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Nonlinear Lie Triple Higher Derivations on Triangular Algebras by Local Actions: A New Perspective

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Abstract: Let $\mathcal{R}$ be a commutative ring with unity and $\mathcal{T}$ be a triangular algebra over $\mathcal{R}$. Let a sequence $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ of nonlinear mappings $\delta_n : \mathcal{T} \to \mathcal{T}$ is a Lie triple higher derivation by local actions satisfying the equation. Under some mild conditions on $\mathcal{T}$, we prove in this paper that every Lie triple higher derivation by local actions on the triangular algebras is proper. As an application, we shall give a characterization of Lie triple higher derivations by local actions on upper triangular matrix algebras and nest algebras, respectively.

Keywords: Lie triple higher derivation; faithful bimodule; higher derivation; local action; triangular algebras

MSC: 16W25; 15A78; 17B40; 16N60

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1. Introduction

Throughout this paper, we assume that $\mathcal{R}$ is a commutative ring with unity and $\mathcal{A}$ is an algebra over $\mathcal{R}$. $\mathcal{Z}(\mathcal{A})$ is the center of $\mathcal{A}$. Let us denote the Lie product of arbitrary elements $x, y, z \in \mathcal{R}$ by $[x, y] = xy - yx$. Suppose that an additivity (resp. nonlinear) mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a (resp. nonlinear) derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$ and is said to be a (resp. nonlinear) Lie triple derivation if

$$\delta([[x,y],z]) = [[\delta(x),y],z] + [[x,\delta(y)],z] + [[x,y],\delta(z)] \quad (1).$$

for all $x, y, z \in \mathcal{A}$. If d is a derivation of $\mathcal{A}$ and f is an $\mathcal{R}$-linear (additive) map from $\mathcal{A}$ into its center, then $d + f$ is a Lie triple derivation if and only if f annihilates all second commutators $[[x, y], z]$. A Lie triple derivation of the form $d + f$, where d is a derivation and f is central-valued map, will be called **proper** Lie triple derivation. Otherwise, a Lie triple derivation will be called **improper**. Due to the renowned Herstein's Lie-type mapping research program, Lie triple derivation have been studied extensively both by algebraists and analysts, see [1–13], etc.

Recently, many mathematicians have studied the structural properties of derivations of rings or operator algebras completely determined by some elements concerning products. This is actually to study "*local behaviours*" of linear (nonlinear) mappings. There is a fairly substantial literature on so-called local mappings for operator algebras, starting with the papers of Larson and Sourour [12,14]. Assume that $\theta : \mathcal{A} \to \mathcal{A}$ is a linear (resp. nonlinear) mapping and $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a map. If Ω is a proper subset of $\mathcal{A}$ and relation (1) holds for any $x_1, x_2, x_3 \in \mathcal{A}$ with $F(x_1, x_2, x_3) \in \Omega$, then θ is called a linear (nonlinear) Lie triple derivation by local action of $\mathcal{A}$. In the last few decades, Lie triple derivation by local actions satisfying the Relation (1) on rings and algebras has been studied by many authors [15–17]. Liu in [15] studied the structure of Lie triple derivations by local actions satisfying the condition $xy \in \Omega = \{0, p\}$ in Relation (1), where p is a fixed nontrivial projection of factor von Neumann algebra M with a dimension greater than 1. He showed that every Lie triple derivations by local actions is **proper**, i.e., every Lie triple derivation by local actions is of the form $d + f$, where d being a derivation and f being central-valued mapping. For von Neumann algebra with no central abelian projections $\mathcal{M}$, Liu in [16] obtain a similar result with [15]. In 2021, Zhao [17] considered the structure of Lie triple derivations by local actions satisfying the condition $xyz = 0 \in \Omega = \{0\}$ in Relation (1) on a triangular algebra. He proved that every Lie triple derivation by local actions is **proper**. Motivated by the above works, we will discuss the structure form of the nonlinear Lie triple higher derivations by local actions satisfying the condition $xyz = 0$ in Relation (2) on triangular algebras.

There are many interesting generalizations of (Lie triple) derivation, one of them being (Lie triple) higher derivation (see [18–24]). Let us first recall some basic facts related to Lie triple higher derivations. Let $\mathcal{N}$ be the set of all non-negative integers and $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ be a family of $\mathcal{R}$-linear (resp. nonlinear) mapping on $\mathcal{A}$ such that $\delta_0 = id_{\mathcal{A}}$. Δ is called:

(a) a (resp. nonlinear) higher derivation if

$$\delta_n(xy) = \sum_{i+j=n} \delta_i(x)\delta_j(y)$$

for all $x, y \in \mathcal{A}$ and for each $n \in \mathcal{N}$;

(b) a (resp. nonlinear) Lie higher derivation if

$$\delta_n([x, y]) = \sum_{i+j=n} [\delta_i(x), \delta_j(y)]$$

for all $x, y \in \mathcal{A}$ and for each $n \in \mathcal{N}$;

(c) a (resp. nonlinear) Lie triple higher derivation if

$$\delta_n([[x, y], z]) = \sum_{i+j=n} [[\delta_i(x), \delta_j(y)], \delta_j(z)] \qquad (2)$$

for all $x, y, z \in \mathcal{A}$ and for each $n \in \mathcal{N}$.

Assume that $\{d_n\}_{n\in\mathcal{N}}$ is a higher derivation on $\mathcal{A}$, and $\{\tau_n\}_{n\in\mathcal{N}}$ is a sequence of nonlinear ($\mathcal{R}$-linear) mappings form $\mathcal{A}$ to its center vanishing on $[[x, y], z]$ with $\tau_0 = 0$. For each non-negative integer n, we set

$$\delta_n = d_n + \tau_n. \qquad (3)$$

Then, it is obvious that $\{\delta_n\}_{n\in\mathcal{N}}$ is a Lie triple higher derivation. A Lie triple higher derivation $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ of the form (3) is called **proper**. It is obvious that Lie higher derivations and higher derivations are usual Lie triple derivations and derivations for $n = 1$, respectively. Lie-type derivations are an active subject of research in algebras which may not be associative or commutative (see [6]). In fact, many researchers have made substantial contributions related to this topic, such as [14–17,21,25], etc. For example, Zhao [17] investigated nonlinear Lie triple derivation by local actions at zero products on triangular algebras. Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra over $\mathcal{R}$, where A and B are unital algebras over $\mathcal{R}$, and M is a faithful (A, B)-bimodule. He studied nonlinear mappings $\delta : \mathcal{T} \to \mathcal{T}$ that act like Lie triple derivations on certain subsets of $\mathcal{T}$:

$$\delta([[x, y], z]) = [[\delta(x), y], z] + [[x, \delta(y)], z] + [[x, y], \delta(z)]$$

for all $x, y, z \in \mathcal{T}$ with $xyz = 0$. He showed that, under certain conditions on a triangular algebra $\mathcal{T}$, any such nonlinear mapping $\delta : \mathcal{T} \to \mathcal{T}$ is the sum of an additive derivation $d : \mathcal{T} \to \mathcal{T}$ and a nonlinear central mapping $\tau : \mathcal{T} \to \mathcal{T}$ vanishing as $[[x, y], z]$ with $xyz = 0$. Inspired by the results above, it is natural to consider some Lie triple higher derivations by local actions on zero products on triangular algebras.

In this paper, we investigate the Lie triple higher derivation by local actions at zero products on triangular algebras. Let $\mathcal{T}$ be a triangular algebra over a commutative ring $\mathcal{R}$. Under some mild conditions on $\mathcal{T}$, we prove that, if a family $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ of *nonlinear* mappings on $\mathcal{T}$ satisfies the condition

$$\delta_n([[x,y],z]) = \sum_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)] \tag{4}$$

for all $x, y \in \mathcal{T}$ with $xyz = 0$, then there exists a higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ and a nonlinear mapping $\tau_n : \mathcal{T} \to \mathcal{T}$ on $\mathcal{T}$ vanishing all $[[x,y],z]$ with $xyz = 0$ such that

$$\delta_n(x) = d_n(x) + \tau_n(x).$$

Then, we immediately apply the obtained results to the background of nest algebras and describe Lie triple higher derivations by local actions on these algebras. Our results also generalize the existing results ([17], Theorems 2.1 and 2.2) in triangular algebra.

2. Triangular Algebras

In this section, we give some notions that will be needed in what follows.

Triangular algebras were first introduced in [26]. Here, we offer a definition of a triangular algebra. Let $\mathcal{R}$ be a commutative ring with identity. Let A, B be an associative algebras over $\mathcal{R}$ with idengtity 1_A and 1_B, respectively. Let M be a faithful (A, B)-bimodule, that is, for $a \in A$, $aM = \{0\}$ implies $a = 0$ and, for $b \in B$, $Mb = \{0\}$ implies $b = 0$. We denote the *triangular algebra* consisting of A, B, and M by

$$\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}.$$

Then, $\mathcal{T}$ is an associative and noncommutative $\mathcal{R}$-algebra, the most common examples of triangular algebras are upper matrix algebras and nest algebras (see [26,27] for details). Furthermore, the center $\mathcal{Z}(\mathcal{T})$ of $\mathcal{T}$ is (see [26,28])

$$\mathcal{Z}(\mathcal{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| am = mb, \ \forall\, m \in M \right\}. \tag{5}$$

Let us define two natural $\mathcal{R}$-linear projections $\pi_A : \mathcal{T} \to A$ and $\pi_B : \mathcal{T} \to B$ by

$$\pi_A : \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \longmapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \longmapsto b.$$

It is easy to see that $\pi_A(\mathcal{Z}(\mathcal{T}))$ is a subalgebra of $\mathcal{Z}(A)$ and that $\pi_B(\mathcal{Z}(\mathcal{T}))$ is a subalgebra of $\mathcal{Z}(B)$. Furthermore, a unique algebraic isomorphism $\tau\colon \pi_A(\mathcal{Z}(\mathcal{T})) \longrightarrow \pi_B(\mathcal{Z}(\mathcal{T}))$ exists such that $am = m\tau(a)$ for all $a \in \pi_A(\mathcal{Z}(\mathcal{T}))$ and for all $m \in M$.

3. Main Theorem

This section is aimed at studying Lie triple higher derivations for a zero product on triangular algebras. More precisely, we will give the higher version corresponding to ([17], Theorems 2.1 and 2.2).

Theorem 1. *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra. Suppose that a sequence $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ of mappings $\delta_n : \mathcal{T} \to \mathcal{T}$ is a nonlinear map*

$$\delta_n([[x,y],z]) = \sum_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$$

for all $x, y, z \in \mathcal{T}$ with $xyz = 0$. If $\pi_A(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ and $\pi_A(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$, then, every nonlinear mapping δ_n is almost additive on $\mathcal{T}$, that is,

$$\delta_n(x+y) - \delta_n(x) - \delta_n(y) \in \mathcal{Z}(\mathcal{T})$$

for all $x, y \in \mathcal{T}$.

In order to prove our main results, we begin with the following theorem coming from ([17], Theorem 2.1):

Theorem 2 ([17] Theorem 2.1)**.** *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra. Suppose that a mapping $\delta_1 : \mathcal{T} \to \mathcal{T}$ is a nonlinear map satisfying*

$$\delta_1([[x,y],z]) = \sum_{i+j+k=1} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$$

for all $x, y, z \in \mathcal{T}$ with $xyz = 0$. Then, nonlinear mapping δ_1 is almost additive on $\mathcal{T}$, that is,

$$\delta_1(x+y) - \delta_1(x) - \delta_1(y) \in \mathcal{Z}(\mathcal{T}).$$

For convenience, let us write $A_{11} = A$, $A_{22} = B$ and $A_{12} = M$; then, triangular algebra $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ can be rewritten by $\mathcal{T} = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right]$.

Proof. Assume that a sequence $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ of nonlinear mappings $\delta_n : \mathcal{T} \to \mathcal{T}$ is a Lie triple higher derivation by local actions on triangular algebras $\mathcal{T} = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right]$. We shall use the method of induction for n. For $n = 1$, $\delta_1 : \mathcal{T} \to \mathcal{T}$ is a Lie triple derivation by local actions. According to Theorem 2, we obtain that a nonlinear Lie triple derivation δ_1 by local actions satisfies the following properties:

$$\mathfrak{C}_1 = \begin{cases} \delta_1(0) = 0, \delta_1(a_{ii} + a_{ij}) - \delta_1(a_{ii}) - \delta_1(a_{ij}) \in \mathcal{Z}(\mathcal{T}); \\ \delta_1(a_{ij} + a'_{ij}) = \delta_1(a_{ii}) + \delta_1(a'_{ij}); \delta_1(a_{ii} + a_{ii}) - \delta_1(a_{ii}) - \delta_1(a_{ii}) \in \mathcal{Z}(\mathcal{T}); \\ \delta_1(a_{ii} + a_{ij} + a_{jj}) - \delta_1(a_{ii}) - \delta_1(a_{ij}) - \delta_1(a_{jj}) \in \mathcal{Z}(\mathcal{T}) \end{cases}$$

for all $a_{ij} \in A$ with $i < j \in \{1, 2\}$.

We assume that the result holds for all $1 < s < n$, $n \in \mathbb{N}$. Then, nonlinear Lie triple higher derivation $\{\delta_l\}_{l=0}^{l=s}$ satisfies the following:

$$\mathfrak{C}_s = \begin{cases} \delta_s(0) = 0, \delta_s(a_{ii} + a_{ij}) - \delta_s(a_{ii}) - \delta_s(a_{ij}) \in \mathcal{Z}(\mathcal{T}); \\ \delta_s(a_{ij} + a'_{ij}) = \delta_s(a_{ii}) + \delta_s(a'_{ij}); \delta_s(a_{ii} + a_{ii}) - \delta_s(a_{ii}) - \delta_s(a_{ii}) \in \mathcal{Z}(\mathcal{T}); \\ \delta_s(a_{ii} + a_{ij} + a_{jj}) - \delta_s(a_{ii}) - \delta_s(a_{ij}) - \delta_s(a_{jj}) \in \mathcal{Z}(\mathcal{T}) \end{cases}$$

for all $a_{ij} \in A$ with $i < j \in \{1, 2\}$.

Our aim is to show that the above conditions $\mathfrak{C}_s$ also hold for n. The proof will be realized via a series of claims. □

Claim 1: With notations as above, we have $\delta_n(0) = 0$.

With the help of condition $\mathfrak{C}_s$, we find that

$$\delta_n(0) = \delta_n([[0,0],0]) = \sum_{i+j+k=n} [[\delta_i(0), \delta_j(0)], \delta_k(0)] = 0.$$

Claim 2: With notations as above, we have

(i) $\delta_n(a_{11} + a_{12}) - \delta_n(a_{11}) - \delta_n(a_{12}) \in \mathcal{Z}(\mathcal{T})$;
(ii) $\delta_n(a_{22} + a_{12}) - \delta_n(a_{22}) - \delta_n(a_{12}) \in \mathcal{Z}(\mathcal{T})$

for all $a_{ij} \in A$ with $i \leq j \in \{1,2\}$.

In order to maintain the integrity of the proof, we give the proof of all cases. Let us consider the case: $(i,j) = (1,2)$.

It is clear that $m_{12}(a_{11} + m'_{12})p_1 = m_{12}a_{11}p_1 = 0 = m_{12}m'_{12}p_1$ for all $a_{11} \in A_{11}$ and $m_{12}, m'_{12} \in A_{12}$. Then, on the one hand, we have

$$\delta_n(a_{11}m_{12}) = \delta_n([[m_{12}, a_{11} + m'_{12}], p_1]) = \sum_{i+j+k=n} [[\delta_i(m_{12}), \delta_j(a_{11} + m'_{12})], \delta_k(p_1)]$$

and, on the other hand, we have

$$\begin{aligned}\delta_n(a_{11}m_{12}) &= \delta_n([[m_{12}, a_{11}], p_1]) + \delta_n([[m_{12}, m'_{12}], p_1]) \\ &= \sum_{i+j+k=n} [[\delta_i(m_{12}), \delta_j(a_{11})], \delta_k(p_1)] + \sum_{i+j+k=n} [[\delta_i(m_{12}), \delta_j(m'_{12})], \delta_k(p_1)] \\ &= \sum_{i+j+k=n} [[\delta_i(m_{12}), \delta_j(a_{11}) + \delta_j(m'_{12})], \delta_k(p_1)].\end{aligned}$$

By observing the two equations above and condition $\mathfrak{C}_s$ for all $0 \leq s \leq n-1$, we have

$$\begin{aligned}0 &= \sum_{i+j+k=n} [[\delta_i(m_{12}), \delta_n(a_{11} + m'_{12}) - (\delta_j(a_{11}) + \delta_j(m'_{12}))], \delta_k(p_1)] \\ &= \sum_{i+j+k=n, j\neq n} [[\delta_i(m_{12}), \delta_j(a_{11} + m'_{12}) - (\delta_j(a_{11}) + \delta_j(m'_{12}))], \delta_k(p_1)] \\ &\quad + [[m_{12}, \delta_n(a_{11} + m'_{12}) - (\delta_n(a_{11}) + \delta_n(m'_{12}))], p_1] \\ &= [[m_{12}, \delta_n(a_{11} + m'_{12}) - (\delta_n(a_{11}) + \delta_n(m'_{12}))], p_1] \\ &= p_1(\delta_n(a_{11} + m'_{12}) - (\delta_n(a_{11}) + \delta_n(m'_{12})))m_{12} - m_{12}(\delta_n(a_{11} + m'_{12}) - (\delta_n(a_{11}) + \delta_n(m'_{12})))p_2\end{aligned}$$

for all $a_{11} \in A_{11}$ and $m_{12}, m'_{12} \in A_{12}$. Then, it follows from the center of algebra $\mathcal{T}$ that

$$p_1(\delta_n(a_{11} + m'_{12}) - (\delta_n(a_{11}) + \delta_n(m'_{12})))p_1 + p_2(\delta_n(a_{11} + m'_{12}) - (\delta_n(a_{11}) + \delta_n(m'_{12})))p_2 \in \mathcal{Z}(\mathcal{T})$$

for all $a_{11} \in A_{11}$ and $m_{12} \in A_{12}$.

In the following, we prove $p_1(\delta_n(a_{11} + m'_{12}) - (\delta_n(a_{11}) + \delta_n(m'_{12})))p_2 = 0$ for all $a_{11} \in A_{11}$ and $m_{12} \in A_{12}$. With the help of $p_2(a_{11} + m_{12})p_1 = 0 = p_2a_{11}p_1$, we have

$$\delta_n(m_{12}) = \delta_n([[p_2, a_{11} + m_{12}], p_1]) = \sum_{i+j+k=n} [[\delta_n(p_2), \delta_n(a_{11} + m_{12})], \delta_n(p_1)]$$

for all $a_{11} \in A_{11}$ and $m_{12} \in A_{12}$. On the other hand, we have

$$\begin{aligned}\delta_n(m_{12}) &= \delta_n([[p_2, m_{12}], p_1]) + \delta_n([[p_2, a_{11}], p_1]) \\ &= \sum_{i+j+k=n} [[\delta_i(p_2), \delta_j(m_{12})], \delta_k(p_1)] + \sum_{i+j+k=n} [[\delta_i(p_2), \delta_j(a_{11})], \delta_k(p_1)] \\ &= \sum_{i+j+k=n} [[\delta_i(p_2), \delta_j(a_{11}) + \delta_j(m_{12})], \delta_k(p_1)]\end{aligned}$$

for all $a_{11} \in A_{11}$ and $m_{12} \in A_{12}$. With the help of the two equations above and relation $\mathfrak{C}_s$, we have

$$\begin{aligned}0 &= \sum_{i+j+k=n} [[\delta_i(p_2), \delta_j(a_{11} + m_{12}) - (\delta_j(a_{11}) + \delta_j(m_{12}))], \delta_k(p_1)] \\ &= \sum_{i+j+k=n, j\neq n} [[\delta_i(p_2), \delta_j(a_{11} + m_{12}) - (\delta_j(a_{11}) + \delta_j(m_{12}))], \delta_k(p_1)] \\ &\quad + [[p_2, \delta_n(a_{11} + m_{12}) - (\delta_n(a_{11}) + \delta_n(m_{12}))], p_1] \\ &= [[p_2, \delta_n(a_{11} + m_{12}) - (\delta_n(a_{11}) + \delta_n(m_{12}))], p_1] \\ &= p_1(\delta_n(a_{11} + m'_{12}) - (\delta_n(a_{11}) + \delta_n(m'_{12})))p_2\end{aligned}$$

for all $a_{11} \in A_{11}$ and $m_{12} \in A_{12}$. In the following, we prove that the conclusion (i) holds.

For conclusion (ii), taking into accounts the relations $m_{12}(b_{22} + m'_{12})p_1 = m_{12}m'_{12}p_1 = m_{12}b_{22}p_1 = 0$, by an analogous manner, one can show that the conclusion

$$\delta_n(b_{22} + m'_{12}) - (\delta_n(b_{22}) + \delta_n(m'_{12})) \in \mathcal{Z}(\mathcal{T})$$

holds for all $b_{22} \in A_{22}$ and $m_{12}, m'_{12} \in A_{12}$.

Claim 3: With notations as above, we have $\delta_n(m_{12} + m'_{12}) = \delta_n(m_{12}) + \delta_n(m'_{12})$ for all $m_{12}, m'_{12} \in M_{12}$.

Thanks to relation $\mathfrak{C}_s$ for all $1 \leq s < n$ and $(-m_{12} - p_1)(p_2 + m'_{12})p_1 = 0$, we have

$$\begin{aligned}
\delta_n(m_{12} + m'_{12}) &= \delta_n([[-m_{12} - p_1, p_2 + m'_{12}], p_1]) \\
&= \sum_{i+j+k=n} [[\delta_i(-m_{12} - p_1), \delta_j(p_2 + m'_{12})], \delta_k(p_1)] \\
&= \sum_{i+j+k=n} [[\delta_i(-m_{12}) + \delta_i(-p_1), \delta_j(p_2) + \delta_j(m'_{12})], \delta_k(p_1)] \\
&= \sum_{i+j+k=n} [[\delta_i(-m_{12}), \delta_j(p_2)], \delta_k(p_1)] + \sum_{i+j+k=n} [[\delta_i(-m_{12}), \delta_j(m'_{12})], \delta_k(p_1)] \\
&+ \sum_{i+j+k=n} [[\delta_i(-p_1), \delta_j(p_2)], \delta_k(p_1)] + \sum_{i+j+k=n} [[\delta_i(\delta_i(-p_1), \delta_j(m'_{12})], \delta_k(p_1)] \\
&= \delta_n([[-m_{12}, p_2], p_1]) + \delta_n([[-m_{12}, m'_{12}], p_1]) + \delta_n([[-p_2, p_1], p_1]) + \delta_n([[-p_1, m_{12}], p_1]) \\
&= \delta_n(m_{12}) + \delta_n(m'_{12}),
\end{aligned}$$

that is, $\delta_n(m_{12} + m'_{12}) = \delta_n(m_{12}) + \delta_n(m'_{12})$ for all $m_{12}, m'_{12} \in A_{12}$.

Claim 4: With notations as above, we have

(i) $\delta_n(a_{11} + a'_{11}) = \delta_n(a_{11}) + \delta_n(a'_{11})$;
(ii) $\delta_n(a_{22} + a'_{22}) = \delta_n(a_{22}) + \delta_n(a'_{22})$

for all $a_{ii}, a'_{ii} \in A_{ii}$ with $i \in \{1, 2\}$.

We only prove the statements (i). The statement (ii) can be proved in a similar way. Because of relations $m_{12}p_1(a_{11} + a'_{11}) = m_{12}p_1a_{11} = 0 = m_{12}p_1a'_{11}$, we arrive at

$$\delta_n((a_{11} + a'_{11})m_{12}) = \delta_n([m_{12}, p_1], (a_{11} + a'_{11})]) = \sum_{i+j+k=n} [\delta_i(m_{12}), \delta_j(p_1)], \delta_k(a_1 + a'_{11})],$$

on the other hand, we have

$$\begin{aligned}
\delta_n((a_{11} + a'_{11})m_{12}) &= \delta_n(a_{ii}m_{12}) + \delta_n(a'_{11}m_{12}) \\
&= \sum_{i+j+k=n} [[\delta_i(m_{12}), \delta_j(p_1)], \delta_k(a_{11})] + \sum_{i+j+k=n} [\delta_i(m_{12}), \delta_j(p_1)], \delta_k(a'_{11})] \\
&= \sum_{i+j+k=n} [[\delta_i(m_{12}), \delta_j(p_1)], \delta_k(a_{11}) + \delta_k(a'_{11})].
\end{aligned}$$

for all $a_{11} \in A_{11}, m_{12} \in A_{12}$. On comparing the above two relations and together with condition $\mathfrak{C}_s$ for all $1 \leq s < n$, we see that

$$\begin{aligned}
0 &= \sum_{i+j+k=n} [[\delta_i(m_{12}), \delta_j(p_1)], \delta_k(a_{11} + a'_{11}) - (\delta_k(a_{11}) + \delta_k(a'_{11}))] \\
&= \sum_{i+j+k=n, k \neq n} [[\delta_i(m_{12}), \delta_j(p_1)], \delta_k(a_{11} + a'_{11}) - (\delta_k(a_{11}) + \delta_k(a'_{11}))] \\
&+ [[m_{12}, p_1], \delta_n(a_{11} + a'_{11}) - (\delta_n(a_{11}) + \delta_n(a'_{11}))] \\
&= [[m_{12}, p_1], \delta_n(a_{11} + a'_{11}) - (\delta_n(a_{11}) + \delta_n(a'_{11}))],
\end{aligned}$$

that is,

$$p_1(\delta_n(a_{11} + a'_{11}) - (\delta_n(a_{11}) + \delta_n(a'_{11})))m_{12} = m_{12}(\delta_n(a_{ii} + a'_{11}) - (\delta_n(a_{11}) + \delta_n(a'_{11})))p_2.$$

It follows from the center of triangular algebra $\mathcal{T}$ and the above equation that

$$p_1(\delta_n(a_{11}+a'_{11})-(\delta_n(a_{11})+\delta_n(a'_{11})))p_1 \oplus p_2(\delta_n(a_{ii}+a'_{11})-(\delta_n(a_{11})+\delta_n(a'_{11})))p_2 \in \mathcal{Z}(\mathcal{T}). \tag{6}$$

In the following, we prove

$$p_1(\delta_n(a_{11}+a'_{11})-(\delta_n(a_{11})+\delta_n(a'_{11})))p_2 = 0$$

for all $a_{11}, a'_{11} \in A_{11}$.

Benefitting from $(a_{11}+a'_{11})p_2p_2 = a_{11}p_2p_2 = a'_{11}p_2p_2 = 0$, we have

$$0 = \delta_n([[a_{11}+a'_{11}, p_2], p_2]) = \sum_{i+j+k=n} [[\delta_i(a_{11}+a'_{11}), \delta_j(p_2)], \delta_k(p_2)]$$

and

$$\begin{aligned} 0 &= \delta_n([[a_{11}, p_2], p_2]) + \delta_n([[a'_{11}, p_2], p_2]) \\ &= \sum_{i+j+k=n} [[\delta_i(a_{11}), \delta_j(p_2)], \delta_k(p_2)] + \sum_{i+j+k=n} [[\delta_i(a'_{11}), \delta_j(p_2)], \delta_k(p_2)] \\ &= \sum_{i+j+k=n} [[\delta_i(a_{11}) + \delta_i(a'_{11}), \delta_j(p_2)], \delta_k(p_2)]. \end{aligned}$$

By combining the above two equations with condition $\mathfrak{C}_s$ for all $1 \le s < n$, we can obtain

$$\begin{aligned} 0 = &\sum_{i+j+k=n, i\neq n} [[\delta_i(a_{11}+a'_{11}) - (\delta_i(a_{11})+\delta_i(a'_{11})), \delta_j(p_2)], \delta_k(p_2)] \\ &+ [[\delta_n(a_{11}+a'_{11}) - (\delta_n(a_{11})+\delta_n(a'_{11})), p_2], p_2] \\ &= [[\delta_n(a_{11}+a'_{11}) - (\delta_n(a_{11})+\delta_n(a'_{11})), p_2], p_2], \end{aligned}$$

that is,

$$p_1(\delta_n(a_{11}+a'_{11})-(\delta_n(a_{11})+\delta_n(a'_{11})))p_2 = 0 \tag{3.2}$$

Combining Equations (6) and (7), this claim holds.

Claim 5: With notations as above, we have $\delta_n(a_{11}+m_{12}+b_{22}) - \delta_n(a_{11}) - \delta_n(m_{12}) - \delta_n(b_{22}) \in \mathcal{Z}(\mathcal{T})$ for all $a_{11} \in A_{11}, m_{12} \in A_{12}, a_{22} \in A_{22}$.

For arbitrary $a_{11} \in A_{11}, m_{12} \in A_{12}, a_{22} \in A_{22}$, in view of $(a_{11}+m_{12}+b_{22})m'_{12}p_1 = 0$, we have

$$\begin{aligned} \delta_n(m_{12}b_{22} - a_{11}m_{12}) &= \delta_n([[a_{11}+m_{12}+b_{22}, m'_{12}], p_1]) \\ &= \sum_{i+j+k=n} [[\delta_i(a_{11}+m_{12}+b_{22}), \delta_j(m'_{12})], \delta_k(p_1)] \end{aligned}$$

and

$$\begin{aligned} \delta_n(m_{12}b_{22} - a_{11}m_{12}) &= \delta_n(m_{12}b_{22}) + \delta_n(-a_{11}m_{12}) \\ &= \delta_n([[[a_{11}, m'_{12}], p_1]]) + \delta_n([[m_{12}, m'_{12}], p_1]) + \delta_n([[b_{22}, m'_{12}], p_1]) \\ &= \sum_{i+j+k=n} [[\delta_i(a_{11}) + \delta_i(m_{12}) + \delta_i(b_{22}), \delta_j(m'_{12})], \delta_k(p_1)]. \end{aligned}$$

Let us set $W_i = \delta_i(a_{11}+m_{12}+b_{22}) - (\delta_i(a_{11}) + \delta_i(m_{12}) + \delta_i(b_{22}))$. Taking into account the above equation and inductive hypothesis $\mathfrak{C}_s$ for all $1 \le s \le n$, we have

$$\begin{aligned} 0 &= \sum_{i+j+k=n} [[W_i, \delta_j(m'_{12})], \delta_k(p_1)] \\ &= \sum_{i+j+k=n, i\neq n} [[W_i, \delta_j(m'_{12})], \delta_k(p_1)] + [[W_n, m'_{12}], p_1] = [[W_n, m'_{12}], p_1] = p_1W_nm'_{12} - m'_{12}W_np_2, \end{aligned}$$

that is, $p_1W_nm'_{12} - m'_{12}W_np_2 = 0$, i.e.,

$$p_1W_np_1 \oplus p_2W_np_2 \in \mathcal{Z}(\mathcal{T}). \tag{8}$$

In the following part, we prove $p_1 W_n p_2 = 0$. It is clear that $(a_{11} + m_{12} + b_{22})(-p_1)p_2 = 0$, and then

$$\delta_n(m_{12}) = \delta_n([a_{11} + m_{12} + b_{22}, -p_1], p_2]) = \sum_{i+j+k=n} [\delta_i(a_{11} + m_{12} + b_{22}), \delta_j(-p_1)], \delta_k(p_2)]$$

and

$$\begin{aligned} \delta_n(m_{12}) &= \delta_n([a_{11}, -p_1], p_2]) + \delta_n([m_{12}, -p_1], p_2]) + \delta_n([b_{22}, -p_1], p_2]) \\ &= \sum_{i+j+k=n} [[\delta_i(a_{11}) + \delta_i(m_{12}) + \delta_i(b_{22}), \delta_j(-p_1)], \delta_k(p_2)]. \end{aligned}$$

According to the above two equations and inductive hypothesis $\mathfrak{C}_s$ for all $1 \le s \le n$, we can obtain

$$\begin{aligned} 0 &= \sum_{i+j+k=n} [[W_i, \delta_j(-p_1)], \delta_k(p_2)] \\ &= \sum_{i+j+k=n, i\neq n} [[W_i, \delta_j(-p_1)], \delta_k(p_2)] + [[W_n, -p_1], p_2] \\ &= [[W_n, -p_1], p_2] \end{aligned}$$

that is,

$$p_1 W_n p_2 = 0. \tag{9}$$

It follows from Equations (8) and (9) that the claim holds.

Next, we give the proof of this theorem. For arbitrary $x = a_{11} + m_{12} + b_{22}$ and $y = a'_{11} + m'_{12} + b'_{22}$, we have

$$\begin{aligned} \delta_n(x+y) &= \delta_n(a_{11} + a'_{11} + m_{12} + m'_{12} + b_{22} + b'_{22}) \\ &= \delta_n(a_{11} + a'_{11}) + \delta_n(m_{12} + m'_{12}) + \delta_n(b_{22} + b'_{22}) + Z_1 \\ &= \delta_n(a_{11}) + \delta_n(a'_{11}) + \delta_n(m_{12}) + \delta_n(m'_{12}) + \delta_n(b_{22}) + \delta_n(b'_{22}) + Z_1 + Z_2 + Z_3 \\ &= \delta_n(x) + \delta_n(y) + Z_1 + Z_2 + Z_3 + Z_4 + Z_5, \end{aligned}$$

which implies that $\delta_n(x+y) - \delta_n(x) - \delta_n(y) \in \mathcal{Z}(\mathcal{T})$.

Based on the almost additive of δ_n on $\mathcal{T}$, we give the main result in this section reading as follows:

Theorem 3. *Let* $\mathcal{T} = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$ *be a triangular algebra satisfying*

(i) $\pi_{A_{11}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})_{11}$ *and* $\pi_{A_{22}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})_{22}$
(ii) *For any* $a_{11} \in A_{11}$, *if* $[a_{11}, A_{11}] \in \mathcal{Z}(\mathcal{A})_{11}$, *then* $a_{11} \in \mathcal{Z}(\mathcal{A})$ *or for any* $a_{22} \in A_{22}$, *if* $[a_{22}, A_{22}] \in \mathcal{Z}(\mathcal{A}_{22})$, *then* $a_{22} \in \mathcal{Z}(\mathcal{A}_{22})$.

Suppose that a sequence $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ *of mappings* $\delta_n : \mathcal{T} \to \mathcal{T}$ *is a nonlinear map satisfying*

$$\delta_n([[x,y],z]) = \sum_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$$

for all $x, y, z \in \mathcal{T}$ *with* $xyz = 0$. *Then, for every* $n \in \mathbb{N}$,

$$\delta_n(x) = \chi_n(x) + f_n(x)$$

for all $x \in \mathcal{T}$, *where a sequence* $\Upsilon = \{\chi_n\}_{n\in\mathbb{N}}$ *of additive mapping* $\chi_n : \mathcal{T} \to \mathcal{T}$ *is a higher derivation, and* $f_n : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ *is a nonlinear mapping such that* $f_n([[x,y],z]) = 0$ *for any* $x, y, z \in \mathcal{T}$ *with* $xyz = 0$.

Proof. In order to obtain this theorem, we will use an induction method for the component index n. For $n = 1$, δ_1 is a Lie triple derivation on $\mathcal{T}$ by local action, by ([17], Theorem 2.2), it follows that there exists an additive derivation ϖ_1 and a nonlinear center mapping f_1

satisfying $f_1([[x,y],z]) = 0$ for any $x,y,z \in \mathcal{T}$ with $xyz = 0$ such that $\delta_1(x) = d_1(x) + f_1(x)$ for all $x \in \mathcal{T}$. Moreover, δ_1 and d_1 satisfy the following properties:

$$\mathfrak{F}_1 = \begin{cases} \delta_1(0) = 0, \delta_1(A_{ii}) \subseteq A_{ii} + A_{ij} + \mathcal{Z}(\mathcal{T}), \delta_1(A_{12}) \subseteq A_{12}, \delta_1(p_i) \subseteq A_{12} + \mathcal{Z}(\mathcal{T}); \\ d_1(A_{ii}) \subseteq A_{ii} + A_{ij}, d_1(A_{12}) \subseteq A_{12}; f_1([[x,y],z]) = 0 \end{cases}$$

for $i \leq j \in \{1,2\}$ and for any $x,y,z \in \mathcal{T}$ with $xyz = 0$.

We assume that the result holds for s for all $1 < s < n$, $n \in \mathbb{N}$. Then, there exists an additive derivation ϖ_s and a nonlinear center mapping f_s satisfying $f_s([[x,y],z]) = 0$ for any $x,y,z \in \mathcal{T}$ with $xyz = 0$ such that $\delta_s(x) = d_s(x) + f_s(x)$ for all $x \in \mathcal{T}$. Moreover, δ_s and d_s satisfy the following properties:

$$\mathfrak{F}_s = \begin{cases} \delta_s(0) = 0, \delta_s(A_{ii}) \subseteq A_{ii} + A_{ij} + \mathcal{Z}(\mathcal{T}), \delta_s(A_{12}) \subseteq A_{12}, \delta_s(p_i) \subseteq A_{12} + \mathcal{Z}(\mathcal{T}); \\ d_s(A_{ii}) \subseteq A_{ii} + A_{ij}, d_s(A_{12}) \subseteq A_{12}; f_s([[x,y],z]) = 0 \end{cases}$$

for $i \leq j \in \{1,2\}$ and for any $x,y,z \in \mathcal{T}$ with $xyz = 0$.

The induction process can be realized through a series of lemmas. □

Claim 6: With notations as above, we have

(i) $\delta_n(A_{12}) \subseteq A_{12}$;
(ii) $p_1\delta_n(p_1)p_1 \oplus p_2\delta_n(p_1)p_2 \in \mathcal{Z}(\mathcal{T})$ and $p_1\delta_n(p_2)p_1 \oplus p_2\delta_n(p_2)p_2 \in \mathcal{Z}(\mathcal{T})$;
(iii) $\delta_n(p_1) \in M_{12} + \mathcal{Z}(\mathcal{T})$.

In fact, it is clear that $m_{12}p_1p_1 = 0$ for $m_{12} \in A_{12}$, according to the proof of ([17], Claim 7), we know that $\delta_1(A_{12}) \subseteq A_{12}$.

Because of $m_{12}p_1p_1 = 0$ for $m_{12} \in A_{12}$, with the help of condition $\mathfrak{F}_s$ for all $1 < s < n$, we have

$$\begin{aligned} \delta_n(m_{12}) &= \delta_n([[m_{12}, p_1], p_1]) \\ &= \sum_{i+j+k=n} [[\delta_i(m_{12}), \delta_j(p_1)], \delta_k(p_1)] \\ &= \sum_{i+j+k=n, 0 \leq i,j,k<n} [[\delta_i(m_{12}), \delta_j(p_1)], \delta_k(p_1)] + [[\delta_n(m_{12}), p_1], p_1] + [[m_{12}, \delta_n(p_1)], p_1] + [[m_{12}, p_1], \delta_n(p_1)] \\ &= [[\delta_n(m_{12}), p_1], p_1] + [[m_{12}, \delta_n(p_1)], p_1] + [[m_{12}, p_1], \delta_n(p_1)] \\ &= p_1\delta_n(m_{12})p_2 + p_1\delta_n(p_1)m_{12} + \delta_n(p_1)m_{12} - 2m_{12}\delta_n(p_1) \end{aligned}$$

and then we can obtain that $\delta_n(m_{12}) \in A_{12}$. Multiplying by p_1 on the left side and p_2 on the right side of the above equation, we can obtain that $p_1\delta_n(p_1)m_{12} = m_{12}\delta_n(p_1)p_2$ for all $m_{12} \in A_{12}$. It follows from the definition of center that

$$p_1\delta_n(p_1)p_1 \oplus p_2\delta_n(p_1)p_2 \in \mathcal{Z}(\mathcal{T}).$$

Because of $p_2m_{12}p_1 = 0$, adopt the same discussion as relations $\delta_n(m_{12}) = \delta_n([[p_2, m_{12}], p_1])$, we can prove that $p_1\delta_n(p_2)p_1 \oplus p_2\delta_n(p_2)p_2 \in \mathcal{Z}(\mathcal{T})$ holds.

Claim 7: With notations as above, we have

(i) $\delta_n(a_{11}) \in A_{11} + A_{12} + \mathcal{Z}(\mathcal{T})$, where $p_2\delta_n(a_{11})p_2 \in \mathcal{Z}(A_{11})$;
(ii) $\delta_n(a_{22}) \in A_{22} + A_{12} + \mathcal{Z}(\mathcal{T})$, where $p_1\delta_n(a_{22})p_1 \in \mathcal{Z}(A_{22})$

for all $a_{ii} \in A_{ii}$ with $i \in \{1,2\}$.

In fact, it is clear that $a_{22}a_{11}a_{12} = 0$ for all $a_{ij} \in A_{ij}$ for all $i, j \in \{1, 2\}$. Then, according to condition $\mathfrak{F}_s$, we have

$$\begin{aligned}0 =&\delta_n([[a_{22}, a_{11}], a_{12}]) = \sum_{i+j+k=n} [[\delta_i(a_{22}), \delta_j(a_{11})], \delta_k(a_{12})] \\ =& \sum_{i+j+k=n, 0\le i,j,k<n} [[\delta_i(a_{22}), \delta_j(a_{11})], \delta_k(a_{12})] + [[\delta_n(a_{22}), a_{11}], a_{12}] + [[a_{22}, \delta_n(a_{11})], a_{12}] \\ =&[[\delta_n(a_{22}), a_{11}], a_{12}] + [[a_{22}, \delta_n(a_{11})], a_{12}] \\ =&[[p_1\delta_n(a_{22})p_1, a_{11}], a_{12}] + [[a_{22}, p_2\delta_n(a_{11})p_2], a_{12}]\end{aligned}$$

for all $a_{ij} \in A_{ij}$ for all $i \le j \in \{1, 2\}$. Furthermore, we obtain

$$[p_1\delta_n(a_{22})p_1, a_{11}] \oplus [a_{22}, p_2\delta_n(a_{11})p_2] \in \mathcal{Z}(\mathcal{T})$$

for all $a_{ii} \in A_{ii}$ for all $i \in \{1, 2\}$. With the help of assumption (ii), we have

$$[p_1\delta_n(a_{22})p_1, a_{11}] \in \mathcal{Z}(A_{11});\ [a_{22}, p_2\delta_n(a_{11})p_2] \in \mathcal{Z}(A_{22})$$

and then

$$p_1\delta_n(a_{22})p_1 \in \mathcal{Z}(A_{11});\ p_2\delta_n(a_{11})p_2 \in \mathcal{Z}(A_{22})$$

for all $a_{ii} \in A_{ii}$ for all $i \in \{1, 2\}$. Furthermore, we have

$$\begin{aligned}\delta_n(a_{11}) = &\, p_1\delta_n(a_{11})p_1 - \tau^{-1}(p_2\delta_n(a_{11})p_2) + p_1\delta_n(a_{11})p_2 \\ &+ \tau^{-1}(p_2\delta_n(a_{11})p_2) + p_2\delta_n(a_{11})p_2 \in A_{11} + A_{12} + \mathcal{Z}(\mathcal{T})\end{aligned}$$

and

$$\begin{aligned}\delta_n(a_{22}) = &\, p_2\delta_n(a_{22})p_2 - \tau(p_1\delta_n(a_{22})p_1) + p_1\delta_n(a_{22})p_2 \\ &+ p_1\delta_n(a_{22})p_1 + \tau^{-1}(p_1\delta_n(a_{22})p_1) \in A_{22} + A_{12} + \mathcal{Z}(\mathcal{T})\end{aligned}$$

for all $a_{ij} \in A_{ij}$ with $i \le j \in \{1, 2\}$. Then, we can conclude that this claim can be established.

Now, we define mapping $f_{n1}(a_{11}) = \tau^{-1}(p_2\delta_n(a_{11})p_2) + p_2\delta_n(a_{11})p_2$ and $f_{n2}(a_{22}) = p_1\delta_n(a_{22})p_1 + \tau(p_1\delta_n(a_{22})p_1)$ for all $a_{11} \in A_{11}$ and $a_{22} \in A_{22}$. It follows from Claim 7 that $f_{n1} : A_{11} \to \mathcal{Z}(A_{11})$ such that $f_{n1}([[a_{11}, b_{11}], c_{11}]) = 0$ for all $a_{11}, b_{11}, c_{11} \in A_{11}$ with $a_{11}b_{11}c_{11} = 0$ and $f_{n2} : A_{22} \to \mathcal{Z}(A_{22})$ such that $f_{n2}([[a_{22}, b_{22}], c_{22}]) = 0$ for all $a_{22}, b_{22}, c_{22} \in A_{22}$ with $a_{22}b_{22}c_{22} = 0$. Now set

$$\begin{aligned}f_n(x) = f_{n1}(a_{11}) + f_{n2}(a_{22}) = &\,\tau^{-1}(p_2\delta_n(a_{11})p_2) + p_2\delta_n(a_{11})p_2 \\ &+ p_1\delta_n(a_{22})p_1 + \tau^{-1}(p_1\delta_n(a_{22})p_1)\end{aligned} \tag{10}$$

for all $x = a_{11} + a_{12} + a_{22} \in \mathcal{T}$. It is clear that $f_n(x) \in \mathcal{Z}(\mathcal{T})$ and $f_n([[x, y], z]) = 0$ with $xyz = 0$ for all $x, y, z \in \mathcal{T}$. Define a new mapping

$$\varpi_n(x) = \delta_n(x) - f_n(x) \tag{11}$$

for all $x \in \mathcal{T}$.

Taking into account Claim 6 and Claim 7 and together with Equations (10) and (11), we can easily obtain the following Claim 8.

Claim 8: With notations as above, we have

(1) $\varpi_n(0) = 0$, $\varpi_n(a_{12}) = \delta_n(a_{12}) \in A_{12}$
(2) $\varpi_n(a_{11}) \in A_{11} + A_{12}$ and $\varpi_n(a_{22}) \in A_{22} + A_{12}$,

for all $a_{ij} \in A_{ij}$ with $i \le j \in \{1, 2\}$.

Claim 9: With notations as above, we have

(i) $\varpi_n(a_{11}a_{12}) = a_{11}\varpi_n(a_{12}) + \varpi_n(a_{11})a_{12} + \sum_{i+j=n, 0<i,j<n} d_i(a_{11})d_j(a_{12})$;
(ii) $\varpi_n(a_{12}a_{22}) = a_{12}\varpi_n(a_{22}) + \varpi_n(a_{12})a_{22} + \sum_{i+j=n, 0<i,j<n} d_i(a_{12})d_j(a_{22})$

for all $a_{ij} \in A_{ij}$ with $i \leq j \in \{1,2\}$.

Now, we only prove the conclusion (i), The conclusion (ii) can be proved by similar methods. It follows from $a_{12}a_{11}p_1 = 0$ and the induction hypothesis $\mathfrak{F}_s$ for all $1 \leq s \leq n-1$ that

$$\begin{aligned}
\varpi_n(a_{11}a_{12}) &= \delta_n(a_{11}a_{12}) = \delta_n([[a_{12}, a_{11}], p_1]) \\
&= \sum_{i+j+k=n} [[\delta_i(a_{12}), \delta_j(a_{11})], \delta_k(p_1)] \\
&= [[\delta_n(a_{12}), a_{11}], p_1] + [[a_{12}, \delta_n(a_{11})], p_1] + [[a_{12}, a_{11}], \delta_n(p_1)] \\
&+ \sum_{i+j+k=n, 0\leq i,j,k<n} [[\delta_i(a_{12}), \delta_j(a_{11})], \delta_k(p_1)] \\
&= a_{11}\varpi_n(a_{12}) + \varpi_n(a_{11})a_{12} + \sum_{i+j+k=n, 0\leq i,j,k<n} [[d_i(a_{12}), d_j(a_{11})], d_k(p_1)] \\
&= a_{11}\varpi_n(a_{12}) + \varpi_n(a_{11})a_{12} + \sum_{i+j=n, 0<i,j<n} [[d_i(a_{12}), d_j(a_{11})], p_1] \\
&= a_{11}\varpi_n(a_{12}) + \varpi_n(a_{11})a_{12} + \sum_{i+jk=n, 0<i,j<n} d_j(a_{11})d_i(a_{12})
\end{aligned}$$

for all $a_{st} \in A_{st}$ with $s \leq t \in \{1,2\}$.

Adopting the same discussion as relations $\varpi_n(a_{12}a_{22}) = \delta_n(a_{12}a_{22}) = \delta_n([[a_{22}, a_{12}], p_1])$ with $a_{22}a_{12}p_1 = 0$, we can prove

$$\varpi_n(a_{12}a_{22}) = a_{12}\varpi_n(a_{22}) + \varpi_n(a_{12})a_{22} + \sum_{i+j=n, 0<i,j<n} d_i(a_{12})d_j(a_{22})$$

for all $a_{st} \in A_{st}$ with $s \leq t \in \{1,2\}$.

Claim 10: With notations as above, we have

(i) $\varpi_n(a_{11}a'_{11}) = \varpi_n(a_{11})a'_{11} + a_{11}\varpi_n(a'_{11})p_2 + \sum_{i+j=n, 0<i,j<n} d_i(a_{11})d_j(a'_{11})$;

(ii) $\varpi_n(b_{22}b'_{22}) = \varpi_n(b_{22})b'_{22} + b_{22}\varpi_n(b'_{22})p_2 + \sum_{i+j=n, 0<i,j<n} d_i(b_{22})d_j(b'_{22})$

for all $a_{ii}, a'_{ii} \in A_{ii}$ with $i \in \{1,2\}$.

For conclusion (i), for arbitrary $a_{11}, a'_{11} \in A_{11}$ and $a_{12} \in A_{12}$, by conclusion (i) in Claim 9, we have

$$\begin{aligned}
\varpi_n(a_{11}a'_{11}a_{12}) &= a_{11}a'_{11}\varpi_n(a_{12}) + \varpi_n(a_{11}a'_{11})a_{12} \\
&+ \sum_{i+j=n, 0<i,j<n} d_i(a_{11}a'_{11})d_j(a_{12}) \\
&= a_{11}a'_{11}\varpi_n(a_{12}) + \varpi_n(a_{11}a'_{11})a_{12} \\
&+ \sum_{i+j=n, 0<i,j<n} (\sum_{i_1+i_2=i, 0<i<n} d_i(a_{11})d_i(a'_{11}))d_j(a_{12}) \\
&= a_{11}a'_{11}\varpi_n(a_{12}) + \varpi_n(a_{11}a'_{11})a_{12} \\
&+ \sum_{i_1+i_2+j=n, 0<i_1,i_2,j<n} d_i(a_{11})d_i(a'_{11})d_j(a_{12})
\end{aligned} \tag{12}$$

and

$$
\begin{aligned}
\varpi_n(a_{11}a'_{11}a_{12}) &= a_{11}\varpi_n(a'_{11}a_{12}) + \varpi_n(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=n,0<i,j<n} d_i(a_{11})d_j(a'_{11}a_{12}) \\
&= a_{11}a'_{11}\varpi_n(a_{12}) + a_{11}\varpi_n(a'_{11})a_{12} + \varpi_n(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=n,0<i,j<n} a_{11}d_i(a'_{11})d_j(a_{12}) + \sum_{i+j=n,0<i,j<n} d_i(a_{11})d_j(a'_{11}a_{12}) \\
&= a_{11}a'_{11}\varpi_n(a_{12}) + a_{11}\varpi_n(a'_{11})a_{12} + \varpi_n(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=n,0<i,j<n} a_{11}d_i(a'_{11})d_j(a_{12}) + \sum_{i+j=n,0<i,j<n} d_i(a_{11})(\sum_{j_1+j_2=j,0<j<n} d_{j_1}(a'_{11})d_{j_2}(a_{12})) \qquad (13) \\
&= a_{11}a'_{11}\varpi_n(a_{12}) + a_{11}\varpi_n(a'_{11})a_{12} + \varpi_n(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=n,0<i,j<n} a_{11}d_i(a'_{11})d_j(a_{12}) + \sum_{i+j_1+j_2=n,0<i,j_1,j_2<n} d_i(a_{11})d_{j_1}(a'_{11})d_{j_2}(a_{12}) \\
&= a_{11}a'_{11}\varpi_n(a_{12}) + a_{11}\varpi_n(a'_{11})a_{12} + \varpi_n(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=n,0<i,j<n} d_i(a_{11})d_j(a'_{11})a_{12} + \sum_{i+j_1+j_2=n,0<i,j_1,j_2<n} d_i(a_{11})d_{j_1}(a'_{11})d_{j_2}(a_{12})
\end{aligned}
$$

for all $a_{tt}, a'_{tt} \in A_{tt}$ with $t \in \{1,2\}$.

Combining Equation (12) with Equation (13) leads to

$$\varpi_n(a_{11}a'_{11})a_{12} = (\varpi_n(a_{11})a'_{11} + a_{11}\varpi_n(a'_{11}) + \sum_{i+j=n,0<i,j<n} d_i(a_{11})d_j(a'_{11}))a_{12}$$

for all $a_{tt}, a'_{tt} \in A_{tt}$ with $t \in \{1,2\}$.

Since $\varpi_n(A_{11}) \subseteq A_{11} + A_{12}$ and A_{12} is faithful as a left A_{11}-module, the above relation implies that

$$\varpi_n(a_{11}a'_{11})p_1 = \{\varpi_n(a_{11})a'_{11} + a_{11}\varpi_n(a'_{11}) + \sum_{i+j=n,0<i,j<n} d_i(a_{11})d_j(a'_{11})\}p_1 \qquad (14)$$

for all $a_{11}, a'_{11} \in A_{11}$.

On the other hand, by $a_{11}p_2p_2 = 0$ for all $a_{11} \in A_{11}$, we arrive at

$$
\begin{aligned}
0 = \delta_n([[a_{11}, p_2], p_2]) &= [[\delta_n(a_{11}), p_2], p_2] + [[a_{11}, \delta_n(p_2)], p_2] \\
&+ \sum_{i+j+k=n,0\leq i,j,k<n} [[\delta_i(a_{11}), \delta_j(p_2)], \delta_k(p_2)] \\
&= [[\varpi_n(a_{11}), p_2], p_2] + [[a_{11}, \varpi_n(p_2)], p_2] \\
&+ \sum_{i+j+k=n,0\leq i,j,k<n} [[d_i(a_{11}), d_j(p_2)], d_k(p_2)]
\end{aligned}
$$

for all $a_{11}, a'_{11} \in A_{11}$.

Since $\varpi_n(A_{11}) \subseteq A_{11} + A_{12}, \varpi_n(p_2) \in A_{12}$ and $d_i(p_2) \in A_{12}$, the above equation implies that

$$0 = \varpi_n(a_{11})p_2 + a_{11}\varpi_n(p_2) + \sum_{i+j=n,0\leq i,j<n} d_i(A_{11})d_j(p_2)$$

for all $a_{11}, a'_{11} \in A_{11}$. On substituting a_{11} by $a_{11}a'_{11}$ in the above equation, we obtain

$$\begin{aligned}
0 &= \varpi_n(a_{11}a'_{11})p_2 + a_{11}a'_{11}\varpi_n(p_2) + \sum_{i+j=n,0\leq i,j<n} d_i(a_{11}a'_{11})d_j(p_2) \\
&= \varpi_n(a_{11}a'_{11})p_2 + a_{11}a'_{11}\varpi_n(p_2) \\
&+ \sum_{i+j=n,0\leq i,j<n} (\sum_{i_1+i_2=i,0\leq i_1,i_2<n} d_{i_1}(a_{11})d_{i_2}(a'_{11}))d_j(p_2) \\
&= \varpi_n(a_{11}a'_{11})p_2 + a_{11}a'_{11}\varpi_n(p_2) \\
&+ \sum_{i_1+i_2+j=n,0\leq i_1,i_2,j<n} d_{i_1}(a_{11})d_{i_2}(a'_{11})d_j(p_2)
\end{aligned}$$

for all $a_{11}, a'_{11} \in A_{11}$. Therefore, we have

$$\begin{aligned}
&p_1(\varpi_n(a_{11}a'_{11})p_2 + a_{11}a'_{11}\varpi_n(p_2) \\
&+ \sum_{i_1+i_2+j=n,0\leq i_1,i_2,j<n} d_{i_1}(a_{11})d_{i_2}(a'_{11})d_j(p_2))p_2 = 0
\end{aligned} \tag{15}$$

Again, note that $a'_{11}p_2p_2 = 0$ for all $a'_{11} \in A_{11}$, we have

$$\begin{aligned}
0 = \delta_n([[a'_{11}, p_2], p_2]) &= \sum_{i+j+k=n} [[\delta_i(a'_{11}), \delta_j(p_2)], \delta_k(p_2)] = [[\varpi_n(a'_{11}), p_2], p_2] + [[a'_{11}, \varpi_n(p_2)], p_2] \\
&+ \sum_{i+j+k=n,0\leq i,j,k<n} [[d_i(a'_{11}), d_j(p_2)], d_k(p_2)] \\
&= [[\varpi_n(a'_{11}), p_2], p_2] + [[a'_{11}, \varpi_n(p_2)], p_2] \\
&+ \sum_{i+j+k=n,0\leq i,j,k<n} [[d_i(a'_{11}), d_j(p_2)], d_k(p_2)].
\end{aligned}$$

This gives us

$$0 = \varpi_n(a'_{11})p_2 + a'_{11}\varpi_n(p_2) + \sum_{i+j=n,0\leq i,j<n} d_p(a'_{11})d_q(p_2) \tag{16}$$

Now left multiplying a_{11} in Equation (16) and combining it with Equation (15) gives

$$\varpi_n(a_{11}a'_{11})p_2 + \sum_{i+j+k=n,0\leq i,0<j} d_i(a_{11})d_j(a'_{11})d_j(p_2) = a_{11}\varpi_n(a'_{11})p_2.$$

This implies that

$$\varpi_n(a_{11}a'_{11})p_2 + \sum_{i=1}^{n} d_i(a_{11}) \sum_{j+k=n-i,0\leq i} d_j(a'_{11})d_j(p_2) = a_{11}\varpi_n(a'_{11})p_2.$$

Now, using the condition $\mathfrak{F}_s$, we find that

$$\varpi_n(a_{11}a'_{11})p_2 - \sum_{i=1}^{n-1} d_i(A_{11})d_{n-i}(A'_{11})p_2 = a_{11}\varpi_n(a'_{11})p_2,$$

which gives

$$\varpi_n(a_{11}a'_{11})p_2 = a_{11}\varpi_n(a'_{11})p_2 + \sum_{i=1}^{n-1} d_i(A_{11})d_{n-i}(A'_{11})p_2.$$

Hence,

$$\varpi_n(a_{11}a'_{11})p_2 = \{\varpi_n(a_{11})a'_{11} + a_{11}\varpi_n(a'_{11})p_2 + \sum_{i+j=n,0<i,j<n} d_i(a_{11})d_{n-i}(a'_{11})\}p_2. \tag{17}$$

Now, adding Equations (14) and (17), we have

$$\varpi_n(a_{11}a'_{11}) = \varpi_n(a_{11})a'_{11} + a_{11}\varpi_n(a'_{11})p_2 + \sum_{i+j=n,0<i,j<n} d_i(a_{11})d_j(a'_{11}).$$

Adopting the same discussion, we have

$$\varpi_n(b_{22}b'_{22}) = \varpi_n(b_{22})b'_{22} + b_{22}\varpi_n(b'_{22})p_2 + \sum_{i+j=n,0<i,j<n} d_i(b_{22})d_j(b'_{22})$$

for all $b_{22}, b'_{22} \in A_{22}$.

Remark 1. *Now, we establish a mapping* $g_n : T \to \mathcal{Z}(\mathcal{T})$ *by*

$$g_n(x) = \varpi_n(x) - \varpi_n(p_1xp_1) - \varpi_n(p_1xp_2) - \varpi_n(p_2xp_2)$$

and $g_n([[x,y],z]) = 0$ *with* $xyz = 0$ *for all* $x,y,z \in \mathcal{T}$. *Then, define a mapping* $\chi_n(x) = \varpi_n(x) - g_n(x)$ *for all* $x \in \mathcal{T}$. *It is easy to verify that*

$$\chi_n(a_{11} + a_{12} + a_{22}) = \chi_n(a_{11}) + \chi_n(a_{12}) + \chi_n(a_{22}).$$

From the definition of χ_n *and* g_n, *we find that*

$$\varphi_n(x) = \varpi_n(x) + f_n(x) = \chi_n(x) + g_n(x) + f_n(x) = \chi_n(x) + h_n(x),$$

where $h_n(x) = g_n(x) + f_n(x)$ *for all* $x \in \mathcal{T}$.

Claim 11: With notations as above, we obtain that $\{\chi_n\}_{i=0}^{i=n}$ is an additive higher derivation on triangular algebras $\mathcal{T}$.

Suppose that $x, y \in \mathcal{T}$ such that $x = a_{11} + a_{12} + a_{22}$ and $y = a'_{11} + a'_{12} + a'_{22}$ where $a_{ij}, a'_{ij} \in A_{ij}$ with $i \leqslant j \in \{1,2\}$. Then,

$$\begin{aligned}
\chi_n(x+y) &= \chi_n((a_{11} + a_{12} + a_{22}) + (a'_{11} + a'_{12} + a'_{22})) \\
&= \chi_n((a_{11} + a'_{11}) + (a_{12} + a'_{12}) + (a_{22} + a'_{22})) \\
&= \varpi_n(a_{11} + a'_{11}) + \varpi_n(a_{12} + a'_{12}) + \varpi_n(a_{22} + a'_{22}) \\
&= \varpi_n(a_{11}) + \varpi_n(a'_{11}) + \varpi_n(a_{12}) + \varpi_n(a'_{12}) + \varpi_n(a_{22}) + \varpi_n(a'_{22}) \\
&= \chi_n(a_{11} + a_{12} + a_{22}) + \chi_n(a'_{11} + a'_{12} + a'_{22})) \\
&= \chi_n(x) + \chi_n(y).
\end{aligned}$$

By Claims 4 and 5, we have

$$\begin{aligned}
\chi_n(xy) &= \chi_n((a_{11} + a_{12} + a_{22})(a'_{11} + a'_{12} + a'_{22})) \\
&= \chi_n(a_{11}a'_{11} + a_{11}a'_{12} + a_{12}a'_{22} + a_{22}a'_{22}) \\
&= \varpi_n(a_{11})a'_{11} + a_{11}\varpi_n(a'_{11}) + \sum_{i+j=n,0<i<n} d_i(a_{11})d_j(a'_{11}) \\
&+ \varpi_n(a_{11})a'_{12} + a_{11}\varpi_n(a'_{12}) + \sum_{i+j=n,0<i<n} d_i(a_{11})d_j(a'_{12}) \\
&+ \varpi_n(a_{12})a'_{22} + a_{12}\varpi_n(a'_{22}) + \sum_{i+j=n,0<i<n} d_i(a_{12})d_j(a'_{22}) \\
&+ \varpi_n(a_{22})a'_{22} + a_{22}\varpi_n(a'_{22}) + \sum_{i+j=n,0<i<n} d_i(a_{22})d_j(a'_{22}).
\end{aligned} \tag{18}$$

On the other hand, we have

$$\begin{aligned}
&\chi_n(x)y + x\chi_n(y) + \sum_{i+j=n,0<i<n} \chi_i(x)\chi_j(y) \\
&= \chi_n(a_{11}+a_{12}+a_{22})y + x\chi_n(a'_{11}+a'_{12}+a'_{22}) + \sum_{i+j=n,0<i<n} \chi_i(x)\chi_j(y) \\
&= (\varpi_n(a_{11})+\varpi_n(a_{12})+\varpi_n(a_{22}))y + \sum_{i+j=n,0<p<n} d_i(a_{11})d_i(a'_{11}) \\
&+ x(\varpi_n(a'_{11})+\varpi_n(a'_{12})+\varpi_n(a'_{22})) + \sum_{i+j=n,0<i<n} d_i(a_{11})d_i(a'_{12}) + \sum_{i+j=n,0<i<n} d_i(a_{11})d_i(a'_{22}) \\
&+ \sum_{i+j=n,0<i<n} d_i(a_{12})d_i(a'_{11}) + \sum_{i+j=n,0<i<n} d_i(a_{12})d_i(a'_{12}) + \sum_{i+j=n,0<p<n} d_i(a_{12})d_i(a'_{22}) \\
&+ \sum_{i+j=n,0<i<n} d_i(a_{22})d_i(a'_{11}) + \sum_{i+j=n,0<i<n} d_i(a_{22})d_i(a'_{12}) + \sum_{i+j=n,0<p<n} d_i(a_{22})d_i(a'_{22}).
\end{aligned}$$

Taking into account the induction hypothesis $\mathfrak{C}_s$ and claim 8–10, we calculate that

$$\begin{aligned}
&\chi_n(x)y + x\chi_n(y) + \sum_{i+j=n,0<i<n} d_i(x)d_j(y) \\
&= \varpi_n(a_{11})a'_{11} + \varpi_n(a_{11})a'_{12} + \varpi_n(a_{12})a'_{22} + \varpi_n(a_{22})a'_{22} \\
&+ a_{11}\varpi_n(a'_{11}) + a_{11}\varpi_n(a'_{12}) + a_{12}\varpi_n(a'_{22}) + a_{22}\varpi_n(a'_{22}) \\
&+ \sum_{i+j=n,0<i<n} d_i(a_{11})d_j(a'_{11}) + \sum_{i+j=n,0<i<n} d_i(a_{11})d_j(a'_{12}) \\
&+ \sum_{i+j=n,0<i<n} d_i(a_{12})d_j(a'_{22}) + \sum_{i+j=n,0<i<n} d_i(a_{22})d_j(a'_{22}).
\end{aligned} \tag{19}$$

Combining Equations (18) and (19), we obtain

$$\chi_n(xy) = \chi_n(x)y + x\chi_n(y) + \sum_{i+j=n,0<i<n} \chi_i(x)\chi_j(y)$$

for all $x, y \in \mathcal{T}$. This shows that each χ_n satisfies the Leibniz formula of higher order on $\mathcal{T}$.

Finally, we need to prove that each h_n vanishes $[[x,y],z]$ with $xyz = 0$ for all $x, y, z \in \mathcal{T}$. Note that h_n maps into $\mathcal{Z}(\mathcal{T})$, $\{d_n\}_{i=0}^n$ is an additive higher derivation of $\mathcal{T}$. Therefore, $\{\chi_n\}_{i=0}^n$ is an additive higher derivation of $\mathcal{T}$. Therefore,

$$f_n([[x,y],z]) = \delta_n([[x,y],z]) - \chi_n([[x,y],z]) = 0$$

with $xyz = 0$ for all $x, y, z \in \mathcal{T}$. We lastly complete the proof of the main theorem.

In particular, we have the following corollary:

Corollary 1 ([17], Theorem 2.2)**.** *Let* $\mathcal{T} = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$ *be a triangular algebra satisfying*

(i) $\pi_{A_{11}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{11})$ *and* $\pi_{A_{22}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{22})$

(ii) *For any* $a_{11} \in A_{11}$*, if* $[a_{11}, A_{11}] \in \mathcal{Z}(\mathcal{A}_{11})$*, then* $a_{11} \in \mathcal{Z}(\mathcal{A}_{11})$ *or for any* $a_{22} \in A_{22}$*, if* $[a_{22}, A_{22}] \in \mathcal{Z}(\mathcal{A}_{22})$*, then* $a_{22} \in \mathcal{Z}(\mathcal{A}_{22})$*.*

Suppose $\delta_1 : \mathcal{T} \to \mathcal{T}$ *is a nonlinear map satisfying*

$$\delta_1([[x,y],z]) = \sum_{i+j+k=1} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$$

for all $x, y, z \in \mathcal{T}$ *with* $xyz = 0$*. Then, there exist an additive derivation* ϖ_1 *of* $\mathcal{T}$ *and a nonlinear map* $\tau_1 : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ *such that*

$$\delta_1(x) = \varpi_1(x) + \tau_1(x)$$

for all $x \in \mathcal{T}$*, where* $\tau_1([[x,y],z])$ *for any* $x, y, z \in \mathcal{T}$ *with* $xyz = 0$*.*

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