left[9^x\left(4x^2 + 6x - 7\right) - 20x^2 - 14x + 7\right] \\
& - \left(-9^{x+1} + 32x^2 + 40x + 9\right)\left[9^x\left(4x^2 + 6x - 7\right)\ln 9 + 9^x(8x + 6) - 40x - 14\right] \\
= & 9^{2x+1}(8x + 6) + 9^x\left[9\left(20x^2 + 14x - 7\right) - \left(4x^2 + 6x - 7\right)\left(32x^2 + 40x + 9\right)\right]\ln 9 \\
& + 9^x\left[(64x + 40)\left(4x^2 + 6x - 7\right) - 9(40x + 14) - (8x + 6)\left(32x^2 + 40x + 9\right)\right] \\
& - (64x + 40)\left(20x^2 + 14x - 7\right) + (40x + 14)\left(32x^2 + 40x + 9\right) \\
= & 9^{2x+1}(8x + 6) + 9^x\left(352x + 128x^2 - 352x^3 - 128x^4\right)\ln 9 \\
& + 9^x \times 4\left(-115 - 220x + 8x^2\right) + 406 + 808x + 352x^2 \\
= & 2 \times 9^x\left[9^{x+1}(3 + 4x) + \left(176x + 64x^2 - 176x^3 - 64x^4\right)\ln 9 - 230 - 440x + 16x^2\right] \\
& + 406 + 808x + 352x^2.
\end{aligned}$$

Let

$$m(x) = 9^{x+1}(3 + 4x) + \left(176x + 64x^2 - 176x^3 - 64x^4\right)\ln 9 - 230 - 440x + 16x^2.$$

Then

$$\begin{aligned}
m'(x) &= 9^{x+1}\ln 9(3 + 4x) + 4 \times 9^{x+1} + \left(176 + 128x - 528x^2 - 256x^3\right)\ln 9 - 440 + 32x, \\
m'(2) &= 4219\ln 9 + 2412 \\
&> 0, \\
m''(x) &= \ln^2 9 \times 9^{x+1}(3 + 4x) + 8\ln 9 \times 9^{x+1} + \left(128 - 1056x - 768x^2\right)\ln 9 + 32, \\
m''(2) &= 8019\ln^2 9 + 776\ln 9 + 32 \\
&> 0, \\
m^{(3)}(x) &= \ln^3 9 \times 9^{x+1}(3 + 4x) + 12\ln^2 9 \times 9^{x+1} + (-1056 - 1536x)\ln 9, \\
m^{(3)}(2) &= 8019\ln^3 9 + 8748\ln^2 9 - 2112\ln 9 \\
&> 0, \\
m^{(4)}(x) &= \ln^4 9 \times 9^{x+1}(3 + 4x) + 16\ln^3 9 \times 9^{x+1} - 1536\ln 9 \\
&> \ln^4 9 \times 9^{x+1}(3 + 4x) + 11664\ln^3 9 - 1536\ln 9 \\
&> 0
\end{aligned}$$

on $[2, \infty)$. Therefore, the function $m(x)$ is increasing on $[2, \infty)$ and

$$m(2) = 6973 - 1824\ln 9 > 1501 > 0.$$

Hence, it follows that $\ell(x) > 0$ and the function $k(x)$ is increasing on $[2, \infty)$.

According to (10), we can observe that c_n is increasing for $n \in \mathbb{N}$ and $n \geq 2$. Thus, based on Lemma 2, the function $h_1(x) = \frac{A(x)}{B(x)}$ is increasing on $(0, \infty)$.

The limits in (8) are straightforward. The proof of Lemma 5 is complete. □

3. Necessary and Sufficient Conditions

Now we are in a position to state and prove our main results.

Theorem 1. *Let $x, r \in \mathbb{R}$.*

1. *When $r \geq \frac{8}{25}$, the double inequality (4) holds if and only if $\alpha \leq 0$ and $\beta \geq \frac{1}{6}$.*
2. *When $r < 0$, the right-hand side of the inequality (4) holds if and only if $\beta \leq \frac{1}{6}$.*

Proof. Let

$$F(x) = \frac{\operatorname{sinhc}^r x - 1}{\cosh^{2r} x - 1} \triangleq \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \operatorname{sinhc}^r x - 1$ and $f_2(x) = \cosh^{2r} x - 1$. Then

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sinh^{r-2} x(x\cosh x - \sinh x)}{2x^{r+1}\cosh^{2r-1} x}$$

and

$$\begin{aligned}
\left[\frac{f_1'(x)}{f_2'(x)}\right]' &= \frac{r-1}{2}\left(\frac{\sinh x}{x\cosh^2 x}\right)^{r-2} \frac{x - \sinh x\cosh x - x\sinh^2 x}{x^2\cosh^3 x}\frac{x\cosh x - \sinh x}{x^2 \sinh x\cosh x} \\
&\quad + \frac{1}{2}\left(\frac{\sinh x}{x\cosh^2 x}\right)^{r-1} \frac{2\sinh^2 x\cosh x - x^2\cosh^3 x - x\sinh x}{x^3\sinh^2 x\cosh^2 x} \\
&= \frac{1}{2}\left(\frac{\sinh x}{x\cosh^2 x}\right)^{r-2} \frac{1}{x^4\sinh x\cosh^4 x}\big[(r-1)\big(x - \sinh x\cosh x - x\sinh^2 x\big) \\
&\quad \times (x\cosh x - \sinh x) + \big(2\sinh^2 x\cosh x - x\sinh x - x^2\cosh^3 x\big)\big] \\
&= \frac{1}{2}\left(\frac{\sinh x}{x\cosh^2 x}\right)^{r-2} \frac{\big(x - \sinh x\cosh x - x\sinh^2 x\big)(x\cosh x - \sinh x)}{x^4\sinh x\cosh^4 x} \\
&\quad \times \left[r - 1 + \frac{2\sinh^2 x\cosh x - x\sinh x - x^2\cosh^3 x}{\big(x - \sinh x\cosh x - x\sinh^2 x\big)(x\cosh x - \sinh x)}\right] \\
&= \frac{1}{2}\left(\frac{\sinh x}{x\cosh^2 x}\right)^{r-2} \frac{B(x)}{x^4\sinh x\cosh^4 x}[r - 1 + h_1(x)].
\end{aligned}$$

Based on the result (9) in the proof of Lemma 5, we can observe that the function $B(x) < 0$.

When $r \geq \frac{8}{25}$ and $x \in (0, \infty)$, we have $r - 1 + h_1(x) > 0$, and then $\frac{f_1'(x)}{f_2'(x)}$ is decreasing on $(0, \infty)$. Accordingly, by Lemma 1, the function $F(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x) - f_1(0^+)}{f_2(x) - f_2(0^+)}$ is decreasing on $(0, \infty)$.

When $r < 0$ and $x \in (0, \infty)$, we have $r - 1 + h_1(x) < 0$, and then $\frac{f_1'(x)}{f_2'(x)}$ is increasing on $(0, \infty)$. Accordingly, based on Lemma 1, the function $F(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x) - f_1(0^+)}{f_2(x) - f_2(0^+)}$ is increasing on $(0, \infty)$.

It is straightforward that $\lim_{x \to 0^+} F(x) = \frac{1}{6}$. The proof of Theorem 1 is thus complete. □

Corollary 1. *Let $r > 0$ and $x \in \mathbb{R}$. Then the inequality*

$$\frac{1}{\operatorname{sinhc}^r x} < 1 - \alpha + \alpha\left(\frac{1}{\cosh x}\right)^{2r}$$

holds if and only if $\alpha \leq \frac{1}{6}$.

Corollary 2. *Let $x \in \mathbb{R}$. Then*

$$\frac{1}{\cosh^2 x} < \frac{1}{\operatorname{sinhc} x} < \frac{5}{6} + \frac{1}{6\cosh^2 x} < 1 < \operatorname{sinhc} x < \frac{5}{6} + \frac{\cosh^2 x}{6} < \cosh^2 x.$$

Corollary 3. *Let $t \neq 0$. Then*

$$\frac{1}{1+t^2} < \frac{\operatorname{arcsinh} t}{t} < \frac{5}{6} + \frac{1}{6(1+t^2)} < 1 < \frac{t}{\operatorname{arcsinh} t} < \frac{5}{6} + \frac{1+t^2}{6} < 1+t^2.$$

Theorem 2. *Let $r \in \mathbb{R}$. For $x \in \left(0, \frac{\pi}{2}\right)$,*

1. *when $r \geq \frac{1}{2}$, the double inequality (5) holds if and only if $\alpha \geq 1 - \left(\frac{2}{\pi}\right)^r$ and $\beta \leq \frac{1}{6}$;*
2. *when $0 < r \leq \frac{8}{25}$, the double inequality (5) holds if and only if $\alpha \geq \frac{1}{6}$ and $\beta \leq 1 - \left(\frac{2}{\pi}\right)^r$;*
3. *when $r < 0$, then the right-hand side inequality in (5) holds if and only if $\beta \geq \frac{1}{6}$.*

Proof. Let

$$G(x) = \frac{\operatorname{sinc}^r x - 1}{\cos^{2r} x - 1} \triangleq \frac{g_1(x)}{g_2(x)},$$

where $g_1(x) = \operatorname{sinc}^r x - 1$ and $g_2(x) = \cos^{2r} x - 1$. Then

$$\frac{g_1'(x)}{g_2'(x)} = -\frac{1}{2}\left(\frac{\sin x}{x\cos^2 x}\right)^{r-1} \frac{x\cos x - \sin x}{x^2 \sin x \cos x}$$

and

$$\begin{aligned}
\left[\frac{g_1'(x)}{g_2'(x)}\right]' =& \frac{r-1}{2}\left(\frac{\sin x}{x\cos^2 x}\right)^{r-2} \frac{x - \sin x\cos x + x\sin^2 x}{x^2\cos^3 x} \frac{\sin x - x\cos x}{x^2 \sin x\cos x} \\
&+ \frac{1}{2}\left(\frac{\sin x}{x\cos^2 x}\right)^{r-1} \frac{x^2\cos^3 x + x\sin x - 2\sin^2 x\cos x}{x^3\sin^2 x\cos^2 x} \\
=& \frac{1}{2}\left(\frac{\sin x}{x\cos^2 x}\right)^{r-2} \frac{1}{x^4 \sin x\cos^4 x}\big[(r-1)\left(x - \sin x\cos x + x\sin^2 x\right) \\
&\times (\sin x - x\cos x) + \left(x^2\cos^3 x + x\sin x - 2\sin^2 x\cos x\right)\big] \\
=& \frac{1}{2}\left(\frac{\sin x}{x\cos^2 x}\right)^{r-2} \frac{2x\sin x - \sin^2 x\cos x - x^2\cos x - x^2\sin^2 x\cos x}{x^4\sin x\cos^4 x} \\
&\times \left(r + \frac{2x^2\cos x - x\sin x - \sin^2 x\cos x}{2x\sin x - \sin^2 x\cos x - x^2\cos x - x^2\sin^2 x\cos x}\right) \\
=& \frac{1}{2}\left(\frac{\sin x}{x\cos^2 x}\right)^{r-2} \frac{2x\sin x - \sin^2 x\cos x - x^2\cos x - x^2\sin^2 x\cos x}{x^4\sin x\cos^4 x}[r + u(x)],
\end{aligned}$$

where

$$\begin{aligned}
u(x) &= \frac{2x^2\cos x - x\sin x - \sin^2 x\cos x}{2x\sin x - \sin^2 x\cos x - x^2\cos x - x^2\sin^2 x\cos x} \\
&= \frac{\frac{2x^2}{\sin^2 x} - \frac{2x}{\sin 2x} - 1}{\frac{4x}{\sin 2x} - 1 - \frac{x^2}{\sin^2 x} - x^2} \triangleq \frac{D(x)}{E(x)}
\end{aligned}$$

with

$$D(x) = \frac{2x^2}{\sin^2 x} - \frac{2x}{\sin 2x} - 1 \quad \text{and} \quad E(x) = \frac{4x}{\sin 2x} - 1 - \frac{x^2}{\sin^2 x} - x^2.$$

By virtue of (6) and (7), we have

$$\begin{aligned}
D(x) &= 2x^2\left[\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}\right] - \left[1 + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|(2x)^{2n}\right] - 1 \\
&= \sum_{n=1}^{\infty} \frac{2^{2n+1}(2n-1)}{(2n)!}|B_{2n}|x^{2n} - \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|(2x)^{2n}
\end{aligned}$$

$$= \sum_{n=2}^{\infty} \frac{2^{2n}(4n - 2^{2n})}{(2n)!} |B_{2n}| x^{2n} \triangleq \sum_{n=2}^{\infty} d_n x^{2n}$$

and

$$\begin{aligned} E(x) = 2\left[1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}|(2x)^{2n}\right] - x^2 \left[\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n-2}\right] - x^2 - 1 \\ = \sum_{n=1}^{\infty} \frac{(2^{2n+1} - 2n - 3)2^{2n}}{(2n)!} |B_{2n}| x^{2n} - x^2 \\ = \sum_{n=2}^{\infty} \frac{(2^{2n+1} - 2n - 3)2^{2n}}{(2n)!} |B_{2n}| x^{2n} \triangleq \sum_{n=2}^{\infty} e_n x^{2n}, \end{aligned}$$

where

$$d_n = \frac{2^{2n}(4n - 2^{2n})}{(2n)!} |B_{2n}| \quad \text{and} \quad e_n = \frac{(2^{2n+1} - 2n - 3)2^{2n}}{(2n)!} |B_{2n}| > 0.$$

Since the sequence $c_n = \frac{d_n}{e_n} = \frac{4n - 2^{2n}}{2^{2n+1} - 2n - 3}$ for $n = 2, 3, \ldots$ is decreasing, according to Lemma 2, the function $u(x) = \frac{D(x)}{E(x)}$ is decreasing from $(0, \frac{\pi}{2})$ onto $(-\frac{1}{2}, -\frac{8}{25})$. When $r \geq \frac{1}{2}$, the function $\frac{g_1'(x)}{g_2'(x)}$ is increasing on $(0, \frac{\pi}{2})$, and based on Lemma 1, the function $G(x) = \frac{g_1(x)}{g_2(x)} = \frac{g_1(x) - g_1(0^+)}{g_2(x) - g_2(0^+)}$ is increasing on $(0, \frac{\pi}{2})$. When $r \leq \frac{8}{25}$, the function $\frac{g_1'(x)}{g_2'(x)}$ is decreasing on $(0, \frac{\pi}{2})$, and according to Lemma 1, the function $G(x) = \frac{g_1(x)}{g_2(x)} = \frac{g_1(x) - g_1(0^+)}{g_2(x) - g_2(0^+)}$ is decreasing on $(0, \frac{\pi}{2})$.

It is straightforward that $\lim_{x \to 0^+} G(x) = \frac{1}{6}$. The proof of Theorem 2 is thus complete. □

Corollary 4. *Let $r > 0$ and $|x| < \pi/2$. Then the inequality*

$$\frac{1}{\operatorname{sinc}^r x} < 1 - \alpha + \alpha \left(\frac{1}{\cos x}\right)^{2r}$$

holds if and only if $\alpha \geq \frac{1}{6}$.

Corollary 5. *Let $|x| \leq \frac{\pi}{2}$. Then*

$$\cos^2 x < \cos x < \operatorname{sinc} x < \frac{5}{6} + \frac{\cos^2 x}{6} < 1 < \frac{1}{\operatorname{sinc} x} < \frac{5}{6} + \frac{1}{6\cos^2 x} < \frac{1}{\cos^2 x}.$$

Corollary 6. *Let $t \in (0, 1)$. Then*

$$1 - t^2 < \frac{t}{\arcsin t} < \frac{5}{6} + \frac{1 - t^2}{6} < 1 < \frac{\arcsin t}{t} < \frac{5}{6} + \frac{1}{6(1 - t^2)} < \frac{1}{1 - t^2}.$$

4. Applications of Necessary and Sufficient Conditions

In this section, using Theorems 1 and 2 , we can obtain the following inequalities.

Theorem 3. *Let $s, t > 0$ with $s \neq t$. When $r \geq \frac{8}{25}$, the double inequality*

$$\alpha C^r(s,t) + (1-\alpha)A^r(s,t) < M^r(s,t) < \beta C^r(s,t) + (1-\beta)A^r(s,t) \qquad (11)$$

holds if and only if $\alpha \leq \frac{1}{2^r - 1} \frac{1 - \ln^r(1+\sqrt{2})}{\ln^r(1+\sqrt{2})}$ and $\beta \geq \frac{1}{6}$; when $r < 0$, the inequality (11) holds if and only if $\alpha \geq \frac{1}{2^r - 1} \frac{1 - \ln^r(1+\sqrt{2})}{\ln^r(1+\sqrt{2})}$ and $\beta \leq \frac{1}{6}$.

Proof. Without loss of generality, we assume that $s > t > 0$. Let $u = \frac{s-t}{s+t}$. Then $u \in (0,1)$ and

$$\frac{M^r(s,t) - A^r(s,t)}{C^r(s,t) - A^r(s,t)} = \frac{\frac{u^r}{\operatorname{arcsinh}^r u} - 1}{(1+u^2)^r - 1}.$$

Let $t = \sinh\theta$. Then $\theta \in \left(0, \ln\left(1+\sqrt{2}\right)\right)$ and

$$\frac{M^r(s,t) - A^r(s,t)}{C^r(s,t) - A^r(s,t)} = \frac{\frac{\sinh^r\theta}{\theta^r} - 1}{\cosh^{2r}\theta - 1} \triangleq F(\theta).$$

Using Theorem 1, we can observe that, when $r \geq \frac{8}{25}$, the function $F(\theta)$ is decreasing on the interval $\left(0, \ln\left(1+\sqrt{2}\right)\right)$, whereas $F(\theta)$ is increasing on $\left(0, \ln\left(1+\sqrt{2}\right)\right)$ for $r < 0$.

According to L'Hospital's rule, we have

$$\lim_{\theta\to 0^+} F(\theta) = \frac{1}{6} \quad \text{and} \quad \lim_{\theta\to\ln(1+\sqrt{2})^-} F(\theta) = \frac{1}{2^r - 1}\,\frac{1-\ln^r\left(1+\sqrt{2}\right)}{\ln^r\left(1+\sqrt{2}\right)}.$$

The proof of Theorem 3 is thus complete. □

Theorem 4. *Let $s, t > 0$ with $s \neq t$. Then the double inequality*

$$\alpha H^r(s,t) + (1-\alpha)A^r(s,t) < P^r(s,t) < \beta H^r(s,t) + (1-\beta)A^r(s,t)$$

holds if and only if

$$\begin{cases} \text{for } r \geq \frac{1}{2}, & \alpha \geq 1 - \left(\frac{2}{\pi}\right)^r \text{ and } \beta \leq \frac{1}{6}; \\ \text{for } 0 < r \leq \frac{8}{25}, & \alpha \geq \frac{1}{6} \text{ and } \beta \leq 1 - \left(\frac{2}{\pi}\right)^r; \\ \text{for } r < 0, & \alpha \leq 0 \text{ and } \beta \geq \frac{1}{6}. \end{cases}$$

Proof. Without the loss of generality, we assume that $s > t > 0$. Let $v = \frac{s-t}{s+t}$. Then $v \in (0,1)$ and

$$\frac{P^r(s,t) - A^r(s,t)}{H^r(s,t) - A^r(s,t)} = \frac{\frac{v^r}{\arcsin^r v} - 1}{(1-v^2)^r - 1}.$$

Let $v = \sin\theta$. Then $\theta \in \left(0, \frac{\pi}{2}\right)$ and

$$\frac{P^r(s,t) - A^r(s,t)}{H^r(s,t) - A^r(s,t)} = \frac{\frac{\sin^r\theta}{\theta^r} - 1}{\cos^{2r}\theta - 1} \triangleq G(\theta).$$

By virtue of Theorem 2, we can observe that, when $r \in (-\infty, 0) \cup \left(0, \frac{8}{25}\right]$, the function $G(\theta)$ is decreasing on $\left(0, \frac{\pi}{2}\right)$, whereas $G(\theta)$ is increasing on $\left(0, \frac{\pi}{2}\right)$ for $r \geq \frac{1}{2}$.

Using L'Hospital's rule, we obtain the limits $\lim_{\theta\to 0^+} G(\theta) = \frac{1}{6}$ and

$$\lim_{\theta\to(\pi/2)^-} G(\theta) = \begin{cases} 1 - \left(\frac{2}{\pi}\right)^r, & r > 0; \\ 0, & r < 0. \end{cases}$$

The proof of Theorem 4 is thus complete. □

Corollary 7. *For all $s, t > 0$ with $s \neq t$,*

1. *The double inequality*

$$\frac{\alpha_1}{H(s,t)} + \frac{1-\alpha_1}{A(s,t)} < \frac{1}{P(s,t)} < \frac{\beta_1}{H(s,t)} + \frac{1-\beta_1}{A(s,t)}$$

holds if and only if

$$\alpha_1 \le 2\big[1-\ln\big(1+\sqrt{2}\,\big)\big] = 0.237253\ldots \quad \textit{and} \quad \beta_1 \ge \frac{1}{6};$$

2. *The double inequality*

$$\frac{\alpha_2}{H^2(s,t)} + \frac{1-\alpha_2}{A^2(s,t)} < \frac{1}{P^2(s,t)} < \frac{\beta_2}{H^2(s,t)} + \frac{1-\beta_2}{A^2(s,t)}$$

holds if and only if $\alpha_2 \le 0$ *and* $\beta_2 \ge \frac{1}{6}$;

3. *The double inequality*

$$\alpha_3 H(s,t) + (1-\alpha_3)A(s,t) < P(s,t) < \beta_3 H(s,t) + (1-\beta_3)A(s,t)$$

holds if and only if

$$\alpha_3 \ge 1 - \frac{2}{\pi} = 0.36338\ldots, \quad \textit{and} \quad \beta_3 \le \frac{1}{6};$$

4. *The double inequality*

$$\alpha_4 H^2(s,t) + (1-\alpha_4)A^2(s,t) < P^2(s,t) < \beta_4 H^2(s,t) + (1-\beta_4)A^2(s,t)$$

holds if and only if

$$\alpha_4 \ge 1 - \left(\frac{2}{\pi}\right)^2 = 0.594715\ldots \quad \textit{and} \quad \beta_4 \le \frac{1}{6}.$$

Corollary 8. *For all* $s,t>0$ *with* $s \ne t$, *then*

$$\begin{aligned} H(s,t) &< \left(1-\frac{2}{\pi}\right)H(s,t) + \frac{2}{\pi}A(s,t) < P(s,t) < \frac{1}{6}H(s,t) + \frac{5}{6}A(s,t) \\ &< A(s,t) < \frac{1-\ln\big(1+\sqrt{2}\,\big)}{\ln\big(1+\sqrt{2}\,\big)}C(s,t) + \frac{2\ln\big(1+\sqrt{2}\,\big)-1}{\ln\big(1+\sqrt{2}\,\big)}A(s,t) \\ &< M(s,t) < \frac{1}{6}C(s,t) + \frac{5}{6}A(s,t) < C(s,t). \end{aligned} \tag{12}$$

5. Remarks

Remark 1. *When taking* $r = -2,-1,1,2$ *in Theorem 1, we can obtain the results reported in [13,23].*

Remark 2. *The inequality chain* (12) *improves the left-hand sides of inequalities* (1) *and* (2).

Remark 3. *From* $\sinh(z\,\mathrm{i}) = \mathrm{i}\sin z$, *it follows that* $\operatorname{sinhc}(z\,\mathrm{i}) = \operatorname{sinc} z$. *This relation is possibly available to simplify proofs of the main results in this paper.*

Remark 4. *In [33–36], series expansions of the functions*

$$\left(\frac{\arcsin t}{t}\right)^r, \quad \left(\frac{\operatorname{arcsinh} t}{t}\right)^r, \quad \left[\frac{(\arccos x)^2}{2(1-x)}\right]^r,$$

$$\left[\frac{(\operatorname{arccosh} x)^2}{2(1-x)}\right]^r, \quad (\arccos t)^r, \quad (\operatorname{arccosh} t)^r$$

for $r \in \mathbb{R}$ were established. These series expansions are possibly available to prove the main results presented in this paper.

6. Conclusions

In this paper, we have established some inequalities for the trigonometric functions and hyperbolic functions. These results can trigger further investigations on inequalities involving trigonometric and hyperbolic functions. The techniques used in this paper are suitable for proving and establishing many other inequalities involving the Neuman–Sándor mean, the Seiffert mean, the Toader mean, and so on.

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